Stochastic Processes and their Applications 80 (1999) 103–128

Two limit theorems for queueing systems around the convergence of stochastic integrals with respect to renewal processes

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Received 22 January 1997; received in revised form 10 August 1998; accepted 4 September 1998

Abstract

Two limit theorems on asymptotic behaviors of some processes related to some queueing systems are investigated. In the first result (Theorem 1), sticky diffusions appear as limit processes for queues with vacations. In the second result (Theorem 2), limiting behavior of occupation times and counting processes related to open queueing networks is discussed. The core of the arguments for obtaining our results is to discuss the convergence of stochastic integrals with respect to renewal processes. © 1999 Published by Elsevier Science B.V. All rights reserved.

Keywords: Convergence of stochastic integrals; Renewal processes; Queueing systems with vacations; Sticky diffusion limits; Occupation time problems for open queueing networks

1. Introduction

In this paper we present two results (Theorems 1 and 2) on limiting behavior of some processes related to some type of queueing systems. In the first result (Theorem 1), we consider a single server queueing system which takes a vacation or goes under repair when the system becomes vacant, and consider to approximate such systems by reflecting or sticky diffusions. In the second result (Theorem 2), we investigate the limiting behavior of occupation times and counting processes related to open queueing networks which operates under heavy traffic condition. The two results have no direct connection. However these results have a common feature that each queueing system is based on the model in which arrival and departure processes are renewal processes and hence queueing processes are not Markovian and we must deal with the convergence problem of stochastic integrals with respect to renewal processes, and that to discuss this convergence problem constitutes a main part in the arguments for obtaining our results. For example, in Theorem 1 we discuss the convergence of the following form of stochastic integrals:

$$\mathscr{I}_n(t) = \int_0^t 1(X_n(s-) > 0)\,\mathrm{d}\frac{1}{\phi_n}\tilde{A}_n(s).$$

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PII: S0304-4149(98)00073-8

Here $(A_n(t))$ is a sequence of renewal processes (arrival processes) and $\tilde{A}_n(t)=A_n(t)-\lambda_n t$ where λ_n is the arrival rate. We have a situation $(X_n,(1/\phi_n)\tilde{A}_n)\to_{\mathscr{L}}(X,\tilde{A})$ where X is a sticky Brownian motion and $\tilde{A}$ is a Brownian motion. We want to show the convergence:

$$\mathscr{I}_n(t)\to_{\mathscr{L}}\int_0^t 1(X(s)>0)\,\mathrm{d}\tilde{A}(s).$$

Similar problems were considered in Yamada (1993, 1994) for Markov processes. In this case the stochastic integrals are martingales, and this made us to use the stochastic calculus approach as in Jacod and Shiryaev (1987). Similar problems are also discussed in, for example, Jakubowsky et al. (1989), Kurtz and Protter (1991) and Slominsky (1989). They discussed the convergence of the stochastic integrals of the type $\int_0^t Z_n(s)\,\mathrm{d}X_n(s)$ where (X_n) are semimartingales and the convergence $(Z_n,X_n)\to_{\mathscr{L}}(Z,X)$ is assumed. Because of this joint convergence assumption, their results seem not applicable to our problems. Our basic approach is, by using a typical property of renewal processes, to decompose the above integral as the sum of two processes $M_n(t)$ and $N_n(t)$, say, $\mathscr{I}_n(t)=M_n(t)+N_n(t)$ such that $M_n(t)$ is a martingale and $N_n(t)$ becomes negligible as n tends to infinity (see Lemmas 2.1 and 2.2). Thus, the convergence of $\mathscr{I}_n(t)$ is reduced to that of $M_n(t)$ and hence the stochastic calculus approach is again available. Although the device for decomposing the integral $\mathscr{I}_n(t)$ as above is very simple, it seems this decomposition has not been used in other works. We also remark that the importance of such decomposition was stressed out in the invited lecture delivered by Kurtz in the Conference on Applied Probability (1995, Atlanta), and that our decomposition method seems applicable to stochastic integrals with respect to more general processes such as regenerative processes and seems useful in lifting up Markov property from models we build. However, we treat only the stochastic integrals of the type for which, as we see in Eq. (1.1), X_n appearing in the integrand is only one dimensional. This restriction is due to the following reasons; In Example 1, to estimate occupation times of the limit process X of X_n, we use space–time formula for local time of the process X which is a one-dimensional semimartingales. In Example 2, a version of Dynkin's formula is used, but this works only when the occupation function is a function of one-dimensional variable, say, the queue length at a service station (see the definition of $\mathscr{B}_n^i(t)$ for scaled occupation time processes given just above Theorem 2).

The usual presentation of the results of the problems like ours may be to give first general results on the convergence of stochastic integrals and then to show applications to some specific problems. However, our arguments depend in some points on the special structure of the models and the author was not able to give such a presentation. At the same time it should be pointed out that our approach, though oriented to specific problems, seems applicable to other problems widely and that in spite of the importance especially in applications, convergence problems for stochastic integrals with respect to renewal processes have not been discussed fully.

We denote by $D([0,\infty),\mathbb{R}^d)$ the space of functions $f:[0,\infty)\to\mathbb{R}^d$ that are right-continuous and admit left limits, and we endow this space with Skorohod's J_1 topology. We sometimes use the simple notation D if it causes no confusion. Also "$\to_{\mathrm{P}}$"

and "$\to_{\mathscr{L}}$" denote convergence in probability and in law, respectively. Finally, for $z \in D([0,\infty),\mathbb{R}^d)$, $\Delta z(t)$ denotes the jumpsize of z at t, i.e., $\Delta z(t) = z(t) - z(t-)$.

2. Example 1 – sticky diffusion limits for queueing systems with vacations

We consider the following sequence of state-dependent single-server queueing systems:

$$Q_n(t) = Q_n(0) + \int_0^t 1(Q_n(s-) > 0)\,\mathrm{d}A_n(s) + V_n\left(\int_0^t 1(Q_n(s) = 0)\,\mathrm{d}s\right)$$
$$-D_n\left(\int_0^t 1(Q_n(s) > 0)\,\mathrm{d}s\right), \quad n \geqslant 1.$$

Here $Q_n(t)$ is the number of customers at the service station at time t for the nth queue. The above equation tells us the queueing system works as follows: As long as the system is not empty, the arrival stream $A_n(t)$, which represents the number of persons who come to the system up to time t, is accepted as the customer arrival to the queue. This is manifested by the stochastic intregral $\int_0^t 1(Q_n(s-) > 0)\,\mathrm{d}A_n(s)$. When the system becomes empty, it takes a vacation, and after a while, there is an arrival of a customer and this ends the vacation. Let $V_n(t)$ be a renewal process formed by i.i.d. (independent and identically distributed) sequence of vacation lengths. Then $V_n(\int_0^t 1(Q_n(s) = 0)\,\mathrm{d}s)$ represents the number of customers arriving at the end of vacation up to time t. Note that even during vacations, there are arrivals of persons due to $A_n(t)$, but they do not join the queue. The sequence of service times constitutes a renewal process $D_n(t)$ and a customer leaves the system only when it is not empty. This situation is represented by $D_n(\int_0^t 1(Q_n(s) > 0)\,\mathrm{d}s)$. $A_n(t), V_n(t)$ and $D_n(t)$ are renewal processes defined respectively as follows:

$$A_n(t) = \max\left\{k;\ \sum_{l=1}^{k} \xi_n(l) \leqslant t\right\},$$

where $(\xi_n(l))$ is a sequence of i.i.d. (independent and identically distributed) random variables and is interpreted as inter-arrival times and we let $\lambda_n = 1/E\xi_n(l)$:

$$V_n(t) = \max\left\{k;\ \sum_{l=1}^{k} v_n(l) \leqslant t\right\},$$

where $(v_n(l))$ is a sequence of i.i.d. random variables which represent the lengths of idle periods; let $v_n = 1/E\,v_n(l)$:

$$D_n(t) = \max\left\{k;\ \sum_{l=1}^{k} S_n(l) \leqslant t\right\},$$

where $(S_n(l))$ is a sequence of i.i.d. random variables representing the service times for customers; let $\mu_n = 1/ES_n(l)$.

Hereafter, we consider the situation that λ_n, μ_n and v_n tend to infinity as $n \to \infty$ and assume that the system operates under heavy traffic situation, that is

(A1) (heavy traffic condition) There exists a sequence (ϕ_n) such that $\phi_n \to \infty$ and

$$\frac{1}{\phi_n}(\lambda_n - \mu_n) \to c$$

as n tends to infinity.

Define

$$X_n(t) = \frac{1}{\phi_n} Q_n(t).$$

Our problem is to investigate the asymptotic behavior of $X_n(t)$ as n tends to infinity, and, to this end, we make further assumptions:

(A2) (1) There exists $\lambda > 0$ such that

$$\frac{\lambda_n}{(\phi_n)^2} \to \lambda > 0$$

and, for each t,

$$\frac{A_n(t)}{(\phi_n)^2} \to_{\mathrm{P}} \lambda t.$$

(2) Let

$$\chi_n(t) = \frac{1}{\phi_n} \sum_{l=1}^{[(\phi_n)^2 t]} (1 - \lambda_n \xi_n(l)),$$
$$\tilde{V}_n(t) = V_n(t) - v_n t,$$
$$\tilde{D}_n(t) = D_n(t) - \mu_n t.$$

Then

$$\left(\chi_n, \frac{1}{\sqrt{v_n}} \tilde{V}_n, \frac{1}{\phi_n} \tilde{D}_n \right) \to_{\mathscr{L}} (\chi, \tilde{V}, \tilde{D})$$

in $D([0,\infty), \mathbb{R}^3)$ where $(\chi, \tilde{D})$ is a two-dimensional Brownian motion with $\langle \chi \rangle (t) = \sigma^2 t$ and $\tilde{V}$ is a continuous process. ($\langle \chi \rangle (t)$ is a (predictable) quadratic process of the process χ.)

(3) For each nth system, arrival and departure processes for busy cycles have no common jumps with probability one. (This condition is satisfied, for example, if A_n and D_n are independent and inter-arrival times or service times have a density.) Note that condition (A2) implies

$$\left(\frac{1}{\phi_n} \tilde{A}_n, \frac{1}{\sqrt{v_n}} \tilde{V}_n, \frac{1}{\phi_n} \tilde{D}_n \right) \to_{\mathscr{L}} (\tilde{A}, \tilde{V}, \tilde{D}),$$

where $\tilde{A}(t) = \chi(\lambda t)$ and $\tilde{A}$ and $\tilde{D}$ are independent Brownian motions.

(A3) (condition for instantaneous or sticky reflection)

$$\frac{v_n}{\phi_n} \to \rho, \quad 0 < \rho \leqslant \infty, \quad \text{and} \quad \frac{\sqrt{v_n}}{\phi_n} \to v, \quad 0 \leqslant v < \infty.$$

We then have the following result:

Theorem 1. *Under assumptions* (A1)–(A3), *if* $X_n(0) \to_{\mathscr{L}} X(0)$, *then* $X_n(t) \to_{\mathscr{L}} X(t)$ *in* $D([0,\infty),\mathbb{R}^1)$ *where* $X(t)$ *is the unique solution of the following Skorohod equation*:

$$X(t) = X(0) + \int_0^t c1(X(s)>0)\,\mathrm{d}s + \int_0^t 1(X(s)>0)\,\mathrm{d}\tilde{A}(s)$$

$$-\tilde{D}\left(\int_0^t 1(X(s)>0)\,\mathrm{d}s\right) + \xi(t), \quad X(t) \geqslant 0 \tag{2.1}$$

where $\xi(t)$ *is non-decreasing with* $\xi(0)=0$, *and satisfies*

$$0 = \int_0^t 1(X(s)>0)\,\mathrm{d}\xi(s)$$

and

$$\int_0^t 1(X(s)=0)\,\mathrm{d}s = \frac{1}{\rho}\xi(t).$$

To show this theorem, we proceed as follows. $X_n(t)$ satisfies the following Skorohod equation:

$$X_n(t) = X_n(0) + Z_n(t) + \xi_n(t), \tag{2.2}$$

$$Z_n(t) = \int_0^t 1(X_n(s-)>0)\,\mathrm{d}\frac{1}{\phi_n}\tilde{A}_n(s) + \frac{1}{\phi_n}\tilde{V}_n\left(\int_0^t 1(X_n(s)=0)\,\mathrm{d}s\right)$$

$$-\frac{1}{\phi_n}\tilde{D}_n\left(\int_0^t 1(X_n(s)>0)\,\mathrm{d}s\right) + \int_0^t 1(X_n(s)>0)\frac{1}{\phi_n}(\lambda_n-\mu_n)\,\mathrm{d}s,$$

$$\xi_n(t) = \frac{v_n}{\phi_n}\int_0^t 1(X_n(s)=0)\,\mathrm{d}s, \quad n \geqslant 1.$$

$$\int_0^t 1(X_n(s)>0)\,\mathrm{d}\xi_n(s) = 0.$$

Then we prove tightness of (Z_n), which implies tightness of (Z_n, X_n, ξ_n) as well. Then we show that any weak limit (X,ξ) of (X_n,ξ_n) satisfies Eq. (2.1). In proving tightness of (Z_n) and identifying the weak limit (X,ξ) as is shown in Theorem 1, the main part of the discussion consists in showing the tightness of the stochastic integrals $(\mathscr{I}_n)$ with respect to renewal processes:

$$\mathscr{I}_n(t) = \int_0^t 1(X_n(s-)>0)\,\mathrm{d}\frac{1}{\phi_n}\tilde{A}_n(s)$$

and showing the convergence:

$$\mathscr{I}_n(t) \to_{\mathscr{L}} \int_0^t 1(X(s)>0)\,\mathrm{d}\tilde{A}(s) \quad \text{in } D.$$

The basic fact for this is that we can decompose the process $\mathscr{I}_n$ as the sum of a stochastic integral, which is a martingale, and a process which becomes negligible as n tends to infinity. This will be shown in the next Lemma 2.1.

Remark 1. As for the uniqueness in law sense of the solution of Eq. (2.1), see Ikeda and Watanabe (1981), (Theorem IV.7.2). When $\rho=\infty$, $X(t)$ is a reflecting Brownian motion, i.e., $\int_0^\infty 1(X(s)=0)\,\mathrm{d}s=0$; when $\rho<\infty$, $X(t)$ is a sticky Brownian motion; i.e. the boundary 0 is a sticky point and $\int_0^\infty 1(X(s)=0)\,\mathrm{d}s>0$.

Remark 2. According to assumption (A3), $\mathscr{I}_n$ and the process $(1/\phi_n)\tilde{D}_n(\int_0^t 1(X_n(s)>0)\,\mathrm{d}s)$ have no common jumps.

Let us define a stopping time for any process $z\in D([0,\infty),R^1)$ by

$$\tau_z(t)=\inf\{s, s\geqslant t; \Delta z(s)\neq 0\}.$$

Then, we have

Lemma 2.1. *We write $\mathscr{I}_n(t)$ as*

$$\mathscr{I}_n(t)=M_n(t)-N_n(t)$$

where

$$M_n(t)=\int_0^{\tau_{A_n}(t)} 1(X_n(s-)>0)\,\mathrm{d}\tilde{A}_n(s)/\phi_n,$$

$$N_n(t)=\int_t^{\tau_{A_n}(t)} 1(X_n(s-)>0)\,\mathrm{d}\tilde{A}_n(s)/\phi_n.$$

Then,
(1) *$M_n(t)$ is a martingale with respect to the filtration $\mathscr{F}_n(t)$ which is defined as*

$$\begin{aligned}\mathscr{F}_n(t)=\sigma\Bigg(&Q_n(0),\ X_n(s),\ 0\leqslant s\leqslant t,\ \ \xi_n(l),\ \ 1\leqslant l\leqslant A_n(t)+1,\\ &S_n(l),\ 1\leqslant l\leqslant D_n\left(\int_0^t 1(X_n(s)>0)\,\mathrm{d}s\right)+1,\\ &v_n(l),\quad 1\leqslant l\leqslant V_n\left(\int_0^t 1(X_n(s)=0)\,\mathrm{d}s\right)+1\Bigg).\end{aligned}$$

(2) $\sup_{0\leqslant t\leqslant T}|N_n(t)|\to_{\mathrm{P}} 0$ *as n tends to infinity for an arbitrary T.*

Proof. (1) First we note that

$$\tilde{A}_n(\tau_{A_n}(t))=A_n(\tau_{A_n}(t))-\lambda_n\tau_{A_n}(t)$$

is an $\mathscr{F}_n(t)$ martingale. Indeed,

$$\tilde{A}_n(\tau_{A_n}(t)) = A_n(t) + 1 - \lambda_n \sum_{l=1}^{A_n(t)+1} \xi_n(l)$$
$$= \sum_{l=1}^{A_n(t)+1} (1 - \lambda_n \xi_n(l)) \tag{2.3}$$

and it suffices to note that $A_n(t) + 1$ is an $\mathscr{F}_n(t)$ stopping time (see Ross, 1983, Ch. 3,3.3, p. 60) and $\{1 - \lambda_n \xi(l)\}_{l \geqslant 1}$ is an i.i.d. sequence. Then $M_n(t)$ is an $\mathscr{F}_n(t)$ martingale since $M_n(t)$ is a stochastic integral with respect to a martingale $\tilde{A}_n(\tau_{A_n}(t))$:

$$M_n(t) = \int_0^t 1(X_n(\tau_{A_n}(s)-) > 0)\, \mathrm{d}\tilde{A}_n(\tau_{A_n}(s))/\phi_n.$$

(2) We have

$$\sup_{0 \leqslant t \leqslant T} |N_n(t)| = \sup_{0 \leqslant t \leqslant T} \left| \int_t^{\tau_{A_n}(t)} 1(X_n(s-) > 0)\, \mathrm{d}\tilde{A}_n/\phi_n(s) \right|$$
$$\leqslant \sup_{0 \leqslant t \leqslant T} \int_t^{\tau_{A_n}(t)} \mathrm{d}\|\tilde{A}_n/\phi_n\|(s) \leqslant \sup_{0 \leqslant t \leqslant T} \frac{1}{\phi_n}(1 + \lambda_n \xi_n(A_n(t) + 1))$$
$$\leqslant \sup_{0 \leqslant t \leqslant T} \left(\frac{2}{\phi_n} + \frac{1}{\phi_n} |\Delta \tilde{A}_n(\tau_{A_n}(\cdot))(t)| \right),$$

where $\|\cdot\|$ denotes the total variation of a sample path of processes. In obtaining the last inequality in the above equation, we note that

$$\Delta \tilde{A}_n(\tau_{A_n}(\cdot))(t) = 1 - \lambda_n \xi_n(A_n(t) + 1).$$

Noting Eq. (2.3) and using (A2) (1) and (2), we have

$$\frac{1}{\phi_n} \tilde{A}_n(\tau_{A_n}(t)) = \frac{1}{\phi_n} \sum_{l=1}^{A_n(t)+1} (1 - \lambda_n \xi_n(l)) \to_{\mathscr{L}} \chi(\lambda t). \tag{2.4}$$

Then the continuity of $\chi(t)$ implies

$$\sup_{0 \leqslant t \leqslant T} \frac{1}{\phi_n} |\Delta \tilde{A}_n(\tau_{A_n}(\cdot))(t)| \to_{\mathrm{P}} 0. \tag{2.5}$$

Thus

$$\sup_{0 \leqslant t \leqslant T} \int_t^{\tau_{A_n}(t)} \mathrm{d} \left\| \frac{\tilde{A}_n}{\phi_n} \right\| (s) \to_{\mathrm{P}} 0. \qquad \square$$

Remark 3. We can show, by using the same argument as in the proof of Lemma 2.1, that

$$\sup_{0 \leqslant t \leqslant T} |\tau_{A_n}(t) - t| \to_{\mathrm{P}} 0.$$

Indeed,

$$\begin{aligned}\sup_{0\leqslant t\leqslant T} |\tau_{A_n}(t)-t| &\leqslant \sup_{0\leqslant t\leqslant T} \xi_n(A_n(t)+1)\\ &= \sup_{0\leqslant t\leqslant T} \frac{1}{\lambda_n}(1-\Delta\tilde{A}_n(\tau_{A_n}(\cdot))(t))\\ &\leqslant \frac{1}{\lambda_n}+\frac{\phi_n^2}{\lambda_n}\frac{1}{\phi_n}\sup_{0\leqslant t\leqslant T}\left|\frac{1}{\phi_n}\Delta\tilde{A}_n(\tau_n^A(\cdot))(t)\right| \to_{\mathrm{P}} 0.\end{aligned}$$

where the last convergence is due to Eq. (2.5) and (A2)(1).

For an arbitrary sequence of processes (z_n) where $z_n \in D([0,\infty),\mathbb{R}^d)$, define $z_n^R(t)$ by

$$z_n^R(t)=z_n(t) \quad \text{if } t<H_n(R);\ = z_n(H_n^R-), \quad \text{if } t\geqslant H_n^R,$$

where H_n^R is defined as

$$H_n^R = \inf\{t;|\Delta(1/\phi_n)\tilde{A}_n(\tau_{A_n}(\cdot))(t)|>R\}.$$

We note that by Eq. (2.5), $H_n(R)\to_{\mathrm{P}}\infty$ as $n\to\infty$.

Lemma 2.2. *We have the following results*:
(1) $(X_n^R(t),M_n^R(t))$ *is tight in* $D([0,\infty),\mathbb{R}^2)$
(2) *Let* $(X_R,\mathscr{M}_R)$ *be any weak limit of* (X_n^R,M_n^R). *Then*

$$\mathscr{M}_R(t)=\int_0^t 1(X_R(s)>0)\,\mathrm{d}\tilde{A}(s).$$

Proof of (1). We will show that (M_n^R) is C-tight and (X_n^R) is tight. The tightness of (M_n^R) is a consequence of Lemma 2.3, which will be given at the end of this section, and we will show that the following conditions of Lemma 2.3 hold:
(a) $[M_n^R]$ is C-tight, where $[\cdot]$ expresses the optional quadratic process (see Jacod and Shiryaev, 1987, Ch. I, Section 4e).
(b) $E\sup_{0\leqslant t\leqslant T}\Delta[M_n^R](t)\to 0$.

Proof of (*a*): We note that $M_n^R(t)$ can be written as

$$M_n^R(t)=\int_0^t 1(X_n^R(\tau_{A_n}(s)-)>0)1(s<H_n(R))\,\mathrm{d}\frac{1}{\phi_n}\tilde{A}_n(\tau_{A_n}(s)).$$

We then have

$$[M_n^R](t)=\int_0^t 1(X_n^R(\tau_{A_n}(s)-)>0)1(s<H_n(R))\,\mathrm{d}\frac{1}{(\phi_n)^2}[\tilde{A}_n(\tau_{A_n}(\cdot))](s). \tag{2.6}$$

Putting

$$L_n(t)=\int_0^t 1(s<H_n(R))\,\mathrm{d}\frac{1}{\phi_n}\tilde{A}_n(\tau_{A_n}(s))$$

we have

$$[M_n^R](t) - [M_n^R](s) \leqslant \int_s^t 1(u < H_n(R))\,\mathrm{d}\frac{1}{\phi_n^2}[\tilde{A}_n(\tau_{A_n}(\cdot))](u)$$
$$= [L_n](t) - [L_n](s).$$

But, since $H_n(R) \to \infty$ as n tends to ∞, $L_n \to_{\mathscr{L}} \tilde{A}$. Moreover, $|\Delta L_n(t)| \leqslant R$ identically. Thus, $[L_n] \to_{\mathscr{L}} [\tilde{A}]$ (see Theorem 6.1, Corollary 6.6 in Jacod and Shiryaev, 1987). Since $[\tilde{A}] = \langle \tilde{A} \rangle$ is a continuous process, we conclude $([M_n^R])$ is C-tight.

Proof of (*b*): We have, from Eq. (2.6),

$$\Delta[M_n^R](t) = 1(X_n^R(\tau_{A_n}(t)-) > 0)1(t < H_n(R))\frac{1}{(\phi_n)^2}(1 - \lambda_n \xi_n(A_n(t)+1))^2.$$

Note that if $t < H_n(R)$,

$$\left|\Delta \frac{1}{\phi_n}\tilde{A}_n(\tau_{A_n}(\cdot))(t)\right| \leqslant R,$$

which implies

$$\left|\frac{1}{\phi_n}(1 - \lambda_n \xi_n(A_n(t)+1))\right| \leqslant R.$$

Thus, for an arbitrary $T > 0$,

$$\sup_{0 \leqslant t \leqslant T} |\Delta[M_n^R](t)| \leqslant R^2.$$

On the other hand, by (a), we have

$$\sup_{0 \leqslant t \leqslant T} \Delta[M_n^R](t) \to_{\mathrm{P}} 0.$$

Hence, we have, by the bounded convergence theorem,

$$E \sup_{0 \leqslant t \leqslant T} \Delta[M_n^R](t) \to 0.$$

Now we have proved (a) and (b), and (M_n^R) is tight. Actually it is C-tight since Eq. (2.5) implies $\sup_{0 \leqslant T} |\Delta M_n^R(t)| \to_{\mathrm{P}} 0$.

Next we prove the tightness of (X_n^R). $X_n^R(t)$ satisfies the following Skorohod equation:

$$X_n^R(t) = X_n(0) + Z_n^R(t) + \xi_n^R(t),$$

where $\xi_n^R(t)$ is increasing with $\xi_n^R(0) = 0$, and

$$\int_0^t 1(X_n^R(s) > 0)\,\mathrm{d}\xi_n^R(s) = 0.$$

Write $Z_n(t)$ as $Z_n(t) = \mathscr{I}_n(t) + \bar{Z}_n(t)$. Then

$$Z_n^R(t) = \mathscr{I}_n^R(t) + \bar{Z}_n^R(t).$$

Since

$$\mathscr{I}_n^R(t) = M_n^R(t) - N_n^R(t)$$

and since M_n^R is C-tight as we have shown, due to Lemma 2.1(2), $(\mathscr{I}_n^R)$ is C-tight. While, $(\bar{Z}_n^R)$ is clearly tight by assumption (A2). It follows that (Z_n^R) is tight. Thus (X_n^R, ξ_n^R) is also tight owing to the continuity property of the solutions of Skorohod equation (2.2) (Liptser and Shiryaev, 1989, Ch. 10, Theorem 1).

Proof of (2). For the proof of assertion (2), it suffices to show the following:
(c) $\mathscr{M}_R(t)$ is a continuous martingale,
(d) it holds that

$$\langle \mathscr{M}_R \rangle(t) = \int_0^t 1(X_R(s) > 0)\, \mathrm{d}\langle \tilde{A} \rangle(s).$$

Proof of (*c*): That $\mathscr{M}_R(t)$ is a martingale is a consequence of the fact that $M_n^R(t)$ is a martingale for each $n \geqslant 1$ and $|\Delta M_n^R(t)| \leqslant R$ identically (see Proposition 1.17, IX, in Jacod and Shiryaev, 1987). To see that $\mathscr{M}_R(t)$ is continuous, it suffices to note that it was previously shown that $\sup_{0 \leqslant t \leqslant T} |\Delta M_n^R(t)| \to_{\mathrm{P}} 0$. Thus $\sup_{0 \leqslant t \leqslant T} |\Delta \mathscr{M}_R(t)| = 0$, which implies the continuity of the path of the process $\mathscr{M}_R(t)$.

Proof of (*d*): To prove (d), we will show that the following hold:
(e) For any Borel set A in $\mathbb{R}^1$ with Lebesgue measure zero,

$$\int_0^t 1(X_R(s) \in A)\, \mathrm{d}\langle \mathscr{M}_R \rangle(s) = 0.$$

(f) We have

$$\int_0^t 1(X_R(s) > 0)\, \mathrm{d}\langle \tilde{A} \rangle(s) = \int_0^t 1(X_R(s) > 0)\, \mathrm{d}\langle \mathscr{M}_R \rangle(s).$$

With (e) and (f), we get the result (d) as we see in the following calculations:

$$\begin{aligned}\int_0^t 1(X_R(s) > 0)\, \mathrm{d}\langle \tilde{A} \rangle(s) &= \int_0^t 1(X_R(s) > 0)\, \mathrm{d}\langle \mathscr{M}_R \rangle(s) \quad \text{(from (f))} \\ &= \langle \mathscr{M}_R \rangle(t) - \int_0^t 1(X_R(s) = 0)\, \mathrm{d}\langle \mathscr{M}_R \rangle(s) \\ &= \langle \mathscr{M}_R \rangle(t) \quad \text{(from (e))}\end{aligned}$$

Proof of (*e*): Let

$$\left(X_n^R(t), \xi_n^R(t), M_n^R(t), \int_0^t 1(X_n^R(s) > 0)\, \mathrm{d}s \right) \to_{\mathscr{L}} (X_R(t), \xi_R(t), \mathscr{M}_R(t), \delta(t))$$

in $D([0,\infty), R^4)$. Then $X_R(t)$ satisfies the following equation:

$$X_R(t) = X(0) + \mathscr{M}_R(t) - \tilde{D}(\delta(t)) + c\delta(t) + \xi_R(t)$$

where $\xi_R(t)$ is non-decreasing and $\xi_R(0) = 0$, and

$$\int_0^t 1(X_R(s) > 0)\, \mathrm{d}\xi_R(s) = 0.$$

In obtaining the above result, we have used the fact that $v_n^R(t) \to_{\mathscr{L}} 0$ where

$$v_n(t) = \frac{1}{\phi_n} \tilde{V}_n \left(\int_0^t 1(X_n(s)=0)\, \mathrm{d}s \right).$$

Indeed, first consider the case $\rho = \infty$ in assumption (A3). We know that $\xi_n^R(t)$ can be written as

$$\xi_n^R(t) = \frac{v_n}{\phi_n} \int_0^t 1(s < H_n(R)) 1(X_n^R(s)=0)\, \mathrm{d}s$$

and that (ξ_n^R) is tight. Then $\rho = \infty$ implies

$$\int_0^t 1(s < H_n(R)) 1(X_n^R(s)=0)\, \mathrm{d}s \to_{\mathscr{L}} 0.$$

On the other hand, we note that

$$v_n^R(t) = \frac{1}{\phi_n} \tilde{V}_n \left(\int_0^t 1(X_n^R(s)=0) 1(s < H_n(R))\, \mathrm{d}s \right)$$

and $(1/\phi_n)\tilde{V}_n \to_{\mathscr{L}} v\tilde{V}$. Thus, $v_n^R(t) \to_{\mathscr{L}} v\tilde{V}(0) = 0$. Next consider the case where $\rho < \infty$. In this case, since $\sqrt{v_n}/\phi_n \to 0$, $(1/\phi_n)\tilde{V}_n \to_{\mathscr{L}} 0$ and $v_n^R(t) \to_{\mathscr{L}} 0$.

Now we note that $\mathscr{M}_R(t)$ and $\tilde{D}(\delta(t))$ are orthogonal martingales. Indeed we will show that $\langle \mathscr{M}_R, \tilde{D}(\delta(\cdot)) \rangle = 0$. Owing to Remark 2, $M_n^R(t)$ and $(1/\phi_n)\tilde{D}_n(\tau_n^D(\int_0^t 1(X_n^R(s) > 0)\, \mathrm{d}s))$ are orthogonal and hence we have

$$\left\langle M_n^R, (1/\phi_n)\tilde{D}_n \left(\tau_n^D \left(\int_0^{\cdot} 1(X_n^R(s) > 0)\, \mathrm{d}s \right) \right) \right\rangle (t) = 0.$$

(Note that $\tilde{D}_n(t)$ is not a martingale, but $\tilde{D}_n(\tau_n^D(t))$ is a martingale.) Letting n tend to infinity, by a standard argument, we deduce $\langle \mathscr{M}_R, \tilde{D}(\delta(\cdot)) \rangle = 0$. Then, since $X_R(t)$ is a semimartingale, it has a local time $L_x^t(X_R)$ and, for any Borel set A in R with Lebesgue measure zero, we have

$$(*) \quad \int_0^t 1(X_R(s) \in A)\, \mathrm{d}\langle \mathscr{M}_R - \tilde{D}(\delta(\cdot)) \rangle(s) = \int_0^t 1_A(x) L_x^t(X_R)\, \mathrm{d}x = 0$$

(see Jacod, 1979, p. 188). Since $\mathscr{M}_R$ and $\tilde{D}(\delta(\cdot))$ are orthogonal, it follows that

$$\int_0^t 1(X_R(s) \in A)\, \mathrm{d}\langle \mathscr{M}_R \rangle(s) = 0.$$

This completes the proof of (e).

Proof of (f): First, we remark the following fact. Suppose $X_R(s) > 0$. Then, since we may assume that with probability one $X_n^R(\tau_{A_n}(t)) \to X_R(t)$ uniformly on t-compact sets, there exists a $\varepsilon > 0$ such that $X_n^R(\tau_{A_n}(u)) > 0$ for all $u \in [s - \varepsilon, s + \varepsilon]$ and for all sufficiently large n. On the other hand, since $H_n(R) \to \infty$ as $n \to \infty$, We have

$$[L_n](t) = \int_0^t 1(s < H_n(R))\, \mathrm{d}\frac{1}{(\phi_n)^2} [\tilde{A}_n(\tau_{A_n}(\cdot))](s) \to_{\mathscr{L}} \langle \tilde{A} \rangle(t)$$

(see the proof of (a)). Thus we have, by Eq. (2.6),

$$\begin{aligned}\int_0^t 1(X_n^R(\tau_{A_n}(s)-) &= 0)1(s<H_n(s))\,\mathrm{d}\frac{1}{(\phi_n)^2}[\tilde{A}_n(\tau_{A_n}(\cdot))](s)\\ &= [L_n](t)-[M_n^R](t)\\ &\to_{\mathscr{L}} \langle\tilde{A}\rangle(t)-\langle\mathscr{M}_R\rangle(t)(\equiv\theta(t)).\end{aligned}$$

To obtain the last convergence, we note that $M_n^R \to_{\mathscr{L}} \mathscr{M}_R$ and $|\Delta M_n^R(t)|\leqslant R$ imply $[M_n^R]\to_{\mathscr{L}}[\mathscr{M}_R]$ (see Theorem 6.1, Corollary 6.6 in Jacod and Shiryaev, 1987). Now, since the above convergence may be assumed as uniform convergence on compact t-sets with probability one, we have

$$\begin{aligned}\int_{s-\varepsilon}^{s+\varepsilon} 1(X_n(\tau_{A_n}(s)-) = 0)1(s<H_n(R))\,\mathrm{d}\frac{1}{(\phi_n)^2}[\tilde{A}_n(\tau_{A_n}(\cdot))](s)\\ \to\theta(s+\varepsilon)-\theta(s-\varepsilon).\end{aligned}$$

But the left-hand side in the above convergence is zero for sufficiently large n as was remarked at the beginning of the proof of (f). Hence $\theta(s+\varepsilon)-\theta(s-\varepsilon)=0$ and the conclusion of (f) follows. □

With the help of Lemmas 2.1 and 2.2, it is now easy to prove Theorem 1.

Proof of Theorem 1. Let us consider Eq. (2.2). We have shown that (Z_n^R) is tight and this implies the joint tightness of (X_n^R, Z_n^R, ξ_n^R) (see the proof of (1)(b) of Lemma 2.2). Next, we will show that

$$\delta(t)=\int_0^t 1(X_R(s)>0)\,\mathrm{d}s, \tag{2.7}$$

where $\delta(t)$ was the weak limit of $\int_0^t 1(X_n^R(s)>0)\,\mathrm{d}s$ (see the proof of (e)). Indeed, from the discussion in the proof of (e)(see (∗)) we see that

$$\int_0^t 1(X_R(s)\in A)\,\mathrm{d}<\tilde{D}(\delta(\cdot))>(s)=0$$

for any Borel set A with Lebesgue measure zero. Then this implies

$$\int_0^t 1(X_R(s)\in A)\,\mathrm{d}\delta(s)=0.$$

On the other hand, we can show, as in the proof of (f), that

$$\int_0^t 1(X_R(s)>0)\,\mathrm{d}\tilde{\delta}(s)=0$$

where $\tilde{\delta}(t)=t-\delta(t)$. From these two equations, we get Eq. (2.7).

Now combining all previously obtained facts, we see that the limit process X_R satisfies the following equation:

$$X_R(t) = X(0) + \int_0^t 1(X_R(s) > 0)\, d\tilde{A}(s) - \tilde{D}\left(\int_0^t 1(X_R(s) > 0)\, ds\right)$$
$$+ \int_0^t c1(X_R(s) > 0)\, ds + \xi_R(t),$$

where $\xi_R(t)$ is non-decreasing and $\xi_R(0) = 0$, and

$$\int_0^t 1(X_R(s) > 0)\, d\xi_R(s) = 0$$

Hence, by the uniqueness in law of the solution of Eq. (2.1), $X_R(t)$ is equivalent in law to $X(t)$. Thus, we have shown that

$$(X_n^R(t), M_n^R(t)) \to_{\mathscr{L}} (X(t), \int_0^t 1(X(s) > 0)\, d\tilde{A}(s))$$

On the other hand, since $H_n(R) \to_{\mathrm{P}} \infty$, we have

$$\sup_{0 \leqslant t \leqslant T} |M_n^R(t) - M_n(t)| \to_{\mathrm{P}} 0,$$

$$\sup_{0 \leqslant t \leqslant T} |X_n^R(t) - X_n(t)| \to_{\mathrm{P}} 0.$$

Thus,

$$(X_n(t), M_n(t)) \to_{\mathscr{L}} (X(t), \int_0^t 1(X(s) > 0)\, d\tilde{A}(s))$$

and

$$(X_n(t), \mathscr{I}_n(t)) \to_{\mathscr{L}} (X(t), \int_0^t 1(X(s) > 0)\, d\tilde{A}(s)). \qquad \square$$

Lemma 2.3. *Let (M_n) be a sequence of locally square integrable martingales. Then the sequence (M_n) is tight in $D([0,\infty), R)$ if $([M_n])$ is C-tight and $E \sup_{s \leqslant T} \Delta[M_n](s) \to 0$ for any $T \geqslant 0$.*

Proof. The proof is almost the same as in Jacod and Shiryaev (1987), (VI, Theorem 4.13, p. 322); we note M_n^2 is L-dominated by $[M_n]$ and use the second Lenglart inequality (3.32) in Jacod and Shiryaev (1987), (I, Lemma 3.30, p. 35). $\square$

3. Example 2: A limit theorem for occupation times and counting processes for busy cycles in queueing networks – an approach by a Dynkin's formula

We consider a sequence of open queueing networks with K stations described in Reiman (1984). Let $Q_n^i(t)$ be the queue length at time t at station i for the nth network. Then $Q_n(t) = (Q_n^1(t), Q_n^2(t), \ldots, Q_n^K(t))$ satisfies the following equation: for each

$i, 1 \leqslant i \leqslant K$,

$$Q_n^i(t) = Q_n^i(0) + A_n^i(t) + \sum_{j=1}^{K} \bar{D}_n^{ji}(t) - \bar{D}_n^i(t),$$

$$\bar{D}_n^i(t) = D_n^i \left(\int_0^t 1(Q_n^i(s) > 0)\, \mathrm{d}s \right),$$

$$\bar{D}_n^{ji}(t) = D_n^{ji} \left(\int_0^t 1(Q_n^j(s) > 0)\, \mathrm{d}s \right).$$

In the above equation, $A_n^i(t)$ is an arrival process to station i and is a renewal process defined by

$$A_n^i(t) = \max \left\{ k; \sum_{l=1}^{k} u_n^i(l) \leqslant t \right\},$$

where $(u_n^i(l))$ is a sequence of random variables with $\lambda_n^i = 1/Eu_n^i(1)$. $D_n^i(t)$ is a potential departure process from station i and is a renewal process defined by

$$D_n^i(t) = \max \left\{ k;\ \sum_{l=1}^{k} v_n^i(l) \leqslant t \right\}$$

with $\mu_n^i = 1/Ev_n^i(l)$. Finally, $D_n^{ji}(t)$ is a potential stream of customers from station j to station i and is defined as

$$D_n^{ji}(t) = \sum_{l=1}^{D_n^j(t)} \xi_{ji}(l),$$

where $(\xi_j(l)), 1 \leqslant j \leqslant K$, are sequences of i.i.d. K-dimensional vector random variables taking the values $\{e_1, \ldots, e_K\}$ where e_i is the K-dimensional vector whose ith component is 1 and others are 0. We let $P(\xi_j(l) = e_i) = p_{ji}$. The matrix $P = (p_{ij})$ is called a routing matrix.

We assume the following conditions:

(B1) (Heavy traffic condition): For $1 \leqslant i \leqslant K$,

$$\sqrt{n} \left(\lambda_n^i + \sum_{j=1}^{K} p_{ji} \mu_n^j - \mu_n^i \right) \to c_i$$

and

$$\lambda_n^i \to \lambda_i, \qquad \mu_n^i \to \mu_i$$

(B2) (1) For each t and $i \geqslant 1$,

$$\frac{A_n^i(t)}{n} \to_{\mathrm{P}} \lambda_i t, \qquad \frac{D_n^i(t)}{n} \to_{\mathrm{P}} \mu_i t$$

(2) For $1 \leqslant i \leqslant K$, put

$$U_n^i(t) = \sum_{l=1}^{[nt]} (1 - \lambda_n^i u_n^i(l)), \qquad U_n(t) = (U_n^1(t), \ldots, U_n^K(t)),$$

$$\tilde{D}_n^i(t) = D_n^i(t) - \mu_n^i t, \qquad \tilde{D}_n(t) = (\tilde{D}_n^1(t), \dots, \tilde{D}_n^K(t)),$$

$$R_n(t) = \left(\sum_{l=1}^{[nt]} (\xi_1(l) - p(1)), \dots, \sum_{l=1}^{[nt]} (\xi_K(l) - p(K)) \right),$$

where $p(i) = (p_{i1}, \dots, p_{iK})$.

Then

$$\left(\frac{1}{\sqrt{n}} U_n(t), \frac{1}{\sqrt{n}} \tilde{D}_n(nt), \frac{1}{\sqrt{n}} R_n(t) \right) \to_{\mathscr{L}} (\tilde{U}(t), \tilde{D}(t), R(t)) \quad \text{in } D$$

where $\tilde{U}$, $\tilde{D}$, and R are Brownian motions with $\langle \tilde{U}_i \rangle(t) = \sigma_i^2 t$, $1 \leqslant i \leqslant K$.

Remark. (B2) implies that

$$\left(\frac{1}{\sqrt{n}} \tilde{A}_n(nt), \frac{1}{\sqrt{n}} \tilde{D}_n(nt), \frac{1}{\sqrt{n}} R_n(t) \right) \to_{\mathscr{L}} (\tilde{A}(t), \tilde{D}(t), R(t)),$$

where $\tilde{A}(t) = (\tilde{U}_1(\lambda_1 t), \dots, \tilde{U}_K(\lambda_K t))$.

(B3) The routing matrix P has spectral radius strictly smaller than unity.

Then we have the following result (Reiman, 1984).

Proposition 1 (Diffusion approximation theorem). *Assume the conditions* (B1)–(B3) *and assume that* $X_n(0) \to_{\mathscr{L}} X(0)$. *Let us consider a sequence of scaled processes* (X_n, I_n) *defined by*

$$X_n(t) = \left(\frac{1}{\sqrt{n}} Q_n^1(nt), \dots, \frac{1}{\sqrt{n}} Q_n^K(nt) \right), \quad n \geqslant 1,$$

$$I_n(t) = \left(\frac{1}{\sqrt{n}} \mu_n^1 \int_0^{nt} 1(Q_n^i(s) = 0)\, \mathrm{d}s, \dots, \frac{1}{\sqrt{n}} \mu_n^K \int_0^{nt} 1(Q_n^K(s) = 0)\, \mathrm{d}s \right), \quad n \geqslant 1.$$

Then $(X_n, I_n) \to_{\mathscr{L}} (X, I)$ *in* $D([0,\infty), \mathbb{R}^2)$ *where the limit process* $(X(t), I(t))$ *is the unique solution of the following Skorohod equation*:

$$X_i(t) = X_i(0) + c_i t + \tilde{A}_i(t) + \sum_{j=1}^K \tilde{D}_{ji}(t) - \tilde{D}_i(t)$$
$$+ I_i(t) - \sum_{j=1}^K p_{ji} I_j(t), \quad X_i(t) \geqslant 0, \ 1 \leqslant i \leqslant K,$$

where

$$\tilde{D}_{ji}(t) = \tilde{R}_{ji}(\mu_j t) + p_{ji} \tilde{D}_j(t)$$

and $R_{ji}(t)$ *is the ith component of the K-dimensional vector process* $R^j(t)$ *where* $R(t) = (R^1(t), \dots, R^K(t))$ *is the weak limit process of* $R_n(t)$.

Moreover $I_i(t), 1 \leqslant i \leqslant K$, are non-decreasing with $I_i(0)=0$ and satisfy

$$\int_0^t 1(X_i(s)>0)\,\mathrm{d}I_i(s)=0.$$

Our problem in this section is to investigate the limiting behavior of occupation times for the above queueing networks. That is, for each station i, we define a sequence of scaled processes $(\mathscr{B}_n^i)$ defined by

$$\mathscr{B}_n^i(t)=\frac{1}{\sqrt{n}}\int_0^{nt} f(Q_n^i(s))\,\mathrm{d}s,$$

where f is assumed to have a compact support and f may be different for each station. A typical example of f is $f(x)=1$ $(x \leqslant \varepsilon)$; $\mathscr{B}_n^i(t)$ expresses the scaled process of occupation time at station i when the queue length $Q_n^i(t)$ is under ε.

Then we have

Theorem 2. *Let $\Sigma(f)=\sum_{q=0}^{\infty} f(q)$ and $\bar{\lambda}_i=\lambda_i+\sum_{j=1}^{K} p_{ji}\mu_j$ for $1 \leqslant i \leqslant K$. Then under assumptions* (B1)–(B3), *$\mathscr{B}_n \rightarrow_{\mathscr{L}} \mathscr{B}$ in $D([0,\infty),\mathbb{R}^K)$ where the ith component $\mathscr{B}_i(t)$ of the process $\mathscr{B}(t)$ is defined as*

$$\mathscr{B}_i(t)=\frac{1}{\bar{\lambda}_i}(\Sigma(f)(1-p_{ii})+p_{ii}f(0))I_i(t).$$

Corollary 1. *Let f be as in Theorem 2, and let for each i $(1 \leqslant i \leqslant K)$,*

$$\mathscr{C}_n^i(t)=\frac{1}{\sqrt{n}}\int_0^{nt} f(Q_n^i(s-))\,\mathrm{d}\mathscr{A}_n^i(s),$$

where

$$\mathscr{A}_n^i(t)=A_n^i(t)+\sum_{j=1}^{K} D_n^{ji}\left(\int_0^t 1(Q_n^j(s)>0)\,\mathrm{d}s\right).$$

(That is, $\mathscr{A}_n^i(t)$ is the arrival process to station i.) Then $\mathscr{C}_n(t) \rightarrow_{\mathscr{L}} \mathscr{C}(t)$ in $D([0,\infty),\mathbb{R}^K)$ where the ith component of the process $\mathscr{C}(t)$ is given by

$$\mathscr{C}_i(t)=(\Sigma(f)(1-p_{ii}))I_i(t).$$

Let, as a special case, $f(Q)=1(Q=0)$. Then $\int_0^t f(Q_n^i(s-)=0)\,\mathrm{d}\mathscr{A}_n^i(s)$ expresses the number of busy cycles up to time t at station i. In this case, $\mathscr{C}_i(t)=(1-p_{ii})I_i(t)$.

The proof of this theorem can be done in the same way as in Yamada (1993), where we have used a Dynkin's formula for Markov processes. Though our process $Q_n(t)$ is not Markovian, we can still use a similar approach and the general idea is as follows: For simplicity, we consider the single station case, and let $Q_n(t)$ be a sequence of G/G/1 processes with arrival rate λ_n and service rate μ_n. For an arbitrary function F defined on $\{0,1,2,\ldots\}$, we define a sequence of the processes $M_n(t)$ by

$$M_n(t) \equiv F(Q_n(t))-F(Q_n(0))-\int_0^t \mathscr{L}_n F(Q_n(s))\,\mathrm{d}s,$$

where $\mathscr{L}_n$ is an operator on a function space defined by

$$\mathscr{L}_n F(Q) = (F(Q+1) - F(Q))\lambda_n - (F(Q) - F(Q-1))1(Q>0)\mu_n.$$

Thus if we can find a function F satisfying $\mathscr{L}_n F(q) = f(q)$, the investigation of limiting behavior of occupation time $1/\sqrt{n}\int_0^{nt} f(Q_n(s))\,\mathrm{d}s$ is reduced to that of limiting behavior of $((1/\sqrt{n})F(Q_n(nt))$ and $(1/\sqrt{n})M_n(nt))$. Suppose $(Q_n(t))$ is a Markov process. Then $\mathscr{L}_n$ is the generator of the process $Q_n(t)$ and Dynkin's formula tells us that $(M_n(t))$ is a sequence of martingales and hence we can apply the stochastic calculus approach to the investigation of limiting behavior of the sequence of the martingales $(1/\sqrt{n})M_n(nt),\ n\geqslant 1$. Unfortunately, the process $Q_n(t)$ is not a Markov process, and hence $M_n(t)$ is no more martingale. However, as in Section 2, the process $M_n(t)$ is expressed as a stochastic integral with respect to renewal processes and $(1/\sqrt{n})M_n(nt)$ can be expressed as

$$\frac{1}{\sqrt{n}}M_n(nt) = m_n(t) + l_n(t)$$

such that (m_n) is a sequence of martingales and (l_n) converges in law to the null process. Thus, similar arguments as in Section 2 can be applied. In this step the convergence of martingales (m_n), which are stochastic integrals with respect to renewal processes, and the identification of the limit process are the main part of the discussion in the proof of Theorem 2 and this is contained in Step 4.

Proof of Theorem 2. For an arbitrary function F on $\{0,1,2,\ldots\}$, we have

$$\begin{aligned} F(Q_n^i(t)) &= F(Q_n^i(0)) + \sum_{s\leqslant t,\,\Delta Q_n^i(s)\neq 0} (F(Q_n^i(s) - F(Q_n^i(s-)) \\ &= F(Q_n^i(0)) + \int_0^t \delta F(Q_n^i(s-)+1)\,\mathrm{d}\Big(A_n^i(s) + \sum_{j=1}^K \bar{D}_n^{ji}(s)\Big) \\ &\quad - \int_0^t \delta F(Q_n^i(s-))\,\mathrm{d}\bar{D}_n^i(s) \end{aligned}$$

where $\delta F(Q) = F(Q) - F(Q-1)$ for $Q\geqslant 1$.

We put

$$\bar{\lambda}_n^i = \lambda_n^i + \sum_{j=1}^K \mu_n^j p_{ji},$$

$$B_n^i(t) = \int_0^t 1(Q_n^i(s)>0)\,\mathrm{d}s,$$

$$\tilde{D}_n^{ji}(t) = D_n^{ji}(t) - \mu_n^j p_{ji} t.$$

Then

$$F(Q_n^i(t)) = F(Q_n^i(0)) + \int_0^t (\delta F(Q_n^i(s)+1)\bar{\lambda}_n^i - \delta F(Q_n^i(s))\mu_n^i 1(Q_n^i(s)>0))\,\mathrm{d}s$$
$$+ M_n(t) - \int_0^t \delta F(Q_n^i(s)+1)\sum_{j=1}^K \mu_n^j p_{ji} 1(Q_n^j(s)=0)\,\mathrm{d}s, \tag{3.1}$$

where

$$M_n(t) = \int_0^t \delta F(Q_n^i(s-)+1)\,\mathrm{d}\left(\tilde{A}_n^i(s) + \sum_{j=1}^K \tilde{D}_n^{ji}(B_n^j(s))\right)$$
$$- \int_0^t \delta F(Q_n^i(s-))\,\mathrm{d}\tilde{D}_n^i(B_n^i(s)).$$

Since f has a compact support, there exists a number Q_0 such that $f(Q)=0$ if $Q>Q_0$. We choose a function F such that $F(0)=0$ and

$$\delta F(Q) = \frac{1}{\bar{\lambda}_n^i}\left\{ f(Q-1) + \frac{\mu_n^i}{\bar{\lambda}_n^i} f(Q-2) + \cdots + \left(\frac{\mu_n^i}{\bar{\lambda}_n^i}\right)^{Q-1} f(0)\right\}$$

for $Q \leqslant Q_0+1$ and $\delta F(Q) = \delta F(Q_0+1)$ for $Q>Q_0+1$. Then F satisfies

$$\delta F(Q+1)\bar{\lambda}_n^i - \delta F(Q)\mu_n^i 1(Q>0) = f(Q)$$

for $Q \leqslant Q_0+1$. Then Eq. (3.1) can be written as

$$F(Q_n^i(t)) = F(Q_n^i(0)) + \int_0^t f(Q_n^i(s))\,\mathrm{d}s + \int_0^t 1(Q_n^i(s)>Q_0)\delta F(Q_0+1)(\bar{\lambda}_n^i - \mu_n^i)\,\mathrm{d}s$$
$$+ M_n(t) - \sum_{j=1}^K p_{ji}\int_0^t \delta F(Q_n^i(s)+1)\mu_n^j 1(Q_n^j(s)=0)\,\mathrm{d}s.$$

This leads to the following equation:

$$\frac{1}{\sqrt{n}}F(\sqrt{n}X_n^i(t)) = \frac{1}{\sqrt{n}}F(\sqrt{n}X_n^i(0)) + \mathscr{B}_n^i(t)$$
$$+ \int_0^t 1\left(X_n^i(s) > \frac{Q_0}{\sqrt{n}}\right)\delta F(Q_0+1)\sqrt{n}(\bar{\lambda}_n^i - \mu_n^i)\,\mathrm{d}s + \frac{1}{\sqrt{n}}M_n(nt)$$
$$- \sum_{j=1}^K p_{ji}\int_0^t \delta F(\sqrt{n}X_n^i(s)+1)\,\mathrm{d}I_n^j(s), \tag{3.2}$$

where

$$I_n^j(t) = \int_0^t \sqrt{n}\mu_n^j 1(X_n^j(s)=0)\,\mathrm{d}s.$$

Thus, the limit process of $\mathscr{B}_n^i$ can be obtained by considering the limit processes of the other terms in Eq. (3.2). Especially the investigation of the limit of the process

$(1/\sqrt{n})M_n(nt)$, which is the scaled stochastic integral with respect to renewal processes, is the main part of our discussion as in Section 2. Hereafter, in view of Skorohod's representation theorem, we may assume that the convergence in (B2) holds w.p.1. We may also assume that in Reiman's diffusion approximation theorem (Proposition 1), $(X_n, I_n) \to (X, I)$ w.p.1. Note that since in these convergences the limit processes are continuous, the above assumption implies the uniform convergence on compact t-sets (abbreviated as u.o.c.).

Step 1. In this step we note that the following facts hold:
(a) $\delta F(Q_0+1) \to (1/\bar{\lambda}_i)\sum(f)$ as $n \to \infty$. (Note that $\delta F(\cdot)$ depends on n.)
(b) $\delta F(\sqrt{n}X_n^i(s)+1) \to (1/\bar{\lambda}_i)\sum(f)$ as $n \to \infty$ if $X_i(s) > 0$.
(c) As $n \to \infty$, for each t,

$$\frac{1}{\sqrt{n}}F(\sqrt{n}X_n^i(t)) \to \frac{1}{\bar{\lambda}_i}\sum(f)X_i(t).$$

All these facts are deduced easily from the definition of F.

Step 2. Note that w.p.1, $X_i(t) > 0$ for a.e. t. Hence w.p.1, $1(X_n^i(s) > (Q_0/\sqrt{n})) \to 1$ for a.e. s. Thus we have that w.p.1,

$$\int_0^t 1\left(X_n^i(s) > \frac{Q_0}{\sqrt{n}}\right)\delta F(Q_0+1)\sqrt{n}(\bar{\lambda}_n^i - \mu_n^i)\,\mathrm{d}s \to \frac{1}{\bar{\lambda}_i}\sum(f)c_i t$$

for any t.

Step 3. We will show that w.p.1,

$$\int_0^t \delta F(\sqrt{n}X_n^i(s)+1)\,\mathrm{d}I_n^j(s) \to \frac{1}{\bar{\lambda}_i}\left(\sum(f)I_j(t)1(i \neq j) + f(0)I_i(t)1(i=j)\right)$$

for any t. We have

$$\int_0^t \delta F(\sqrt{n}X_n^i(s)+1)\,\mathrm{d}I_n^i(s) = \delta F(1)I_n^i(t) \to \frac{1}{\bar{\lambda}_i}f(0)I_i(t) \quad \text{u.o.c}$$

as n tends to ∞. We also have, by (b) of Step 1, that w.p.1,

$$\delta F(\sqrt{n}X_n^i(s)+1)1(X_i(s)>0) \to 1(X_i(s)>0)(1/\bar{\lambda}_i)\sum(f)$$

for all $s \geqslant 0$. We also note that the function $t \to 1(X_i(t) > 0)$ is approximated by a sequence of step functions. Indeed we let, for each k, $[0,\infty) = \bigcup_{l=1}^{\infty} I_k^l$ where $I_k^l = [(l-1)/2^k, l/2^k)$. We define a sequence of step functions (f_k) as follows: For an arbitrarily fixed s, suppose $s \in I_k^l$. Then $f_k(s) = 0$ if there exists a $u \in I_k^l$ such that $X_i(u) = 0$, and $f_k(s) = 1$ if $X_i(u) > 0$ for all $u \in I_k^l$. Then since $X_i(\cdot)$ is continuous, it is evident that $f_k(s) \to 1(X_i(s) > 0)$ as $k \to \infty$ for all s. Hence with these facts, by Lemma 3.1, which is to appear in the last part of this section, we have

$$\int_0^t \delta F(\sqrt{n}X_n^i(s)+1)1(X_i(s)>0)\,\mathrm{d}I_n^j(s) \to \frac{1}{\bar{\lambda}_i}\sum(f)\int_0^t 1(X_i(s)>0)\,\mathrm{d}I_j(s).$$

However, owing to Reiman and Williams (1988), (Theorem 1), we have

$$\int_0^t 1(X_i(s)>0)\,\mathrm{d}I_j(s) = I_j(t) - \int_0^t 1(X_i(s)=0)\,\mathrm{d}I_j(s) = I_j(t)1 \quad (i \neq j).$$

Similarly, since $\delta F(Q)$ is bounded uniformly in n (i.e., there exists a constant C such that $|\delta F(Q)| \leqslant C$ for all n and Q), using Lemma 3.1 we have, if $i \neq j$,

$$\left| \int_0^t \delta F(\sqrt{n} X_n^i(s) + 1) 1(X_i(s) = 0)\, \mathrm{d}I_n^j(s) \right|$$
$$\leqslant C \int_0^t 1(X_i(s) = 0)\, \mathrm{d}I_n^j(s) \to C \int_0^t 1(X_i(s) = 0)\, \mathrm{d}I_j(s) = 0, \quad \text{u.o.c.}$$

Combining these facts, we have the conclusion for Step 3.

Step 4 (*Convergence of stochastic integrals*): We will show that

$$\frac{1}{\sqrt{n}} M_n(nt) \to_{\mathscr{L}} \frac{1}{\bar{\lambda}_i} \sum (f) \left(\tilde{A}_i(t) + \sum_{j=1}^K \tilde{D}_{ji}(t) - \tilde{D}_i(t) \right) \quad \text{in } D([0,\infty), R^1).$$

Note that

$$\frac{1}{\sqrt{n}} M_n(nt) = \int_0^t \delta F(\sqrt{n} X_n^i(s-) + 1)\, \mathrm{d} \left(\frac{1}{\sqrt{n}} \tilde{A}_n^i(ns) + \sum_{j=1}^K \frac{1}{\sqrt{n}} \tilde{D}_n^{ji}(B_n^j(ns)) \right)$$
$$- \int_0^t \delta F(\sqrt{n} X_n^i(s-))\, \mathrm{d} \frac{1}{\sqrt{n}} \tilde{D}_n^i(B_n^i(ns)).$$

We must consider the convergence of stochastic integrals in the above equation, and, for example, we will consider the convergence of the stochastic integral:

$$W_n(t) \equiv \int_0^t \delta F(\sqrt{n} X_n^i(s-) + 1)\, \mathrm{d} \frac{1}{\sqrt{n}} \tilde{D}_n^{ji}(B_n^j(ns)). \tag{3.3}$$

We proceed in the same way as in Section 2. We put

$$\theta_{n,j}(t) = \int_0^t 1(X_n^j(s) > 0)\, \mathrm{d}s.$$

Then we have

$$W_n(t) = \int_0^{\theta_{n,j}(t)} \delta F(\sqrt{n} X_n^i(\theta_{n,j}^{-1}(s)-) + 1)\, \mathrm{d} \frac{1}{\sqrt{n}} \tilde{D}_n^{ji}(ns).$$

Since w.p.1 $\theta_{n,j}(t) \to t$ u.o.c., it suffices to consider the convergence

$$\bar{W}_n(t) \equiv \int_0^t \delta F(\sqrt{n} X_n^i(\theta_{n,j}^{-1}(s)-) + 1)\, \mathrm{d} \frac{1}{\sqrt{n}} \tilde{D}_n^{ji}(ns).$$

Recalling the definition of $\tilde{D}_n^{ji}(t)$, we define, as in Section 2, a stopping time $\tau_n^j(t)$ by

$$\tau_n^j(t) = \inf(s;\ s \geqslant t,\ \Delta D_n^j(ns) \neq 0).$$

We then have the following decomposition:

$$\bar{W}_n(t) = W_n^1(t) - W_n^2(t),$$

where

$$W_n^1(t) = \int_0^{\tau_n^j(t)} \delta F(\sqrt{n}X_n^i(\theta_{n,j}^{-1}(s)-)+1)\,\mathrm{d}\frac{1}{\sqrt{n}}\tilde{D}_n^{ji}(ns)$$
$$= \int_0^{t} \delta F(\sqrt{n}X_n^i(\theta_{n,j}^{-1}(\tau_n^j(s))-)+1)\,\mathrm{d}\frac{1}{\sqrt{n}}\tilde{D}_n^{ji}(n\tau_n^j(s)),$$
$$W_n^2(t) = \int_t^{\tau_n^j(t)} \delta F(\sqrt{n}X_n^i(\theta_{n,j}^{-1}(s)-)+1)\,\mathrm{d}\frac{1}{\sqrt{n}}\tilde{D}_n^{ji}(ns).$$

Under assumption (B2), as in Section 2 we have

$$\sup_{0\leqslant t\leqslant T} |W_n^2(t)| \to_{\mathrm{P}} 0.$$

We also note that $W_n^1(t)$ is an $\mathscr{F}_n(t)$-martingale, where the filtration $\mathscr{F}_n(t)$ is defined by

$$\mathscr{F}_n(t) = \sigma(Q_n(0),\ u_n^i(l),\ 1\leqslant l\leqslant A_n^i(nt)+1,\ v_n^i(l),\ 1\leqslant l\leqslant D_n^i(nt)+1,$$
$$\xi_{ij}(l),\ 1\leqslant l\leqslant D_n^{ij}(nt)+1,\ 1\leqslant i,\ j\leqslant K).$$

Thus the convergence of the stochastic integral $W_n(t)$ in Eq. (3.3) is reduced to that of the martingale $W_n^1(t)$. However, since it holds that $\tau_n^j(s)\to s$, $\theta_{n,j}(s)\to s$ u.o.c. and $X_i(s)>0$ for a.e. s,

$$\delta F(\sqrt{n}X_n^i(\theta_{n,j}^{-1}(\tau_n^j(s))-)+1) - \delta F(\sqrt{n}X_n^i(s-))\to 0, \quad \text{for a.e. } s.$$

(See Step 1(b).). Then noting that $1/\sqrt{n}\tilde{D}_n^{ji}(n\tau_n^j(t)) \to_{\mathscr{L}} \tilde{D}_{ji}(t)$ in D, it is easy to see that $W_n^1(t)$ is the sum of $\int_0^t \delta F(\sqrt{n}X_n^i(s-))\,\mathrm{d}(1/\sqrt{n})\tilde{D}_n^{ji}(n\tau_n^j(s))$ and a process which is convergent to the null process as n tends to infinity. Thus summarizing the above discussion, we come to the conclusion that to show the convergence of $(1/\sqrt{n})M_n(nt)$, it suffices to show the convergence:

$$\mathscr{A}_n(t) \equiv \int_0^t \delta F(\sqrt{n}X_n^i(s-))\,\mathrm{d}\alpha_n^i(s) \to_{\mathscr{L}} \mathscr{A}(t) \equiv \frac{1}{\bar{\lambda}_i}\Sigma(f)\left(\tilde{A}_i(t)+\sum_{j=1}^K \tilde{D}_{ji}(t)-\tilde{D}_i(t)\right),$$

where

$$\alpha_n^i(t) = \frac{1}{\sqrt{n}}\tilde{A}_n^i(n\gamma_n^i(t)) + \sum_{j=1}^K \frac{1}{\sqrt{n}}\tilde{D}_n^{ji}(n\tau_n^j(t)) - \frac{1}{\sqrt{n}}\tilde{D}_n^i(n\tau_n^i(t)).$$
$$\gamma_n^i(t) = \inf(s, s\geqslant t, \Delta\tilde{A}_n^i(ns)\neq 0).$$

As in Section 2, define a stopping time $H_n(R)$ by

$$H_n^R(t) = \inf(t;\ |\Delta\alpha_n^i(t)|>R)$$

and we will show the convergence:

$$\mathscr{A}_n^R(t) \equiv \int_0^t \delta F(\sqrt{n}X_n^i(s-))1(s<H_n(R))\,\mathrm{d}\alpha_n^i(s) \to_{\mathscr{L}} \mathscr{A}(t) = \frac{1}{\bar{\lambda}_i}\Sigma(f)\alpha_i(t), \tag{3.4}$$

where $\alpha_i(t) = \tilde{A}_i(t) + \sum_{j=1}^K \tilde{D}_{ji}(t) - \tilde{D}_i(t)$.

Write $\mathscr{A}_n^R(t)$ as

$$\begin{aligned}\mathscr{A}_n^R(t) &= \int_0^t \delta F(\sqrt{n}X_n^i(s-))1(X_i(s)>0)1(s<H_n(R))\,\mathrm{d}\alpha_n^i(s)\\ &\quad+\int_0^t \delta F(\sqrt{n}X_n^i(s-))1(X_i(s)=0)1(s<H_n(R))\,\mathrm{d}\alpha_n^i(s)\\ &\equiv \mathscr{A}_n^{R,1}(t)+\mathscr{A}_n^{R,2}(t).\end{aligned}$$

We note that by assumption (B2) and the fact $\tau_n^i(t)\to t$ u.o.c. as n tends to infinity, $\alpha_n^i \to_{\mathscr{L}} \alpha_i$. Since $1(s<H_n(R))\to 1$ as n tends to infinity, this implies that

$$\int_0^t 1(s<H_n(R))\,\mathrm{d}\alpha_n^i(s) \to_{\mathscr{L}} \alpha_i(t).$$

Moreover,

$$\left|\Delta \int_0^t 1(s<H_n(R))\,\mathrm{d}\alpha_n^i(s)\right| \leqslant R.$$

Hence,

$$\int_0^t 1(s<H_n(R))\,\mathrm{d}[\alpha_n^i](s) \to_{\mathscr{L}} [\alpha_i](t)$$

(see Jacod, 1987, VI, Theorem 6.1, Cororally 6.6, p. 342). Since this convergence may be assumed to be the uniform convergence on compact t-sets and since $[\alpha_i](t)=C_0t$ where C_0 is a positive constant, by Lemma 3.1,

$$\begin{aligned}[\mathscr{A}_n^{R,2}](t) &= \int_0^t (\delta F(\sqrt{n}X_n^i(s-)))^2 1(X_i(s)=0)1(s<H_n(R))\,\mathrm{d}[\alpha_n^i](s)\\ &\leqslant C^2\int_0^t 1(X_i(s)=0)1(s<H_n(R))\,\mathrm{d}[\alpha_n^i](s)\\ &\to C^2\int_0^t 1(X_i(s)=0)\,\mathrm{d}[\alpha_i](s)=0 \quad \text{u.o.c.}\end{aligned}$$

as n tends to infinity (recall that C was the bound for $|\delta F(Q)|$). Thus $\mathscr{A}_n^{R,2}\to_{\mathscr{L}} 0$. It follows that to see Eq. (3.4) it suffices to show the convergence $\mathscr{A}_n^{R,1}\to_{\mathscr{L}}\mathscr{A}$. But for this, since

$$|\Delta\mathscr{A}_n^{R,1}(t)| \leqslant \frac{1}{\sqrt{n}}(K+2),$$

it suffices to show $[\mathscr{A}_n^{R,1}](t)\to_{\mathrm{P}}[\mathscr{A}](t)$ for each t. We have

$$[\mathscr{A}_n^{R,1}](t) = \int_0^t (\delta F(\sqrt{n}X_n^i(s-)))^2 1(X_i(s)>0)1(s<H_n(R))\,\mathrm{d}[\alpha_n^i](s).$$

Since, by Step 1(b) in the proof of Theorem 2,

$$(\delta F(\sqrt{n}X_n^i(s-)))^2 1(X_i(s)>0) \to \left(\frac{1}{\bar{\lambda}_i}\Sigma(f)\right)^2 1(X_i(s)>0),$$

for all s, by Lemma 3.1 we have, for each t,

$$[\mathscr{A}_n^{R,1}](t) \to \left(\frac{1}{\bar{\lambda}_i}\Sigma(f)\right)^2 \int_0^t 1(X_i(s)>0)\,\mathrm{d}[\alpha_i](s) = \left(\frac{1}{\bar{\lambda}_i}\Sigma(f)\right)^2 [\alpha_i](t) = [\mathscr{A}](t).$$

Thus we have shown (3.4), and we come to the conclusion of Step 4.

Step 5: Combining the results in Steps 1–4, from Eq. (3.2) it follows that for each t,

$$\mathscr{B}_n^i(t) \to \frac{1}{\bar{\lambda}_i}\sum(f)\left[X_i(t) - X_i(0) - c_i t - \tilde{A}_i(t) - \sum_{j=1}^K \tilde{D}_{ji}(t) + \tilde{D}_i(t) + \sum_{j=1, j\neq i}^K p_{ji} I_j(t)\right]$$
$$+ \frac{1}{\bar{\lambda}_i} p_{ii} f(0) I_i(t) = \frac{1}{\bar{\lambda}_i}\sum(f)(1-p_{ii})I_i(t) + \frac{1}{\bar{\lambda}_i} p_{ii} f(0) I_i(t).$$

Thus, we conclude that

$$\mathscr{B}_i(t) = \frac{1}{\bar{\lambda}_i}(\Sigma(f)(1-p_{ii}) + f(0)p_{ii})I_i(t). \qquad \square$$

Proof of Corollary. We have

$$\mathscr{C}_n^i(t) = \frac{1}{\sqrt{n}} \int_0^{nt} f(Q_n^i(s))(\lambda_n^i + \sum_{j=1}^K p_{ji}\mu_n^j)\,\mathrm{d}s$$
$$- \sum_{j=1}^K \frac{1}{\sqrt{n}} \int_0^{nt} f(Q_n^i(s)) p_{ji}\mu_n^j 1(Q_n^j(s)=0)\,\mathrm{d}s$$
$$+ \frac{1}{\sqrt{n}} \int_0^{nt} f(Q_n^i(s-))\,\mathrm{d}\tilde{A}_n^i(s)$$
$$+ \sum_{j=1}^K \frac{1}{\sqrt{n}} \int_0^{nt} f(Q_n^i(s-))\,\mathrm{d}\tilde{D}_n^{ji}\left(\int_0^s 1(Q_n^j(u)>0)\,\mathrm{d}u\right)$$
$$\equiv Z_n^1(t) - Z_n^2(t) + Z_n^3(t) + Z_n^4(t).$$

By Theorem 2, we have

$$Z_n^1(t) \to_{\mathscr{L}} (\Sigma(f)(1-p_{ii}) + p_{ii} f(0))I_i(t).$$

As for $Z_n^3(t)$, we have

$$Z_n^3(t) = \int_0^t f(\sqrt{n}X_n^i(s-))\,\mathrm{d}\frac{1}{\sqrt{n}}\tilde{A}_n^i(ns).$$

Since f has a compact support and $X_i(s)>0$ for a.e. s, $f(\sqrt{n}X_n^i(s-)) \to 0$ for a.e.s. Moreover, $(1/\sqrt{n})\tilde{A}_n(nt) \to_{\mathscr{L}} \tilde{A}(t)$. Hence by using the same argument as in Section 1 (Lemmas 2.1 and 2.2) and Section 2 (Step 4), $Z_n^3(t) \to_{\mathscr{L}} 0$. A similar argument also yields the convergence: $Z_n^4(t) \to_{\mathscr{L}} 0$. As for the convergence of $Z_n^2(t)$, we have, using the same argument as in Step 3 in the proof of Theorem 2,

$$Z_n^2(t) = \sum_{j=1}^{K} p_{ji} \int_0^t f(\sqrt{n}X_n^i(s))\sqrt{n}\mu_n^j 1(X_n^j(s)=0)\,\mathrm{d}s$$

$$= p_{ii} f(0) I_n^i(t) + \sum_{j=1, j\neq i}^{K} p_{ji} \int_0^t f(\sqrt{n}X_n^i(s)) 1(X_i(s)>0)\,\mathrm{d}I_n^j(s)$$

$$+ \sum_{j=1, j\neq i}^{K} \int_0^t f(\sqrt{n}X_n^i(s)) 1(X_i(s)=0)\,\mathrm{d}I_n^j(s) \to_{\mathscr{L}} p_{ii} f(0) I_i(t).$$

In obtaining the above convergence, note that $f(\sqrt{n}X_n^i(s))1(X_i(s)>0)\to 0$ as $n\to\infty$ since f has a compact support. □

Lemma 3.1. *Let us assume that* (1) $y_n(t)$, $n\geqslant 1$, *and* $y(t)$, *both of which belong to* $D([0,\infty),R^1)$, *are non-decreasing functions with* $y_n(0)=y(0)=0$, $y_n(t)\to y(t)$ *uniformly on compact t-sets*, (2) $f_n(t)\to f(t)$ *for each t and* f_n *and* f *are bounded. Then we have*

$$\int_0^t |f_n(s) - f(s)|\,\mathrm{d}y_n(s) \to 0 \quad u.o.c.$$

Moreover, suppose that there exists a sequence of step functions $(f_k(t))$ *such that* $f(t) = \lim_{k\to\infty} f_k(t)$ *for each t.* (*A function f is said to be a step function if it has the form*

$$f(t) = c_i, \quad \varepsilon_{i-1} < t < \varepsilon_i$$

for a subdivision of $[0,\infty)$.) *Then we have*

$$\int_0^t f_n(s)\,\mathrm{d}y_n(s) \to \int_0^t f(s)\,\mathrm{d}y(s) \quad u.o.c.$$

Proof. To prove the first convergence, let t be fixed. Given an arbitrary $\varepsilon>0$, there exists a Borel set $A\subset[0,t]$ and an integer N such that $y(A)<\varepsilon$ and $|f_n(s)-f(s)|<\varepsilon$ for all $s\in A^c$ and all $n\geqslant N$ where $y(A)$ represents the measure of the set A induced by $y(\cdot)$ (see Royden, 1968, (Ch. 3, Section 6, Proposition 23, p. 71)). Note that there exists a finite union of intervals B such that $A\subset B$ and $y(B)<\varepsilon$. Then we have for $n\geqslant N$,

$$\int_0^t |f_n(s) - f(s)|\,\mathrm{d}y_n(s)$$

$$= \int_0^t |f_n(s) - f(s)| 1_{B^c}(s)\,\mathrm{d}y_n(s) + \int_0^t |f_n(s) - f(s)| 1_B(s)\,\mathrm{d}y_n(s)$$

$$\leqslant \varepsilon y_n(t) + 2C y_n(B),$$

where C is the bound for f_n and f. Since $y_n(B) \to y(B)$ and $y_n(t) \to y(t)$, we have the first convergence. To see the second convergence, owing to the first convergence it suffices to show

$$\int_0^t f(s)\,\mathrm{d}y_n(s) \to \int_0^t f(s)\,\mathrm{d}y(s) \quad \text{u.o.c.}$$

We have

$$\int_0^t f(s)\,\mathrm{d}y_n(s) = \int_0^t (f(s) - f_k(s))\,\mathrm{d}y_n(s) + \int_0^t f_k(s)\,\mathrm{d}y_n(s).$$

Then using the same argument as in the proof of the first convergence, for any $\varepsilon > 0$, there exists a K such that for $k \geqslant K$,

$$\int_0^t |f(s) - f_k(s)|\,\mathrm{d}y_n(s) < \varepsilon \quad \text{uniformly in } n$$

and

$$\int_0^t |f(s) - f_k(s)|\,\mathrm{d}y(s) \leqslant \varepsilon.$$

Then

$$\begin{aligned}
\Delta_n(t) \equiv \left| \int_0^t f(s)\,\mathrm{d}y_n(s) - \int_0^t f(s)\,\mathrm{d}y(s) \right| &\leqslant \left| \int_0^t f(s)\,\mathrm{d}y_n(s) - \int_0^t f_k(s)\,\mathrm{d}y_n(s) \right| \\
&+ \left| \int_0^t f_k(s)\,\mathrm{d}y_n(s) - \int_0^t f_k(s)\,\mathrm{d}y(s) \right| + \left| \int_0^t f_k(s)\,\mathrm{d}y(s) - \int_0^t f(s)\,\mathrm{d}y(s) \right| \\
&\leqslant 2\varepsilon + \left| \int_0^t f_k(s)\,\mathrm{d}y_n(s) - \int_0^t f_k(s)\,\mathrm{d}y(s) \right|.
\end{aligned}$$

Letting n tend to infinity, for a step function $f_k(s)$ we have

$$\int_0^t f_k(s)\,\mathrm{d}y_n(s) - \int_0^t f_k(s)\,\mathrm{d}y(s) \to 0 \quad \text{u.o.c.}$$

Hence

$$\limsup_n \Delta_n(t) \leqslant 2\varepsilon \quad \text{u.o.c.}$$

Since ε was arbitrary, we have $\lim_n \Delta_n(t) = 0$ u.o.c. □

Acknowledgements

The author is grateful to Prof. T. Kurtz for the helpful discussion at the Conference on Applied Probability (1995, Atlanta). He also thanks a referee for the careful reading of the paper and useful suggestions.

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