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Navier–Stokes equations and forward–backward SDEs on the group of diffeomorphisms of a torus

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Abstract

We establish a connection between the strong solution to the spatially periodic Navier–Stokes equations and a solution to a system of forward–backward stochastic differential equations (FBSDEs) on the group of volume-preserving diffeomorphisms of a flat torus. We construct representations of the strong solution to the Navier–Stokes equations in terms of diffusion processes.

Keywords: Navier–Stokes equations; Forward–backward SDEs; Diffeomorphism group

1. Introduction

The classical Navier–Stokes equations read as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}u(t,x) = -(u,\nabla)u(t,x) + \nu\Delta u(t,x) - \nabla p(t,x), \\ &\operatorname{div} u = 0, \\ &u(0,x) = -u_0(x), \end{aligned} \tag{1}$$

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where $u_0(x)$ is a divergence-free smooth vector field. We fix a time interval $[0, T]$, and rewrite Eqs. (1) with respect to the function

$$\tilde{u}(t,x) = -u(T-t,x).$$

Problem (1) is equivalent to the following:

$$\begin{aligned}
&\frac{\partial}{\partial t}\tilde{u}(t,x) = -(\tilde{u},\nabla)\tilde{u}(t,x) - \nu\Delta\tilde{u}(t,x) - \nabla\tilde{p}(t,x),\\
&\operatorname{div}\tilde{u} = 0,\\
&\tilde{u}(T,x) = u_0(x),
\end{aligned} \tag{2}$$

where $\tilde{p}(t,x) = p(T-t,x)$.

In what follows, system (2) will be referred to as the backward Navier–Stokes equations. To this system we associate a certain system of forward–backward stochastic differential equations on the group of volume-preserving diffeomorphisms of a flat torus. For simplicity, we work in two dimensions. However, the generalization of most of the results to the case of n dimensions is straightforward. The necessary constructions and non-straightforward generalizations related to the n-dimensional case are considered in the Appendix.

Assuming the existence of a solution of (2) with the final data in the Sobolev space H^α for sufficiently large α, we construct a solution of the associated system of FBSDEs. Conversely, if we assume that a solution of the system of FBSDEs exists, then the solution of the Navier–Stokes equations can be obtained from the solution of the FBSDEs. In fact, the constructed FBSDEs on the group of volume-preserving diffeomorphisms can be regarded as an alternative object to the Navier–Stokes equations for studying the properties of the latter.

The connection between forward–backward SDEs and quasi-linear PDEs in finite dimensions has been studied by many authors, for example in [9,18,22].

Our construction uses the approach originating in the work of Arnold [3] which states that the motion of a perfect fluid can be described in terms of geodesics on the group of volume-preserving diffeomorphisms of a compact manifold. The necessary differential-geometric structures were developed in later work by Ebin and Marsden [10]. We note here that [3,10] deal only with differential geometry on the group of maps without involving probability.

The associated system of FBSDEs is solved using the existence of a solution to (2), and by applying results from the works of Gliklikh [12–15]. The latter works use, in turn, the approach to stochastic differential equations on Banach manifolds developed by Dalecky and Belopolskaya [5], and started by McKean [19]. Conversely, a solution of (2) is obtained using the existence of a solution to the associated FBSDEs as well as some ideas and constructions from [9]. However, unlike [9], we work in an infinite-dimensional setting.

Representations of the Navier–Stokes velocity field as a drift of a diffusion process were initiated in [24,20]. A different system of stochastic equations (but not a system of two SDEs) associated to the Navier–Stokes system was introduced and studied in [4]. This system also includes an SDE on the group of volume-preserving diffeomorphisms, but is not a system of forward–backward SDEs. Also, we mention here the works [1,2] discussing probabilistic representations of solutions to the Navier–Stokes equations, and the work [6] establishing a stochastic variational principle for the Navier–Stokes equations. Different probabilistic representations of the solution to the Navier–Stokes equations were studied for example in [17,7]. We note that the list of literature on probabilistic approaches to the Navier–Stokes equations as well as connections between finite-dimensional FBSDEs and PDEs cited in this paper is by no means complete.

The method of applying infinite-dimensional forward–backward SDEs in connection to the Navier–Stokes equations is employed, to the authors' knowledge, for the first time.

2. Geometry of the diffeomorphism group of the two-dimensional torus

Let $\mathbb{T}^2 = S^1 \times S^1$ be the two-dimensional torus, and let $H^\alpha(\mathbb{T}^2)$, $\alpha > 2$, be the space of H^α-Sobolev maps $\mathbb{T}^2 \to \mathbb{T}^2$. By G^α we denote the subset of $H^\alpha(\mathbb{T}^2)$ whose elements are C^1-diffeomorphisms. Let G^α_{V} be the subgroup of G^α consisting of diffeomorphisms preserving the volume measure on $\mathbb{T}^2$.

Lemma 1. *Let g be an H^α-map and a local diffeomorphism of a finite-dimensional compact manifold M, F be an H^α-section of the tangent bundle TM. Then, $F \circ g$ is an H^α-map.*

Proof. See [14] (p. 139) or [10] (p. 108). □

Let R_g denote the right translation on G^α, i.e. $R_g(\eta) = \eta \circ g$.

Lemma 2. *The map R_g is C^∞-smooth for every $g \in G^\alpha$. Furthermore, for every $\eta \in G^\alpha$, the tangent map TR_g restricted to the tangent space $T_\eta G^\alpha$ is defined by the formula:*

$$TR_g : T_\eta G^\alpha \to T_{\eta\circ g} G^\alpha, \; X \mapsto X \circ g.$$

Proof. The proof easily follows from the α-lemma (see [10,14,15]). □

Lemma 3. *The groups G^α and G^α_{V} are infinite-dimensional Hilbert manifolds. The group G^α_{V} is a subgroup and a smooth submanifold of G^α.*

Lemma 4. *The tangent space $T_e G^\alpha$ is formed by all H^α-vector fields on $\mathbb{T}^2$. The tangent space $T_e G^\alpha_{\mathrm{V}}$ is formed by all divergence-free H^α-vector fields on $\mathbb{T}^2$.*

The proof of Lemmas 3 and 4 can be found for example in [10,14,15].

Lemma 5. *Let $X \in T_e G^\alpha$ be an H^α-vector field on $\mathbb{T}^2$. Then the vector field $\hat{X}$ on G^α defined by $\hat{X}(g) = X \circ g$ is right-invariant. Furthermore, $\hat{X}$ is C^k-smooth if and only if $X \in H^{\alpha+k}$.*

Proof. The first statement follows from Lemma 2. The proof of the second statement can be found in [10]. □

The vector field $\hat{X}$ on G^α defined in Lemma 5 will be referred to below as the *right-invariant* vector field generated by $X \in T_e G^\alpha$.

Let $g \in G^\alpha$, $X, Y \in T_e G^\alpha$. Consider the weak $(\cdot\,,\,\cdot)_0$ and the strong $(\cdot\,,\,\cdot)_\alpha$ Riemannian metrics on G^α (see [15]):

$$(\hat{X}(g), \hat{Y}(g))_0 = \int_{\mathbb{T}^2} (X \circ g(\theta), Y \circ g(\theta)) \mathrm{d}\theta, \tag{3}$$

$$\begin{aligned}(\hat{X}(g), \hat{Y}(g))_\alpha = & \int_{\mathbb{T}^2} (X \circ g(\theta), Y \circ g(\theta)) \mathrm{d}\theta \\ & + \int_{\mathbb{T}^2} ((d+\delta)^\alpha X \circ g(\theta), (d+\delta)^\alpha Y \circ g(\theta)) \mathrm{d}\theta \end{aligned} \tag{4}$$

where d is the differential, δ is the codifferential, $\hat{X}$ and $\hat{Y}$ are the right-invariant vector fields on G^α generated by the H^α-vector fields X and Y. Metric (3) gives rise to the L_2-topology on the tangent spaces of G^α, and metric (4) gives rise to the H^α-topology on the tangent spaces of G^α (see [15]). If $g \in G^\alpha_{\mathrm{V}}$, then scalar products (3) and (4) do not depend on g. Moreover, for the strong metric on G^α_{V}, we have the following formula:

$$(\hat{X}(g), \hat{Y}(g))_\alpha = \int_{\mathbb{T}^2} (X \circ g(\theta), (1+\Delta)^\alpha Y \circ g(\theta)) \mathrm{d}\theta$$

where $\Delta = (d\delta + \delta d)$ is the Laplace–de Rham operator (see [23]).

Let us introduce the notation:

$$\mathbb{Z}_2^+ = \{(k_1, k_2) \in \mathbb{Z}^2 : k_1 > 0 \text{ or } k_1 = 0, k_2 > 0\};$$
$$k = (k_1, k_2) \in \mathbb{Z}_2^+, \qquad \bar{k} = (k_2, -k_1), \qquad |k| = \sqrt{k_1^2 + k_2^2}, \qquad k \cdot \theta = k_1\theta_1 + k_2\theta_2,$$
$$\theta = (\theta_1, \theta_2) \in \mathbb{T}^2, \qquad \nabla = \left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}\right), \qquad (\bar{k}, \nabla) = k_2 \frac{\partial}{\partial\theta_1} - k_1 \frac{\partial}{\partial\theta_2},$$

and the vectors

$$\bar{A}_k(\theta) = \frac{1}{|k|^{\alpha+1}} \cos(k\cdot\theta) \begin{pmatrix} k_2 \\ -k_1 \end{pmatrix}, \qquad \bar{B}_k(\theta) = \frac{1}{|k|^{\alpha+1}} \sin(k\cdot\theta) \begin{pmatrix} k_2 \\ -k_1 \end{pmatrix},$$
$$\bar{A}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \bar{B}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $\{A_k(g), B_k(g)\}_{k\in\mathbb{Z}_2^+\cup\{0\}}$ be the right-invariant vector fields on G^α generated by $\{\bar{A}_k, \bar{B}_k\}_{k\in\mathbb{Z}_2^+\cup\{0\}}$, i.e.

$$A_k(g) = \bar{A}_k \circ g, \qquad B_k(g) = \bar{B}_k \circ g, \quad g \in G^\alpha,$$
$$A_0 = \bar{A}_0, \quad B_0 = \bar{B}_0.$$

By ω-lemma (see [14]), A_k and B_k are C^∞-smooth vector fields on G^α.

Lemma 6. *The vectors $A_k(g)$, $B_k(g)$, $k \in \mathbb{Z}_2^+ \cup \{0\}$, $g \in G^\alpha_{\mathrm{V}}$, form an orthogonal basis of the tangent space $T_g G^\alpha_{\mathrm{V}}$ with respect to both the weak and the strong inner products in $T_g G^\alpha_{\mathrm{V}}$. In particular, the vectors $\bar{A}_k$, $\bar{B}_k$, $k \in \mathbb{Z}_2^+ \cup \{0\}$, form an orthogonal basis of the tangent space $T_e G^\alpha_{\mathrm{V}}$. Moreover, the weak and the strong norms of the basis vectors are bounded by the same constant.*

Proof. It suffices to prove the lemma for the strong norm. Let us compute $\Delta^\alpha \bar{A}_k$. Note that the vectors $\frac{k}{|k|}$ and $\frac{\bar{k}}{|k|}$ form an orthonormal basis of $\mathbb{R}^2$. Let us observe that by the identity $(\bar{k}, \nabla)\cos(k\cdot\theta) = 0$, $\delta\bar{A}_k = 0$. Hence $d\delta\bar{A}_k = 0$ which implies $\Delta\bar{A}_k = \delta d\bar{A}_k$. We obtain:

$$\bar{A}_k = \frac{1}{|k|^\alpha} \cos(k\cdot\theta) \frac{\bar{k}}{|k|},$$
$$\mathrm{d}\,\bar{A}_k = -\frac{1}{|k|^{\alpha-1}} \sin(k\cdot\theta) \frac{k}{|k|} \wedge \frac{\bar{k}}{|k|},$$
$$\Delta\bar{A}_k = \delta d\bar{A}_k = \frac{1}{|k|^{\alpha-2}} \cos(k\cdot\theta) \frac{\bar{k}}{|k|} = |k|^2 \bar{A}_k,$$

$$\Delta^\alpha \bar{A}_k = |k|^\alpha \cos(k\cdot\theta)\frac{\bar{k}}{|k|} = |k|^{2\alpha}\bar{A}_k.$$

This and the volume-preserving property of $g \in G^\alpha_{\mathrm{V}}$ imply that

$$(B_m(g), A_k(g))_\alpha = (\bar{B}_m, \bar{A}_k)_\alpha = (1+|k|^{2\alpha})(\bar{B}_m, \bar{A}_k)_{L_2} = 0,$$
$$\|A_k(g)\|^2_\alpha = \|\bar{A}_k\|^2_\alpha = \left(1+|k|^{2\alpha}\right)\|\bar{A}_k\|^2_{L_2} = 2\pi^2\left(|k|^{-2\alpha}+1\right)$$

where $\|\cdot\|_\alpha$ is the norm corresponding to the scalar product $(\cdot,\cdot)_\alpha$. Thus, $2\pi^2 \leqslant \|A_k(g)\|^2_\alpha \leqslant 4\pi^2$. Clearly, for the $\|B_k(g)\|^2_\alpha$ we obtain the same. □

It has been shown, for example, in [10] and [15] that the weak Riemannian metric has the Levi-Civita connection, geodesics, the exponential map, and the spray. Let $\bar{\nabla}$ and $\tilde{\nabla}$ denote the covariant derivatives of the Levi-Civita connection of the weak Riemannian metric (3) on G^α and G^α_{V}, respectively. In [10] (see also [15,14]), it has been shown that

$$\tilde{\nabla} = \mathrm{P}\circ\bar{\nabla}$$

where $\mathrm{P} : TG^\alpha \to TG^\alpha_{\mathrm{V}}$ is defined in the following way: on each tangent space T_gG^α, $\mathrm{P} = P_g$ where $P_g = TR_g \circ P_e \circ TR_{g^{-1}}$, TR_g and $TR_{g^{-1}}$ are tangent maps, and $P_e : T_eG^\alpha \to T_eG^\alpha_{\mathrm{V}}$ is the projector defined by the Hodge decomposition.

Lemma 7. *Let $\hat{U}$ be the right-invariant vector field on G^α generated by an $H^{\alpha+1}$-vector field U on $\mathbb{T}^2$, and let $\hat{V}$ be the right-invariant vector field on G^α generated by an H^α-vector field V on $\mathbb{T}^2$. Then $\bar{\nabla}_{\hat{V}}\hat{U}$ is the right-invariant vector field on G^α generated by the H^α-vector field $\nabla_V U$ on $\mathbb{T}^2$.*

Lemma 8. *Let $\hat{U}$ be the right-invariant vector field on G^α_{V} generated by a divergence-free $H^{\alpha+1}$-vector field U on $\mathbb{T}^2$, and let $\hat{V}$ be the right-invariant vector field on G^α generated by a divergence-free H^α-vector field V on $\mathbb{T}^2$. Then $\tilde{\nabla}_{\hat{V}}\hat{U}$ is the right-invariant vector field on G^α_{V} generated by the divergence-free H^α-vector field $P_e\nabla_V U$ on $\mathbb{T}^2$.*

The proofs of Lemmas 7 and 8 follow from the right-invariance of covariant derivatives on G^α and G^α_{V} (see [15]).

Remark 1. The basis $\{\bar{A}_k, \bar{B}_k\}_{k\in\mathbb{Z}^+_2\cup\{0\}}$ of $T_eG^\alpha_{\mathrm{V}}$ can be extended to a basis of the entire tangent space T_eG^α. Indeed, let us introduce the vectors:

$$\bar{\mathcal{A}}_k(\theta) = \frac{1}{|k|^{\alpha+1}}\cos(k\cdot\theta)\begin{pmatrix}k_1\\k_2\end{pmatrix}, \qquad \bar{\mathcal{B}}_k(\theta) = \frac{1}{|k|^{\alpha+1}}\sin(k\cdot\theta)\begin{pmatrix}k_1\\k_2\end{pmatrix}, \quad k\in\mathbb{Z}^+_2.$$

The system $\bar{A}_k, \bar{B}_k, k\in\mathbb{Z}^+_2\cup\{0\}, \bar{\mathcal{A}}_k, \bar{\mathcal{B}}_k, k\in\mathbb{Z}^+_2$, form an orthogonal basis of T_eG^α. Further let $\mathcal{A}_k$ and $\mathcal{B}_k$ denote the right-invariant vector fields on G^α generated by $\bar{\mathcal{A}}_k$ and $\bar{\mathcal{B}}_k$.

3. The FBSDEs on the group of diffeomorphisms of the two-dimensional torus

Let $h : \mathbb{T}^2 \to \mathbb{R}^2$ be a divergence-free $H^{\alpha+1}$-vector field on $\mathbb{T}^2$, and let $\hat{h}$ be the right-invariant vector field on G^α generated by h. Further let the function $V(s,\cdot)$ be such that there exists a function $p : [t,T] \to H^{\alpha+1}(\mathbb{T}^2,\mathbb{R})$ satisfying $V(s,\cdot) = \nabla p(s,\cdot)$ for all

$s \in [t, T]$. For each $s \in [t, T]$, $\hat{V}(s, \cdot)$ denotes the right-invariant vector field on G^α generated by $V(s, \cdot) \in H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$.

Let E be a Euclidean space spanned on an orthonormal, relative to the scalar product in E, system of vectors $\{e_k^A, e_k^B, e_0^A, e_0^B\}_{k \in \mathbb{Z}_2^+, |k| \leqslant N}$. Consider the map

$$\sigma(g) = \sum_{\substack{k \in \mathbb{Z}_2^+ \cup \{0\}, \\ |k| \leqslant N}} A_k(g) \otimes e_k^A + B_k(g) \otimes e_k^B, \quad g \in G^\alpha,$$

i.e. $\sigma(g)$ is a linear operator $E \to T_g G^\alpha$ for each $g \in G^\alpha$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and W_s, $s \in [t, T]$, be an E-valued Brownian motion:

$$W_s = \sum_{\substack{k \in \mathbb{Z}_2^+ \cup \{0\}, \\ |k| \leqslant N}} (\beta_k^A(s) e_k^A + \beta_k^B(s) e_k^B)$$

where $\{\beta_k^A, \beta_k^B\}_{k \in \mathbb{Z}_2^+ \cup \{0\}, |k| \leqslant N}$ is a sequence of independent Brownian motions. We consider the following system of forward and backward SDEs:

$$\begin{cases} \mathrm{d}Z_s^{t,e} = Y_s^{t,e} \mathrm{d}s + \epsilon \sigma(Z_s^{t,e}) \mathrm{d}W_s, \\ \mathrm{d}Y_s^{t,e} = -\hat{V}(s, Z_s^{t,e}) \mathrm{d}s + X_s^{t,e} \mathrm{d}W_s, \\ Z_t^{t,e} = e; \qquad Y_T^{t,e} = \hat{h}(Z_T^{t,e}). \end{cases} \tag{5}$$

The forward SDE of (5) is an SDE on G_{V}^α where G_{V}^α is considered as a Hilbert manifold. Stochastic differentials and stochastic differential equations on Hilbert manifolds are understood in the sense of Dalecky and Belopolskaya's approach (see [5]). More precisely, we use the results from [14] which interprets the latter approach for the particular case of SDEs on Hilbert manifolds. The stochastic integral in the forward SDE can be explicitly written as follows:

$$\int_t^s \sigma(Z_r^{t,e}) \mathrm{d}W_r = \sum_{k \in \mathbb{Z}_2^+ \cup \{0\}, |k| \leqslant N} \int_t^s A_k(Z_r^{t,e}) \mathrm{d}\beta_k^A(r) + B_k(Z_r^{t,e}) \mathrm{d}\beta_k^B(r). \tag{6}$$

Let us consider the backward SDE:

$$Y_s^{t,e} = \hat{h}(Z_T^{t,e}) + \int_s^T \hat{V}(r, Z_r^{t,e}) \mathrm{d}r - \int_s^T X_r^{t,e} \mathrm{d}W_r. \tag{7}$$

Note that the processes $\hat{V}(s, Z_s^{t,e}) = V(s, \cdot) \circ Z_s^{t,e}$ and $\hat{h}(Z_T^{t,e}) = h \circ Z_T^{t,e}$ are H^α-maps by Lemma 1. Therefore, it makes sense to understand SDE (7) as an SDE in the Hilbert space $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$. Let $\mathcal{F}_s = \sigma(W_r, r \in [0, s])$. We would like to find an $\mathcal{F}_s$-adapted triple of stochastic processes $(Z_s^{t,e}, Y_s^{t,e}, X_s^{t,e})$ solving FBSDEs (5) in the following sense: at each time s, the process $(Z_s^{t,e}, Y_s^{t,e})$ takes values in an H^α-section of the tangent bundle TG_{V}^α. Namely, for each $s \in [t, T]$ and $\omega \in \Omega$, $Z_s^{t,e} \in G_{\mathrm{V}}^\alpha$, $Y_s^{t,e} \in T_{Z_s^{t,e}} G_{\mathrm{V}}^\alpha$. Therefore, the forward SDE is well-posed on both G^α and G_{V}^α, and can be written in the Dalecky–Belopolskaya form:

$$\mathrm{d}Z_s^{t,e} = \exp_{Z_s^{t,e}}\{Y_s^{t,e} \mathrm{d}s + \epsilon \sigma(Z_s^{t,e}) \mathrm{d}W_s\} \quad \text{or}$$
$$\mathrm{d}Z_s^{t,e} = \widetilde{\exp}_{Z_s^{t,e}}\{Y_s^{t,e} \mathrm{d}s + \epsilon \sigma(Z_s^{t,e}) \mathrm{d}W_s\}$$

where $\exp$ and $\widetilde{\exp}$ are the exponential maps of the Levi-Civita connection of the weak Riemannian metrics (3) on G^α and resp. G_{V}^α. Below, we will show that using either of these representations leads to the same solution of FBSDEs (5).

Finally, the process $X_s^{t,e}$ takes values in the space of linear operators $\mathcal{L}(E, H^\alpha(\mathbb{T}^2, \mathbb{R}^2))$, i.e.

$$X_s^{t,e} = \sum_{k \in \mathbb{Z}_2^+ \cup \{0\}, |k| \leqslant N} X_s^{kA} \otimes e_k^A + X_s^{kB} \otimes e_k^B \tag{8}$$

where the processes X_s^{kA} and X_s^{kB} take values in $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$.

Remark 2. The results obtained below also work in the situation when the Brownian motion W_s is infinite dimensional (as in [8]). Namely, when $W_s = \sum_{k \in \mathbb{Z}_2^+ \cup \{0\}} a_k \beta_k^A \otimes e_k^A + b_k \beta_k^B \otimes e_k^B$ where $a_k, b_k, k \in \mathbb{Z}_2^+ \cup \{0\}$, are real numbers satisfying $\sum_{k \in \mathbb{Z}_2^+ \cup \{0\}} |a_k|^2 + |b_k|^2 < \infty$. However, this requires an additional analysis on the solvability of the forward SDE based on the approach of Dalecky and Belopolskaya [5] since the results of Gliklikh [12,14,15] applied below are obtained for the case of a finite-dimensional Brownian motion.

4. Constructing a solution of the FBSDEs

4.1. The forward SDE

Let us consider the backward Navier–Stokes equations in $\mathbb{R}^2$:

$$\begin{aligned} &y(s,\theta) = h(\theta) + \int_s^T \big[\nabla p(r,\theta) + \big(y(r,\theta), \nabla\big) y(r,\theta) + \nu \Delta y(r,\theta)\big] \mathrm{d}r, \\ &\operatorname{div} y(s,\theta) = 0 \end{aligned} \tag{9}$$

where $s \in [t, T]$, $\theta \in \mathbb{T}^2$, Δ and ∇ are the Laplacian and the gradient.

Assumption 1. Let us assume that on the interval $[t, T]$ there exists a solution $\big(y(s,\,\cdot\,), p(s,\,\cdot\,)\big)$ to (9) such that the functions $p : [t, T] \to H^{\alpha+1}(\mathbb{T}^2, \mathbb{R})$ and $y : [t, T] \to H^{\alpha+1}(\mathbb{T}^2, \mathbb{R}^2)$ are continuous.

Clearly, $y(s,\,\cdot\,) \in T_e G_{\mathrm{V}}^\alpha$. Let $\{Y_s^{t;kA}, Y_s^{t;kB}\}_{k \in \mathbb{Z}_2^+ \cup \{0\}}$ be the coordinates of $y(s,\,\cdot\,)$ with respect to the basis $\{\bar{A}_k, \bar{B}_k\}_{k \in \mathbb{Z}_2^+ \cup \{0\}}$, i.e.

$$y(s,\theta) = \sum_{k \in \mathbb{Z}_2^+ \cup \{0\}} Y_s^{t;kA} \bar{A}_k(\theta) + Y_s^{t;kB} \bar{B}_k(\theta).$$

Let $\hat{Y}_s(\,\cdot\,)$ denote the right-invariant vector field on G^α generated by the solution $y(s,\,\cdot\,)$, i.e. $\hat{Y}_s(g) = y(s,\,\cdot\,) \circ g$. On each tangent space $T_g G^\alpha$, the vector $\hat{Y}_s(g)$ can be represented by a series converging in the H^α-topology:

$$\hat{Y}_s(g) = \sum_{k \in \mathbb{Z}_2^+ \cup \{0\}} Y_s^{t;kA} A_k(g) + Y_s^{t;kB} B_k(g). \tag{10}$$

In this paragraph we will study the SDE:

$$d Z_s^{t,e} = \hat{Y}_s(Z_s^{t,e}) \mathrm{d}s + \epsilon \sigma(Z_s^{t,e}) \mathrm{d}W_s. \tag{11}$$

Later, in Theorem 6, we will show that the solution $Z_s^{t,e}$ to (11) and the process $Y_s^{t,e} = \hat{Y}_s(Z_s^{t,e})$ are the first two processes in the triple $(Z_s^{t,e}, Y_s^{t,e}, X_s^{t,e})$ that solves FBSDEs (5).

Theorem 1. *There exists a unique strong solution $Z_s^{t,e}$, $s \in [t, T]$, to (11) on G_V^α, with the initial condition $Z_t^{t,e} = e$.*

Proof. Below, we verify the assumptions of Theorem 13.5 of [15]. The latter theorem will imply the existence and uniqueness of the strong solution to (11). Note that, if sum (6) representing the stochastic integral $\int_t^s \sigma(Z_s^{t,e}) \mathrm{d}W_s$ contains only the terms $A_0(\beta_0^A(s) - \beta_0^A(t))$ and $B_0(\beta_0^B(s) - \beta_0^B(t))$, i.e., informally speaking, if the Brownian motion runs only along the constant vectors A_0 and B_0, then the statement of the theorem follows from Theorem 28.3 of [15]. If sum (6) contains also terms with A_k and B_k, $k \in \mathbb{Z}_2^+$, or, informally, when the Brownian motion runs also along non-constant vectors A_k and B_k, $k \in \mathbb{Z}_2^+$, then the assumptions of Theorem 13.5 of [15] require the boundedness of A_k and B_k with respect to the strong norm. The latter fact holds by Lemma 6.

Hence, all the assumptions of Theorem 13.5 of [15] are satisfied. Indeed, the proof of Theorem 28.3 of [15] shows that the Levi-Civita connection of the weak Riemannian metric (3) on G_V^α is compatible (see Definition 13.7 of [15]) with the strong Riemannian metric (4). The function $\sigma(g) = \sum_{k \in \mathbb{Z}_2^+ \cup \{0\}, |k| \leqslant N} A_k(g) \otimes e_k^A + B_k(g) \otimes e_k^B$ is C^∞-smooth since A_k and B_k are C^∞-smooth. Moreover, by Lemma 6, $\sigma(g)$ is bounded on G_V^α. Next, since $y : [t, T] \to H^{\alpha+1}(\mathbb{T}^2, \mathbb{R}^2)$ is continuous, then it is also bounded with respect to (at least) the H^α-norm. Hence, the generated right-invariant vector field $\hat{Y}_s(g)$ is bounded in s with respect to the strong metric (4), and it is at least C^1-smooth in g. The boundedness of $\hat{Y}_s$ in g follows from the volume-preserving property of g. $\square$

Theorem 2. *There exists a unique strong solution $Z_s^{t,e}$, $s \in [t, T]$, to (11) on G^α, with the initial condition $Z_t^{t,e} = e$. This solution coincides with the solution to SDE (11) on G_V^α.*

Proof. Consider the identical imbedding $\imath : G_V^\alpha \to G^\alpha$. By results of [5] (Proposition 1.3, p. 146; see also [15], p. 64), the stochastic process $\imath(Z_s^{t,e}) = Z_s^{t,e}$, $s \in [t, T]$, is a solution to SDE (11) on G^α, i.e. with respect to the exponential map exp. This easily follows from the fact that $T\imath : TG_V^\alpha \to TG^\alpha$, where T is the tangent map, is the identical imbedding, and that $\imath\big(\exp(X)\big) = \tilde{\exp}(T\imath \circ X)$. The solution $Z_s^{t,e}$ to (11) on G^α is unique. This follows from the uniqueness theorem for SDE (11) considered on the manifold G^α equipped with the weak Riemannian metric. Indeed, $\sigma(g)$ and $\hat{Y}_s(g)$ are bounded with respect to the weak metric (3) since the functions $\bar{A}_k$, $\bar{B}_k$, $k \in \mathbb{Z}_2^+ \cup \{0\}$, are bounded on $\mathbb{T}^2$, and $y(\cdot\,, \cdot)$ is bounded on $[t, T] \times \mathbb{T}^2$. Moreover $\sigma(g)$ is C^∞-smooth and $\hat{Y}_s$ is at least C^1-smooth on G^α. $\square$

One can also consider (11) as an SDE with values in the Hilbert space $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$.

Theorem 3. *There exists a unique strong solution $Z_s^{t,e}$ to the $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$-valued SDE (11) on $[t, T]$, with the initial condition $Z_t^{t,e} = e$ where e is the identity of G_V^α. This solution coincides with the solution to SDE (11) on G_V^α or G^α.*

Proof. By Theorem 1, SDE (11) on G_V^α has a unique strong solution $Z_s^{t,e}$ on $[t, T]$. Let us prove that the solution $Z_s^{t,e}$ to (11) solves this SDE considered as an SDE in $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$. Consider the identical imbedding $\imath_V : G_V^\alpha \to H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$, $g \mapsto g$. Applying Itô's formula to $\imath_V$, and taking into account that $A_k(g)\,\imath_V(g) = \nabla_{\bar{A}_k}\theta \circ g = A_k(g)$ and that $A_k(g)A_k(g)\,\imath_V(g) = A_k(g)A_k(g) = 0$, we obtain that the solution $Z_s^{t,e}$ to (11) on G_V^α solves the $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$-valued SDE (11). Note that by the uniqueness theorem for SDEs in Hilbert spaces, SDE (11) can have only one solution in $L_2(\mathbb{T}^2, \mathbb{R}^2)$. This proves the uniqueness of its solution in $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$ as well. Thus the solutions to (11) on G^α, G^α_{V}, and in $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$ coincide. □

Let us find the representations of SDE (11) in normal coordinates on G^α and G^α_{V}. First, we prove the following lemma.

Lemma 9. *The following equality holds:*

$$\int_t^s \sigma(Z_r^{t,e}) \circ \mathrm{d}W_r = \int_t^s \sigma(Z_r^{t,e}) \mathrm{d}W_r,$$

i.e. instead of the Itô stochastic integral in (11) we can write the Stratonovich stochastic integral $\int_t^s \sigma(Z_r^{t,e}) \circ \mathrm{d}W_r$.

Proof. We have:

$$\sigma(Z_r^{t,e}) \circ \mathrm{d}W_r = \sigma(Z_r^{t,e})\mathrm{d}W_r + \sum_{k \in \mathbb{Z}_2^+ \cup \{0\}, |k| \leqslant N} \mathrm{d}A_k(Z_r^{t,e})\mathrm{d}\beta_k^A(r) + \mathrm{d}B_k(Z_r^{t,e})\mathrm{d}\beta_k^B(r).$$

Hence, we have to prove that $\mathrm{d}A_k(Z_r^{t,e})\mathrm{d}\beta_k^A(r) = 0$ and $\mathrm{d}B_k(Z_r^{t,e})\mathrm{d}\beta_k^B(r) = 0$. For simplicity of notation we use the notation A_ν for both of the vector fields A_k and B_k and the notation $\bar{A}_\nu$ for $\bar{A}_k$ and $\bar{B}_k$, $k \in \mathbb{Z}_2^+ \cup \{0\}$. Also, we use the notation $\beta_\nu(s)$ for the Brownian motions $\{\beta_k^A(s), \beta_k^B(s)\}_{k \in \mathbb{Z}_2^+ \cup \{0\}, |k| \leqslant N}$. We obtain:

$$d(\bar{A}_\nu \circ Z_s^{t,e}) = \sum_\gamma A_\gamma(Z_s^{t,e})\big(\bar{A}_\nu \circ Z_s^{t,e}\big) \circ \mathrm{d}\beta_\gamma(s) + Y_s^{t,e}\big(\bar{A}_\nu \circ Z_s^{t,e}\big)\mathrm{d}t.$$

This implies

$$d(\bar{A}_\nu \circ Z_s^{t,e}) \cdot \mathrm{d}\beta_\nu = A_\nu(Z_s^{t,e})\big(\bar{A}_\nu \circ Z_s^{t,e}\big)\mathrm{d}s = 0$$

which holds by the identity $(\bar{k}, \nabla)\cos(k \cdot \theta) = (\bar{k}, \nabla)\sin(k \cdot \theta) = 0$ or by differentiating of constant vector fields. □

Let $\bar{Z}_s^t = \{Z_s^{t;kA}, Z_s^{t;kB}\}_{k \in \mathbb{Z}_2^+ \cup \{0\}}$ be the vector of local coordinates of the solution $Z_s^{t,e}$ to (11) on G^α_{V}, i.e. the vector of normal coordinates provided by the exponential map $\tilde{\exp} : T_e G^\alpha_{\mathrm{V}} \to G^\alpha_{\mathrm{V}}$. Let U_e be the canonical chart of the map $\tilde{\exp}$.

Theorem 4 (*SDE (11) in Local Coordinates*). *Let*

$$\tau = \inf\{s \in [t, T] : Z_s^{t,e} \notin U_e\}. \tag{12}$$

On the interval $[t, \tau]$*, SDE (11) has the following representation in local coordinates:*

$$\begin{aligned} Z_{s \wedge \tau}^{t,kA} &= \int_t^{s \wedge \tau} Y_r^{t;kA}\mathrm{d}r + \delta_k \epsilon(\beta_k^A(s \wedge \tau) - \beta_k^A(t)), \\ Z_{s \wedge \tau}^{t,kB} &= \int_t^{s \wedge \tau} Y_r^{t;kB}\mathrm{d}r + \delta_k \epsilon(\beta_k^B(s \wedge \tau) - \beta_k^B(t)) \end{aligned} \tag{13}$$

where $\delta_k = 1$ *if* $|k| \leqslant N$*, and* $\delta_k = 0$ *if* $|k| > N$.

Proof. Let $\bar{g} = \{g^{kA}, g^{kB}\}_{k \in \mathbb{Z}_2^+ \cup \{0\}}$ be local coordinates in the neighborhood U_e provided by the map $\tilde{\exp}$. Let $f \in \mathrm{C}^\infty(G^\alpha_{\mathrm{V}})$, and let $\tilde{f} : T_e G^\alpha_{\mathrm{V}} \to \mathbb{R}$ be such that $\tilde{f} = f \circ \tilde{\exp}$. Since $\tilde{\exp}$ is a

C^∞-map (see [10]), then $\tilde{f} \in C^\infty(U_0)$, where $U_0 = \tilde{\exp}^{-1} U_e$. Note that $\frac{\partial}{\partial g^{kA}} \tilde{f}(\bar{g}) = A_k(g) f(g)$ and $\frac{\partial}{\partial g^{kB}} \tilde{f}(\bar{g}) = B_k(g) f(g)$. By Itô's formula, we obtain:

$$
\begin{aligned}
f(Z^{t,e}_{s\wedge\tau}) - f(e) &= \tilde{f}(\bar{Z}^{t,0}_{s\wedge\tau}) - \tilde{f}(0) \\
&= \int_t^{s\wedge\tau} \mathrm{d}r \sum_{k\in\mathbb{Z}^+_2\cup\{0\}} \frac{\partial \tilde{f}}{\partial g^{kA}}(\bar{Z}^t_r) Y^{t;kA}_r + \int_t^{s\wedge\tau} \epsilon \sum_{k\in\mathbb{Z}^+_2\cup\{0\}} \delta_k \frac{\partial \tilde{f}}{\partial g^{kA}}(\bar{Z}^t_r) Y^{t;kA}_r \circ \mathrm{d}\beta^A_k(r) \\
&\quad + \int_t^{s\wedge\tau} \mathrm{d}r \sum_{k\in\mathbb{Z}^+_2\cup\{0\}} \frac{\partial \tilde{f}}{\partial g^{kB}}(\bar{Z}^t_r) Y^{t;kB}_r + \int_t^{s\wedge\tau} \epsilon \sum_{k\in\mathbb{Z}^+_2\cup\{0\}} \delta_k \frac{\partial \tilde{f}}{\partial g^{kB}}(\bar{Z}^t_r) Y^{t;kB}_r \circ \mathrm{d}\beta^B_k(r) \\
&= \int_t^{s\wedge\tau} \mathrm{d}r \sum_{k\in\mathbb{Z}^+_2\cup\{0\}} \big(Y^{t;kA}_r A_k(Z^{t,e}_r) f(Z^{t,e}_r) + Y^{t;kB}_r B_k(Z^{t,e}_r) f(Z^{t,e}_r)\big) \\
&\quad + \int_t^{s\wedge\tau} \epsilon \sum_{k\in\mathbb{Z}^+_2\cup\{0\}} \delta_k \big(A_k(Z^{t,e}_r) f(Z^{t,e}_r) \circ \mathrm{d}\beta^A_k(r) + B_k(Z^{t,e}_r) f(Z^{t,e}_r) \circ \mathrm{d}\beta^B_k(r)\big).
\end{aligned}
$$

Using representations (10) and (6) we obtain:

$$
f(Z^{t,e}_{s\wedge\tau}) - f(e) = \int_t^{s\wedge\tau} \hat{Y}_r(Z^{t,e}_r) f(Z^{t,e}_r)\mathrm{d}r + \int_t^{s\wedge\tau} \epsilon\sigma(Z^{t,e}_r) f(Z^{t,e}_r) \circ \mathrm{d}W_r.
$$

This shows that the process

$$
\exp\Big\{ \sum_{k\in\mathbb{Z}^+_2\cup\{0\}} Z^{t,kA}_{s\wedge\tau} \bar{A}_k + Z^{t,kB}_{s\wedge\tau} \bar{B}_k \Big\}
$$

solves SDE (11) on the interval $[t, \tau]$. □

Let

$$
\check{\bar{Z}}^t_s = \{\check{Z}^{t;kA}_s, \check{Z}^{t;kB}_s, \check{Z}^{t;k\mathcal{A}}_s, \check{Z}^{t;k\mathcal{B}}_s, \check{Z}^{t;0A}_s, \check{Z}^{t;0B}_s\}_{k\in\mathbb{Z}^+_2}
$$

be the vector of local coordinates of the solution $Z^{t,e}_s$ to (11) on G^α, i.e. the vector of normal coordinates provided by the exponential map $\exp : T_e G^\alpha \to G^\alpha$. Further let $\check{U}_e$ be the canonical chart of the map exp.

Theorem 5. *Let*

$$
\check{\tau} = \inf\{s \in [t, T] : Z^{t,e}_s \notin \check{U}_e\}.
$$

Then, a.s. $\check{\tau} = \tau$, *where the stopping time* τ *is defined by* (12), *and on* $[t, \tau]$, $\check{Z}^{t;kA}_s = Z^{t;kA}_s$, $\check{Z}^{t;kB}_s = Z^{t;kB}_s$, $k \in \mathbb{Z}^2_+ \cup \{0\}$, $\check{Z}^{t;k\mathcal{A}}_s = \check{Z}^{t;k\mathcal{B}}_s = 0$, $k \in \mathbb{Z}^+_2$, *a.s.*

Proof. Let us introduce additional local coordinates $g^{k\mathcal{A}}$, $g^{k\mathcal{B}}$, $k \in \mathbb{Z}^+_2$, and perform the same computation as in the proof of Theorem 4. We have to take into account that $Y^{k\mathcal{A}}_s = Y^{k\mathcal{B}}_s = 0$, $k \in \mathbb{Z}^+_2$, and that the components of the Brownian motion are non-zero only along divergence-free and constant vector fields. We obtain that the coordinate process $\check{\bar{Z}}^t_s$ verifies SDEs (13) and the equations $\check{Z}^{t;k\mathcal{A}}_s = \check{Z}^{t;k\mathcal{B}}_s = 0$, $k \in \mathbb{Z}^+_2$. □

4.2. The backward SDE and the solution of the FBSDEs

We have the following result:

Theorem 6. *Let $\hat{Y}_s$ be the right-invariant vector field generated by the solution $y(s,\,\cdot\,)$ to the backward Navier–Stokes equations (9). Further let $Z_s^{t,e}$ be the solution to SDE (11) on G_{V}^{α}. Then there exists an $\epsilon > 0$ such that the triple of stochastic processes*

$$Z_s^{t,e},\ Y_s^{t,e} = \hat{Y}_s(Z_s^{t,e}),\ X_s^{t,e} = \epsilon\sigma(Z_s^{t,e})\hat{Y}_s(Z_s^{t,e})$$

solves FBSDEs (5) on the interval $[t,T]$.

Remark 3. The expression $\sigma(Z_s^{t,e})\hat{Y}_s(Z_s^{t,e})$ means the following:

$$\sigma(Z_s^{t,e})\hat{Y}_s(Z_s^{t,e}) = \sum_{k\in\mathbb{Z}_2^+\cup\{0\},|k|\leqslant N} A_k(Z_s^{t,e})\hat{Y}_s(Z_s^{t,e})\otimes e_k^A + B_k(Z_s^{t,e})\hat{Y}_s(Z_s^{t,e})\otimes e_k^B$$

where $\hat{Y}_s(\,\cdot\,)$ is regarded as a function $G_{\mathrm{V}}^{\alpha}\to H^{\alpha}(\mathbb{T}^2,\mathbb{R}^2)$, and $A_k(g)\hat{Y}_s(g)$ means differentiation of $\hat{Y}_s : G_{\mathrm{V}}^{\alpha}\to H^{\alpha}(\mathbb{T}^2,\mathbb{R}^2)$ along the vector field A_k at the point $g\in G_{\mathrm{V}}^{\alpha}$. Let γ_ξ be the geodesic in G_{V}^{α} such that $\gamma_0 = e$ and $\gamma_0' = \bar{A}_k$. We obtain:

$$\begin{aligned} A_k(g)\hat{Y}_s(g)(\theta) &= \frac{\mathrm{d}}{\mathrm{d}\xi}\hat{Y}_s(\gamma_\xi\circ g)(\theta)|_{\xi=0} = R_g\frac{\mathrm{d}}{\mathrm{d}\xi}y(s,\gamma_\xi\theta)|_{\xi=0} \\ &= R_g\nabla_{\bar{A}_k}y(s,\theta) = \bar{\nabla}_{A_k}\hat{Y}_s(g)(\theta). \end{aligned} \tag{14}$$

Thus,

$$X_s^{t,e} = \epsilon\sum_{k\in\mathbb{Z}_2^+\cup\{0\},|k|\leqslant N}[\nabla_{\bar{A}_k}y(s,\,\cdot\,)\otimes e_k^A + \nabla_{\bar{B}_k}y(s,\,\cdot\,)\otimes e_k^B]\circ Z_s^{t,e}, \tag{15}$$

and the stochastic integral in (7) can be represented as

$$\begin{aligned} &\int_s^T X_r^{t,e}\mathrm{d}W_r \\ &= \epsilon\sum_{k\in\mathbb{Z}_2^+\cup\{0\},|k|\leqslant N}\int_s^T\nabla_{\bar{A}_k}y(r,\,\cdot\,)\circ Z_r^{t,e}\mathrm{d}\beta_k^A(r) + \int_s^T\nabla_{\bar{B}_k}y(r,\,\cdot\,)\circ Z_r^{t,e}\mathrm{d}\beta_k^B(r). \end{aligned}$$

In particular, if $N=0$,

$$\int_s^T X_r^{t,e}\mathrm{d}W_r = \epsilon\left(\int_s^T\frac{\partial}{\partial\theta_1}y(r,\,\cdot\,)\circ Z_r^{t,e}\mathrm{d}\beta_0^A(r) + \int_s^T\frac{\partial}{\partial\theta_2}y(r,\,\cdot\,)\circ Z_r^{t,e}\mathrm{d}\beta_0^B(r)\right).$$

A result similar to Lemma 10 was obtained in [6].

Lemma 10 (*The Laplacian of a Right-invariant Vector Field*). *Let $\hat{V}$ be the right-invariant vector field on $G^{\tilde{\alpha}}$ generated by an $H^{\tilde{\alpha}+2}$-vector field V on $\mathbb{T}^2$. Further let $\epsilon>0$ be such that $\frac{\epsilon^2}{2}\Big(1+\frac{1}{2}\sum_{k\in\mathbb{Z}_2^+,|k|\leqslant N}\frac{1}{|k|^{2\alpha}}\Big) = \nu$. Then for all $g\in G^{\tilde{\alpha}}$,*

$$\frac{\epsilon^2}{2}\sum_{\substack{k\in\mathbb{Z}_2^+\cup\{0\},\\|k|\leqslant N}}\big(\bar{\nabla}_{A_k}\bar{\nabla}_{A_k} + \bar{\nabla}_{B_k}\bar{\nabla}_{B_k}\big)\hat{V}(g) = \nu\,\Delta V\circ g. \tag{16}$$

Here $\tilde{\alpha}$ is an integer which is not necessary equal to α.

Proof. By the right-invariance of the vector fields $\bar{\nabla}_{A_k}\bar{\nabla}_{A_k}\hat{V}$ and $\bar{\nabla}_{B_k}\bar{\nabla}_{B_k}\hat{V}$ (Lemma 7), it suffices to show (16) for $g = e$. We observe that

$$(\bar{k}, \nabla)\cos(k\cdot\theta) = (\bar{k}, \nabla)\sin(k\cdot\theta) = 0.$$

Then, for $k \in \mathbb{Z}_2^+$, $\theta \in \mathbb{T}^2$,

$$\begin{aligned}\bar{\nabla}_{A_k}\bar{\nabla}_{A_k}\hat{V}(e)(\theta) &= \frac{1}{|k|^{2\alpha+2}}\cos(k\cdot\theta)(\bar{k},\nabla)\big[\cos(k\cdot\theta)(\bar{k},\nabla)V(\theta)\big]\\ &= \frac{1}{|k|^{2\alpha+2}}\cos(k\cdot\theta)^2(\bar{k},\nabla)^2 V(\theta).\end{aligned}$$

Similarly, $\bar{\nabla}_{B_k}\bar{\nabla}_{B_k}\hat{V}(e)(\theta) = \frac{1}{|k|^{2\alpha+2}}\sin(k\cdot\theta)^2(\bar{k},\nabla)^2V(\theta)$. Hence, for each $k \in \mathbb{Z}_2^+$,

$$(\bar{\nabla}_{A_k}\bar{\nabla}_{A_k} + \bar{\nabla}_{B_k}\bar{\nabla}_{B_k})\,\hat{V}(e)(\theta) = \frac{1}{|k|^{2\alpha+2}}(\bar{k},\nabla)^2V(\theta). \tag{17}$$

Note that for each $k \in \mathbb{Z}_2^+$, either $\bar{k}$ or $-\bar{k}$ is in $\mathbb{Z}_2^+$, and

$$(\bar{k},\nabla)^2 + (k,\nabla)^2 = |k|^2\Delta.$$

Summation over $k \in \mathbb{Z}_2^+$, $|k| \leqslant N$, in (17), and coupling the terms numbered by k and $\bar{k}$ (or $-\bar{k}$) gives:

$$\sum_{k\in\mathbb{Z}_2^+,|k|\leqslant N}(\bar{\nabla}_{A_k}\bar{\nabla}_{A_k} + \bar{\nabla}_{B_k}\bar{\nabla}_{B_k})\hat{V}(e)(\theta) = \frac{1}{2}\sum_{k\in\mathbb{Z}_2^+,|k|\leqslant N}\frac{1}{|k|^{2\alpha}}\,\Delta V(\theta).$$

Note that $(\bar{\nabla}_{A_0}\bar{\nabla}_{A_0} + \bar{\nabla}_{B_0}\bar{\nabla}_{B_0})\hat{V}(e)(\theta) = \Delta V(\theta)$. Finally, we obtain:

$$\sum_{\substack{k\in\mathbb{Z}_2^+\cup\{0\},\\|k|\leqslant N}}(\bar{\nabla}_{A_k}\bar{\nabla}_{A_k} + \bar{\nabla}_{B_k}\bar{\nabla}_{B_k})\hat{V}(e)(\theta) = \Big(1 + \frac{1}{2}\sum_{k\in\mathbb{Z}_2^+,|k|\leqslant N}\frac{1}{|k|^{2\alpha}}\Big)\Delta V(\theta).$$

The lemma is proved. □

Corollary 1. *Let the function $\varphi : \mathbb{T}^2 \to \mathbb{R}^2$ be C^2-smooth. Further let $A_k(g)[\varphi\circ g]$ and $B_k(g)[\varphi\circ g]$, $k \in \mathbb{Z}_2^+$, mean the differentiation of the function $G^{\tilde{\alpha}} \to L_2(\mathbb{T}^2,\mathbb{R}^2)$, $g \mapsto \varphi\circ g$ along A_k and resp. B_k. Then for all $g \in G^{\tilde{\alpha}}$,*

$$\frac{\epsilon^2}{2}\sum_{\substack{k\in\mathbb{Z}_2^+\cup\{0\},\\|k|\leqslant N}}\big(A_k(g)A_k(g) + B_k(g)B_k(g)\big)[\varphi\circ g] = \nu\,\Delta\varphi\circ g. \tag{18}$$

Proof. The computation that we made in (14) but applied to $\varphi\circ g$ implies that

$$A_k(g)[\varphi\circ g] = \Big[\frac{1}{|k|^{\alpha+1}}\cos(k\cdot\theta)(\bar{k},\nabla)\varphi(\theta)\Big]\circ g.$$

Similarly, we compute $B_k(g)[\varphi\circ g]$. Now we just have to repeat the proof of Lemma 10 to come to (18). □

Lemma 11. *Let Φ_r, $r \in [t,T]$, $t \in [0,T)$, be an $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$-valued stochastic process whose trajectories are integrable, and let ϕ_T be an $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$-valued random element so that* *both Φ_r and ϕ_T possess finite expectations. Then there exists an $\mathcal{F}_s$-adapted $H^\alpha(\mathbb{T}^2, \mathbb{R}^2) \times \mathcal{L}\big(E, H^\alpha(\mathbb{T}^2, \mathbb{R}^2)\big)$-valued pair of stochastic processes (Y_s, X_s) solving the BSDE*

$$Y_s = \phi_T + \int_s^T \Phi_r \, \mathrm{d}r - \int_s^T X_r \, \mathrm{d}W_r \tag{19}$$

on $[t, T]$. The Y_s-part of the solution has the representation

$$Y_s = \mathbb{E}\left[\phi_T + \int_s^T \Phi_r \, \mathrm{d}r | \mathcal{F}_s\right], \tag{20}$$

and therefore is unique. The X_s-part of the solution is unique with respect to the norm $\|X_s\|^2 = \int_t^T \|X_s\|^2_{\mathcal{L}(E, H^\alpha(\mathbb{T}^2, \mathbb{R}^2))} \, \mathrm{d}s$.

The proof of the lemma uses some ideas from [21].

Proof. Representation (20) follows from (19). Let us extend the process Y_s to the entire interval $[0, T]$ by setting $Y_s = Y_t$ for $s \in [0, t]$, and note that the extended process Y_s is a solution of the SDE

$$Y_s = \phi_T + \int_s^T \mathbb{I}_{[t,T]} \Phi_r \, \mathrm{d}r - \int_s^T X_r \mathrm{d}W_r$$

on $[0, T]$. Let $X_s \in \mathcal{L}\big(E, H^\alpha(\mathbb{T}^2, \mathbb{R}^2)\big)$, $s \in [0, T]$, be such that

$$\mathbb{E}\left[\phi_T + \int_0^T \mathbb{I}_{[t,T]} \, \Phi_r \, \mathrm{d}r - Y_0 \,|\, \mathcal{F}_s\right] = \int_0^s X_r \, \mathrm{d}W_r. \tag{21}$$

The process X_s exists by the martingale representation theorem. Indeed, on the right-hand side of (21) we have a Hilbert space valued martingale.

By Theorem 6.6 of [16], each component of the $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$-valued martingale on the right-hand side of (21) can be represented as a sum of real-valued stochastic integrals with respect to the Brownian motions $\{\beta_k^A(s), \beta_k^B(s)\}_{k \in \mathbb{Z}_2^+ \cup \{0\}, |k| \leqslant N}$. Hence, there exist $\mathcal{F}_s$-adapted stochastic processes $\{X_s^{kA}, X_s^{kB}\}_{k \in \mathbb{Z}_2^+ \cup \{0\}, |k| \leqslant N}$ such that

$$\mathbb{E}\left[\phi_T + \int_0^T \mathbb{I}_{[t,T]} \Phi_r \, \mathrm{d}r - Y_0 | \mathcal{F}_s\right] = \sum_{\substack{k \in \mathbb{Z}_2^+ \cup \{0\}, \\ |k| \leqslant N}} \int_0^s X_r^{kA} \, \mathrm{d}\beta_k^A(r) + \int_0^s X_r^{kB} \, \mathrm{d}\beta_k^B(r).$$

Let the process X_s be defined by (8) via the processes X_s^{kA} and X_s^{kB}, $k \in \mathbb{Z}_2^+ \cup \{0\}$, $|k| \leqslant N$. Itô's isometry shows that $\mathbb{E} \int_0^T \|X_r\|^2_{\mathcal{L}(E, H^\alpha(\mathbb{T}^2, \mathbb{R}^2))} < \infty$. Note that for all $s \in [0, t]$, $\int_0^s X_r \mathrm{d}W_r = \int_0^t X_r \mathrm{d}W_r$. This shows that $X_s = 0$ for almost all $\omega \in \Omega$ and almost all $s \in [0, t]$, and therefore can be chosen equal to zero on $[0, t]$. Thus, (21) takes the form:

$$\mathbb{E}\left[\phi_T + \int_t^T \Phi_r \, \mathrm{d}r - Y_t | \mathcal{F}_s\right] = \int_t^s X_r \, \mathrm{d}W_r. \tag{22}$$

It is easy to verify that the pair (Y_s, X_s) defined by (20) and (22) solves BSDE (19). To prove the uniqueness, note that any $\mathcal{F}_s$-adapted solution to (19) takes the form (20), (22). Moreover, if the processes X_s and X'_s satisfy (22), then

$$\int_t^T \|X_s - X'_s\|^2_{\mathcal{L}(E, H^\alpha(\mathbb{T}^2,\mathbb{R}^2))}\, \mathrm{d}r = \left\| \int_t^T (X_s - X'_s)\, \mathrm{d}W_r \right\|^2_{H^\alpha(\mathbb{T}^2,\mathbb{R}^2)} = 0. \quad \square$$

Proof of Theorem 6. Let us consider BSDE (7) as an $L_2(\mathbb{T}^2, \mathbb{R}^2)$-valued SDE, and $\hat{Y}_s$ as a function $G^\alpha_{\mathrm{V}} \to L_2(\mathbb{T}^2, \mathbb{R}^2)$. Since for each $s \in [t, T]$, $y(s, \cdot) \in H^{\alpha+1}(\mathbb{T}^2, \mathbb{R}^2)$ and $\alpha > 2$ by assumption, then $\hat{Y}_s : G^\alpha_{\mathrm{V}} \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ is at least C^2-smooth. Eqs. (2) show that the function $\partial_s y(\cdot, \cdot) : [t, T] \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ is continuous since ∇p, Δy, and $(y, \nabla y)$ are continuous functions $[t, T] \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ by Assumption 1. Taking into account that the diffeomorphisms of G^α_{V} are volume-preserving, we conclude that for each fixed $g \in G^\alpha_{\mathrm{V}}$, $\partial_s \hat{Y}_s(g) : [t, T] \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ is a continuous function. Hence, $\hat{Y}_\bullet : [t, T] \times G^\alpha_{\mathrm{V}} \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ is C^1-smooth in $s \in [t, T]$ and C^2-smooth in $g \in G^\alpha_{\mathrm{V}}$. Itô's formula is therefore applicable to $\hat{Y}_s(Z^{t,e}_s)$. Below we use the fact that $Z^{t,e}_s$ is a solution to forward SDE (11) and the identity $\frac{\partial \hat{Y}_s}{\partial s}(Z^{t,e}_s) = \frac{\partial y(s,\cdot)}{\partial s} \circ Z^{t,e}_s$. For the latter derivative we substitute the right-hand side of the first equation of (2). The notation $\hat{X}(g)[\hat{Y}_s(g)]$ (sometimes without square brackets) means differentiation of the function $\hat{Y}_s : G^\alpha_{\mathrm{V}} \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ along the right-invariant vector field $\hat{X}$ on G^α_{V} at the point $g \in G^\alpha_{\mathrm{V}}$. The same argument as in Remark 3 implies that $\hat{X}(g)[\hat{Y}_s(g)] = \bar{\nabla}_{\hat{X}} \hat{Y}_s(g)$. Taking into account this argument, we obtain:

$$\begin{aligned} \hat{Y}_s(Z^{t,e}_s) - \hat{h}(Z^{t,e}_T) &= -\int_s^T \partial_r \hat{Y}_r(Z^{t,e}_r)\, \mathrm{d}r - \int_s^T \mathrm{d}r\, \hat{Y}_r(Z^{t,e}_r)[\hat{Y}_r(Z^{t,e}_r)] \\ &\quad - \int_s^T \mathrm{d}r\, \frac{\epsilon^2}{2} \sum_{k \in \mathbb{Z}^+_2 \cup \{0\}, |k| \leqslant N} \left[A_k(Z^{t,e}_r) A_k(Z^{t,e}_r) \hat{Y}_r(Z^{t,e}_r) + B_k(Z^{t,e}_r) B_k(Z^{t,e}_r) \hat{Y}_r(Z^{t,e}_r) \right] \\ &\quad - \int_s^T \epsilon\, \sigma(Z^{t,e}_r) \hat{Y}_r(Z^{t,e}_r)\, \mathrm{d}W_r. \end{aligned} \tag{23}$$

Note that

$$\hat{Y}_r(Z^{t,e}_r)[\hat{Y}_r(Z^{t,e}_r)] = [(y(r, \cdot), \nabla) y(r, \cdot)] \circ Z^{t,e}_r.$$

Also, let us observe that

$$\begin{aligned} &\frac{\epsilon^2}{2} \sum_{k \in \mathbb{Z}^+_2 \cup \{0\}, |k| \leqslant N} \left[A_k(Z^{t,e}_r) A_k(Z^{t,e}_r) \hat{Y}_r(Z^{t,e}_r) + B_k(Z^{t,e}_r) B_k(Z^{t,e}_r) \hat{Y}_r(Z^{t,e}_r) \right] \\ &= \frac{\epsilon^2}{2} \sum_{k \in \mathbb{Z}^+_2 \cup \{0\}, |k| \leqslant N} \left[\bar{\nabla}_{A_k} \bar{\nabla}_{A_k} \hat{Y}_r(Z^{t,e}_r) + \bar{\nabla}_{B_k} \bar{\nabla}_{B_k} \hat{Y}_r(Z^{t,e}_r) \right] \\ &= \nu[\Delta y(s, \cdot)] \circ Z^{t,e}_r \end{aligned}$$

where the latter equality holds by Lemma 10, and $\epsilon > 0$ is chosen so that $\frac{\epsilon^2}{2}\Big(1 + \frac{1}{2} \sum_{k \in \mathbb{Z}^+_2, |k| \leqslant N} \frac{1}{|k|^{2\alpha}}\Big) = \nu$. Note that the terms $\bar{\nabla}_{A_k} \bar{\nabla}_{A_k} \hat{Y}_r(Z^{t,e}_r)$ and $\bar{\nabla}_{B_k} \bar{\nabla}_{B_k} \hat{Y}_r(Z^{t,e}_r)$ are elements of $TG^{\alpha-1}$, and therefore are well defined in $L_2(\mathbb{T}^2, \mathbb{R}^2)$. Continuing (23), we obtain:

$$\hat{Y}^t_s(Z^{t,e}_s) - \hat{h}(Z^{t,e}_T)$$

$$= \int_s^T \mathrm{d}r\Big[\hat{V}(r, Z_r^{t,e}) + [(y(r,\,\cdot\,),\nabla)y(r,\,\cdot\,)]\circ Z_r^{t,e} + \nu[\Delta y(r,\,\cdot\,)]\circ Z_r^{t,e}\Big]$$
$$- \int_s^T [(y(r,\,\cdot\,),\nabla)y(r,\,\cdot\,)]\circ Z_r^{t,e}\,\mathrm{d}r - \int_s^T \nu[\Delta y(r,\,\cdot\,)]\circ Z_r^{t,e}\,\mathrm{d}r$$
$$- \int_s^T \epsilon\,\sigma(Z_r^{t,e})\hat{Y}_r(Z_r^{t,e})\,\mathrm{d}W_r$$
$$= \int_s^T \hat{V}(r, Z_r^{t,e})\,\mathrm{d}r - \int_s^T \epsilon\,\sigma(Z_r^{t,e})\hat{Y}_r(Z_r^{t,e})\,\mathrm{d}W_r. \tag{24}$$

Thus the pair of stochastic processes $(\hat{Y}_s(Z_s^{t,e}), \epsilon\,\sigma(Z_s^{t,e})\hat{Y}_s(Z_s^{t,e}))$ is a solution to BSDE (7) in $L_2(\mathbb{T}^2,\mathbb{R}^2)$. It is $\mathcal{F}_s$-adapted since $Z_s^{t,e}$ is $\mathcal{F}_s$-adapted. By Lemma 11, we know that there exists a unique $\mathcal{F}_s$-adapted solution $(Y_s^{t,e}, X_s^{t,e})$ to (7) in $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$. Clearly, $(Y_s^{t,e}, X_s^{t,e})$ is also a unique $\mathcal{F}_s$-adapted solution to (7) in $L_2(\mathbb{T}^2,\mathbb{R}^2)$. Hence, $Y_s^{t,e} = \hat{Y}_s(Z_s^{t,e})$ and $\int_t^T \|X_s^{t,e} - \epsilon\,\sigma(Z_s^{t,e})(\hat{Y}_s(Z_s^{t,e}))\|^2_{\mathcal{L}(E,H^\alpha(\mathbb{T}^2,\mathbb{R}^2))}\,\mathrm{d}s = 0$, and therefore the pair of stochastic processes $\big(\hat{Y}_s(Z_s^{t,e}), \epsilon\,\sigma(Z_s^{t,e})\hat{Y}_s(Z_s^{t,e})\big)$ is a unique $\mathcal{F}_s$-adapted solution to BSDE (7) in $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$. The theorem is proved. □

5. Some identities involving the Navier–Stokes solution

The backward SDE allows us to obtain the representation below for the Navier–Stokes solution. Also, it easily implies the well-known energy identity for the Navier–Stokes equations.

5.1. Representation of the Navier–Stokes solution

Theorem 7. *Let $t \in [0,T]$, and let $Z_s^{t,e}$ be the solution to SDE (11) on $[t,T]$ with the initial condition $Z_t^{t,e} = e$. Then the following representation holds for the solution $y(t,\,\cdot\,)$ to (9).*

$$y(t,\,\cdot\,) = \mathbb{E}\Big[\hat{h}(Z_T^{t,e}) + \int_t^T \nabla p(s,\,\cdot\,)\circ Z_s^{t,e}\mathrm{d}s\Big].$$

Proof. Note that $\hat{Y}_t(Z_t^{t,e}) = y(t,\,\cdot\,)$, and $\mathbb{E}[\int_t^T X_r^{t,e}\mathrm{d}W_r] = 0$. Taking the expectation from the both parts of (7) at time $s = t$ we obtain the above representation. □

5.2. A simple derivation of the energy identity

Itô's formula applied to the squared $L_2(\mathbb{T}^2,\mathbb{R}^2)$-norm of $Y_s^{t,e}$ gives:

$$\|Y_s^{t,e}\|^2_{L_2} = \|\hat{h}(Z_T^{t,e})\|^2_{L_2} + 2\int_s^T (Y_r^{t,e}, \hat{V}(Z_r^{t,e}))_{L_2}\mathrm{d}r$$
$$-2\int_s^T (Y_r^{t,e}, X_r^{t,e}\mathrm{d}W_r)_{L_2} - \int_s^T \|X_s^{t,e}\|^2_{L_2}\mathrm{d}r. \tag{25}$$

Using representation (15) for the process $X_s^{t,e}$ we obtain:

$$\|X_s^{t,e}\|^2_{L_2} = \epsilon^2\left[\sum_{k\in\mathbb{Z}_2^+\cup\{0\},|k|\leqslant N} \|\nabla_{\bar{A}_k} y(s,\,\cdot\,)\|^2_{L_2} + \|\nabla_{\bar{B}_k} y(s,\,\cdot\,)\|^2_{L_2}\right]$$

$$= \epsilon^2 \left[\sum_{k \in \mathbb{Z}_2^+, |k| \leqslant N} \frac{1}{|k|^{2\alpha+2}} \|(\bar{k}, \nabla y(s,\,\cdot))\|_{L_2}^2 + \|\nabla y(s,\,\cdot)\|_{L_2}^2 \right]$$

$$= \epsilon^2 \Big[\frac{1}{2} \sum_{k \in \mathbb{Z}_2^+, |k| \leqslant N} \frac{1}{|k|^{2\alpha+2}} \big(\|(\bar{k}, \nabla y(s,\,\cdot))\|_{L_2}^2 + \|(k, \nabla y(s,\,\cdot))\|_{L_2}^2 \big) + \|\nabla y(s,\,\cdot)\|_{L_2}^2 \Big]$$

$$= \epsilon^2 \Big(1 + \frac{1}{2} \sum_{k \in \mathbb{Z}_2^+, |k| \leqslant N} \frac{1}{|k|^{2\alpha}} \Big) \|\nabla y(s,\,\cdot)\|_{L_2}^2 = 2\,\nu\,\|\nabla y(s,\,\cdot)\|_{L_2}^2.$$

Taking the expectation in (25) and using the volume-preserving property of $Z_s^{t,e}$, we obtain:

$$\|y(s,\,\cdot)\|_{L_2}^2 + 2\nu \int_s^T \|\nabla y(r,\,\cdot)\|_{L_2}^2 \mathrm{d}r = \|h\|_{L_2}^2.$$

6. Constructing the solution to the Navier–Stokes equations from a solution to the FBSDEs

Let us prove now a result which is, in some sense, a converse of Theorem 6. In this section we consider (5) as a system of forward and backward SDEs in the Hilbert space $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$, where $\alpha \geq 3$. As before, let $\hat{V}(s, Z_s^{t,e})$ denote $\nabla p(s,\,\cdot) \circ Z_s^{t,e}$, and let $\mathcal{F}_s$ denote the filtration $\sigma\{W_r, r \in [0, s]\}$.

Theorem 8. *Assume, for an $H^{\alpha+1}$-smooth function $p(s,\,\cdot)$, $s \in [0, T]$, and for any $t \in (0, T)$, the existence of an $\mathcal{F}_s$-adapted solution $(Z_s^{t,e}, Y_s^{t,e}, X_s^{t,e})$ to (5) on $[t, T]$ such that the processes $Z_s^{t,e}$ and $Y_s^{t,e}$ have a.s. continuous trajectories and such that $Z_s^{t,e}$ take values in G_{V}^α. Then there exists $T_0 > 0$ such that for all $T < T_0$ there exists a deterministic function $y(s,\,\cdot) \in T_e G_{\mathrm{V}}^\alpha$ on $[0, T]$, such that a.s. on $[t, T]$ the relation $Y_s^{t,e} = y(s,\,\cdot) \circ Z_s^{t,e}$ holds. Moreover, the pair of functions (y, p) solves the backward Navier–Stokes equations (9) on $[0, T]$.*

Lemma 12–18 are the steps in the proof of Theorem 8.

Lemma 12. *For all $t \in [0, T)$ and for any $\mathcal{F}_t$-measurable G_{V}^α-valued random variable ξ, the triple of stochastic processes*

$$(Z_s^{t,\xi}, Y_s^{t,\xi}, X_s^{t,\xi}) = (Z_s^{t,e} \circ \xi, Y_s^{t,e} \circ \xi, X_s^{t,e} \circ \xi) \tag{26}$$

is $\mathcal{F}_s$-adapted and solves the FBSDEs

$$\begin{cases} Z_s^{t,\xi} = \xi + \int_t^s Y_r^{t,\xi}\,\mathrm{d}r + \int_t^s \sigma(Z_r^{t,\xi})\,\mathrm{d}W_r \\ Y_s^{t,\xi} = h(Z_T^{t,\xi}) + \int_s^T \hat{V}(r, Z_r^{t,\xi})\mathrm{d}r - \int_s^T X_r^{t,\xi}\,\mathrm{d}W_r \end{cases} \tag{27}$$

on the interval $[t, T]$ in the space $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$.

Proof. Let us apply the operator R_ξ of the right translation to the both sides of FBSDEs (5). We only have to prove that we are allowed to write R_ξ under the signs of both stochastic integrals in (5). Let us prove that it is true for an $\mathcal{F}_t$-measurable stepwise function $\xi = \sum_{i=1}^\infty g_i \mathbb{I}_{A_i}$, where $g_i \in G_{\mathrm{V}}^\alpha$ and the sets A_i are $\mathcal{F}_t$-measurable. Indeed, let s and S be such that $t \leqslant s < S \leqslant T$, and let Φ_r be an $\mathcal{F}_r$-adapted stochastically integrable process. We obtain:

$$\int_s^S \Phi_r \, \mathrm{d}W_r \circ \sum_{i=1}^\infty g_i \mathbb{I}_{A_i} = \sum_{i=1}^\infty \mathbb{I}_{A_i} \int_s^S \Phi_r \circ g_i \, \mathrm{d}W_r = \sum_{i=1}^\infty \int_s^S \mathbb{I}_{A_i} \Phi_r \circ g_i \, \mathrm{d}W_r$$
$$= \int_s^S \Phi_r \circ \sum_{i=1}^\infty g_i \mathbb{I}_{A_i} \, \mathrm{d}W_r.$$

Next, we find a sequence of $\mathcal{F}_t$-measurable stepwise functions converging to ξ in the space of continuous functions $\mathrm{C}(\mathbb{T}^2, \mathbb{R}^2)$. This is possible due to the separability of $\mathrm{C}(\mathbb{T}^2, \mathbb{R}^2)$. Indeed, let us consider a countable number of disjoint Borel sets O_i^n covering $\mathrm{C}(\mathbb{T}^2, \mathbb{R}^2)$, and such that their diameter in the norm of $\mathrm{C}(\mathbb{T}^2, \mathbb{R}^2)$ is smaller than $\frac{1}{n}$. Let $A_i^n = \xi^{-1}(O_i^n)$ and $g_i^n \in O_i^n \cap G_{\mathrm{V}}^\alpha$. Define $\xi_n = \sum_{i=1}^\infty g_i^n \mathbb{I}_{A_i^n}$. Then it holds that for all $\omega \in \Omega$, $\|\xi - \xi_n\|_{\mathrm{C}(\mathbb{T}^2,\mathbb{R}^2)} < \frac{1}{n}$. Let $I(\Phi)$ and $I(\Phi \circ \xi)$ denote $\int_s^S \Phi_r \, \mathrm{d}W_r$ and resp. $\int_s^S \Phi_r \circ \xi \, \mathrm{d}W_r$. We have to prove that a.s. $I(\Phi) \circ \xi = I(\Phi \circ \xi)$. For this it suffices to prove that

$$\lim_{n\to\infty} \mathbb{E}\|I(\Phi) \circ \xi_n - I(\Phi) \circ \xi\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} = 0, \tag{28}$$

$$\lim_{n\to\infty} \mathbb{E}\|I(\Phi \circ \xi_n) - I(\Phi \circ \xi)\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} = 0. \tag{29}$$

Due to the volume-preserving property of ξ and ξ_n, $\|I(\Phi)\circ\xi_n\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} = \|I(\Phi)\circ\xi\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} = \|I(\Phi)\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}$. Hence, by Lebesgue's theorem, in (28) we can pass to the limit under the expectation sign. Relation (28) holds then by the continuity of $I(\Phi)$ in $\theta \in \mathbb{T}^2$. To prove (29) we observe that by Itô's isometry, the limit in (29) equals to $\lim_{n\to\infty} \mathbb{E} \int_s^S \|\Phi_r \circ \xi_n - \Phi_r \circ \xi\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} \mathrm{d}r$. The same argument that we used to prove (28) implies that we can pass to the limit under the expectation and the integral signs. Relation (29) follows from the continuity of Φ_r in $\theta \in \mathbb{T}^2$.

Hence, $(Z_s^{t,e} \circ \xi, Y_s^{t,e} \circ \xi, X_s^{t,e} \circ \xi)$ is a solution to (27). This solution is clearly $\mathcal{F}_s$-adapted. □

Lemma 13–17 use some ideas and constructions from [9].

Lemma 13. *The map* $[0, T] \times \mathbb{T}^2 \to \mathbb{R}^2$, $(t, \theta) \mapsto Y_t^{t,e}(\theta)$ *is deterministic.*

Proof. Let us extend the solution $(Z_s^{t,e}, Y_s^{t,e}, X_s^{t,e})$ to the interval $[0, t]$ by setting $Z_s^{t,e} = e$, $Y_s^{t,e} = Y_t^{t,e}$, $X_s^{t,e} = 0$ for all $s \in [0, t]$. The extended process solves the problem:

$$\begin{cases} Z_s^{t,e} = e + \int_0^s \mathbb{I}_{[t,T]}(r) Y_r^{t,e} \, \mathrm{d}r + \int_0^s \mathbb{I}_{[t,T]}(r) \sigma(Z_r^{t,e}) \, \mathrm{d}W_r \\ Y_s^{t,e} = h(Z_T^{t,e}) + \int_s^T \mathbb{I}_{[t,T]}(r) \hat{V}(r, Z_r^{t,e}) \, \mathrm{d}r - \int_s^T X_r^{t,e} \, \mathrm{d}W_r. \end{cases} \tag{30}$$

The random vector $Y_0^{t,e}$ is $\mathcal{F}_0$-measurable, and hence is deterministic by Blumenthal's zero–one law. Since $Y_t^{t,e} = Y_0^{t,e}$, the result follows. □

Lemma 14. *There exists a constant* $T_0 > 0$ *such that for* $T < T_0$ *the function* $[0, T] \to H^2(\mathbb{T}^2, \mathbb{R}^2)$, $t \mapsto Y_t^{t,e}$ *is continuous.*

Proof. Let $(Z_s^{t,e}, Y_s^{t,e}, X_s^{t,e})$ and $(Z_s^{t',e}, Y_s^{t',e}, X_s^{t',e})$ be solutions to (27) which start at the identity e at times t and resp. t', and let $t < t'$. These solutions can be regarded as solutions of (30) if we extend them to the entire interval $[0, T]$ as it was described in Lemma 13. The application of Itô's formula to $\|Y_s^{t,e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}$ and the backward SDE of (27) imply that the expectation $\mathbb{E}\|Y_s^{t,e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}$ is bounded. The forward SDE of (30), Gronwall's lemma, and usual stochastic integral estimates imply that there exists a constant $K_1 > 0$ such that

$$\mathbb{E}\|Z_s^{t,e} - Z_s^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} < K_1\Big[\int_0^s \mathbb{I}_{[t,T]}\mathbb{E}\|Y_r^{t,e} - Y_r^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}\mathrm{d}r + (t'-t)\Big].$$

Let us apply Itô's formula to $\|Y_s^{t,e} - Y_s^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}$ when using the backward SDE of (30). Again, Gronwall's lemma, usual stochastic integral estimates and the above estimate for $\mathbb{E}\|Z_s^{t,e} - Z_s^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}$ imply that there exists a constant $K_2 > 0$ such that

$$\mathbb{E}\|Y_s^{t,e} - Y_s^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} < K_2\Big[\int_0^T \mathbb{E}\|Y_r^{t,e} - Y_r^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}\mathrm{d}r + (t'-t)\Big].$$

We take T_0 smaller than $\frac{1}{K_2}$. Then there exists a constant $K > 0$ such that

$$\sup_{s\in[0,T]} \mathbb{E}\|Y_s^{t,e} - Y_s^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} < K(t'-t). \tag{31}$$

Evaluating the right-hand side at the point $s = t$, and taking into account that $Y_t^{t',e} = Y_{t'}^{t',e}$ we obtain that

$$\|Y_t^{t,e} - Y_{t'}^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} < K(t'-t). \tag{32}$$

Differentiating (30) with respect to θ we obtain the following system of forward and backward SDEs:

$$\begin{cases} \nabla Z_s^{t,e} = I + \displaystyle\int_0^s \mathbb{I}_{[t,T]}(r)\nabla Y_r^{t,e}\,\mathrm{d}r + \int_0^s \mathbb{I}_{[t,T]}(r)\nabla\sigma(Z_r^{t,e})\nabla Z_r^{t,e}\,\mathrm{d}W_r \\ \nabla Y_s^{t,g} = \nabla h(Z_T^{t,e})\nabla Z_T^{t,e} + \displaystyle\int_s^T \mathbb{I}_{[t,T]}(r)\nabla\hat{V}(r, Z_r^{t,g})\nabla Z_r^{t,e}\,\mathrm{d}r - \int_s^T \nabla X_r^{t,e}\,\mathrm{d}W_r. \end{cases}$$

Again, standard estimates imply the boundedness of $\mathbb{E}\|\nabla Z_s^{t,e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}$ and $\mathbb{E}\|\nabla Y_s^{t,e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}$. The same argument that we used to obtain (32) as well as the estimate for the $\sup_{s\in[0,T]}\mathbb{E}\|Z_s^{t,e} - Z_s^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)}$, which easily follows from (31), and the forward SDE imply that there exists a constant $L > 0$ such that for all t and t' from the interval $[0, T]$,

$$\|\nabla Y_t^{t,e} - \nabla Y_{t'}^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} < L|t'-t|. \tag{33}$$

Differentiating (30) the second time and using the same argument once again we obtain that there exist a constant $M > 0$ such that for all t and t' belonging to $[0, T]$,

$$\|\nabla\nabla Y_t^{t,e} - \nabla\nabla Y_{t'}^{t',e}\|^2_{L_2(\mathbb{T}^2,\mathbb{R}^2)} < M|t'-t|. \tag{34}$$

Now (32)–(34) imply the continuity of the map $t \mapsto Y_t^{t,e}$ with respect to the $H^2(\mathbb{T}^2, \mathbb{R}^2)$-topology. □

Everywhere below we assume that $T < T_0$ where T_0 is the constant defined in Lemma 14.

Lemma 15. *For every $t \in [0, T)$ and for every $\mathcal{F}_t$-measurable random variable ξ, the solution $(Z_s^{t,\xi}, Y_s^{t,\xi}, X_s^{t,\xi})$ to (27) is unique on $[t, T]$.*

Proof. Let us assume that there exists another solution $(\tilde{Z}_s^{t,\xi}, \tilde{Y}_s^{t,\xi}, \tilde{X}_s^{t,\xi})$ to (27) on $[t, T]$. The same argument as in the proof of Lemma 14 implies the uniqueness of solution to (27). Specifically, the argument that we applied to the pair of solutions $(Z_s^{t,e}, Y_s^{t,e}, X_s^{t,e})$ and $(Z_s^{t',e}, Y_s^{t',e}, X_s^{t',e})$ has to be applied to $(Z_s^{t,\xi}, Y_s^{t,\xi}, X_s^{t,\xi})$ and $(\tilde{Z}_s^{t,\xi}, \tilde{Y}_s^{t,\xi}, \tilde{X}_s^{t,\xi})$, and it has to be taken into account that $t = t'$. □

Lemma 16. *Let the function $y : [0, T] \times \mathbb{T}^2 \to \mathbb{R}^2$ be defined by the formula:*

$$y(t, \theta) = Y_t^{t,e}(\theta). \tag{35}$$

Then, for every $t \in [0, T]$, $y(t, \cdot)$ is H^α-smooth, and a.s.

$$Y_u^{t,e} = y(u, \cdot) \circ Z_u^{t,e}. \tag{36}$$

Proof. Note that (26) implies that if ξ is $\mathcal{F}_t$-measurable then

$$Y_t^{t,\xi} = y(t, \cdot) \circ \xi. \tag{37}$$

Further, for each fixed $u \in [t, T]$, $(Z_s^{t,e}, Y_s^{t,e}, X_s^{t,e})$ is a solution of the following problem on $[u, T]$:

$$\begin{cases} Z_s^{t,e} = Z_u^{t,e} + \int_u^s Y_r^{t,e} \mathrm{d}r + \int_u^s \sigma(Z_r^{t,e}) \mathrm{d}W_r \\ Y_s^{t,e} = h(Z_T^{t,e}) + \int_s^T \hat{V}(r, Z_r^{t,e}) \mathrm{d}r - \int_s^T X_r^{t,e} \mathrm{d}W_r. \end{cases}$$

By uniqueness of solution, it holds that $Y_s^{t,e} = Y_s^{u,Z_u^{t,e}}$ a.s. on $[u, T]$. Next, by (37), we obtain that $Y_u^{u,Z_u^{t,e}} = y(u, \cdot) \circ Z_u^{t,e}$. This implies that there exists a set Ω_u (which depends on u) of full $\mathbb{P}$-measure such that (36) holds everywhere on Ω_u. Clearly, one can find a set $\Omega_{\mathbb{Q}}$, $\mathbb{P}(\Omega_{\mathbb{Q}}) = 1$, such that (36) holds on $\Omega_{\mathbb{Q}}$ for all rational $u \in [t, T]$. But the trajectories of $Z_s^{t,e}$ and $Y_s^{t,e}$ are a.s. continuous. Furthermore, Lemma 14 implies the continuity of $y(t, \cdot)$ in t with respect to (at least) the $L_2(\mathbb{T}^2, \mathbb{R}^2)$-topology. Therefore, (36) holds a.s. with respect to the $L_2(\mathbb{T}^2, \mathbb{R}^2)$-topology. Since both sides of (36) are continuous in $\theta \in \mathbb{T}^2$ it also holds a.s. for all $\theta \in \mathbb{T}^2$. □

Lemma 17. *The function y defined by formula (35) is C^1-smooth in $t \in [0, T]$.*

Proof. Let $\delta > 0$. We obtain:

$$y(t + \delta, \cdot) - y(t, \cdot) = Y_{t+\delta}^{t+\delta,e} - Y_t^{t,e} = Y_{t+\delta}^{t+\delta,e} - Y_{t+\delta}^{t,e} + Y_{t+\delta}^{t,e} - Y_t^{t,e}.$$

Let $\hat{Y}_s$ be the right-invariant vector field on G^α generated by $y(s, \cdot)$. Lemma 16 implies that a.s.

$$Y_{t+\delta}^{t,e} = \hat{Y}_{t+\delta}(Z_{t+\delta}^{t,e}).$$

Thus we obtain that a.s.

$$y(t + \delta, \cdot) - y(t, \cdot) = \big(\hat{Y}_{t+\delta}(e) - \hat{Y}_{t+\delta}(Z_{t+\delta}^{t,e})\big) + (Y_{t+\delta}^{t,e} - Y_t^{t,e}).$$

We use the backward SDE for the second difference and apply Itô's formula to the first difference when considering $\hat{Y}_{t+\delta}$ as a C^2-smooth function $G^\alpha_{\mathrm{V}} \to L_2(\mathbb{T}^2, \mathbb{R}^2)$. We obtain:

$$\hat{Y}_{t+\delta}(Z^{t,e}_{t+\delta}) - \hat{Y}_{t+\delta}(e) = \int_t^{t+\delta} \mathrm{d}r\, \hat{Y}^{t,e}_r(Z^{t,e}_r)[\hat{Y}_{t+\delta}(Z^{t,e}_r)] + \int_t^{t+\delta} \epsilon\, \sigma(Z^{t,e}_r)\, \hat{Y}_{t+\delta}(Z^{t,e}_r)\, \mathrm{d}W_r$$
$$+ \int_t^{t+\delta} \mathrm{d}r \sum_{k\in\mathbb{Z}^+_2\cup\{0\}} [A_k(Z^{t,e}_r)A_k(Z^{t,e}_r) + B_k(Z^{t,e}_r)B_k(Z^{t,e}_r)]\, \hat{Y}_{t+\delta}(Z^{t,e}_r).$$

The same argument as in Theorem 6 implies:

$$\hat{Y}_{t+\delta}(Z^{t,e}_{t+\delta}) - \hat{Y}_{t+\delta}(e) = \int_t^{t+\delta} \mathrm{d}r\, \nabla_{y(r,\,\cdot)}\, y(t+\delta,\,\cdot) \circ Z^{t,e}_r$$
$$+ \int_t^{t+\delta} \mathrm{d}r\, \nu\, \Delta\, y(t+\delta,\,\cdot) \circ Z^{t,e}_r + \int_t^{t+\delta} \epsilon\, \sigma(Z^{t,e}_r)\, \hat{Y}_{t+\delta}(Z^{t,e}_r)\, \mathrm{d}W_r.$$

Further we have:

$$Y^{t,e}_t - Y^{t,e}_{t+\delta} = \int_t^{t+\delta} \mathrm{d}r\, \nabla p(r,\,\cdot) \circ Z^{t,e}_r - \int_t^{t+\delta} X^{t,e}_r\, \mathrm{d}W_r.$$

Finally we obtain that

$$\frac{1}{\delta}\big(y(t+\delta,\,\cdot) - y(t,\,\cdot)\big) = -\frac{1}{\delta}\,\mathbb{E}\Big[\int_t^{t+\delta} \mathrm{d}r\, [\,(y(r,\,\cdot), \nabla)\, y(t+\delta,\,\cdot)$$
$$+ \nu\, \Delta\, y(t+\delta,\,\cdot) + \nabla p(r,\,\cdot)] \circ Z^{t,e}_r\Big]. \tag{38}$$

Note that $Z^{t,e}_r$, $\nabla p(r,\,\cdot)$, and $(y(r,\,\cdot), \nabla)\, y(t+\delta,\,\cdot) \circ Z^{t,e}_r$ are continuous in r a.s. with respect to the $L_2(\mathbb{T}^2, \mathbb{R}^2)$-topology. By Lemma 14, $\nabla\, y(t,\,\cdot)$ and $\Delta\, y(t,\,\cdot)$ are continuous in t with respect to the $L_2(\mathbb{T}^2, \mathbb{R}^2)$-topology. Formula (38) and the fact that $Z^{t,e}_t = e$ imply that in the $L_2(\mathbb{T}^2, \mathbb{R}^2)$-topology

$$\partial_t y(t,\,\cdot) = -[\nabla_{y(t,\,\cdot)}\, y(t,\,\cdot) + \nu\, \Delta\, y(t,\,\cdot) + \nabla p(t,\,\cdot)]. \tag{39}$$

Since the right-hand side of (39) is an $H^{\alpha-2}$-map, so is the left-hand side. This implies that $\partial_t y(t,\,\cdot)$ is continuous in $\theta \in \mathbb{T}^2$. Relation (39) is obtained so far for the right derivative of $y(t,\theta)$ with respect to t. Note that the right-hand side of (39) is continuous in t which implies that the right derivative $\partial_t y(t,\theta)$ is continuous in t on $[0, T)$. Hence, it is uniformly continuous on every compact subinterval of $[0, T)$. This implies the existence of the left derivative of $y(t,\theta)$ in t, and therefore, the existence of the continuous derivative $\partial_t y(t,\theta)$ everywhere on $[0, T]$. □

Lemma 18. *For every $t \in [0, T]$, the function $y(t,\,\cdot) : \mathbb{T}^2 \to \mathbb{R}^2$ is divergence-free. Moreover, the pair (y, p) verifies the backward Navier–Stokes equations.*

Proof. Fix a $t > 0$, and consider the $T_eG^\alpha_{\mathrm{V}}$-valued curve $\gamma_\zeta = \mathbb{E}[\exp^{-1} Z^{t,e}_\zeta]$, $\zeta \geqslant t$, in a neighborhood of the origin of $T_eG^\alpha_{\mathrm{V}}$. The forward SDE of (27) can be represented as an SDE on G^α:

$$\begin{cases} \mathrm{d}Z^{t,e}_s = \exp\{\hat{Y}_s(Z^{t,e}_s)\mathrm{d}s + \sigma(Z^{t,e}_s)\mathrm{d}W_s\}, \\ Z^{t,e}_t = e, \end{cases}$$

where $\hat{Y}_s$ is the right-invariant vector field on G^α generated by $y(s,\cdot)$. This implies that

$$\frac{\partial}{\partial\zeta}\gamma_\zeta\Big|_{\zeta=t} = y(t,\cdot),$$

and therefore $y(t,\cdot)\in T_eG^\alpha_{\mathrm{V}}$. Next, the backward SDE of (27) implies that $Y_T^{t,e} = h(Z_T^{t,e})$. This and relation (36) imply that $y(T,\cdot) = h$. Since we already obtained (39) in Lemma 17 the proof of the lemma is now complete. □

7. The backward SDE as an SDE on a tangent bundle

Let $(Z_s^{t,e}, Y_s^{t,e}, X_s^{t,e})$ be a solution to FBSDEs (5). We will show that the backward SDE can be represented as an SDE on the tangent bundle TG^α_{V} as well as an SDE on TG^α. We will construct a backward SDE in the Dalecky–Belopolskaya form (see [5]) and show that the process $Y_s^{t,e}$ is its unique solution.

7.1. The representation of the backward SDE on TG^α_{V}

Let $y(s,\cdot)$, $s\in[t,T]$, be the solution to the backward Navier–Stokes equations (9). Let $\hat{Y}_s$ be the right-invariant vector field on G^α_{V} generated by $y(s,\cdot)$. The connection map on the manifold G^α_{V} generates the connection map on the manifold TG^α_{V} as it was shown in [5], p. 58 (see also [11]). As before, we consider the Levi-Civita connection of the weak Riemannian metric (3) on G^α_{V}. Let $\overline{\exp}$ denote the exponential map of the generated connection on TG^α_{V}. More precisely, $\overline{\exp}$ is given as follows:

$$\overline{\exp}_{\binom{x}{a}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\gamma_\alpha(1)\\ \eta_\beta(1)\end{pmatrix}$$

where $\binom{\gamma_\alpha(t)}{\eta_\beta(t)}$ is the geodesic curve on TG^α_{V} with the initial data $\gamma'_\alpha(0) = \alpha$, $\eta'_\beta(0) = \beta$, $\gamma_\alpha(0) = x$, $\eta_\beta(0) = a$. Let the vector fields A_k^{H} and B_k^{H} be the horizontal lifts of A_k and B_k onto TTG^α_{V}. Further let $\partial_s\hat{Y}_s^\ell$ be the vertical lift of $\partial_s\hat{Y}_s$ onto TTG^α_{V}. Let us consider the backward SDE on TG^α_{V}:

$$\begin{aligned} dY_s^{t,e} &= \overline{\exp}_{Y_s^{t,e}}\Big\{\partial_s\hat{Y}_s^\ell(Y_s^{t,e})\mathrm{d}s + \mathrm{S}(Y_s^{t,e})\mathrm{d}s \\ &\quad + \epsilon\sum_{k\in\mathbb{Z}_2^+\cup\{0\},|k|\leqslant N}\left[A_k^{\mathrm{H}}(Y_s^{t,e})\otimes e_k^A + B_k^{\mathrm{H}}(Y_s^{t,e})\otimes e_k^A\right]\mathrm{d}W_s\Big\}, \qquad (40)\\ Y_T^{t,e} &= \hat{h}(Z_T^{t,e}) \end{aligned}$$

where S is the geodesic spray of the Levi-Civita connection of the weak Riemannian metric (3) on G^α_{V} (see [14] or [15]), and $Z_s^{t,e}$, $s\in[t,T]$, is the solution to (11) on G^α_{V} with the initial condition $Z_t^{t,e} = e$.

Theorem 9. *There exists a solution to (40) on $[t,T]$. Moreover, if $\partial_s y(s,\cdot)\in H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$, then this solution is unique and coincides with the $Y_s^{t,e}$-part of the unique $\mathcal{F}_s$-adapted solution $(Y_s^{t,e}, X_s^{t,e})$ to (7).*

Proof. From the proof of Theorem 6 we know that the pair of stochastic processes $(\hat{Y}_s(Z_s^{t,e}), \epsilon\,\sigma(Z_s^{t,e})\hat{Y}_s(Z_s^{t,e}))$ is the unique $\mathcal{F}_s$-adapted solution to (7) in $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$. Let us prove that $\hat{Y}_s(Z_s^{t,e})$ is a strong solution to (40). First we describe a system of local coordinates $(g^{kA}, X^{kA}, g^{kB}, X^{kB})_{k\in\mathbb{Z}_2^+\cup\{0\}}$ in a neighborhood $U_e g \times T_e G_V^\alpha$ of the point $\hat{X}(g) \in TG_V^\alpha$ where $U_e \subset G_V^\alpha$ is the canonical chart. The vector $\bar{g} = (g^{kA}, g^{kB})_{k\in\mathbb{Z}_2^+\cup\{0\}}$ is the vector of normal coordinates in the neighborhood $U_e g$, $g \in G_V^\alpha$. The vector $\bar{X} = (X^{kA}, X^{kB})_{k\in\mathbb{Z}_2^+\cup\{0\}}$ represents the coordinates of the decomposition of the vector $\hat{X}(g) \in TG_V^\alpha$ in the basis $\{A_k, B_k\}_{k\in\mathbb{Z}_2^+\cup\{0\}}$: $\hat{X}(g) = \sum_{k\in\mathbb{Z}_2^+\cup\{0\}}(X^{kA}A_k(g) + X^{kB}B_k(g))$. Let f be a smooth function on TG_V^α, and let $\tilde{f}(\bar{X}, \bar{g}) = f(\hat{X}(g))$, where $\hat{X}(g) \in TG_V^\alpha$. Let τ be the exit time of the process $Z_r^{t,e}$ from the neighborhood $U_e Z_s^{t,e}$. We will compute the difference $f(Y_s^{t,e}) - f(Y_\tau^{t,e})$ using Itô's formula. Let $(\bar{Z}_r, \bar{Y}_r) = (Z_r^{kA}, Z_r^{kB}, Y_r^{kA}, Y_r^{kB})_{k\in\mathbb{Z}_2^+\cup\{0\}}$ be the vector of local coordinates of the process $\hat{Y}_r(Z_r^{t,e})$ on $[s, \tau]$. Using SDE (40), we obtain:

$$\begin{aligned} f(Y_s^{t,e}) - f(Y_\tau^{t,e}) = &- \sum_{k\in\mathbb{Z}_2^+\cup\{0\}} \int_s^\tau \Big[(Y_r^{kA})' \frac{\partial \tilde{f}(\bar{Y}_r, \bar{Z}_r)}{\partial Y_r^{kA}} + (Y_r^{kB})' \frac{\partial \tilde{f}(\bar{Y}_r, \bar{Z}_r)}{\partial Y_r^{kB}} \\ &+ Y_r^{kA} \frac{\partial \tilde{f}(\bar{Y}_r, \bar{Z}_r)}{\partial Z_r^{kA}} + Y_r^{kB} \frac{\partial \tilde{f}(\bar{Y}_r, \bar{Z}_r)}{\partial Z_r^{kB}} + \frac{\epsilon^2}{2}\delta_k\Big(\frac{\partial^2}{\partial (Z_r^{kA})^2} + \frac{\partial^2}{\partial (Z_r^{kB})^2}\Big)\tilde{f}(\bar{Y}_r, \bar{Z}_r)\Big]\mathrm{d}r \\ &- \epsilon \sum_{k\in\mathbb{Z}_2^+\cup\{0\}, |k|\leqslant N} \int_s^\tau \Big[\frac{\partial \tilde{f}(\bar{Y}_r, \bar{Z}_r)}{\partial Z_r^{kA}} \otimes e_k^A + \frac{\partial \tilde{f}(\bar{Y}_r, \bar{Z}_r)}{\partial Z_r^{kB}} \otimes e_k^A\Big]\mathrm{d}W_r \end{aligned} \tag{41}$$

where $\delta_k = 1$ if $|k| \leqslant N$, and $\delta_k = 0$ otherwise. Since f is a smooth function on TG_V^α, all its restrictions to the tangent spaces of G_V^α are smooth. Hence, one can talk about derivatives of f restricted to a tangent space along the vectors of this tangent space. Namely, the following relation holds:

$$\frac{\partial \tilde{f}(\bar{Y}_r, \bar{Z}_r)}{\partial Y_r^{kA}} = f'(\hat{Y}_r(Z_r^{t,e}))A_k(Z_r^{t,e}).$$

Note that the differentiation of $\tilde{f}$ with respect to Z_r^{kA} and Z_r^{kB} can be regarded as the differentiation of the composite function $f \circ \hat{Y}_r$ along the vectors A_k and B_k. Namely, $\frac{\partial \tilde{f}(\bar{Y}_r, \bar{Z}_r)}{\partial Z_r^{kA}} = A_k(Z_r^{t,e})[(f \circ \hat{Y}_r)(Z_r^{t,e})]$. This implies:

$$\begin{aligned} f(Y_s^{t,e}) - f(\hat{h}(Z_T^{t,e})) = &-\int_s^T \mathrm{d}r \Bigg[\partial_r(f \circ \hat{Y}_r)(Z_r^{t,e}) + \hat{Y}_r(Z_r^{t,e})(f \circ \hat{Y}_r)(Z_r^{t,e}) \\ &+ \frac{\epsilon^2}{2} \sum_{k\in\mathbb{Z}_2^+\cup\{0\}, |k|\leqslant N} \big(A_k(Z_r^{t,e})A_k(Z_r^{t,e}) + B_k(Z_r^{t,e})B_k(Z_r^{t,e})\big)(f \circ \hat{Y}_r)(Z_r^{t,e})\Bigg] \\ &- \epsilon \sum_{k\in\mathbb{Z}_2^+\cup\{0\}, |k|\leqslant N} \int_s^T [A_k(Z_r^{t,e})(f \circ \hat{Y}_r)(Z_r^{t,e}) \otimes e_k^A \\ &+ B_k(Z_r^{t,e})(f \circ \hat{Y}_r)(Z_r^{t,e}) \otimes e_k^B]\mathrm{d}W_r. \end{aligned} \tag{42}$$

We extended the integration to the entire interval $[s, T]$ since the local coordinates no longer appear under the integral signs. This is also possible since (41) holds also with respect to the local coordinates in the neighborhood $U_e Z_\tau^{t,e}$ and a new exit time τ_1. The same argument can be repeated with respect to the local coordinates in the neighborhood $U_e Z^{t,e}_{\tau_1}$, etc. Let us consider now $f \circ \hat{Y}_s$ as a time-dependent function of $g \in G^\alpha_{\mathrm{V}}$. Applying Itô's formula to $(f \circ \hat{Y}_s)(Z^{t,e}_s)$ on the interval $[s, T]$ and using SDE (11) on G^α_{V}, we obtain exactly the above identity. This proves that $Y^{t,e}_s = \hat{Y}_s(Z^{t,e}_s)$ is a strong solution to (40) on TG^α_{V}. By results of [14], $\partial_s \hat{Y}^\ell_s$ is C^1-smooth. Moreover S, A^{H}_k and B^{H}_k, $k \in \mathbb{Z}^+_2$, are C^∞-smooth. Again, by results of [14], the solution of BSDE (40) on TG^α_{V} is unique. □

7.2. *The representation of the backward SDE on TG^α*

Applying Proposition 1.3 (p. 146) of [5] (see also [15], p. 64) to the manifolds TG^α_{V} and TG^α and the identical imbedding $\imath_V : TG^\alpha_{\mathrm{V}} \to TG^\alpha$, we obtain that the process $\imath_V\big(\hat{Y}_s(Z^{t,e}_s)\big) = \hat{Y}_s(Z^{t,e}_s)$ solves the following backward SDE on TG^α:

$$\begin{aligned} dY^{t,e}_s &= \bar{\exp}_{Y^{t,e}_s}\Big\{\partial_s \hat{Y}^{\bar{\ell}}_s(Y^{t,e}_s)\mathrm{d}s + \bar{\mathrm{S}}(Y^{t,e}_s)\mathrm{d}s \\ &\quad + \epsilon \sum_{k\in\mathbb{Z}^+_2\cup\{0\},|k|\leqslant N} \big[A^{\bar{\mathrm{H}}}_k(Y^{t,e}_s)\otimes e^A_k + B^{\bar{\mathrm{H}}}_k(Y^{t,e}_s)\otimes e^A_k\big]\,\mathrm{d}W_s\Big\}, \\ Y^{t,e}_T &= \hat{h}(Z^{t,e}_T) \end{aligned} \tag{43}$$

where $\bar{\mathrm{S}}$ is the geodesic spray of the Levi-Civita connection of the weak Riemannian metric on G^α, $\partial_s \hat{Y}^{\bar{\ell}}_s$ denotes the vertical lift of $\partial_s \hat{Y}_s$ onto TTG^α, $A^{\bar{\mathrm{H}}}_k$ and $B^{\bar{\mathrm{H}}}_k$ denote the horizontal lifts of A_k and B_k onto TTG^α, the process $Z^{t,e}_s$, $s \in [t, T]$, is the solution to (11) on G^α with the initial condition $Z^{t,e}_t = e$. The exponential map $\bar{\exp}$ on TTG^α is defined similarly to the map $\overline{\exp}$ on TTG^α_{V}. Namely, the Levi-Civita connection of the weak Riemannian metric on G^α generates a connection on TG^α. The latter gives rise to the exponential map $\bar{\exp}$ on TTG^α as it was described in Section 7.1. We actually have obtained the following theorem.

Theorem 10. *Backward SDE (43) has a unique strong solution. Moreover, this solution coincides with the unique strong solution to BSDE (40) on TG^α_{V}, and with the $Y^{t,e}_s$-part of the unique $\mathcal{F}_s$-adapted solution $(Y^{t,e}_s, X^{t,e}_s)$ to (7).*

Proof. We have already shown that the process $\hat{Y}_s(Z^{t,e}_s)$ solves BSDE (43). The uniqueness of solution can be proved in exactly the same way as the uniqueness of solution to (40) on TG^α_{V} (see the proof of Theorem 9). □

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Appendix

A.1. *Geometry of the group of volume-preserving diffeomorphisms of the n-dimensional torus*

Let $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$ denote the n-dimensional torus. Let us describe the basis of the tangent space $T_e G^\alpha_{\mathrm{V}}$ of the group G^α_{V} of volume-preserving diffeomorphisms of $\mathbb{T}^n$. We introduce the following notation:

$$\mathbb{Z}_n^+ = \{(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n : k_1 > 0 \text{ or } k_1 = \cdots = k_{i-1} = 0,\ k_i > 0,\ i = 2, \ldots, n\};$$

$$k = (k_1, \ldots, k_n) \in \mathbb{Z}_n^+, \qquad |k| = \sqrt{\sum_{i=1}^n k_i^2}, \qquad k \cdot \theta = \sum_{i=1}^n k_i \theta_i\,,$$

$$\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n, \qquad \nabla = \Big(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \ldots, \frac{\partial}{\partial\theta_n}\Big).$$

For every $k \in \mathbb{Z}_n^+$, $(\bar{k}^1, \ldots, \bar{k}^{n-1})$ denotes an orthogonal system of vectors of length $|k|$ which is also orthogonal to k. Introduce the vector fields on $\mathbb{T}^n$:

$$\bar{A}_k^i = \frac{1}{|k|^{\alpha+1}} \cos(k\cdot\theta)\,\bar{k}^i, \qquad \bar{B}_k^i = \frac{1}{|k|^{\alpha+1}} \sin(k\cdot\theta)\,\bar{k}^i, \quad i = 1, \ldots, n-1,\ k \in \mathbb{Z}_n^+,$$

and the constant vector fields $\bar{A}_0^i$, $i = 1, \ldots, n$, whose ith coordinate is 1 and the other coordinates are 0. Let A_k^i, B_k^i, $i = 1, \ldots, n-1$, $k \in \mathbb{Z}_n^+$, denote the right-invariant vector fields on G_{V}^α generated by $\bar{A}_k^i$, $\bar{B}_k^i$, $i = 1, \ldots, n-1$, $k \in \mathbb{Z}_n^+$, respectively, and let $A_0^i = \bar{A}_0^i$, $i = 1, \ldots, n$, stand for constant vector fields on G_{V}^α. The following lemma is an analog of Lemma 6.

Lemma 19. *The vectors $A_k^i(g)$, $B_k^i(g)$, $k \in \mathbb{Z}_n^+$, $i = 1, \ldots, n-1$, $g \in G_{\mathrm{V}}^\alpha$, A_0^i, $i = 1, \ldots, n$, form an orthogonal basis of the tangent space $T_g G_{\mathrm{V}}^\alpha$ with respect to both the weak and the strong inner products in $T_g G_{\mathrm{V}}^\alpha$. In particular, the vectors $\bar{A}_k^i$, $\bar{B}_k^i$, $k \in \mathbb{Z}_n^+$, $i = 1, \ldots, n-1$, $\bar{A}_0^i$, $i = 1, \ldots, n$, form an orthogonal basis of the tangent space $T_e G_{\mathrm{V}}^\alpha$. Moreover, the weak and the strong norms of the basis vectors are bounded by the same constant.*

The other lemmas of Section 2 hold in the n-dimensional case, with respect to the system A_k^i, B_k^i, $k \in \mathbb{Z}_n^+$, $i = 1, \ldots, n-1$, A_0^i, $i = 1, \ldots, n$, without changes. The index α of the Sobolev space H^α has to be chosen bigger than $\frac{n}{2} + 1$.

A.2. The Laplacian of a right-invariant vector field on $G^\alpha(\mathbb{T}^n)$

One of the most important steps in the proof of Theorems 6 and 8 is Lemma 10, i.e. the computation of the Laplacian of a right-invariant vector field on G^α with respect to the subsystem $\{A_k, B_k\}_{k\in\mathbb{Z}_2^+\cup\{0\}, |k|\leqslant N}$ where N can be fixed arbitrary. Below we prove an n-dimensional analog of this lemma.

Lemma 20. *Let $\hat{V}$ be the right-invariant vector field on $G^{\tilde\alpha}(\mathbb{T}^n)$ generated by an $H^{\tilde\alpha+2}$-vector field V on $\mathbb{T}^n$. Further let $\epsilon > 0$ be such that*

$$\frac{\epsilon^2}{2}\Big(1 + \frac{n-1}{n} \sum_{k\in\mathbb{Z}_n^+, |k|\leqslant N} \frac{1}{|k|^{2\alpha}}\Big) = \nu.$$

Then for all $g \in G^{\tilde\alpha}$,

$$\frac{\epsilon^2}{2}\Big[\sum_{k\in\mathbb{Z}_n^+, |k|\leqslant N} \sum_{i=1}^{n-1} \big(\bar\nabla_{A_k^i}\bar\nabla_{A_k^i} + \bar\nabla_{B_k^i}\bar\nabla_{B_k^i}\big) + \sum_{i=1}^n \bar\nabla_{A_0^i}\bar\nabla_{A_0^i}\Big]\hat{V}(g) = \nu\,\Delta V \circ g.$$

Proof. As it was mentioned in the proof of Lemma 7, it suffices to consider the case $g = e$. We observe that for all $i = 1, \dots, n-1$,

$$(\bar{k}^i, \nabla)\cos(k\cdot\theta) = -\sin(k\cdot\theta)(\bar{k}^i, k) = 0.$$

Similarly, $(\bar{k}^i, \nabla)\sin(k\cdot\theta) = 0$. Then, for $k \in \mathbb{Z}_n^+$, $\theta \in \mathbb{T}^n$,

$$\begin{aligned}\sum_{i=1}^{n-1} \bar{\nabla}_{A_k^i}\bar{\nabla}_{A_k^i}\hat{V}(e)(\theta) &= \frac{1}{|k|^{2\alpha+2}}\sum_{i=1}^{n-1}\cos(k\cdot\theta)(\bar{k}^i,\nabla)\big[\cos(k\cdot\theta)(\bar{k}^i,\nabla)V(\theta)\big]\\ &= \frac{1}{|k|^{2\alpha+2}}\cos(k\cdot\theta)^2\sum_{i=1}^{n-1}(\bar{k}^i,\nabla)^2 V(\theta)\\ &= \frac{1}{|k|^{2\alpha+2}}\cos(k\cdot\theta)^2(|k|^2\Delta - (k,\nabla)^2)V(\theta).\end{aligned}$$

The latter equality holds by the identity $\sum_{i=1}^{n-1}(\bar{k}^i,\nabla)^2 + (k,\nabla)^2 = |k|^2\Delta$ that follows, in turn, from the fact that the system $\left\{\frac{\bar{k}^i}{|k|}, \frac{k}{|k|}\right\}$, $i = 1,\dots,n-1$, forms an orthonormal basis of $\mathbb{R}^n$. Similarly,

$$\sum_{i=1}^{n-1}\bar{\nabla}_{B_k^i}\bar{\nabla}_{B_k^i}\hat{V}(e)(\theta) = \frac{1}{|k|^{2\alpha+2}}\sin(k\cdot\theta)^2(|k|^2\Delta - (k,\nabla)^2)V(\theta).$$

Hence, for each $k \in \mathbb{Z}_n^+$,

$$\sum_{i=1}^{n-1}(\bar{\nabla}_{A_k^i}\bar{\nabla}_{A_k^i} + \bar{\nabla}_{B_k^i}\bar{\nabla}_{B_k^i})\hat{V}(e)(\theta) = \frac{1}{|k|^{2\alpha+2}}(|k|^2\Delta - (k,\nabla)^2)V(\theta). \tag{44}$$

Further we have:

$$\begin{aligned}&\sum_{k\in\mathbb{Z}_n^+, |k|\leqslant N}\frac{1}{|k|^{2\alpha+2}}(k,\nabla)^2 = \frac{1}{2}\sum_{k\in\mathbb{Z}_n, |k|\leqslant N}\frac{1}{|k|^{2\alpha+2}}(k,\nabla)^2\\ &\quad= \frac{1}{2}\sum_{k\in\mathbb{Z}_n, |k|\leqslant N}\frac{1}{|k|^{2\alpha+2}}\sum_{i=1}^{n}k_i^2\partial_i^2 + \sum_{k\in\mathbb{Z}_n, |k|\leqslant N}\frac{1}{|k|^{2\alpha+2}}\sum_{i\neq j}k_ik_j\partial_i\partial_j\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial\theta_i}$, and due to the factor $\frac{1}{2}$ we perform the summation over all $k \in \mathbb{Z}_n$. Clearly, the second sum is zero. To show this, we have to specify the way of summation. Let us collect in a group the terms $k_ik_j\partial_i\partial_j$ attributed to those $k \in \mathbb{Z}_n$ whose coordinates except the ith and the jth coincide, while the ith and the jth coordinates satisfy the following rules: they are obtained from k_i and k_j attributed to one of the vectors of the group by means of an arbitrary assignment of a sign. This operation specifies four vectors. The other four vectors are obtained from the first four vectors of the group by means of the permutation of the ith and the jth coordinates. In total, we get eight vectors in the group. Clearly, the summands $k_ik_j\partial_i\partial_j$ attributed to these vectors cancel each other. Let us compute the first sum.

$$\sum_{k\in\mathbb{Z}_n, |k|\leqslant N}\frac{1}{|k|^{2\alpha+2}}\sum_{i=1}^{n}k_i^2\partial_i^2 = \sum_{i=1}^{n}\Big[\sum_{k\in\mathbb{Z}_n, |k|\leqslant N}\frac{1}{|k|^{2\alpha+2}}k_i^2\Big]\partial_i^2.$$

Note that

$$\sum_{k\in\mathbb{Z}_n,|k|=const} k_1^2 = \cdots = \sum_{k\in\mathbb{Z}_n,|k|=const} k_n^2 = \frac{1}{n} \sum_{k\in\mathbb{Z}_n,|k|=const} |k|^2.$$

This implies:

$$\sum_{k\in\mathbb{Z}_n,|k|\leqslant N} \frac{1}{|k|^{2\alpha+2}} \sum_{i=1}^n k_i^2 \partial_i^2 = \frac{1}{n} \sum_{k\in\mathbb{Z}_n,|k|\leqslant N} \frac{1}{|k|^{2\alpha}} \Delta = \frac{2}{n} \sum_{k\in\mathbb{Z}_n^+,|k|\leqslant N} \frac{1}{|k|^{2\alpha}} \Delta.$$

Together with (44) it gives:

$$\sum_{k\in\mathbb{Z}_n^+,|k|\leqslant N} \sum_{i=1}^{n-1} (\bar\nabla_{A_k^i} \bar\nabla_{A_k^i} + \bar\nabla_{B_k^i} \bar\nabla_{B_k^i}) \hat V(e)(\theta) = \frac{n-1}{n} \sum_{k\in\mathbb{Z}_n^+,|k|\leqslant N} \frac{1}{|k|^{2\alpha}} \Delta V(\theta).$$

We also have to take into consideration the term

$$\sum_{i=1}^n \bar\nabla_{A_0^i} \bar\nabla_{A_0^i} \hat V(e)(\theta) = \Delta V(\theta).$$

Finally, we obtain:

$$\Big[\sum_{k\in\mathbb{Z}_n^+,|k|\leqslant N} \sum_{i=1}^{n-1} (\bar\nabla_{A_k^i} \bar\nabla_{A_k^i} + \bar\nabla_{B_k^i} \bar\nabla_{B_k^i}) + \sum_{i=1}^n \bar\nabla_{A_0^i} \bar\nabla_{A_0^i}\Big] \hat V(e)(\theta)$$
$$= \Big(1 + \frac{n-1}{n} \sum_{k\in\mathbb{Z}_n^+,|k|\leqslant N} \frac{1}{|k|^{2\alpha}}\Big) \Delta V(\theta).$$

The lemma is proved. □

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