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Hitting times for the perturbed reflecting random walk

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Abstract

We consider a nearest neighbor random walk on $\mathbb{Z}$ which is reflecting at 0 and perturbed when it reaches its maximum. We compute the law of the hitting times and derive many corollaries, especially invariance principles with (rather) explicit descriptions of the asymptotic laws. We also obtain some results on the almost sure asymptotic behavior. As a by-product one can derive results on the reflecting Brownian motion perturbed at its maximum.

Keywords: Perturbed random walk; Once reinforced random walk; Perturbed Brownian motion; Hitting times; Invariance principle; Recurrence; Law of the iterated logarithm

1. Introduction and statement of the results

Processes with reinforcement have already generated an important amount of literature. Pemantle gives in [13] a very pleasant survey with lots of references. Reinforced random walks on a graph where introduced by Diaconis in 1987 with an edge reinforcement scheme. Other reinforcement schemes were introduced later, for instance sequence-type reinforcement as in [7]. Many questions remain open concerning reinforced random walks, especially in dimension greater than 1. In the present paper we stay in dimension 1 and we concentrate on the simplest case: the once reinforced random walk which is a random walk perturbed when reaching its extrema and more particularly its variant obtained by reflection at 0. This walk will be called a perturbed reflecting random walk (*PRRW*). Let us give a precise definition.

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For any real valued process $(X_n)_{n\geq 0}$, we denote by $\mathcal{F}_n^X$ the σ-algebra generated by $X_0, X_1, \ldots, X_n$ and we set $\overline{X}_n = \max\{X_0, X_1, \ldots, X_n\}$. The PRRW with reinforcement parameter $r \in (-1, 1)$ is a process $(X_n)_{n\geq 0}$ taking its values on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ such that, for every $n \geq 0$, $X_{n+1} \in \{X_n - 1, X_n + 1\}$ and the transition probability $\mathbb{P}\left(X_{n+1} = X_n + 1 | \mathcal{F}_n^X\right)$ is equal to

- $1/2$ if $0 < X_n < \overline{X}_n$
- $(1-r)/2$ if $X_n = \overline{X}_n$ and $n \geq 1$
- 1 if $X_n = 0$

and moreover we suppose $X_0 = 0$. Of course the case $r = 0$ corresponds to the reflecting standard random walk (*RSRW*). We will also use the quantity $\beta = (1+r)/(1-r)$ to simplify some formulas. We interpret the case $r > 0$ as a self attractive walk called in the literature a reinforced random walk—whereas for $r < 0$ the walk is self repulsive and often called a negatively reinforced random walk. We summarize it in the array below.

r	$-1 \nearrow 0$	0	$0 \nearrow 1$
$\beta = (1+r)/(1-r)$	$0 \nearrow 1$	1	$1 \nearrow +\infty$
Terminology	negatively reinforced	standard	reinforced
Interpretation	self repellent		self attracting

(1)

Davis [8,9] has shown that a random walk perturbed when reaching its extrema converges, after the same rescaling as in Donsker's Theorem, toward a continuous time process called perturbed Brownian motion. This process has been studied by many authors, see for instance [12,3, 16,8,9,14,4,5] and the references therein. In our case where reflection at 0 is added, the continuous time limit is the solution of the equation

$$W_t = B_t + \alpha \sup_{s\leq t} W_s + \frac{1}{2} L_t^W \tag{2}$$

where $(L_t^W)_{t\geq 0}$ is the local time process at level 0.

The goal of the present paper is to study the PRRW via an excursion point of view. Since the PRRW behaves as a standard random walk when it is below the maximum, we concentrate on the study of the maximum process. This leads to the study of the hitting time process $(T_n)_{n\geq 0}$ defined, as usual, by $T_n = \inf\{k \geq 0;\ X_k = n\}$. Our starting point is an elementary representation of these hitting times using the excursions below the already visited levels (see Section 3). Most of the results in the paper are in fact derived from this representation and we will state them in the rest of this section.

We start with an invariance principle for the rescaled hitting time process.

For a process with trajectories in the space $\mathbb{D}([0, +\infty), \mathbb{R})$ of càdlàg functions, "convergence in law" means weak convergence of probability laws on this space endowed with the usual Skorohod topology.

Theorem 1. *Let $(\tau_t^n)_{t\geq 0}$ be the rescaled process of the hitting times of the PRRW defined by*

$$\tau_t^n = \frac{1}{n^2} T_{[nt]} \tag{3}$$

where $[\cdot]$ denotes the integer part.

Then, as $n \to +\infty$, the process $(\tau_t^n)_{t\geq 0}$ converges in law to a process $(\tau_t)_{t\geq 0}$ with independent non-negative increments whose laws are given, for $0 < s < t$, by the Laplace transform

$$\mathbb{E}\left[e^{-\frac{\mu^2}{2}(\tau_t-\tau_s)}\right] = \left(\frac{\cosh(\mu\, s)}{\cosh(\mu\, t)}\right)^{\beta}. \tag{4}$$

This process has strictly increasing trajectories and is self-similar:

$$\forall a > 0, \quad (\tau_{a\,t})_{t\geq 0} \overset{(\mathrm{d})}{=} (a^2\, \tau_t)_{t\geq 0}. \tag{5}$$

Moreover, for any $t > 0$, the density of τ_t on $(0, +\infty)$ is equal to

$$\phi_{\tau_t}(x) = \frac{2^{\beta}}{\sqrt{2\pi}} \sum_{k=0}^{+\infty} \binom{-\beta}{k} \frac{(\beta+2k)\,t}{x^{3/2}} e^{-\frac{(\beta+2k)^2\,t^2}{2x}} \tag{6}$$

(see (40) for the definition of generalized binomial coefficients). The process $(\tau_t)_{t\geq 0}$ can be represented as

$$\tau_t = \int_0^t \int_{\mathbb{R}_+} x\, \mathcal{N}(ds\, dx) \tag{7}$$

where $\mathcal{N}(ds\, dx)$ is a Poisson point measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $f_s(x)\, ds\, dx$ where

$$f_s(x) = \frac{\pi^2}{4\, s^3}\, \beta \sum_{n=1}^{+\infty} (2n-1)^2\, e^{-\frac{(2n-1)^2\pi^2}{8}\,\frac{x}{s^2}}. \tag{8}$$

The proof of this theorem is given in Section 4. Note that in the case $\beta = 1$ of the RSRW, $(\tau_t)_{t\geq 0}$ is the hitting times process of a reflecting Brownian motion. Formulas (4) and (6) are given for instance in [2] as Formulas 3.1.1.2 and 3.1.1.4. For the latter note that the binomial coefficient appearing in (6) equals simply $(-1)^k$ in that case.

Since the maximum process $(\overline{X}_n)_{n\geq 0}$ can be obtained by inversion of the hitting times $(T_n)_{n\geq 1}$, we will be able to state later an invariance principle for the rescaled maximum and the limit will be the inverse of $(\tau_t)_{t\geq 0}$. In preparation for this result let us introduce and study this process.

Proposition 2. *Let $(Y_s)_{s\geq 0}$ be the non-decreasing process defined by*

$$Y_s = \inf\{t;\ \tau_t > s\} \tag{9}$$

where $(\tau_t)_{t\geq 0}$ is defined in Theorem 1. This process has continuous trajectories. It is self-similar:

$$\forall a > 0, \quad (Y_{a\,s})_{s\geq 0} \overset{(\mathrm{d})}{=} (\sqrt{a}\, Y_s)_{s\geq 0}. \tag{10}$$

Concerning the marginal laws, we have

$$\forall s > 0, \quad Y_s \overset{(\mathrm{d})}{=} \sqrt{\frac{1}{\tau_{1/\sqrt{s}}}} \overset{(\mathrm{d})}{=} \sqrt{\frac{s}{\tau_1}} \tag{11}$$

and, for any $s > 0$, *the variable* Y_s *admits the density on* $\mathbb{R}_+$ *given by:*

$$\phi_{Y_s}(x) = \frac{2\,s}{x^3}\,\phi_{\tau_1}\left(\frac{s}{x^2}\right) \tag{12}$$

$$= \frac{2^{\beta+1}}{\sqrt{2\pi}}\sum_{k=0}^{+\infty}\binom{-\beta}{k}\frac{(\beta+2k)}{\sqrt{s}}e^{-\frac{(\beta+2k)^2x^2}{2s}}. \tag{13}$$

The occupation measure of $(Y_s)_{s\geq 0}$ *has the following Laplace transform: for any measurable non-negative function* φ,

$$-\log\mathbb{E}\left(e^{-\int_0^{+\infty}\varphi(Y_s)\,ds}\right) = \beta\int_0^{+\infty}\sqrt{2\,\varphi(s)}\,\tanh\left(s\,\sqrt{2\,\varphi(s)}\right)\,ds. \tag{14}$$

For any $s \geq 0$, *the variable* Y_s *admits moments of every order* $p \geq 1$ *and for* $\beta < p$ *this moment is given by the formula*

$$\mathbb{E}[Y_s^p] = \frac{2^{\beta+\frac{p+1}{2}}}{\sqrt{2\pi}}\,\Gamma\left(\frac{p+1}{2}\right)\left(\sum_{k=0}^{+\infty}\binom{-\beta}{k}\frac{1}{(\beta+2k)^p}\right)s^{p/2}. \tag{15}$$

A corollary of Theorem 1 is an invariance principle for the maximum of the PRRW.

Theorem 3. *Let* $(X_n)_{n\geq 1}$ *be a PRRW as before. Denote by* $(Y_t^n)_{t\geq 0}$ *the rescaled maximum process, defined as:*

$$Y_t^n = \frac{1}{\sqrt{n}}\overline{X}_{[nt]}.$$

Then the process $(Y_t^n)_{t\geq 0}$ *converges in law to the process* $(Y_t)_{t\geq 0}$ *defined in Proposition 2.*

This result can be extended to the whole rescaled process $(X_t^n)_{t\geq 0}$ defined by $X_t^n = \frac{1}{\sqrt{n}}X_{[nt]}$, as proved in [9]. In our approach we could re-obtain this result by proving that the couple $(Y_t^n, Y_t^n - X_t^n)_{t\geq 0}$ converges in law to $(Y_t, Z_t)_{t\geq 0}$ where the conditional law of $(Z_t)_{t\geq 0}$ knowing $(Y_t)_{t\geq 0}$ is that of a Brownian motion on $[0, +\infty)$, reflecting at Y_t and conditioned to return at 0 at the increasing times of Y_t. This process can be constructed in the following way: introduce $\mathcal{D} = \{d;\ \tau_d^- \neq \tau_d\}$; define $(e_d,\ d \in \mathcal{D})$ such that e_d is a Brownian excursion between 0 and $\tau_d - \tau_d^-$ reflecting at d and theses excursions are mutually independent; for any $s > 0$ and d such that $s \in (\tau_d^-, \tau_d]$, set $Z_s = e_d(s - \tau_d^-)$. However completely developing the argument seems too lengthy and we will not do it in the present paper.

Note that Theorem 3 is a consequence of the results of [9], however one interest of our approach lies in the description of the limit $(Y_t)_{t\geq 0}$. A consequence is a better understanding of the solution of (2) as stated below. Concerning Eq. (2) one can refer for instance to [4] for existence and unicity results.

Theorem 4. *Let* $(W_t)_{t\geq 0}$ *be a solution of* (2) *for* $\alpha \in (-\infty, 1)$ *and, for* $s \geq 0$, $\overline{W}_s = \sup_{t\leq s} W_t$. *Then the process* $(\overline{W}_s)_{s\geq 0}$ *has the same law as the process* $(Y_s)_{s\geq 0}$ *studied in Proposition 2 with* $\beta = 1 - \alpha$.

The estimates we have obtained for the previous theorems can be used to describe – at least partially – the almost sure behavior of the PRRW. As it is well known, the almost sure asymptotic behavior of a RSRW is given by the famous law of the iterated logarithm and the so called Chung's law of the iterated logarithm [6]. The latter states that, for a RSRW $(X_n)_{n\geq 0}$,

$$\liminf_{n\to+\infty} \frac{\overline{X}_n}{\sqrt{\frac{n}{\log^{(2)} n}}} = \frac{\pi}{\sqrt{8}} \tag{16}$$

almost surely. We used the notation $\log^{(2)} x = \log(\log(x))$ for the iterated natural logarithm. A comparison argument – such as the one formalized in Proposition 18 – entails that the inequality $\geq$ holds for the PRRW in the self-repulsive case $r \in (-1, 0]$ and that the inequality $\leq$ holds in the self attractive case $r \in [0, 1)$. We do not know if equality still holds in the self repulsive case – we conjecture no – but we can show it in the self attractive case as stated in the following result which is proved in Section 6.

Theorem 5. *For any $r \in [0, 1)$ the PRRW with reinforcement parameter r satisfies Chung's law of the iterated logarithm as stated in (16).*

The "classical" law of the iterated logarithm says – as it is well known – that for a RSRW $(X_n)_{n\geq 0}$, almost surely,

$$\limsup_{n\to+\infty} \frac{X_n}{\sqrt{n \, \log^{(2)} n}} = \sqrt{2}. \tag{17}$$

For the PRRW the same remark as above applies i.e. the inequality $\geq$ holds in the self-repulsive case and that the inequality $\leq$ holds in the self attractive case. We go a little further with the following result which is obviously upgradeable. It shows in the self attractive case that in contrast to Chung's law, the behavior is different from the standard case, at least when the reinforcement is strong enough.

Proposition 6. *For any $r \in (1/9, 1)$ i.e. $\beta > 5/4$, the PRRW $(X_n)_{n\geq 0}$ with such a reinforcement parameter r satisfies,*

$$\limsup_{n\to+\infty} \frac{X_n}{\sqrt{n \, \log^{(2)} n}} \leq \frac{1}{\sqrt{2\,(\beta-1)}}. \tag{18}$$

For any $r \in (-1, 0)$ i.e. $\beta \in (0, 1)$, the PRRW $(X_n)_{n\geq 0}$ with such a reinforcement parameter r satisfies, for every $\varepsilon > 0$,

$$\limsup_{n\to+\infty} \frac{X_n}{\sqrt{n \, \log^{(2)} n \, (\log^{(3)} n)^{\varepsilon}}} = 0. \tag{19}$$

Note that (18) is still true for $\beta \in (1, \ 5/4)$ but in that case the bound on the right hand side is not as good as the obvious bound $\sqrt{2}$.

As another application of the study of the hitting times, we state a result concerning the return time to 0 and the maximum before this time for the PRRW. For the SRW, this return time to 0 is of course a.s. finite but has an infinite mean since recurrence – but not positive recurrence – holds. For the PRRW it is well known that recurrence still holds, whatever the value of the reinforcement parameter but it is remarkable that above a critical value for this reinforcement parameter "positive recurrence" occurs.

Proposition 7. *Let* $(X_n)_{n\geq 0}$ *be a PRRW with reinforcement parameter* r *and* $\zeta = \inf\{n > 0, X_n = 0\}$*. Then*

$$\mathbb{E}(\zeta) < +\infty \Leftrightarrow r > 1/3 \quad (\Leftrightarrow \beta > 2).$$

The proof is rather elementary and given in Section 7. We also get the following proposition which shows a new dichotomy in behavior according to the reinforcement parameter.

Proposition 8. *Let* $(X_n)_{n\geq 0}$ *be a PRRW with reinforcement parameter* r*; let* $\zeta = \inf\{n > 0, X_n = 0\}$ *as before and* $M = \max\{|X_n|, n \leq \zeta\}$*. Then*

$$\mathbb{E}(M) < +\infty \Leftrightarrow r > 0 \quad (\Leftrightarrow \beta > 1)$$

and for these values $\mathbb{E}(M) = \beta/(\beta - 1)$.

In fact the joint law of (ζ, M) can be described rather explicitly (see Proposition 19). A natural question is the asymptotic conditional law which turns out to be simple as stated in the following.

Theorem 9. *The conditional law of* $\frac{\zeta}{m^2}$ *knowing* $M = m$ *converges, as* $m \to +\infty$*, to the law of a variable* Z *having the following Laplace transform*

$$\mathbb{E}\left(e^{-\frac{\lambda^2}{2} Z}\right) = \left(\frac{\lambda}{\sinh \lambda}\right)^{1+\beta}. \tag{20}$$

This theorem is a generalization of the standard case $\beta = 1$, where Z is distributed as the length of a Brownian excursion conditioned to have height 1 and for which the above formula is well known.

The rest of the paper is devoted to the proofs of the results stated in the present section, beginning with a section of lemmas.

2. Preliminary lemmas

We start with elementary results on the RSRW or standard random walk (*SRW*). For the sake of completeness we sometimes give a sketch of proof. The notations cosh, sinh and tanh refer to the usual functions of hyperbolic trigonometry. Let $(S_n)_{n\geq 0}$ be a RSRW starting at level $k > 0$ and conditioned on $S_1 = k - 1$. The variable

$$L = \inf\{n > 0;\ S_n = k\} \tag{21}$$

denotes the length of an excursion below level k for the RSRW.

Lemma 10 (*Laplace Transform of* L). *For any* $\lambda \in \mathbb{R}$*, we have*

$$\mathbb{E}\left[(\cosh \lambda)^{-L}\right] = 1 - \tanh(\lambda)\ \tanh(k\lambda). \tag{22}$$

Moreover, for $|\lambda| < \frac{\pi}{2k}$*, we have*

$$\mathbb{E}\left[(\cos \lambda)^{-L}\right] = 1 + \tan(\lambda)\ \tan(k\lambda). \tag{23}$$

Proof. The variable L has the same law as $1+T$ where T is the hitting time of $\{-k,k\}$ for a SRW $(S_n)_{n\geq 0}$, starting from $k-1$. But for any $\lambda \in \mathbb{R}$, the process $\left(\frac{e^{\lambda S_n}}{(\cosh\lambda)^n}\right)_{n\geq 0}$ is a martingale, bounded up to time T. The stopping theorem entails, setting $z = 1/\cosh(\lambda)$,

$$e^{\lambda(k-1)} = e^{\lambda k}\,\mathbb{E}\left[z^T\,\mathbf{1}_{\{S_T=k\}}\right] + e^{-\lambda k}\,\mathbb{E}\left[z^T\,\mathbf{1}_{\{S_T=-k\}}\right].$$

Changing λ into $-\lambda$ leads to a supplementary equation. Then solving the system formed by these two equations, we obtain

$$\mathbb{E}\left[z^T\,\mathbf{1}_{\{S_T=k\}}\right] = \frac{\sinh(\lambda(2k-1))}{\sinh(2\lambda k)} \quad \text{and} \quad \mathbb{E}\left[z^T\,\mathbf{1}_{\{S_T=-k\}}\right] = \frac{\sinh(\lambda)}{\sinh(2\lambda k)}.$$

It follows by usual hyperbolic trigonometry that

$$\mathbb{E}[z^T] = \cosh(\lambda) - \frac{\sinh(\lambda k)\,\sinh(\lambda)}{\cosh(\lambda k)}$$

where the last equality follows from usual hyperbolic trigonometry. But $\mathbb{E}[z^L] = \mathbb{E}[z^T]/\cosh(\lambda)$ so (22) is proved. □

Formula (23) is obtained by a classical argument of analytic continuation.

Lemma 11 (*Moments of L*)**.** *For the length L defined in* (21) *we have the following mean and variance:*

$$\mathbb{E}(L) = 2\,k, \tag{24}$$

$$\mathbb{V}(L) = \frac{4}{3}\,k(k-1)(2\,k-1). \tag{25}$$

Proof. Use differentiation with respect to λ in (22). □

Lemma 12 (*Time Spent in a Strip by an SRW*)**.** *Let* $(X_n)_{n\geq 0}$ *be a SRW started at* 1 *and* ξ *be the hitting time of* $\{0,k\}$. *Then*

$$J_{+1}(k,\lambda) = \mathbb{E}\left[(\cosh\lambda)^{-(1+\xi)}\,\mathbf{1}_{\{X_\xi=0\}}\right] = 1 - \frac{\tanh\lambda}{\tanh(k\lambda)} \tag{26}$$

and

$$J_{-1}(k,\lambda) = \mathbb{E}\left[(\cosh\lambda)^{-(1+\xi)}\,\mathbf{1}_{\{X_\xi=k\}}\right] = \frac{\tanh\lambda}{\sinh(k\lambda)} \tag{27}$$

and

$$\mathbb{E}\left[(\cosh\lambda)^{-(1+\xi)}\right] = 1 - \tanh\lambda\,\tanh\left(\frac{k\lambda}{2}\right). \tag{28}$$

Moreover

$$\mathbb{E}\left[1+\xi \mid X_\xi = 0\right] = \frac{2\,(k+1)}{3} \quad \textit{and} \quad \mathbb{E}\left[1+\xi \mid X_\xi = k\right] = \frac{2+k^2}{3}. \tag{29}$$

Proof. Same arguments as in Lemma 10. □

Lemma 13. *There exists a constant C such that, for $x \in \mathbb{R}$ and $y \geq 1$,*

$$\left| \left(\frac{1}{\cosh x} \right)^y - e^{-\frac{x^2 y}{2}} \right| \leq C\, y\, x^4. \tag{30}$$

Proof. First we note that there exists a constant C such that for any $x \in \mathbb{R}$,

$$\left| \frac{1}{\cosh x} - e^{-\frac{x^2}{2}} \right| \leq C\, x^4.$$

We note also that, by the an obvious bound on the derivative, we have, for $0 \leq a, b \leq 1$ and $y \geq 1$,

$$\left| a^y - b^y \right| \leq |a - b|\, y.$$

Finally we combine the two inequalities above. $\square$

Lemma 14. *Let f and f_n, $n \geq 1$ be nondecreasing functions belonging to $\mathbb{D}([0, A], [0, B])$ such that f_n converges to f, as $n \to +\infty$, with respect to the Skorohod topology and f is supposed to be strictly increasing. Let $g(t) = \inf\{s,\ f(s) > t\}$ and $g_n(t) = \inf\{s,\ f_n(s) > t\}$ define their respective inverses. Then g_n converges to g with respect to the uniform topology.*

Proof. It is an exercise on Skorohod topology but we give a proof for the sake of completeness. The hypothesis implies the existence of a sequence (λ_n) of continuous strictly increasing functions from $[0, A]$ onto $[0, A]$ and a sequence (ε_n) converging to 0 such that, for every $t \in [0, A]$, we have

$$f(t) - \varepsilon_n \leq f_n(\lambda_n(t)) \leq f(t) + \varepsilon_n \tag{31}$$

and

$$t - \varepsilon_n \leq \lambda_n(t) \leq t + \varepsilon_n. \tag{32}$$

Take $n \geq 1$ and $t \in (\varepsilon_n, A]$. By the definition of g, we see that $f(g(t)) \geq t$. Using (31), we deduce $f_n(\lambda_n(g(t))) \geq t - \varepsilon_n > t - \varepsilon_n - \eta$, for a small $\eta > 0$ and, as a consequence, $g_n(t - \varepsilon_n - \eta) \leq \lambda_n(g(t)) \leq g(t) + \varepsilon_n$, the last bound following from (32). Since f is strictly increasing, it is easy to see that g is continuous. We introduce its oscillation in the usual way $\omega(\eta) = \sup\{|g(x) - g(y)|, |x - y| \leq \eta\}$. We obtain

$$g_n(t) \leq g(t) + \varepsilon_n + \omega(\eta + \varepsilon_n). \tag{33}$$

Besides, for any small $\eta > 0$, the definition of g implies that $f(g(t) - \eta) \leq t$. Using (31), it follows that $f_n(\lambda_n(g(t) - \eta)) \leq t + \varepsilon_n$ which implies that $g_n(t + \varepsilon_n) \geq \lambda_n(g(t) - \eta) \geq g(t) - \eta - \varepsilon_n$, the last equality following from (32). Changing $t + \varepsilon_n$ into t, we get, for $t \geq 2\varepsilon_n$,

$$g_n(t) \geq g(t) - \omega(\varepsilon_n) - \eta - \varepsilon_n. \tag{34}$$

Combining (34) and (33) and letting η tend to 0, we conclude that $|g(t) - g_n(t)| \leq \varepsilon_n + \omega(\varepsilon_n)$ for $t \in (2\varepsilon_n, A)$. Also it is easy to see that $\sup_{t \leq 2\varepsilon_n} g_n(t) = g_n(2\,\varepsilon_n)$ converges to 0 and it is trivial that $\sup_{t \leq 2\varepsilon_n} g(t)$ tends to 0. We conclude that $\sup_{t \in [0,A]} |g(t) - g_n(t)|$ converges to 0 and the proof of the lemma is complete. $\square$

We end this section by recalling some elementary facts that we will use in the sequel.
(1) The classical gamma function Γ satisfies, for $a > 0$, as $x \to +\infty$,

$$\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a. \tag{35}$$

(2) For $a > -1$ and $x > 1$, the following formula holds

$$\sum_{k=1}^{+\infty} \frac{\Gamma(a+k)}{\Gamma(a+k+x)} = \frac{\Gamma(a+1)}{(x-1)\,\Gamma(a+x)} \tag{36}$$

as stated for instance in [10] Formula 8.384 (3) p. 910.
(3) Let G be a random variable following the law given by

$$\forall g \geq 0, \quad \mathbb{P}(G = g) = \frac{1-r}{2}\left(\frac{1+r}{2}\right)^g \tag{37}$$

that we will denote $\mathcal{G}((1-r)/2)$ in the sequel and call geometric law with parameter $(1-r)/2$. Its mean and variance are

$$\mathbb{E}(G) = \frac{1+r}{1-r} = \beta, \qquad \mathbb{V}(G) = \frac{2(1+r)}{(1-r)^2} \tag{38}$$

and the generating function is

$$\mathbb{E}\left(z^G\right) = \frac{1-r}{2-(1+r)\,z}. \tag{39}$$

(4) We recall the usual notation for generalized binomial coefficients

$$\binom{-\beta}{k} = \prod_{j=1}^{k} \frac{-\beta - j + 1}{j} = \frac{(-1)^k}{k!}\frac{\Gamma(k+\beta)}{\Gamma(\beta)} \tag{40}$$

which will appear in the series expansion, valid for $|u| < 1$,

$$(1+u)^{-\beta} = \sum_{k=0}^{+\infty} \binom{-\beta}{k} u^k. \tag{41}$$

3. Excursion representation for the hitting times

Recall that we denote by $(T_n)_{n\geq 0}$ the hitting time process of a PRRW $(X_n)_{n\geq 0}$. We are looking for the limiting law of T_n/n^2. For the RSRW, many methods could apply (see for instance [15] P. 21.5 or T. 23.2 and also Problem 23.10). Here the most appropriate approach seems to decompose T_n using excursion length below the already visited levels, as stated below.

Proposition 15. *The hitting times $(T_n,\ n \geq 2)$ of the PRRW can be represented as sums of independent variables $(Y_k)_{k\geq 1}$ by*

$$T_n = n + \sum_{k=1}^{n-1} Y_k \quad \textit{where } Y_k = \sum_{i=1}^{G_k} L_i^k \tag{42}$$

and

- *the variables* G_k, $k \geq 1$ *are independent and distributed according to the same geometric law* $\mathcal{G}((1-r)/2)$ *defined in* (37).
- *the variables* L_i^k, $k \geq 2$, $i \geq 1$ *are mutually independent and independent of the* G_k*'s.*
- *each variable* L_i^k *has the law of the length of an excursion of the RSRW below level* k *(as studied in Lemma 10).*

We use the usual convention $\sum_1^0 = 0$.

The proof is obvious and omitted. The following corollary will be crucial.

Proposition 16. *The hitting times* (T_n) *of the PRRW have the following mean and variance:*

$$\mathbb{E}(T_n) = n + \beta\, n(n-1) \tag{43}$$

and, for large n,

$$\mathbb{V}(T_n) \sim \beta \frac{2}{3} n^4 \tag{44}$$

and the following Laplace transforms: for any $\lambda > 0$,

$$\mathbb{E}\left[\left(\frac{1}{\cosh\lambda}\right)^{T_n}\right] = \frac{1}{\cosh^n(\lambda)} \prod_{k=1}^{n-1} \frac{1}{1+\beta\,\tanh(\lambda)\,\tanh(k\,\lambda)} \tag{45}$$

and, for any $\lambda \in (-\pi/2n, \pi/2n)$ *such that* $\beta\,\tan(\lambda)\,\tan(k\,\lambda) < 1$, *we have*

$$\mathbb{E}\left[\left(\frac{1}{\cos\lambda}\right)^{T_n}\right] = \frac{1}{\cos^n(\lambda)} \prod_{k=1}^{n-1} \frac{1}{1-\beta\,\tan(\lambda)\,\tan(k\,\lambda)}. \tag{46}$$

Proof. To obtain Formula (43), we take the mean in (42) then use (24) and (38) and the result follows immediately. Similarly Formula (44) is obtained by taking the variance in (42); for the variance of Y_k, we are in the classical situation of a random sum of random variables:

$$\mathbb{V}(Y_k) = \mathbb{E}(G_k)\,\mathbb{V}(L_1^k) + \left(\mathbb{E}(L_1^k)\right)^2\,\mathbb{V}(G_k).$$

We use the Formulas (38), (24), (25) and get (44).

Now let us prove Formula (45). We start again from the representation (42) to get

$$\mathbb{E}\left[(\cosh\lambda)^{-T_n}\right] = \cosh^{-n}(\lambda) \prod_{k=1}^{n-1} \mathbb{E}\left[\mathbb{E}\left[(\cosh\lambda)^{-L_1^k}\right]^{G_k}\right]$$

then use Formulas (22) and (39) and this gives (45). The proof of (46) is similar using (23) instead of (22). $\square$

4. Invariance principle for the hitting times

Proof of Theorem 1. We now give a Proof of Theorem 1. We first show the convergence of finite dimensional marginals. As the independence of the increments of $(\tau_t^n)_{t\geq 0}$ is clear, we only have to prove the convergence in law of $\tau_t^n - \tau_s^n$ for $0 < s < t$. More precisely we want to show that, for any $\mu > 0$,

$$\lim_{n\to+\infty} \mathbb{E}\left[e^{-\frac{\mu^2}{2}\,(\tau_t^n-\tau_s^n)}\right] = \left(\frac{\cosh(\mu\, s)}{\cosh(\mu\, t)}\right)^{\beta}. \tag{47}$$

To do so we start from (42) which gives

$$\tau_t^n - \tau_s^n = \frac{[nt]-[ns]}{n^2} + \frac{1}{n^2}\sum_{k=[ns]}^{[nt]-1} Y_k.$$

By a slight generalization of (45) we have, for any $\lambda > 0$,

$$\mathbb{E}\left[\left(\frac{1}{\cosh\lambda}\right)^{n^2(\tau_t^n-\tau_s^n)}\right] = (\cosh\lambda)^{[ns]-[nt]}\prod_{k=[ns]}^{[nt]-1}\frac{1}{1+\beta\tanh(\lambda)\,\tanh(k\,\lambda)}. \tag{48}$$

Taking $\lambda = \mu/n$ for fixed $\mu > 0$, we get

$$\log\mathbb{E}\left[\left(\frac{1}{\cosh\frac{\mu}{n}}\right)^{n^2(\tau_t^n-\tau_s^n)}\right] = ([ns]-[nt])\,\log\cosh\frac{\mu}{n} - \sum_{k=[ns]}^{[nt]-1}\log\left(1+\beta\tanh\left(\frac{\mu}{n}\right)\,\tanh\left(k\,\frac{\mu}{n}\right)\right).$$

As $n \to +\infty$, the first term on the r.h.s. converges to 0. For the second one an asymptotic expansion of the logarithm at 0 shows that it behaves like

$$\begin{aligned}\beta\sum_{k=[ns]}^{[nt]-1}\tanh\left(\frac{\mu}{n}\right)\,\tanh\left(k\,\frac{\mu}{n}\right) &\sim \beta\,\frac{\mu}{n}\sum_{k=[ns]}^{[nt]-1}\tanh\left(k\,\frac{\mu}{n}\right)\\ &\sim \beta\int_{s\mu}^{t\mu}\tanh(x)\,dx\\ &= \beta\,[\log\cosh(\mu t) - \log\cosh(\mu s)].\end{aligned}$$

Hence

$$\lim_{n\to+\infty}\mathbb{E}\left[\left(\frac{1}{\cosh\frac{\mu}{n}}\right)^{n^2(\tau_t^n-\tau_s^n)}\right] = \left(\frac{\cosh(\mu\, s)}{\cosh(\mu\, t)}\right)^{\beta}. \tag{49}$$

We now apply Inequality (30) of Lemma 13 with $x = \mu/n$ and $y = n^2\,(\tau_t^n - \tau_s^n)$ to get

$$\mathbb{E}\left(\left|\left(\frac{1}{\cosh\frac{\mu}{n}}\right)^{n^2(\tau_t^n-\tau_s^n)} - e^{-\frac{\mu^2}{2}\,(\tau_t^n-\tau_s^n)}\right|\right) \le \frac{\mathbb{E}(\tau_t^n-\tau_s^n)}{n^2} = \frac{\mathbb{E}(T_{[nt]}-T_{[ns]})}{n^4}.$$

But (43) proves that the right hand side converges to 0 as $n \to +\infty$ and combined with (49), it completes the proof of (47) and thus the convergence in law of finite dimensional marginals.

To show the tightness of the laws of the processes $\left((\tau_t^n)_{t\ge0},\, n\ge1\right)$ we can use for instance the criterion stated in [1, Theorem 15.6] which consists, for any $T > 0$, in finding a nondecreasing continuous function F such that, for all $0 \le t_1 \le t \le t_2 \le T$ and all n large enough,

$$\mathbb{E}\left[(\tau_t^n - \tau_{t_1}^n)\,(\tau_{t_2}^n - \tau_t^n)\right] \le [F(t_2) - F(t_1)]^2. \tag{50}$$

By (43), we have

$$\mathbb{E}\left[\tau_t^n - \tau_s^n\right] = \frac{[nt]-[ns]}{n}\left(\frac{1-\beta}{n} + \beta\,\frac{[nt]+[ns]}{n}\right).$$

Combining this with the independence of the increments, we obtain, for $0 \le t_1 \le t \le t_2 \le T$,

$$\mathbb{E}\left[(\tau_t^n - \tau_{t_1}^n)\,(\tau_{t_2}^n - \tau_t^n)\right] \le (1+2\beta T)^2\left(\frac{[n\,t_2]-[n\,t_1]}{n}\right)^2.$$

We deduce that (50) is satisfied with $F(t) = 2\,(1+2\beta T)\,t$. Indeed if $t_2 - t_1 \ge 1/n$, it follows from the inequality above. If $t_2 - t_1 \le 1/n$ then either $n\,t_1$ and $n\,t$ lie in the same interval of the form $[i, i+1)$ (for a certain integer i) or else $n\,t_2$ and $n\,t$ do; in either of these cases the left hand side in (50) vanishes.

To obtain the expression of the density of τ_t as given in (6) we first re-express the Laplace transform of τ_t: for $\mu \ge 0$,

$$\begin{aligned}\mathbb{E}\left[e^{-\mu\,\tau_t}\right] &= \left(\cosh(\sqrt{2\mu}\,t)\right)^{-\beta}\\ &= \frac{2^\beta\, e^{-\beta\sqrt{2\mu}\,t}}{(1+e^{-2\sqrt{2\mu}\,t})^\beta}\\ &= 2^\beta \sum_{k=0}^{+\infty}\binom{-\beta}{k}\, e^{-(\beta+2k)\sqrt{2\mu}\,t}.\end{aligned} \tag{51}$$

The last line follows from (41) and holds for $\mu > 0$. But a classical result on Laplace transforms states that, for all $a > 0$, $\mu \ge 0$,

$$e^{-a\sqrt{2\mu}} = \int_0^{+\infty} e^{-\mu\,x}\,\frac{a}{\sqrt{2\pi}\,x^{3/2}}\,e^{-\frac{a^2}{2x}}\,dx.$$

Using this equality in (51) and then inverting the integral and the sum gives the expression of the density of τ_t given in (6).

Now we want to show the representation given in (7). Since this formula obviously defines a process with independent increments it suffices to check that, for all $0 < s < t$,

$$\mathbb{E}\left(e^{-\frac{\mu^2}{2}\int_s^t\int_{\mathbb{R}_+} x\,\mathcal{N}(du\,dx)}\right) = \left(\frac{\cosh(\mu\,s)}{\cosh(\mu\,t)}\right)^\beta. \tag{52}$$

But the exponential formula for Poisson measures entails

$$\mathbb{E}\left(e^{-\frac{\mu^2}{2}\int_s^t\int_{\mathbb{R}_+} x\,\mathcal{N}(du\,dx)}\right) = \exp\left[-\int_s^t\int_{\mathbb{R}_+}\left(1-e^{-\frac{\mu^2}{2}x}\right)\,f_u(x)\,du\,dx\right]$$

where $f_s(x)$ is the intensity function defined by (8). So it suffices to check that

$$\int_0^t\int_{\mathbb{R}_+}\left(1-e^{-\frac{\mu^2}{2}x}\right)\,f_s(x)\,ds\,dx = \beta\,\log\cosh(\mu\,t).$$

By changing $\mu^2/2$ into ν and deriving with respect to t, then doing an integration by parts, this is equivalent to

$$\int_0^{+\infty} e^{-\nu\,y}\int_y^{+\infty} f_t(x)\,dx\,dy = 2\,\beta\,\frac{\tanh(\sqrt{2\,\nu}\,t)}{\sqrt{2\,\nu}}.$$

But noting that $f_t(x) = f_1(x/t^2)/t^3$ and doing straightforward changes of variables, we see that it suffices to check the formula for $t = 1$. Note also that, by the definition of $f_1(x)$ given in (8), we have

$$\int_y^{+\infty} f_1(x)\, dx = 2\beta \sum_{n=1}^{+\infty} e^{-\frac{(2n-1)^2\pi^2}{8}\, y}$$

so it suffices to check that

$$\int_0^{+\infty} e^{-\nu\, y} \left(\sum_{n=1}^{+\infty} e^{-\frac{(2n-1)^2\pi^2}{8}\, y} \right) dy = \frac{\tanh(\sqrt{2\,\nu})}{\sqrt{2\,\nu}}.$$

Doing straightforward integration and setting $x = (2\sqrt{2\nu})/\pi$, this formula is equivalent to

$$\tanh \frac{\pi\, x}{2} = \frac{4\, x}{\pi} \sum_{n=1}^{+\infty} \frac{1}{x^2 + (2n-1)^2} \tag{53}$$

which is a classical expansion in series of simple fractions, see [10] Formula 1.421(2) p. 44. □

5. Explicit formulas for the process Y

Proof of Proposition 2. This section is devoted to the proof of Proposition 2 concerning the process $(Y_s)_{s\geq 0}$. The self-similarity expressed in (10) is a direct consequence of the self similarity of $(\tau_t)_{t\geq 0}$. Let us pass to (11). Let φ be a continuously differentiable function on $\mathbb{R}_+$ with $\varphi(0) = 0$. Then

$$\begin{aligned}
\mathbb{E}\,[\varphi(Y_s)] &= \int_0^{+\infty} \varphi(y)\, \mathbb{P}(Y_s \in dy) = \int_0^{+\infty} \varphi'(t)\, \mathbb{P}(Y_s \geq t)\, dt \\
&= \int_0^{+\infty} \varphi'(t)\, \mathbb{P}(\tau_t \leq s)\, dt = \int_0^{+\infty} \varphi'(t) \int_0^{s/t^2} \mathbb{P}(\tau_1 \in dy)\, dt \\
&= \int_0^{+\infty} \varphi\left(\sqrt{\frac{s}{y}}\right) \mathbb{P}(\tau_1 \in dy) = \mathbb{E}\left[\varphi\left(\sqrt{\frac{s}{\tau_1}}\right)\right].
\end{aligned}$$

We have successively used Fubini's Theorem, the definition (9), the scaling (5), and again Fubini's Theorem. Since the equality holds for a law determining class of functions φ, we deduce (10). Then it is straightforward to deduce Formulas (12) and (13).

Now we want to prove Formula (14). We first note that, for φ a nonnegative measurable function on $\mathbb{R}_+$,

$$\int_0^{+\infty} \varphi(t)\, d\tau_t = \int_0^{+\infty} \varphi(Y_s)\, ds.$$

This is straightforward when $\varphi = \mathbf{1}_{[0,a]}$, $a > 0$ and the general case follows by a monotone class argument. Using this formula and the representation (7) of τ_t in terms of a Poisson measure and finally the exponential formula for Poisson measures, we get

$$\begin{aligned}
-\log \mathbb{E}\left(e^{-\int_0^{+\infty} \varphi(Y_s)\, ds}\right) &= -\log \mathbb{E}\left(e^{-\int_0^{+\infty}\int_0^{+\infty} \varphi(t)\, x\, \mathcal{N}(dt\, dx)}\right) \\
&= \int_0^{+\infty}\int_0^{+\infty} f_t(x) \left(1 - e^{-x\,\varphi(t)}\right) dt\, dx.
\end{aligned}$$

The explicit value of the intensity function $f_t(x)$ given in (8) allows us to compute the integral above. Denoting $\mathcal{I}$ the set of odd integers, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} f_t(x)\,\left(1-e^{-x\,\varphi(t)}\right)\,dt\,dx \\
&= 2\beta\int_0^{+\infty}\left(\int_0^{+\infty}\sum_{n\in\mathcal{I}}\frac{n^2\pi^2}{8\,t^2}\,e^{-\frac{n^2\pi^2}{8t^2}\,x}\left(1-e^{-x\,\varphi(t)}\right)dx\right)\frac{dt}{t} \\
&= 2\beta\int_0^{+\infty}\left(\sum_{n\in\mathcal{I}}\frac{\varphi(t)}{\frac{n^2\pi^2}{8t^2}+\varphi(t)}\right)\frac{dt}{t}
\end{aligned}$$

where the last equality is obtained via term by term integration. Using Formula (53), we compute the sum appearing above and get the desired formula (14).

Now we prove Formula (15) for the moments of Y_s. By scaling we may restrict to $s = 1$. We start from the density given by (13) and get

$$\mathbb{E}(Y_1^p) = \int_0^{+\infty} x^p\,\phi_{Y_1}(x)\,dx = \frac{2^{\beta+1}}{\sqrt{2\pi}}\int_0^{+\infty}\sum_{k=0}^{+\infty}\binom{-\beta}{k}(\beta+2k)\,x^p\,e^{-\frac{(\beta+2k)^2x^2}{2}}\,dx.$$

The sought-after formula (15) simply follows by inverting the sum and the integral. However (35) entails that, for $k \to +\infty$,

$$\left|\binom{-\beta}{k}\right| = \frac{\Gamma(\beta+k)}{\Gamma(\beta)\,\Gamma(k+1)} \sim \frac{1}{\Gamma(\beta)}k^{\beta-1}$$

so that this inversion can be justified by Fubini's Theorem only when $p > \beta$. But of course, the existence of moments for large p implies the existence for all $p \geq 1$. □

6. Laws of the iterated logarithm for the PRRW

Now we want to prove Theorem 5 and Proposition 6 and as before we proceed via the hitting times $(T_n)_{n\geq 0}$. The representation (42) obtained in Proposition 15 makes T_n the sum of independent variables Y_k and it is natural to try the laws of the iterated logarithm that have been proved in this framework – see for instance [11] and the references therein – but the hypotheses of many of these theorems seem difficult to check in our context and the criterion involving moments (Corollary 6.1 of [11]) does not apply since one can check that $\mathbb{E}(Y_k^3) \sim c\,k^5$. Our strategy is to take advantage of the (rather) explicit form of the Laplace transforms obtained previously and deduce tail estimates. Our result on hitting times is as follows.

Theorem 17. *For the PRRW the hitting times $(T_n)_{n\geq 1}$ satisfy*

$$\limsup_{n\to+\infty}\frac{T_n}{n^2\,\log^{(2)} n} \leq \frac{8}{\pi^2}. \tag{54}$$

Moreover if $\beta \in (0, 1)$ we have, for every $\varepsilon > 0$,

$$\liminf_{n\to+\infty}\frac{T_n}{\frac{n^2}{\log^{(2)} n\,(\log^{(3)} n)^{\varepsilon}}} = +\infty \tag{55}$$

and if $\beta > 1$, we have

$$\liminf_{n\to+\infty} \frac{T_n}{\frac{n^2}{\log^{(2)} n}} \geq 2(\beta - 1). \tag{56}$$

Proof. We start with (54). We take $q > 1$ and, as before, denote $[\cdot]$ the integer part. For the moment $(a_n)_{n\geq 1}$ is any positive sequence and $(\lambda_n)_{n\geq 1}$ is a sequence of real numbers in $(0, \pi/2)$. We begin using the so-called "Markov inequality":

$$\mathbb{P}\left(\frac{T_{[q^n]} - [q^n]}{q^{2n}\, a_n} \geq 1\right) \leq (\cos\lambda_n)^{q^{2n}\, a_n}\, \mathbb{E}\left[\left(\frac{1}{\cos\lambda_n}\right)^{T_{[q^n]}-[q^n]}\right]. \tag{57}$$

We fix $\varepsilon > 0$ and take $\lambda_n = \mu/q^n$ with $\mu = \frac{\pi(1-\varepsilon)}{2} \in (0, \pi/2)$. With this choice it is straightforward to check that

$$\lim_{n\to+\infty} \sup_{1\leq k<[q^n]} \beta\, \tan(\lambda_n)\, \tan(k\,\lambda_n) = 0 \tag{58}$$

so that we can use (46) which writes as

$$\mathbb{E}\left[\left(\frac{1}{\cos\lambda_n}\right)^{T_{[q^n]}-[q^n]}\right] = \prod_{k=1}^{[q^n]-1} \frac{1}{1 - \beta\, \tan(\lambda_n)\, \tan(k\,\lambda_n)}.$$

We note that for small x, we have $\frac{1}{1-x} \leq \exp((1+\varepsilon)\,x)$. We deduce from the previous inequality that, for large n,

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{\cos\lambda_n}\right)^{T_{[q^n]}-[q^n]}\right] &\leq \exp\left(\beta(1+\varepsilon)\sum_{k=1}^{[q^n]-1} \tan\frac{\mu}{q^n}\, \tan\frac{k\mu}{q^n}\right)\\
&\leq \exp\left(\beta(1+\varepsilon)^2\, \frac{\mu}{q^n} \sum_{k=1}^{[q^n]-1} \tan\frac{k\mu}{q^n}\right)\\
&\leq \exp\left(\beta(1+\varepsilon)^2 \int_0^{\mu} \tan(x)\, dx\right)\\
&\leq \exp\Big(\beta(1+\varepsilon)^2\,(-\log\cos\mu)\Big).
\end{aligned}$$

We have used firstly that, for large n, $\tan(\mu/q^n) \leq (1+\varepsilon)\,(\mu/q^n)$ and secondly a straightforward bound on a Riemann sum. We inject this result in (57) and use moreover the inequality $\cos(x) \leq e^{-\frac{x^2}{2}}$, valid for small x. We get that, for large n,

$$\mathbb{P}\left(\frac{T_{[q^n]} - [q^n]}{q^{2n}\, a_n} \geq 1\right) \leq C\, \exp\Big(-\beta\,(1+\varepsilon)^2 \log\sin\frac{\varepsilon\,\pi}{2}\Big)\, \exp\left(-\frac{\mu^2}{2}\, a_n\right). \tag{59}$$

The choice

$$a_n = \frac{2}{\mu^2}(1+\varepsilon)\log n = \frac{8}{\pi^2}\, \frac{1+\varepsilon}{(1-\varepsilon)^2}\, \log n$$

ensures that the right hand side in (59) is the term of a convergent series. We deduce by the Borel–Cantelli Lemma that

$$\limsup_{n\to+\infty} \frac{T_{[q^n]} - [q^n]}{q^{2n}\, a_n} \le 1$$

so that

$$\limsup_{n\to+\infty} \frac{T_{[q^n]}}{q^{2n}\, \log^{(2)}(q^n)} \le \frac{8}{\pi^2}\, \frac{1+\varepsilon}{(1-\varepsilon)^2}.$$

Recalling that $\varepsilon > 0$ is arbitrary, the above lim sup is in fact lower than $8/\pi^2$. Then, for any integer k, we use the usual interpolation $[q^n] \le k < [q^{n+1}]$ and the monotonicity property of the T_k's to get

$$\limsup_{k\to+\infty} \frac{T_k}{k^2\, \log^{(2)}(k)} \le q^2\, \frac{8}{\pi^2}$$

and (54) follows by letting $q \downarrow 1$.

We pass to (55) and (56), following similar lines. For $q > 1$, A, $a_n > 0$ and $\lambda_n = \mu_n/q^n$ positive and converging to 0, we write the following inequalities that we will justify below:

$$\mathbb{P}\left(\frac{T_{[q^n]} - [q^n]}{q^{2n}\, a_n} \le A\right) \le (\cosh\lambda_n)^{q^{2n} A\, a_n}\, \mathbb{E}\left[\left(\frac{1}{\cosh\lambda_n}\right)^{T_{[q^n]}-[q^n]}\right] \tag{60}$$

$$= (\cosh\lambda_n)^{q^{2n} A\, a_n} \prod_{k=1}^{[q^n]-1} \frac{1}{1+\beta\, \tanh(\lambda_n)\, \tanh(k\,\lambda_n)} \tag{61}$$

$$\le e^{\frac{\mu_n^2\, A\, a_n}{2}}\, \exp\left[-\beta(1-\eta)^2 \frac{\mu_n}{q^n} \sum_{k=1}^{[q^n]-1} \tanh\left(k\,\frac{\mu_n}{q^n}\right)\right] \tag{62}$$

$$\le \exp\left[\frac{\mu_n^2\, A\, a_n}{2} - \beta(1-\eta)^2 \int_0^{\frac{([q^n]-1)\,\mu_n}{q^n}} \tanh x\, dx\right] \tag{63}$$

$$\le \exp\left[\frac{\mu_n^2\, A\, a_n}{2} - \beta(1-\eta)^2\, \log\cosh\left(\frac{([q^n]-1)\,\mu_n}{q^n}\right)\right] \tag{64}$$

$$\le c\, \exp\left[\mu_n\left(\frac{\mu_n\, A\, a_n}{2} - \beta(1-\eta)^2\, \frac{([q^n]-1)}{q^n}\right)\right] \tag{65}$$

$$\le c\, \exp\left[\mu_n\left(\frac{\mu_n\, A\, a_n}{2} - \beta(1-\eta)^3\right)\right]. \tag{66}$$

Indeed, (60) is a Markov inequality. Equality (61) follows from (45). Then we use, that for x small, $\cosh x \le e^{x^2/2}$ and that, for a fixed small $\eta > 0$ we have, for x small enough, $(1+x)^{-1} \le e^{-(1-\eta)\,x}$ and $\tanh x \ge (1-\eta)\, x$. Recalling that $\lambda_n = \mu_n/q^n$ converges to 0, we get (62), valid for large n. Inequality (63) is obtained by a straightforward bound on the integral and the computation of this integral gives (64). Finally (65), follows simply from $\log\cosh x \ge x - \log 2$ and (66) is a consequence for large n.

In the case $0 < \beta < 1$, we take $a_n = \log^{-1} n\, (\log^{(2)} n)^{-\varepsilon}$ where $\varepsilon > 0$ and $\mu_n = \log n\, (\log^{(2)} n)^{\varepsilon/2}$. We see that the term in (66) is summable in n. By the Borel–Cantelli Lemma, we deduce that $\frac{T_{[q^n]}-[q^n]}{q^{2n}\,a_n} \geq A$ for large n and the assertion (55) follows by standard arguments as before.

In the case $\beta > 1$, take $a_n = \log^{-1} n$ and $\mu_n = \log n = 1/a_n$. For A such that $\frac{A}{2} - \beta < -1$ we can find $\eta > 0$ such that $\frac{A}{2} - \beta\,(1-\eta)^3 < -1$ and the quantity (66) is summable in n. By the same argument as before we deduce

$$\liminf_{n\to+\infty} \frac{T_n}{n^2 (\log^{(2)} n)^{-1}} \geq A$$

and this is true for all $A < 2\,(\beta - 1)$ so that (56) follows. □

We now give a formal statement on the intuitive comparison argument used in the introduction.

Proposition 18. *If $(X_n)_{n\geq 0}$ is a PRRW with reinforcement parameter $r \in [0, 1)$ then one can construct a RSRW $(W_n)_{n\geq 0}$ starting at 0 such that $W_n \geq X_n$ for every $n \geq 0$.*

If $(X_n)_{n\geq 0}$ is a PRRW with reinforcement parameter $r \in (-1, 0]$ then one can construct a RSRW $(W_n)_{n\geq 0}$ starting at 0 such that $W_n \leq X_n$ for every $n \geq 0$.

Proof. We only treat the case $r \geq 0$, the other one is similar. Let $(\eta_n)_{n\geq 0}$ be a sequence of independent Bernoulli variables with mean $r/(1+r)$ and independently, $(\tilde{\eta}_n)_{n\geq 0}$ be a sequence of independent Bernoulli variables with mean $1/2$. We construct $(W_n)_{n\geq 0}$ by setting:

- we set $W_{n+1} - W_n = X_{n+1} - X_n$ if $X_n > 0$ and $W_n > 0$ and one of the following three conditions hold: $X_n < \overline{X}_n$ or $X_{n+1} - X_n = 1$ or $\eta_n = 0$;
- if $X_n > 0$ and $X_n = \overline{X}_n$ and $X_{n+1} - X_n = -1$ and $\eta_n = 1$ we set $W_{n+1} - W_n = 1$;
- if $X_n = 0$ and $W_n > 0$ we set $W_{n+1} - W_n = 2\,\tilde{\eta}_n - 1$;
- finally if $X_n = W_n = 0$ we set $W_{n+1} - W_n = 1 = X_{n+1} - X_n$.

The reader can check that $(W_n)_{n\geq 0}$ is a RSRW and note moreover that, for every $n \geq 0$,

$$X_n \leq W_n \leq X_n + 2\sum_{k=1}^{n-1} \mathbf{1}_{\{X_k = \overline{X}_k = X_{k+1}+1;\ \eta_k = 1\}}. \quad \square$$

Proof of Theorem 5 and Proposition 6. The limsup Result (54) easily implies that, for a PRRW $(X_n)_{n\geq 0}$, whatever the value of $r \in (-1, 1)$, we have almost surely

$$\liminf_{n\to+\infty} \frac{\overline{X}_n}{\sqrt{\frac{n}{\log^{(2)} n}}} \geq \frac{\pi}{\sqrt{8}}.$$

Since the converse inequality is clear in the case $r \in (0, 1)$, by comparison with a RSRW, thanks to Proposition 18, the proof of Theorem 5 is complete. Similarly, (56) implies (18) and (55) implies (19) hence Proposition 6. We leave the details to the reader. □

7. Return time to 0 and maximum

This section is devoted to the proofs of Propositions 7 and 8 but we start by a supplementary statement.

Proposition 19. *Let $(X_n)_{n\geq 0}$ be a PRRW and, as defined in the introduction, $\zeta = \inf\{n > 0, X_n = 0\}$ and $M = \max\{|X_n|, n \leq \zeta\}$. Then the law of M is given, for any $m \geq 1$, by*

$$\mathbb{P}(M = m) = \frac{\beta\, \Gamma(1+\beta)\, \Gamma(m)}{\Gamma(m+1+\beta)} \tag{67}$$

and the conditional law of ζ knowing $M = m$ is given by the following Laplace transform: for $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[(\cosh\lambda)^{-\zeta} | M = m\right] = (\cosh\lambda)^{-m} \frac{m\ \tanh\lambda}{\sinh(m\,\lambda)} \prod_{k=2}^{m} \frac{1}{1 - \frac{\beta}{k+\beta}\ \psi_k(\lambda)} \tag{68}$$

where

$$\psi_k(\lambda) = 1 - \frac{k\ \tanh\lambda}{\tanh k\,\lambda}. \tag{69}$$

Proof. Note first that, for $m \geq 1, \mathbb{P}(M = m) = \mathbb{P}(T_m < \zeta < T_{m+1})$ and let us compute the conditional probability $\mathbb{P}(\zeta > T_{k+1} \mid \zeta > T_k)$ for $k \geq 1$, using the representation leading to (42). This conditional probability is the probability that among the excursions below level k between times T_k and T_{k+1}, none of them hits 0. For each such excursion, the probability of not hitting 0 is $(k-1)/k$ and there is a number G_k of these excursions, still with the notation of (42). As a consequence the sought-after probability is

$$\mathbb{P}(\zeta > T_{k+1} \mid \zeta > T_k) = \mathbb{E}\left[\left(\frac{k-1}{k}\right)^{G_k}\right] = \frac{k}{k+\beta}$$

where the last equality follows from (39). We deduce, for $m \geq 1$,

$$\mathbb{P}(T_m < \zeta < T_{m+1}) = \left(\prod_{k=1}^{m-1} \frac{k}{k+\beta}\right) \frac{\beta}{m+\beta}$$

which can be re-expressed as (67). In the same spirit as the remark before, conditionally on $M = m$, we may write

$$T_m = m + \sum_{k=2}^{m-1} \sum_{i=1}^{G_k} L_i^k \tag{70}$$

where the L_i^k are independent variables distributed as the length of an excursion under level k conditioned not to hit 0 and the G_k are independent variables. However the laws of the G_k's are not the same as in (42) since they are affected by the conditioning. It is easy to show that G_k follows the geometric law with parameter

$$\rho_k = \frac{1-r}{2}\left(1 + \frac{\beta}{k}\right). \tag{71}$$

In particular its generating function is, for suitable values of z

$$\mathbb{E}\left(z^{G_k}\right) = \frac{1}{1 - \frac{1-\rho_k}{\rho_k}(z-1)} = \frac{1}{1 - \beta\,\frac{k-1}{k+\beta}(z-1)}. \tag{72}$$

Using Lemma 12 we get

$$\mathbb{E}\Big[(\cosh\lambda)^{-L_i^k}|M=m\Big]=\frac{J_1(k,\lambda)}{J_1(k,0)}=\frac{1-\frac{\tanh\lambda}{\tanh(k\,\lambda)}}{1-\frac{1}{k}}=1+\frac{1}{k-1}\psi_k(\lambda) \tag{73}$$

where ψ_k is defined in (69). Combining (73) and (72) along the same lines as the proof of (45) in Proposition 16, we deduce

$$\mathbb{E}\Big[(\cosh\lambda)^{-T_m}|M=m\Big]=(\cosh\lambda)^{-m}\prod_{k=2}^{m-1}\frac{1}{1-\frac{\beta}{k+\beta}\;\psi_k(\lambda)}. \tag{74}$$

To continue, note that conditionally on $M=m$, the variable $\zeta-T_m$ is independent of T_m and can be expressed as:

$$\zeta-T_m=\sum_{i=1}^{\tilde{G}_m}\tilde{L}_i^m+D \tag{75}$$

where the variables $\tilde{L}_i^m$ are the lengths of the excursions below level m that are not touching 0, the variable D is the hitting time of 0 for a SRW starting at m, conditioned to go down on the first step and conditioned not to return at m before hitting 0. Moreover $\tilde{G}_m$ is the number of excursions of the PRRW below level m after time T_m before ζ (conditionally on $M=m$ of course). It is easy to see that it is a geometric variable with parameter ρ_m defined in (71). By the same arguments as before we get that, for any $\lambda\in\mathbb{R}$,

$$\mathbb{E}\left[(\cosh\lambda)^{-\sum_{i=1}^{\tilde{G}_m}\tilde{L}_i^m}|M=m\right]=\frac{1}{1-\frac{\beta}{m+\beta}\;\psi_m(\lambda)} \tag{76}$$

and Lemma 12 entails, that, for any $\lambda\in\mathbb{R}$,

$$\mathbb{E}\Big[(\cosh\lambda)^{-D}|M=m\Big]=\frac{J_{-1}(m,\lambda)}{J_{-1}(m,0)}=\frac{m\;\tanh\lambda}{\sinh(m\,\lambda)}. \tag{77}$$

Combining the formulas (74), (76) and (77), we get the announced formula (68) and the proof is complete. We now apply this result. First a Remark. The well known recurrence of the process (X_n) i.e. the almost sure finiteness of ζ can be deduced immediately from (67), by checking that

$$\sum_{m=1}^{+\infty}\mathbb{P}(T_m<\zeta<T_{m+1})=\sum_{m=1}^{+\infty}\frac{\Gamma(m)\;\beta\,\Gamma(1+\beta)}{\Gamma(m+1+\beta)}=1$$

which follows from (36). □

Proof of Proposition 8. The Proof of Proposition 8 is now straightforward. Applying Formula (35) to the expression in (67), we get that, for large m,

$$\mathbb{P}(M=m)\sim c\,m^{-1-\beta} \tag{78}$$

hence the criterion for the finiteness of $\mathbb{E}(M)$ is clear. Moreover, for $\beta>1$, Formula (67) combined with (36) easily gives the value of $\mathbb{E}(M)$. □

Proof of Proposition 7. We now pass to the Proof of Proposition 7. We start with the obvious formula

$$\mathbb{E}(\zeta) = \sum_{m=1}^{+\infty} \mathbb{P}(M = m) \; [\mathbb{E}(T_m \mid M = m) + \mathbb{E}(\zeta - T_m \mid M = m)] . \tag{79}$$

We first compute $\mathbb{E}(T_m \mid M = m)$ using the representation (70). By Lemma 12, the variables L_i^k have a mean equal to $2(1 + k)/3$. Moreover $\mathbb{E}(G_k) = (1 - \rho_k)/\rho_k = \beta\,(k - 1)/(k + \beta)$. It follows that, for $m \geq 2$,

$$\mathbb{E}(T_m \mid M = m) = m + \frac{2\beta}{3} \sum_{k=2}^{m-1} \frac{k^2 - 1}{k + \beta}. \tag{80}$$

Similarly we use Eq. (75) to compute $\mathbb{E}(\zeta - T_m \mid M = m)$. Lemma 12 entails $\mathbb{E}(D) = (2+m^2)/3$ so that we get, for $m \geq 1$,

$$\mathbb{E}(\zeta - T_m \mid M = m) = \frac{2\,\beta\,(m^2 - 1)}{3\,(m + \beta)} + \frac{2 + m^2}{3}. \tag{81}$$

Gathering the expressions in (80) and (81), we obtain that, for large m,

$$\mathbb{E}(T_m \mid M = m) + \mathbb{E}(\zeta - T_m \mid M = m) \sim c\, m^2.$$

Recalling also (78), we see that the series in (79) defining $\mathbb{E}(\zeta)$ converges if and only if $\beta > 2$, as announced. $\square$

Proof of Theorem 9. We now pass to the Proof of Theorem 9. We substitute λ/m for λ in (68) and get

$$\mathbb{E}\left[\left(\cosh\frac{\lambda}{m}\right)^{-\zeta} \middle| M = m\right] = \left[\left(\cosh\frac{\lambda}{m}\right)^{-m} \frac{m\ \tanh(\lambda/m)}{\sinh\lambda}\right]$$
$$\times \exp\left[-\sum_{k=2}^{m} \log\left(1 - \frac{\beta}{k+\beta}\ \psi_k(\lambda/m)\right)\right]. \tag{82}$$

It is clear that the first term in square brackets on the right hand side converges to $\frac{\lambda}{\sinh\lambda}$ as $m \to +\infty$. Recall that

$$\psi_k(\lambda/m) = 1 - \frac{\operatorname{tanhc}\frac{\lambda}{m}}{\operatorname{tanhc}\frac{k\lambda}{m}} \quad \text{where } \operatorname{tanhc}(x) = \frac{\tanh x}{x}.$$

By the Taylor expansion $\operatorname{tanhc}(x) = 1 - (x^2/3) + O(x^4)$, we see easily that $\frac{\beta}{k+\beta}\psi_k(\lambda/m)$ is at most of order $1/m$ for large m, uniformly in $k \in \{2, \dots, m\}$. It follows that, for large m,

$$\sum_{k=2}^{m} \log\left(1 - \frac{\beta}{k+\beta}\ \psi_k(\lambda/m)\right) \sim -\sum_{k=2}^{m} \frac{\beta}{k+\beta}\ \psi_k(\lambda/m)$$
$$\sim -\beta \sum_{k=2}^{m-1} \frac{1}{k+\beta} \left(1 - \frac{1}{\operatorname{tanhc}\frac{k\lambda}{m}}\right)$$

$$\sim -\beta \int_0^1 \frac{1}{x}\left(1 - \frac{x\,\lambda}{\tanh(x\,\lambda)}\right)\,dx$$
$$= -\beta \log \frac{\lambda}{\sinh\lambda}.$$

Reporting this result in (82), we conclude that

$$\lim_{m\to+\infty} \mathbb{E}\left[\left(\cosh\frac{\lambda}{m}\right)^{-\zeta} \middle| M = m\right] = \left(\frac{\lambda}{\sinh\lambda}\right)^{1+\beta}.$$

Using the same argument as the one of Section 4 (see the lines following (49)), we deduce

$$\lim_{m\to+\infty} \mathbb{E}\left[e^{-\frac{\lambda^2}{2}\frac{\zeta}{m^2}} \middle| M = m\right] = \left(\frac{\lambda}{\sinh\lambda}\right)^{1+\beta}$$

and the proof is complete. □

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