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Functional inequalities for nonlocal Dirichlet forms with finite range jumps or large jumps

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Abstract

The paper is a continuation of our paper, Wang and Wang (2013) [13], Chen and Wang [4], and it studies functional inequalities for non-local Dirichlet forms with finite range jumps or large jumps. Let $\alpha \in (0, 2)$ and $\mu_V(dx) = C_V e^{-V(x)}\, dx$ be a probability measure. We present explicit and sharp criteria for the Poincaré inequality and the super Poincaré inequality of the following non-local Dirichlet form with finite range jump

$$\mathscr{E}_{\alpha,V}(f, f) := \frac{1}{2}\iint_{\{|x-y|\leqslant 1\}} \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}}\, dy\, \mu_V(dx);$$

on the other hand, we give sharp criteria for the Poincaré inequality of the non-local Dirichlet form with large jump as follows

$$\mathscr{D}_{\alpha,V}(f, f) := \frac{1}{2}\iint_{\{|x-y|>1\}} \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}}\, dy\, \mu_V(dx),$$

and also derive that the super Poincaré inequality does not hold for $\mathscr{D}_{\alpha,V}$. To obtain these results above, some new approaches and ideas completely different from Wang and Wang (2013), Chen and Wang (0000) are required, e.g. the local Poincaré inequality for $\mathscr{E}_{\alpha,V}$ and $\mathscr{D}_{\alpha,V}$, and the Lyapunov condition for $\mathscr{E}_{\alpha,V}$. In particular, the results about $\mathscr{E}_{\alpha,V}$ show that the probability measure fulfilling the Poincaré inequality and the super Poincaré inequality for non-local Dirichlet form with finite range jump and that for local Dirichlet form enjoy some similar properties; on the other hand, the assertions for $\mathscr{D}_{\alpha,V}$ indicate that even if functional inequalities for non-local Dirichlet form heavily depend on the density of large jump in the

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associated Lévy measure, the corresponding small jump plays an important role for the local super Poincaré inequality, which is inevitable to derive the super Poincaré inequality.

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1. Introduction and main results

Let $C_b^\infty(\mathbb{R}^d)$ be the set of smooth functions with bounded derivatives of every order. This paper is concerned with the following two bilinear forms:

$$\mathscr{E}_{\alpha,V}(f,f) := \frac{1}{2}\iint_{\{|x-y|\leqslant 1\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_V(dx), \quad f \in C_b^\infty(\mathbb{R}^d),$$

and

$$\mathscr{D}_{\alpha,V}(f,f) := \frac{1}{2}\iint_{\{|x-y|>1\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_V(dx), \quad f \in C_b^\infty(\mathbb{R}^d),$$

where $\alpha \in (0,2)$, V is a locally bounded Borel measurable function such that $e^{-V} \in L^1(dx)$, and

$$\mu_V(dx) := \frac{1}{\int e^{-V(x)}\,dx} e^{-V(x)}\,dx =: C_V e^{-V(x)}\,dx$$

is a probability measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. According to [4, Theorem 2.1], both $(\mathscr{E}_{\alpha,V}, C_b^\infty(\mathbb{R}^d))$ and $(\mathscr{D}_{\alpha,V}, C_b^\infty(\mathbb{R}^d))$ are closable bilinear forms on $L^2(\mu_V)$. Therefore, letting $\mathscr{D}(\mathscr{E}_{\alpha,V})$ and $\mathscr{D}(\mathscr{D}_{\alpha,V})$ be the closure of $C_b^\infty(\mathbb{R}^d)$ under the norms

$$\|f\|_{\mathscr{E}_{\alpha,V,1}} := \Big(\|f\|^2_{L^2(\mu_V)} + \mathscr{E}_{\alpha,V}(f,f)\Big)^{1/2}$$

and

$$\|f\|_{\mathscr{D}_{\alpha,V,1}} := \Big(\|f\|^2_{L^2(\mu_V)} + \mathscr{D}_{\alpha,V}(f,f)\Big)^{1/2}$$

respectively, $(\mathscr{E}_{\alpha,V}, \mathscr{D}(\mathscr{E}_{\alpha,V}))$ and $(\mathscr{D}_{\alpha,V}, \mathscr{D}(\mathscr{D}_{\alpha,V}))$ are regular Dirichlet forms on $L^2(\mu_V)$. The Hunt process associated with $(\mathscr{E}_{\alpha,V}, \mathscr{D}(\mathscr{E}_{\alpha,V}))$ is an $\mathbb{R}^d$-valued symmetric jump process with the finite range jump, while the associated Hunt process for $(\mathscr{D}_{\alpha,V}, \mathscr{D}(\mathscr{D}_{\alpha,V}))$ is an $\mathbb{R}^d$-valued symmetric jump process only with the jump larger than 1.

The purpose of this paper is to study the criteria about the Poincaré inequality and the super Poincaré inequality for $(\mathscr{E}_{\alpha,V}, \mathscr{D}(\mathscr{E}_{\alpha,V}))$ and $(\mathscr{D}_{\alpha,V}, \mathscr{D}(\mathscr{D}_{\alpha,V}))$. Recently, functional inequalities have been established in [8,13,4] for non-local Dirichlet form whose jump kernel has full support on $\mathbb{R}^d$, i.e.

$$D_{\rho,V}(f,f) := \frac{1}{2}\iint (f(x)-f(y))^2\,\rho(|x-y|)\,dy\,\mu_V(dx), \tag{1.1}$$

where ρ is a strictly positive measurable function on $\mathbb{R}_+ := (0, \infty)$ such that

$$\int_{(0,\infty)} \rho(r) \left(1 \wedge r^2\right) r^{d-1}\, dr < \infty.$$

Comparing with the methods of obtaining Poincaré type inequalities for $D_{\rho,V}$ in [13,4], in order to get the corresponding functional inequalities for $\mathscr{E}_{\alpha,V}$ and $\mathscr{D}_{\alpha,V}$, there are two fundamental differences.

(1) The efficient approach to yield functional inequalities for $D_{\rho,V}$ is to check the Lyapunov type condition for the generator associated with $D_{\rho,V}$, which heavily depends on the property of ρ. For $D_{\rho,V}$ the Lyapunov function ϕ we choose in [13,4] is of the form $\phi(x) = |x|^\beta$ with some constant $\beta \in (0, 1)$. Similar to [4], one can apply this test function ϕ into the generator of $\mathscr{D}_{\alpha,V}$, and verify the corresponding Lyapunov type condition; however, this test function ϕ is not useful for the generator of $\mathscr{E}_{\alpha,V}$.
(2) Another point on obtaining the Poincaré inequality and the super Poincaré inequality for $D_{\rho,V}$ is to prove the local Poincaré inequality and the local super Poincaré inequality. The local super Poincaré inequality for $D_{\rho,V}$ is derived by the classical Nash inequality of Besov space on $\mathbb{R}^d$ and bounded perturbation of functional inequalities for non-local Dirichlet form, while the local Poincaré inequality is easily obtained for $D_{\rho,V}$ by applying the Cauchy–Schwarz inequality. However we are unable to use these approaches here, since the jump kernel is not positive pointwise for both $\mathscr{E}_{\alpha,V}$ and $\mathscr{D}_{\alpha,V}$.

Due to the above differences and difficulties, obtaining the criteria for the Poincaré inequality and the super Poincaré inequality for $\mathscr{E}_{\alpha,V}$ and $\mathscr{D}_{\alpha,V}$ requires new approaches and ideas, which include the following three points.

(1) The new choice of the Lyapunov function for the generator associated with $\mathscr{E}_{\alpha,V}$, which is efficient to yield the Lyapunov conditions for $\mathscr{E}_{\alpha,V}$, and is completely different from that for $D_{\rho,V}$ (see Lemma 3.3).
(2) The local Poincaré inequality for both $\mathscr{E}_{\alpha,V}$ and $\mathscr{D}_{\alpha,V}$ (see Propositions 2.3 and 2.4), and the local super Poincaré inequality for $\mathscr{E}_{\alpha,V}$ (not for $\mathscr{D}_{\alpha,V}$), where we will use some results on the Sobolev embedding theorem in Besov space, e.g. [3] (see Proposition 2.2).
(3) To show that the super Poincaré inequality does not hold for $\mathscr{D}_{\alpha,V}$ with any locally bounded V (see Section 4).

We are now in a position to state the main results in our paper, which will be split into the following two parts.

1.1. Functional inequalities for $\mathscr{E}_{\alpha,V}$

For any $r > 0$, define

$$k(r) := \inf_{|x|\leqslant r+1} e^{-V(x)}, \qquad K(r) := \sup_{|x|\leqslant r} e^{-V(x)}. \tag{1.2}$$

Theorem 1.1. (1) *Suppose that*

$$\liminf_{|x|\to\infty} \frac{\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}}{\sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-V(z)}} > \frac{1}{\alpha} 2^{2d+1}(e + e^{1/2})(2^\alpha - 1). \tag{1.3}$$

Then the following Poincaré inequality

$$\mu_V\left(f^2\right) \leqslant C_1 \mathscr{E}_{\alpha,V}(f,f), \quad f \in C_b^\infty(\mathbb{R}^d), \mu_V(f) = 0 \tag{1.4}$$

holds for some constant $C_1 > 0$.

(2) *If*

$$\liminf_{|x|\to\infty} \frac{\inf\limits_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}}{\sup\limits_{|x|\leqslant|z|\leqslant|x|+1} e^{-V(z)}} = \infty, \tag{1.5}$$

then there exist constants $C_2, C_3 > 0$ such that the following super Poincaré inequality holds

$$\mu_V(f^2) \leqslant s\mathscr{E}_{\alpha,V}(f,f) + \beta(s)\mu_V(|f|)^2, \quad s > 0, f \in C_b^\infty(\mathbb{R}^d), \tag{1.6}$$

where

$$\begin{aligned}\beta(s) &= C_2((1+s^{-d/\alpha})[\Phi^{-1}(C_3 s^{-1})]^{d+d^2/\alpha} \\ &\quad\times [K(\Phi^{-1}(C_3 s^{-1}))]^{1+d/\alpha}[k(\Phi^{-1}(C_3 s^{-1}))]^{-2-d/\alpha})\end{aligned} \tag{1.7}$$

and

$$\Phi(r) := \inf_{|x|\geqslant r}\left(e^{V(x)} \inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}\right).$$

Though the constant on the right hand side of (1.3) is far from optimal, the criteria in Theorem 1.1 are qualitatively sharp, which can be seen from the following typical examples. For the proofs of examples, see Section 3.2.

Example 1.2. (1) Let

$$\lambda_0 := 2\log\left[\frac{1}{\alpha}2^{2d+1}(e+e^{1/2})(2^\alpha - 1)\right].$$

Then, for any probability measure $\mu_{V_\lambda}(dx) = C_\lambda e^{-\lambda|x|}\,dx$ with $\lambda > \lambda_0$, the Poincaré inequality (1.4) holds.

(2) For probability measure $\mu_{V_\delta}(dx) = C_\delta e^{-(1+|x|^\delta)}\,dx$ with $\delta > 0$, the super Poincaré inequality (1.6) holds if and only if $\delta > 1$, and in this case, it holds with

$$\beta(s) = c_1 \exp\left(c_2\left(1 + \log^{\frac{\delta}{(\delta-1)}}(1+1/s)\right)\right), \quad s > 0 \tag{1.8}$$

for some positive constants c_1 and c_2, and equivalently, the Markov semigroup P_t^{α,V_δ} associated with $\mathscr{E}_{\alpha,V_\delta}$ satisfies

$$\|P_t^{\alpha,V_\delta}\|_{L^1(\mu_{V_\delta})\to L^\infty(\mu_{V_\delta})} \leqslant \lambda_1 \exp\left(\lambda_2\left(1+\log^{\frac{\delta}{\delta-1}}(1+1/t)\right)\right), \quad t > 0$$

for some positive constants λ_1 and λ_2. Moreover, (1.8) is sharp in the sense that (1.6) does not hold with any rate function $\beta(s)$ such that

$$\lim_{s\to 0}\frac{\log\beta(s)}{\log^{\frac{\delta}{\delta-1}}(1+s^{-1})} = 0. \tag{1.9}$$

(3) For probability measure $\mu_{V_\theta}(dx) = C_\theta e^{-|x|\log^\theta(1+|x|)}\,dx$ with $\theta \in \mathbb{R}$, the super Poincaré inequality (1.6) holds if and only if $\theta > 0$, and in this case, it holds with

$$\beta(s) = c_3 \exp\left(1 + e^{c_4 \log^{\frac{1}{\theta}}(1+1/s)}\right), \quad s > 0 \tag{1.10}$$

for some positive constants c_3 and c_4; moreover, (1.10) is sharp in the sense that (1.6) does not hold with any rate function $\beta(s)$ such that

$$\lim_{s\to 0} \frac{\log\log\beta(s)}{\log^{\frac{1}{\theta}}(1+s^{-1})} = 0. \tag{1.11}$$

In particular, the Markov semigroup P_t^{α,V_θ} associated with $\mathscr{E}_{\alpha,V_\theta}$ is ultracontractive if $\theta > 1$, and in this case

$$\|P_t^{\alpha,V_\theta}\|_{L^1(\mu_{V_\theta})\to L^\infty(\mu_{V_\theta})} \leqslant \lambda_3 \exp\left(1 + e^{\lambda_4 \log^{\frac{1}{\theta}}(1+1/t)}\right), \quad t > 0$$

holds with some positive constants λ_3 and λ_4.

Remark 1.3. Example 1.2 above shows that the property of the probability measure μ_V fulfilling the Poincaré inequality and the super Poincaré inequality for $\mathscr{E}_{\alpha,V}(f,f)$ is similar to that for local Dirichlet form $D_V^*(f,f) := \frac{1}{2}\int |\nabla f(x)|^2\,\mu_V(dx)$, e.g. see [10, Chapters 1 and 3]. On the other hand, Example 1.2 also implies that the probability measure μ_V is easier to satisfy some functional inequalities for $\mathscr{E}_{\alpha,V}(f,f)$ than those for $D_V^*(f,f)$. For instance, given the probability measure $\mu_{V_\delta}(dx) = C_\delta e^{-(1+|x|^\delta)}\,dx$ with $\delta > 0$, Example 1.2(2) indicates that the measure μ_{V_δ} satisfies the log-Sobolev inequality for $\mathscr{E}_{\alpha,V_\delta}(f,f)$ if $\delta > 1$; however, μ_{V_δ} satisfies the log-Sobolev inequality for $D_{V_\delta}^*(f,f)$ only if $\delta \geqslant 2$, also see [10, Chapters 3 and 5].

1.2. Functional inequalities for $\mathscr{D}_{\alpha,V}$

Theorem 1.4. (1) *If*

$$\liminf_{|x|\to\infty} \frac{e^{V(x)}}{|x|^{d+\alpha}} > 0, \tag{1.12}$$

then the following weighted Poincaré inequality

$$\int f^2(x) \frac{e^{V(x)}}{1+|x|^{d+\alpha}}\,\mu_V(dx) \leqslant C_1 \mathscr{D}_{\alpha,V}(f,f), \quad f \in C_b^\infty(\mathbb{R}^d),\ \mu_V(f) = 0 \tag{1.13}$$

holds for some constant $C_1 > 0$. In particular, the following Poincaré inequality

$$\mu_V(f^2) \leqslant C_2 \mathscr{D}_{\alpha,V}(f,f), \quad f \in C_b^\infty(\mathbb{R}^d),\ \mu_V(f) = 0 \tag{1.14}$$

holds for some constant $C_2 > 0$.

(2) *For any locally bounded function V, the following super Poincaré inequality*

$$\mu_V(f^2) \leqslant s\mathscr{D}_{\alpha,V}(f,f) + \beta(s)\mu_V(|f|)^2, \quad s > 0,\ f \in C_b^\infty(\mathbb{R}^d) \tag{1.15}$$

does not hold for any rate function $\beta : (0,\infty) \to (0,\infty)$.

We present the following three remarks on Theorem 1.4.

Remark 1.5. (1) The condition (1.12) is sharp for the Poincaré inequality (1.14). For instance, let $\mu_V(dx) := \mu_\varepsilon(dx) = C_\varepsilon(1+|x|)^{-d-\varepsilon}\,dx$ with $\varepsilon > 0$. According to [13, Corollary 1.2], the following Poincaré inequality

$$\mu_V(f^2) \leqslant C_3 D_{\alpha,V}(f,f) := \frac{C_3}{2} \iint \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\, dy\, \mu_V(dx)$$

holds for all $f \in C_b^\infty(\mathbb{R}^d)$ with $\mu_V(f) = 0$, if and only if $\varepsilon \geqslant \alpha$. Note that $\mathscr{D}_{\alpha,V}(f,f) \leqslant D_{\alpha,V}(f,f)$, which along with (1.12) indicates that for the probability measure μ_ε above, the Poincaré inequality (1.14) holds if and only if $\varepsilon \geqslant \alpha$.

(2) The weighted function in the weighted Poincaré inequality (1.13) is

$$w(x) = \frac{e^{V(x)}}{1+|x|^{d+\alpha}},$$

which is optimal in the sense that, the inequality (1.13) fails if we replace $\omega(x)$ above by a positive function $\omega^*(x)$, which satisfies that

$$\liminf_{|x|\to\infty} \frac{\omega^*(x)}{\omega(x)} = \infty.$$

The proof is based on [4, Theorem 1.4] and the fact that $\mathscr{D}_{\alpha,V}(f,f) \leqslant D_{\alpha,V}(f,f)$ for any $f \in C_b^\infty(\mathbb{R}^d)$.

(3) A more important point indicated in Theorem 1.4 is that $\mathscr{D}_{\alpha,V}$ satisfies the weighted Poincaré inequality (1.13) (which is stronger than the Poincaré inequality (1.14)), but not the super Poincaré inequality (1.15). The main reason for this statement is due to the fact that the local super Poincaré inequality does not hold for $\mathscr{D}_{\alpha,V}$, while the local Poincaré inequality holds. That is, to derive the super Poincaré inequality for non-local Dirichlet form, we also need some assumption for the density of small jump for the associated Lévy measure.

The remaining part of this paper is organized as follows. In the next section we present the local super Poincaré inequality for $\mathscr{E}_{\alpha,V}$, and the local Poincaré inequality for both $\mathscr{E}_{\alpha,V}$ and $\mathscr{D}_{\alpha,V}$, which yields the weak Poincaré inequality for $\mathscr{E}_{\alpha,V}$ and $\mathscr{D}_{\alpha,V}$. Section 3 is devoted to functional inequalities for $\mathscr{E}_{\alpha,V}$. We first derive a new Lyapunov type condition for $\mathscr{E}_{\alpha,V}$, which along with the results in Section 2 enables us to prove Theorem 1.1 and also gives us the weighted Poincaré inequality for $\mathscr{E}_{\alpha,V}$ (cf. Proposition 3.4). Then, we study the concentration of measure about the functional inequalities for $\mathscr{E}_{\alpha,V}$, and present the proof of Example 1.2. To illustrate the differences between $\mathscr{E}_{\alpha,V}$ and the non-local Dirichlet forms in [13,4], we also compare these criteria here. In particular, we give a sharp and new example about the Poincaré inequality and the log-Sobolev inequality for $D_{\alpha,\delta,V}$, which is defined in (1.1) by setting $\rho(r) = e^{-\delta r} r^{-(d+\alpha)}$ with $\delta \geqslant 0$ and $\alpha \in (0,2)$. In the last section, we give the proof of Theorem 1.4.

2. The local Poincaré-type inequalities for $\mathscr{E}_{\alpha,V}$ and $\mathscr{D}_{\alpha,V}$

Let $B(x,r)$ be the ball with center $x \in \mathbb{R}^d$ and radius $r > 0$. Let V be a locally bounded measurable function on $\mathbb{R}^d$ such that $e^{-V} \in L^1(dx)$ and $\mu_V(dx) = C_V e^{-V(x)}\, dx$ is a probability measure. For $r > 0$, let $K(r)$ and $k(r)$ be the functions defined by (1.2).

We begin with the following (classical) local super Poincaré inequality for Lebesgue measure, which has been used in the proof of Proposition 2.2.

Lemma 2.1. *There exists a constant $C_1 > 0$ such that the following local super Poincaré inequality holds on any ball $B(0,r)$ with $r > 1$:*

$$\int_{B(0,r)} f^2(x)\,dx \leqslant s\iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\mathbb{1}_{\{|x-y|\leqslant 1\}}\,dy\,dx$$
$$+C_1 r^{d+d^2/\alpha}\left(1+s^{-d/\alpha}\right)\left(\int_{B(0,r+1)}|f(x)|\,dx\right)^2,$$
$$s>0,\ f\in C_b^\infty(\mathbb{R}^d).$$

Proof. For $z\in\mathbb{R}^d$ and $p\geqslant 1$, let $L^p(B(z,1/2),dx)$ be the L^p space with respect to Lebesgue measure for Borel measurable functions defined on the set $B(z,1/2)$. According to [3, (2.3)], for any $\alpha\in(0,d\wedge 2)$, there is a constant $c_1>0$ such that for all $z\in\mathbb{R}^d$ and $f\in C_b^\infty(\mathbb{R}^d)$,

$$\|f\|^2_{L^{2d/(d-\alpha)}(B(z,1/2),dx)} \leqslant c_1\left(\iint_{B(z,1/2)\times B(z,1/2)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx\right.$$
$$\left.+\ \|f\|^2_{L^2(B(z,1/2),dx)}\right).$$

Then, by Wang [10, Corollary 3.3.4(2)], also see [11, Theorem 4.5(2)], for any $\alpha\in(0,d\wedge 2)$, there is a constant $c_2>0$ such that for each $z\in\mathbb{R}^d$ and $f\in C_b^\infty(\mathbb{R}^d)$,

$$\int_{B(z,1/2)} f^2(x)\,dx \leqslant s\iint_{B(z,1/2)\times B(z,1/2)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx$$
$$+c_2\left(1+s^{-d/\alpha}\right)\left(\int_{B(z,1/2)}|f(x)|\,dx\right)^2,\quad s>0. \tag{2.16}$$

On the other hand, according to [3, Propositions 3.1 and 3.3], for any $\alpha\in[d,2)$ (if $d<2$), there is a constant $c_3>0$ such that for all $z\in\mathbb{R}^d$ and $f\in C_b^\infty(\mathbb{R}^d)$,

$$\|f\|^{2(1+\alpha/d)}_{L^2(B(z,1/2),dx)} \leqslant c_3\left(\iint_{B(z,1/2)\times B(z,1/2)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx\right.$$
$$\left.+\ \|f\|^2_{L^2(B(z,1/2),dx)}\right)\|f\|^{2\alpha/d}_{L^1(B(z,1/2),dx)}.$$

By Wang [10, Corollary 3.3.4(2)] again, we know that the inequality (2.16) also holds for $\alpha\in[d,2)$ (possibly with a different constant $c_2>0$). In particular, the constants c_1,c_2,c_3 above do not depend on $z\in\mathbb{R}^d$.

For any $r>1$, we can find a finite set $\Pi_r:=\{z_i\}\subseteq B(0,r)$ such that

$$B(0,r)\subseteq\bigcup_{z_i\in\Pi_r}B(z_i,1/2),\qquad \sharp\,\Pi_r\leqslant c_4 r^d, \tag{2.17}$$

where $\sharp\,\Pi_r$ denotes the number of the element in the set Π_r, and $c_4>0$ is a constant independent of r. Therefore, by (2.16) (note that according to the argument above it holds for all $\alpha\in(0,2)$) and (2.17), we get for each $r>1$ and $f\in C_b^\infty(\mathbb{R}^d)$.

$$\int_{B(0,r)} f^2(x)\,dx \leqslant \sum_{z_i\in\Pi_r}\int_{B(z_i,1/2)} f^2(x)\,dx$$
$$\leqslant \sum_{z_i\in\Pi_r}\left[s\iint_{B(z_i,1/2)\times B(z_i,1/2)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx\right.$$

$$
\begin{aligned}
&+ c_2\left(1+s^{-d/\alpha}\right)\left(\int_{B(z_i,1/2)}|f(x)|\,dx\right)^2\Bigg]\\
&= \sum_{z_i\in\Pi_r}\Bigg[s\iint_{B(z_i,1/2)\times B(z_i,1/2)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\mathbb{1}_{\{|x-y|\leqslant 1\}}\,dy\,dx\\
&+ c_2\left(1+s^{-d/\alpha}\right)\left(\int_{B(z_i,1/2)}|f(x)|\,dx\right)^2\Bigg]\\
&\leqslant c_4r^d s\iint_{B(0,r+1)\times B(0,r+1)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\mathbb{1}_{\{|x-y|\leqslant 1\}}\,dy\,dx\\
&+c_2c_4r^d\left(1+s^{-d/\alpha}\right)\left(\int_{B(0,r+1)}|f(x)|\,dx\right)^2,
\end{aligned}
$$

where in the equality above we have used the fact that for every $x, y \in B(z, 1/2)$ and $z \in \mathbb{R}^d$, $|x-y| \leqslant 1$; and the last inequality follows from the fact that $B(z, 1/2) \subseteq B(0, r+1)$ for each $z \in \Pi_r \subset B(0,r)$ and $\sharp\, \Pi_r \leqslant c_4 r^d$.

The required assertion follows by replacing $c_4r^d s$ with s in the inequality above. □

Now, we turn to the local super Poincaré inequality for $\mathscr{E}_{\alpha,V}$.

Proposition 2.2. *There is a constant $C_2 > 0$ such that for each $r > 1$, $s > 0$ and $f \in C_b^\infty(\mathbb{R}^d)$,*

$$
\int_{B(0,r)} f^2(x)\,\mu_V(dx) \leqslant s\,\mathscr{E}_{\alpha,V}(f,f) + \beta_r(s)\left(\int_{B(0,r+1)}|f(x)|\,\mu_V(dx)\right)^2, \tag{2.18}
$$

where

$$
\beta_r(s) = C_2\frac{r^{d+d^2/\alpha}K(r)^{1+d/\alpha}}{k(r)^{2+d/\alpha}}\left(1+s^{-d/\alpha}\right).
$$

Proof. For any $r > 1$, by Lemma 2.1, we find that for each $f \in C_b^\infty(\mathbb{R}^d)$ and $s > 0$,

$$
\begin{aligned}
\int_{B(0,r)} f^2(x)\,\mu_V(dx) &= C_V\int_{B(0,r)} f^2(x)e^{-V(x)}\,dx\\
&\leqslant C_V K(r)\int_{B(0,r)} f^2(x)\,dx\\
&\leqslant C_V K(r)\Bigg[s\iint_{B(0,r+1)\times B(0,r+1)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\\
&\quad\times\mathbb{1}_{\{|x-y|\leqslant 1\}}\,dy\,dx\\
&\quad+ C_1 r^{d+d^2/\alpha}\left(1+s^{-d/\alpha}\right)\left(\int_{B(0,r+1)}|f(x)|\,dx\right)^2\Bigg]\\
&\leqslant \frac{sK(r)}{k(r)}\iint_{B(0,r+1)\times B(0,r+1)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\\
&\quad\times\mathbb{1}_{\{|x-y|\leqslant 1\}}\,dy\,\mu_V(dx)
\end{aligned}
$$

$$+\frac{C_1 r^{d+d^2/\alpha} K(r)}{C_V k^2(r)} \left(1+s^{-d/\alpha}\right) \left(\int_{B(0,r+1)} |f(x)|\,\mu_V(dx)\right)^2,$$

where C_1 is a positive constant independent of r.

Replacing s with $sk(r)/K(r)$ in the inequality above and according to the definition of $\beta_r(s)$, we arrive at

$$\int_{B(0,r)} f^2(x)\,\mu_V(dx) \leqslant s \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|\leqslant 1\}}\,dy\,\mu_V(dx)$$
$$+\beta_r(s)\left(\int_{B(0,r+1)} |f(x)|\,\mu_V(dx)\right)^2, \quad s>0,$$

which implies the required assertion. $\square$

Next, we will present the local Poincaré inequality for $\mathscr{E}_{\alpha,V}$, which is inspired by the proofs of [2, Theorem 5.1] and [5, Theorem 2.2], see also [7, Section 1].

Proposition 2.3. *There is a constant $C_3>0$ such that for each $r>1$ and $f\in C_b^\infty(\mathbb{R}^d)$,*

$$\int_{B(0,r)} \left(f(x)-\frac{\int_{B(0,r)} f(x)\,\mu_V(dx)}{\mu_V(B(0,r))}\right)^2 \mu_V(dx)$$
$$\leqslant \frac{C_3 K(r) r^{3d}}{k(r)} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|\leqslant 1\}}\,dy\,\mu_V(dx)$$
$$\leqslant \frac{C_3 K(r) r^{3d}}{k(r)} \mathscr{E}_{\alpha,V}(f,f). \tag{2.19}$$

Proof. Let $m(A):=\int_A dx$ be the volume of a Borel set $A\subseteq\mathbb{R}^d$ with respect to Lebesgue measure. For any Borel set A with $m(A)>0$ and $f\in C_b^\infty(\mathbb{R}^d)$, set

$$f_A := \frac{1}{m(A)}\int_A f(x)\,dx.$$

First, there are two positive constants c_1, c_2 such that for any $z\in\mathbb{R}^d$,

$$\int_{B(z,1/6)} \left(f(x)-f_{B(z,1/6)}\right)^2 dx$$
$$=\frac{1}{(m(B(z,1/6)))^2}\int_{B(z,1/6)}\left(\int_{B(z,1/6)}(f(x)-f(y))\,dy\right)^2 dx$$
$$\leqslant c_1\int_{B(z,1/6)}\left(\int_{B(z,1/6)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\right)\left(\int_{B(z,1/6)}|x-y|^{d+\alpha}\,dy\right)dx$$
$$\leqslant c_2\iint_{B(z,1/6)\times B(z,1/6)}\frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\mathbb{1}_{\{|x-y|\leqslant 1\}}\,dy\,dx, \tag{2.20}$$

where the first inequality follows from the Cauchy–Schwarz inequality, and in the second inequality we have used the fact that $|x-y|\leqslant 1$ for every $x,y\in B(z,1/6)$ and $z\in\mathbb{R}^d$.

Second, for any $z_1,z_2\in\mathbb{R}^d$ with $B(z_1,1/6)\bigcap B(z_2,1/6)\neq\emptyset$, there are two constants c_3, $c_4>0$ independent of $z_1,z_2\in\mathbb{R}^d$ such that

$$
\begin{aligned}
&(f_{B(z_1,1/6)} - f_{B(z_2,1/6)})^2 \\
&\quad = \left(\frac{1}{m(B(z_1,1/6))m(B(z_2,1/6))} \int_{B(z_1,1/6)} \int_{B(z_2,1/6)} (f(x) - f(y))\, dy\, dx \right)^2 \\
&\quad \leqslant c_3 \int_{B(z_1,1/6)} \left(\int_{B(z_2,1/6)} \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}}\, dy \right) \left(\int_{B(z_2,1/6)} |x-y|^{d+\alpha}\, dy \right) dx \\
&\quad \leqslant c_4 \iint_{B(z_1,1/2)\times B(z_1,1/2)} \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|\leqslant 1\}}\, dy\, dx. \qquad (2.21)
\end{aligned}
$$

For the first inequality we have also used the Cauchy–Schwartz inequality, and the second inequality follows from the fact that $B(z_1, 1/6) \bigcup B(z_2, 1/6) \subseteq B(z_1, 1/2)$.

As before, for each $r > 1$, we can find a finite set $\Pi_r := \{z_i\} \subseteq B(0, r)$ such that

$$
0 \in \Pi_r, \quad B(0,r) \subseteq \bigcup_{z_i \in \Pi_r} B(z_i, 1/6), \quad \sharp\, \Pi_r \leqslant c_5 r^d,
$$

where $c_5 > 0$ is a constant independent of r.

Next, for a fixed $z \in \Pi_r$, we can find a sequence $\{z_i\}_{i=1}^n \subseteq \Pi_r$ such that $z_1 = z$, $z_n = 0$, $z_i \neq z_j$ if $i \neq j$, and $B(z_i, 1/6) \bigcap B(z_{i+1}, 1/6) \neq \emptyset$ for every $1 \leqslant i \leqslant n-1$. Hence, there exist $c_6, c_7 > 0$ independent of $r > 0$ and $z \in \Pi_r$ such that

$$
\begin{aligned}
&\int_{B(z,1/6)} \left(f(x) - f_{B(0,1/6)}\right)^2 dx \\
&\quad = \int_{B(z,1/6)} \left(\left(f(x) - f_{B(z,1/6)}\right) + \sum_{i=1}^{n-1} \left(f_{B(z_i,1/6)} - f_{B(z_{i+1},1/6)}\right) \right)^2 dx \\
&\quad \leqslant n \left(\int_{B(z,1/6)} \left(f(x) - f_{B(z,1/6)}\right)^2 dx + \sum_{i=1}^{n-1} \int_{B(z,1/6)} \left(f_{B(z_i,1/6)} - f_{B(z_{i+1},1/6)}\right)^2 dx \right) \\
&\quad \leqslant c_6 r^d \sum_{i=1}^{n} \iint_{B(z_i,1/2)\times B(z_i,1/2)} \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|\leqslant 1\}}\, dy\, dx \\
&\quad \leqslant c_7 r^{2d} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|\leqslant 1\}}\, dy\, dx, \qquad (2.22)
\end{aligned}
$$

where in the second inequality we have used (2.20) and (2.21) and the fact that $n \leqslant c_5 r^d$, and the last inequality follows from the facts that $B(z_i, 1/2) \subseteq B(0, r+1)$ for any $z_i \in \Pi_r$ and $n \leqslant c_5 r^d$.

Therefore, by (2.22), for each $r > 1$,

$$
\begin{aligned}
&\int_{B(0,r)} \left(f(x) - \frac{\int_{B(0,r)} f(x)\mu_V(dx)}{\mu_V(B(0,r))} \right)^2 \mu_V(dx) \\
&\quad \leqslant \int_{B(0,r)} \left(f(x) - f_{B(0,1/6)}\right)^2 \mu_V(dx) \\
&\quad \leqslant c_8 K(r) \int_{B(0,r)} \left(f(x) - f_{B(0,1/6)}\right)^2 dx \\
&\quad \leqslant c_8 K(r) \sum_{z_i \in \Pi_r} \int_{B(z_i,1/6)} \left(f(x) - f_{B(0,1/6)}\right)^2 dx
\end{aligned}
$$

$$\leqslant c_9 K(r) r^{3d} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|\leqslant 1\}}\, dy\, dx$$
$$\leqslant \frac{c_{10} K(r) r^{3d}}{k(r)} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|\leqslant 1\}}\, dy\, \mu_V(dx),$$

where c_8, c_9 and c_{10} are some positive constants independent of r. This completes the proof. □

We have derived the local super Poincaré inequality and the local Poincaré inequality for $\mathscr{E}_{\alpha,V}$. In particular, for the local super Poincaré inequality we have used the embedding theorem for subsets of $\mathbb{R}^d$ in the Besov space, but one cannot apply such embedding theorem in the context of $\mathscr{D}_{\alpha,V}$, since the part of the finite range jump in the associated kernel is removed. We believe that the local super Poincaré inequality does not hold for $\mathscr{D}_{\alpha,V}$, see Remark 4.1(2). However, we still can prove the following local Poincaré inequality for $\mathscr{D}_{\alpha,V}$.

Proposition 2.4. *There exists a constant $C_4 > 0$, such that for any $r > 3$ and $f \in C_b^\infty(\mathbb{R}^d)$,*

$$\int_{B(0,r)} \left(f(x) - \frac{\int_{B(0,r)} f(x)\, \mu_V(dx)}{\mu_V(B(0,r))} \right)^2 \mu_V(dx)$$
$$\leqslant \frac{C_4 K(r) r^{2d+\alpha}}{k(r)} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|>1\}}\, dy\, \mu_V(dx)$$
$$\leqslant \frac{C_4 K(r) r^{2d+\alpha}}{k(r)} \mathscr{D}_{\alpha,V}(f,f). \tag{2.23}$$

Proof. Throughout the proof, all the constants $c_i\,(i \geqslant 1)$ are positive and independent of $r > 0$ and $z \in \mathbb{R}^d$. As before, for each $r > 3$, we can find a finite set $\Pi_r := \{z_i\} \subseteq B(0,r)$ such that

$$0 \in \Pi_r, \quad B(0,r) \subseteq \bigcup_{z_i \in \Pi_r} B(z_i, 1/2), \quad \sharp\, \Pi_r \leqslant c_1 r^d.$$

Next, we split the set Π_r as $\Pi_r = \Pi_r^1 \bigcup \Pi_r^2$, where

$$\Pi_r^1 := \{z \in \Pi_r : \operatorname{dist}(B(z,1/2), B(0,1/2)) > 1\},$$
$$\Pi_r^2 := \{z \in \Pi_r : \operatorname{dist}(B(z,1/2), B(0,1/2)) \leqslant 1\},$$

and $\operatorname{dist}(A, B)$ denotes the distance between the subsets A, B in $\mathbb{R}^d$.

For each $z \in \Pi_r^1$, we have

$$\int_{B(z,1/2)} \left(f(x) - f_{B(0,1/2)}\right)^2 dx$$
$$= \frac{1}{(m(B(0,1/2)))^2} \int_{B(z,1/2)} \left(\int_{B(0,1/2)} (f(x)-f(y))\, dy \right)^2 dx$$
$$\leqslant c_2 \int_{B(z,1/2)} \left(\int_{B(0,1/2)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\, dy \right) \left(\int_{B(0,1/2)} |x-y|^{d+\alpha}\, dy \right) dx$$
$$\leqslant c_3 r^{d+\alpha} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|>1\}}\, dy\, dx. \tag{2.24}$$

Here, the first inequality follows from the Cauchy–Schwartz inequality, and in the last inequality we have used the facts that for all $z \in \Pi_r^1$, $B(z, 1/2) \subset B(0, r+1)$, and if $z \in \Pi_r^1$, then for each $x \in B(z, 1/2)$ and $y \in B(0, 1/2)$, $1 < |x-y| \leqslant 2(r+1)$.

For each $z \in \Pi_r^2$, since $r > 3$, there exists $z_0 \in B(0, r)$ such that for each $x \in B(z_0, 1/2)$ and $y \in B(z, 1/2) \bigcup B(0, 1/2)$, it holds that $|x-y| > 1$. Hence,

$$\int_{B(z,1/2)} \left(f(x) - f_{B(0,1/2)}\right)^2 dx \leqslant 2\int_{B(z,1/2)} \left(f(x) - f_{B(z_0,1/2)}\right)^2 dx + 2\int_{B(z,1/2)} \left(f_{B(z_0,1/2)} - f_{B(0,1/2)}\right)^2 dx.$$

Since for $x \in B(z_0, 1/2)$ and $y \in B(z, 1/2)$, $1 < |x-y| \leqslant 2(r+1)$, we can follow the proof of (2.24) and get that

$$\int_{B(z,1/2)} \left(f(x) - f_{B(z_0,1/2)}\right)^2 dx \leqslant c_3 r^{d+\alpha} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \times \mathbb{1}_{\{|x-y|>1\}}\, dy\, dx.$$

On the other hand, according to the argument of (2.21) and noticing that for each $x \in B(z_0, 1/2)$ and $y \in B(0, 1/2)$, $1 < |x-y| \leqslant 2(r+1)$, we have

$$\left(f_{B(0,1/2)} - f_{B(z_0,1/2)}\right)^2 \leqslant c_4 r^{d+\alpha} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \times \mathbb{1}_{\{|x-y|>1\}}\, dy\, dx.$$

Combining both estimates above, we obtain that for each $z \in \Pi_r^2$,

$$\int_{B(z,1/2)} \left(f(x) - f_{B(0,1/2)}\right)^2 dx \leqslant c_5 r^{d+\alpha} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|>1\}}\, dy\, dx. \tag{2.25}$$

Therefore, by (2.24) and (2.25), for each $r > 3$,

$$\begin{aligned}
&\int_{B(0,r)} \left(f(x) - \frac{\int_{B(0,r)} f(x)\mu_V(dx)}{\mu_V(B(0,r))}\right)^2 \mu_V(dx) \\
&\leqslant \int_{B(0,r)} \left(f(x) - f_{B(0,1/2)}\right)^2 \mu_V(dx) \\
&\leqslant c_6 K(r) \int_{B(0,r)} \left(f(x) - f_{B(0,1/2)}\right)^2 dx \\
&\leqslant c_6 K(r) \sum_{z_i \in \Pi_r} \int_{B(z_i,1/2)} \left(f(x) - f_{B(0,1/2)}\right)^2 dx \\
&\leqslant c_7 K(r) r^{2d+\alpha} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|>1\}}\, dy\, dx \\
&\leqslant \frac{c_8 K(r) r^{2d+\alpha}}{k(r)} \iint_{B(0,r+1)\times B(0,r+1)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \mathbb{1}_{\{|x-y|>1\}}\, dy\, \mu_V(dx),
\end{aligned}$$

which completes the proof. □

Remark 2.5. The constants r^{3d} and $r^{2d+\alpha}$ in the local Poincaré inequalities (2.19) and (2.23) are not optimal, and they come from counting the number of elements in Π_r. By taking a cover with some intersection property, we can expect to get better estimates, e.g. see [2, Lemma 5.11]. However, the estimates here are enough for our application.

As a direct consequence of Propositions 2.3 and 2.4, we can derive the following weak Poincaré inequality for $\mathscr{E}_{\alpha,V}$ and $\mathscr{D}_{\alpha,V}$, by the local Poincaré inequality (2.19) and (2.23), respectively.

Proposition 2.6. (1) *There is a constant $C_5 > 0$ such that for every $s > 0$ and $f \in C_b^\infty(\mathbb{R}^d)$ with $\mu_V(f) = 0$,*

$$\mu_V(f^2) \leqslant C_5\alpha_1(s)\mathscr{E}_{\alpha,V}(f,f) + s\|f\|_\infty^2,$$

where

$$\alpha_1(s) := \inf\left\{\frac{r^{3d}K(r)}{k(r)} : \mu_V(B(0,r)^c) \leqslant \frac{s}{1+s} \text{ and } r > 1\right\}.$$

(2) *There is a constant $C_6 > 0$ such that for every $s > 0$ and $f \in C_b^\infty(\mathbb{R}^d)$ with $\mu_V(f) = 0$,*

$$\mu_V(f^2) \leqslant C_6\alpha_2(s)\mathscr{D}_{\alpha,V}(f,f) + s\|f\|_\infty^2,$$

where

$$\alpha_2(s) := \inf\left\{\frac{r^{2d+\alpha}K(r)}{k(r)} : \mu_V(B(0,r)^c) \leqslant \frac{s}{1+s} \text{ and } r > 3\right\}.$$

Proof. The proof is based on [10, Theorem 4.3.1] (see also [9, Theorem 3.1]). Here we only prove assertion (1), since the proof of assertion (2) is similar. First, according to (2.19), there exists a constant $c_1 > 0$ such that for any $r > 1$ and $f \in C_b^\infty(\mathbb{R}^d)$,

$$\mu_V(f^2\mathbb{1}_{B(0,r)}) \leqslant \frac{c_1K(r)r^{3d}}{k(r)}\mathscr{E}_{\alpha,V}(f,f) + \frac{\mu_V(f\mathbb{1}_{B(0,r)})^2}{\mu_V(B(0,r))}.$$

For any $s > 0$, let $r > 1$ such that $\mu_V(B(0,r)^c) \leqslant \frac{s}{1+s}$, i.e. $\mu_V(B(0,r)) \geqslant \frac{1}{1+s}$. Then, for any $f \in C_b^\infty(\mathbb{R}^d)$ with $\mu_V(f) = 0$, one has

$$\mu_V(f\mathbb{1}_{B(0,r)})^2 = \mu_V(f\mathbb{1}_{B(0,r)^c})^2 \leqslant \frac{s^2}{(1+s)^2}\|f\|_\infty^2.$$

Therefore,

$$\begin{aligned}\mu_V(f^2) &= \mu_V(f^2\mathbb{1}_{B(0,r)}) + \mu_V(f^2\mathbb{1}_{B(0,r)^c})\\ &\leqslant \frac{c_1K(r)r^{3d}}{k(r)}\mathscr{E}_{\alpha,V}(f,f) + \frac{\mu_V(f\mathbb{1}_{B(0,r)})^2}{\mu(B(0,r))} + \frac{s}{1+s}\|f\|_\infty^2\\ &\leqslant \frac{c_1K(r)r^{3d}}{k(r)}\mathscr{E}_{\alpha,V}(f,f) + s\|f\|_\infty^2,\end{aligned}$$

which yields the required assertion. $\square$

3. Functional inequalities for Dirichlet forms with finite range jumps

3.1. Lyapunov type condition for $\mathscr{E}_{\alpha,V}$

For any $f, g \in C_b^\infty(\mathbb{R}^d)$, let

$$\mathscr{E}_{\alpha,V}(f,g) = \frac{1}{2}\iint_{\{|x-y|\leqslant 1\}} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}}\,dy\,\mu_V(dx).$$

We define the corresponding truncated Dirichlet form as follows:

$$\widetilde{\mathscr{E}}_{\alpha,V}(f,g) := \frac{1}{2}\iint_{\{1/2\leqslant|x-y|\leqslant 1\}} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}}\,dy\,\mu_V(dx).$$

Let $C_c^\infty(\mathbb{R}^d)$ be the set of smooth functions on $\mathbb{R}^d$ with compact supports. The following result presents the explicit expression for the generator associated with the truncated Dirichlet form $\widetilde{\mathscr{E}}_{\alpha,V}$ on $C_c^\infty(\mathbb{R}^d)$.

Lemma 3.1. *For each $f, g \in C_c^\infty(\mathbb{R}^d)$,*

$$\widetilde{\mathscr{E}}_{\alpha,V}(f,g) = -\int f(x)\widetilde{L}_{\alpha,V}g(x)\,\mu_V(dx) = -\int g(x)\widetilde{L}_{\alpha,V}f(x)\,\mu_V(dx),$$

where

$$\widetilde{L}_{\alpha,V}f(x) := \frac{1}{2}\int_{\{1/2\leqslant|x-y|\leqslant 1\}} (f(y)-f(x))\,\frac{\left(1+e^{V(x)-V(y)}\right)}{|x-y|^{d+\alpha}}\,dy. \tag{3.26}$$

Proof. According to [4, Theorem 2.1], for each $f, g \in C_c^\infty(\mathbb{R}^d)$,

$$\widetilde{\mathscr{E}}_{\alpha,V}(f,g) = -\int f(x)\widetilde{L}^*_{\alpha,V}g(x)\,\mu_V(dx) = -\int g(x)\widetilde{L}^*_{\alpha,V}f(x)\,\mu_V(dx),$$

where

$$\begin{aligned}
\widetilde{L}^*_{\alpha,V}f(x) :=&\ \frac{1}{2}\int_{\{1/2\leqslant|z|\leqslant 1\}} \big(f(x+z)-f(x) \\
&- \nabla f(x)\cdot z\mathbb{1}_{\{|z|\leqslant 1\}}\big)\,\frac{\left(1+e^{V(x)-V(x+z)}\right)}{|z|^{d+\alpha}}\,dz \\
&+ \frac{1}{4}\nabla f(x)\cdot\left[\int_{\{1/2\leqslant|z|\leqslant 1\}} z\left(e^{V(x)-V(x+z)}-e^{V(x)-V(x-z)}\right)\frac{1}{|z|^{d+\alpha}}\,dz\right] \\
=&\ \frac{1}{2}\int_{\{1/2\leqslant|x-y|\leqslant 1\}} (f(y)-f(x))\,\frac{\left(1+e^{V(x)-V(y)}\right)}{|x-y|^{d+\alpha}}\,dy \\
&- \frac{1}{4}\nabla f(x)\cdot\left[\int_{\{1/2\leqslant|z|\leqslant 1\}} z\left(e^{V(x)-V(x+z)}+e^{V(x)-V(x-z)}\right)\frac{1}{|z|^{d+\alpha}}\,dz\right] \\
=:&\ \widetilde{L}_{\alpha,V,1}f(x) + \widetilde{L}_{\alpha,V,2}f(x).
\end{aligned}$$

It is easy to see that for any $f \in C_c^\infty(\mathbb{R}^d)$, $\widetilde{L}_{\alpha,V,1}f(x)$ and $\widetilde{L}_{\alpha,V,2}f(x)$ are well defined. Changing variable from z to $-z$, we can see that for all $x \in \mathbb{R}^d$, $\widetilde{L}_{\alpha,V,2}f(x) = 0$, which gives us the desired expression (3.26). □

According to (3.26), for every $f \in C(\mathbb{R}^d)$ (the set of continuous functions on $\mathbb{R}^d$) and $x \in \mathbb{R}^d$, $\widetilde{L}_{\alpha,V} f(x)$ is well defined, and the function $x \mapsto \widetilde{L}_{\alpha,V} f(x)$ is locally bounded. Then, repeating the proof of [4, Proposition 3.2], we get the following.

Lemma 3.2. *For every $f \in C_c^\infty(\mathbb{R}^d)$ and $\phi \in C(\mathbb{R}^d)$ with $\phi > 0$,*

$$-\int f^2 \frac{\widetilde{L}_{\alpha,V}\phi}{\phi} d\mu_V \leqslant \widetilde{\mathscr{E}}_{\alpha,V}(f,f).$$

Now we present the Lyapunov type condition for $\widetilde{L}_{\alpha,V}$.

Lemma 3.3. *Let $\phi \in C(\mathbb{R}^d)$ such that $\phi > 1$ and $\phi(x) = e^{|x|}$ for $|x| > 1$. If*

$$\liminf_{|x|\to\infty} \frac{\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}}{\sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-V(z)}} > \frac{1}{\alpha} 2^{2d+1}(e+e^{1/2})(2^\alpha - 1), \tag{3.27}$$

then there are positive constants C_1, b and $r_0 > 0$ such that for all $x \in \mathbb{R}^d$,

$$\widetilde{L}_{\alpha,V}\phi(x) \leqslant -C_1 \left(e^{V(x)} \inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)} \right) \phi(x) + b\mathbb{1}_{B(0,r_0)}(x). \tag{3.28}$$

Proof. It is easy to check that $\widetilde{L}_{\alpha,V}\phi$ is locally bounded. Thus, it suffices to prove (3.28) for $|x|$ large enough. First, for $x \in \mathbb{R}^d$ with $|x| \geqslant 2$,

$$\begin{aligned}
\int_{\{1/2\leqslant|z|\leqslant1\}} (\phi(x+z)-\phi(x)) \frac{1}{|z|^{d+\alpha}}\, dz &= \int_{\{1/2\leqslant|z|\leqslant1\}} \left(e^{|x+z|} - e^{|x|}\right) \frac{1}{|z|^{d+\alpha}}\, dz \\
&\leqslant e^{|x|} \int_{\{1/2\leqslant|z|\leqslant1\}} \left(e^{|z|}-1\right) \frac{1}{|z|^{d+\alpha}}\, dz \\
&= c_1 e^{|x|} \\
&\leqslant c_1 \left(e^{V(x)} \sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-V(z)} \right) \phi(x),
\end{aligned}$$

where

$$c_1 := \int_{\{1/2\leqslant|z|\leqslant1\}} \left(e^{|z|}-1\right) \frac{1}{|z|^{d+\alpha}}\, dz.$$

Second, for $x \in \mathbb{R}^d$ with $|x| \geqslant 2$,

$$\begin{aligned}
&\int_{\{1/2\leqslant|z|\leqslant1\}} (\phi(x+z)-\phi(x)) \frac{e^{(V(x)-V(x+z))}}{|z|^{d+\alpha}}\, dz \\
&= e^{V(x)} \left(\int_{\{1/2\leqslant|z|\leqslant1\}} \left(e^{|x+z|}-e^{|x|}\right) \frac{e^{-V(x+z)}}{|z|^{d+\alpha}}\, dz \right) \\
&\leqslant e^{V(x)} \left(\int_{\{1/2\leqslant|z|\leqslant1, |x+z|-|x|\leqslant-1/2\}} \left(e^{|x+z|}-e^{|x|}\right) \frac{e^{-V(x+z)}}{|z|^{d+\alpha}}\, dz \right.
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\{1/2\leqslant|z|\leqslant 1,|x+z|-|x|\geqslant 0\}} \left(e^{|x+z|}-e^{|x|}\right)\frac{e^{-V(x+z)}}{|z|^{d+\alpha}}\,dz\Bigg)\\
&\leqslant e^{V(x)}\Bigg(\int_{\{1/2\leqslant|z|\leqslant 1,|x+z|-|x|\leqslant -1/2\}} \left(e^{|x|-1/2}-e^{|x|}\right)\frac{e^{-V(x+z)}}{|z|^{d+\alpha}}\,dz\\
&+ \int_{\{1/2\leqslant|z|\leqslant 1,|x+z|-|x|\geqslant 0\}} \left(e^{|x|+|z|}-e^{|x|}\right)\frac{e^{-V(x+z)}}{|z|^{d+\alpha}}\,dz\Bigg)\\
&= e^{V(x)}e^{|x|}\Bigg(-\int_{\{1/2\leqslant|z|\leqslant 1,|x+z|-|x|\leqslant -1/2\}} \left(1-e^{-1/2}\right)\frac{e^{-V(x+z)}}{|z|^{d+\alpha}}\,dz\\
&+ \int_{\{1/2\leqslant|z|\leqslant 1,|x+z|-|x|\geqslant 0\}} \left(e^{|z|}-1\right)\frac{e^{-V(x+z)}}{|z|^{d+\alpha}}\,dz\Bigg)\\
&\leqslant e^{V(x)}e^{|x|}\Bigg[-\left(1-e^{-1/2}\right)\\
&\times\left(\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}\right)\int_{\{1/2\leqslant|z|\leqslant 1,|x+z|-|x|\leqslant -1/2\}}\frac{1}{|z|^{d+\alpha}}\,dz\\
&+\left(\sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-V(z)}\right)\int_{\{1/2\leqslant|z|\leqslant 1\}}\left(e^{|z|}-1\right)\frac{1}{|z|^{d+\alpha}}\,dz\Bigg],
\end{aligned}
$$

where in the first inequality we have removed the subset $\{z\in\mathbb{R}^d : 1/2\leqslant|z|\leqslant 1, -1/2 < |x+z|-|x| < 0\}$ in the domain of integral, since the integrand is negative on this subset. For $x\in\mathbb{R}^d$ with $|x|\geqslant 2$, let $z_0=-3x/(4|x|)$. Then,

$$|z_0|=\frac{3}{4}\quad\text{and}\quad |x+z_0|-|x|=-\frac{3}{4}.$$

Hence, for every $z\in B(z_0,\frac{1}{4})$,

$$
\begin{aligned}
&|z|\geqslant|z_0|-\frac{1}{4}\geqslant\frac{1}{2},\qquad |z|\leqslant|z_0|+\frac{1}{4}=1,\\
&|x+z|-|x|\leqslant(|x+z_0|-|x|)+(|x+z|-|x+z_0|)\leqslant-\frac{3}{4}+|z-z_0|\leqslant-\frac{1}{2},
\end{aligned}
$$

which implies that

$$B\left(z_0,\frac{1}{4}\right)\subseteq\left\{z\in\mathbb{R}^d:\frac{1}{2}\leqslant|z|\leqslant 1,\ |x+z|-|x|\leqslant-\frac{1}{2}\right\}.$$

According to both conclusions above, we get that

$$
\begin{aligned}
&\int_{\{1/2\leqslant|z|\leqslant 1\}}(\phi(x+z)-\phi(x))\frac{e^{(V(x)-V(x+z))}}{|z|^{d+\alpha}}dz\\
&\leqslant e^{V(x)}e^{|x|}\Bigg[-\left(1-e^{-1/2}\right)\left(\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}\right)m\left(B\left(z_0,\frac{1}{4}\right)\right)\\
&+\left(\sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-V(z)}\right)\int_{\{1/2\leqslant|z|\leqslant 1\}}\left(e^{|z|}-1\right)\frac{1}{|z|^{d+\alpha}}\,dz\Bigg]
\end{aligned}
$$

$$\leqslant -c_2\phi(x)e^{V(x)}\left(\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}\right)+c_1\phi(x)e^{V(x)}\left(\sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-V(z)}\right),$$

where $m(A)$ is Lebesgue measure for the Borel measurable set A, and

$$c_2 := (1-e^{-1/2})m\left(B\left(0,\frac{1}{4}\right)\right).$$

Combining both estimates above with (3.26), we know that for any $x \in \mathbb{R}^d$ with $|x| \geqslant 2$, it holds that

$$\widetilde{L}_{\alpha,V}\phi(x) \leqslant \frac{1}{2}\left[-c_2\left(\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}\right)+2c_1\left(\sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-V(z)}\right)\right]\phi(x)e^{V(x)}.$$

Therefore, if

$$\liminf_{|x|\to\infty}\frac{\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}}{\sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-V(z)}} > \frac{2c_1}{c_2},$$

then for $|x|$ large enough,

$$\widetilde{L}_{\alpha,V}\phi(x) \leqslant -C_1\phi(x)e^{V(x)}\left(\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}\right)$$

holds with some constant $C_1 > 0$. The required assertion follows from the fact that

$$\frac{2c_1}{c_2} = \frac{2^{2d+1}d}{1-e^{-1/2}}\int_{1/2}^{1}(e^r-1)r^{-1-\alpha}\,dr < \frac{1}{\alpha}2^{2d+1}(e+e^{1/2})(2^\alpha-1)$$

and (3.27). □

Now we present the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof is the same as that of [1, Theorem 2.10] and [4, Theorem 3.6] (see also [13, Theorem 1.1]), and it is based on Lemma 3.3 and the local (super) Poincaré inequality for $\mathscr{E}_{\alpha,V}$. Here, we only show the super Poincaré inequality (1.6). Based on the local Poincaré inequality in Proposition 2.3, the proof for the Poincaré inequality (1.4) is similar and even simpler.

According to Lemma 3.3, there are constants c_1, c_2 and $r_0 > 1$ such that

$$\widetilde{L}_{\alpha,V}\phi(x) \leqslant -c_1\phi(x)e^{V(x)}\left(\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}\right)+c_2\mathbb{1}_{B(0,r_0)}(x),$$

where $\phi(x)$ is the function given in Lemma 3.3.

For $r > 0$, set

$$\Phi(r) = \inf_{|x|\geqslant r}\left[e^{V(x)}\left(\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}\right)\right].$$

By Lemma 3.2, for any $f \in C_c^\infty(\mathbb{R}^d)$ and $r \geqslant r_0$,

$$\int_{B(0,r)^c} f^2(x)\,\mu_V(dx) \leqslant \frac{1}{\Phi(r)}\int f^2(x)\Phi(|x|)\,\mu_V(dx)$$

$$
\begin{aligned}
&\leqslant \frac{1}{\Phi(r)} \int f^2(x) e^{V(x)} \left(\inf_{|x|-1 \leqslant |z| \leqslant |x|-1/2} e^{-V(z)} \right) \mu_V(dx) \\
&\leqslant -\frac{1}{c_1 \Phi(r)} \int \frac{\widetilde{L}_{\alpha,V} \phi(x)}{\phi(x)} f^2(x)\, \mu_V(dx) \\
&\quad + \frac{c_2}{c_1 \Phi(r)} \int_{B(0,r_0)} \frac{f^2(x)}{\phi(x)}\, \mu_V(dx) \\
&\leqslant \frac{c_3}{\Phi(r)} \left[\widetilde{\mathscr{E}}_{\alpha,V}(f,f) + \int_{B(0,r_0)} f^2(x) \mu_V(dx) \right] \\
&\leqslant \frac{c_3}{\Phi(r)} \left[\widetilde{\mathscr{E}}_{\alpha,V}(f,f) + \int_{B(0,r)} f^2(x) \mu_V(dx) \right], \qquad (3.29)
\end{aligned}
$$

where in the fourth inequality we have used the fact that $\phi > 1$.

For every $f \in C_b^\infty(\mathbb{R}^d)$, there is a sequence of functions $\{f_n\}_{n=1}^\infty \subseteq C_c^\infty(\mathbb{R}^d)$ such that

$$
\lim_{n\to\infty} f_n(x) = f(x), \qquad \sup_n \|f_n\|_\infty < \infty, \qquad \sup_n \|\nabla f_n\|_\infty < \infty.
$$

Thus, by the dominated convergence theorem, we get

$$
\begin{aligned}
&\lim_{n\to\infty} \widetilde{\mathscr{E}}_{\alpha,V}(f_n, f_n) = \widetilde{\mathscr{E}}_{\alpha,V}(f,f), \\
&\lim_{n\to\infty} \int_{B(0,r)^c} f_n^2(x)\, \mu_V(dx) = \int_{B(0,r)^c} f^2(x)\, \mu_V(dx),
\end{aligned}
$$

and

$$
\lim_{n\to\infty} \int_{B(0,r)} f_n^2(x)\, \mu_V(dx) = \int_{B(0,r)} f^2(x)\, \mu_V(dx).
$$

Since (3.29) holds for each f_n, letting n tend to infinity and using the estimates above, we show that (3.29) holds for $f \in C_b^\infty(\mathbb{R}^d)$.

Hence, for every $r \geqslant r_0$ and $f \in C_b^\infty(\mathbb{R}^d)$,

$$
\begin{aligned}
\int f^2(x)\, \mu_V(dx) &= \int_{B(0,r)} f^2(x)\, \mu_V(dx) + \int_{B(0,r)^c} f^2(x)\, \mu_V(dx) \\
&\leqslant \frac{c_3}{\Phi(r)} \mathscr{E}_{\alpha,V}(f,f) + \left(1 + \frac{c_3}{\Phi(r)}\right) \int_{B(0,r)} f^2(x)\, \mu_V(dx),
\end{aligned}
$$

where in the inequality above we have used the fact that $\widetilde{\mathscr{E}}_{\alpha,V}(f,f) \leqslant \mathscr{E}_{\alpha,V}(f,f)$ for any $f \in C_b^\infty(\mathbb{R}^d)$.

Applying the local super Poincaré inequality (2.18) into the inequality above, we can obtain that for any $r \geqslant r_0$ and $f \in C_b^\infty(\mathbb{R}^d)$,

$$
\begin{aligned}
\int f^2(x)\, \mu_V(dx) &\leqslant \left(\frac{c_3}{\Phi(r)} + \frac{s}{2} \right) \mathscr{E}_{\alpha,V}(f,f) \\
&\quad + c_4 \left(1 + s^{-d/\alpha}\right) \frac{r^{d+d^2/\alpha} K(r)^{1+d/\alpha}}{k(r)^{2+d/\alpha}} \left(\int |f(x)|\, \mu_V(dx) \right)^2,
\end{aligned}
$$

where we have used the fact that $\sup_{r\geqslant r_0} \Phi(r)^{-1} < \infty$, thanks to (1.5).

If (1.5) holds, then $\lim_{r\to\infty} \Phi(r) = \infty$. By taking $r = \Phi^{-1}(2c_3/s)$ in the estimate above, the required inequality (1.6) follows. □

To close this part, we present the following weighted Poincaré inequality for $\mathscr{E}_{\alpha,V}$. The proof is similar to that of [4, Theorem 3.6], and it is based on the local Poincaré inequality (2.19) and Lemma 3.3. We omit the details here.

Proposition 3.4. *Under (1.3), there exists a constant $C_1 > 0$ such that*

$$\int f^2(x)\left(e^{V(x)}\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V(z)}\right)\mu_V(dx)\leqslant C_1\mathscr{E}_{\alpha,V}(f,f)$$

holds for all $f\in C_b^\infty(\mathbb{R}^d)$ with $\mu_V(f)=0$.

3.2. Concentration of measure about functional inequalities for $\mathscr{E}_{\alpha,V}$

Recall that V is a locally bounded measurable function on $\mathbb{R}^d$ such that $e^{-V}\in L^1(dx)$, and $\mu_V(dx)=C_Ve^{-V(x)}\,dx$ is a probability measure.

Proposition 3.5. (1) *Suppose that there exists a constant $C_1 > 0$ such that the Poincaré inequality holds*

$$\mu_V(f^2)\leqslant C_1\mathscr{E}_{\alpha,V}(f,f),\quad f\in C_b^\infty(\mathbb{R}^d),\mu_V(f)=0.$$

Then there exists a constant $\lambda_0 > 0$ such that

$$\int e^{\lambda_0|x|}\,\mu_V(dx)<\infty.$$

(2) *Assume that the following super Poincaré inequality holds*

$$\mu_V(f^2)\leqslant s\mathscr{E}_{\alpha,V}(f,f)+\beta(s)\mu_V(|f|)^2,\quad s>0,\ f\in C_b^\infty(\mathbb{R}^d),$$

where $\beta:(0,\infty)\to(0,\infty)$ is a decreasing function. Then, for any $\lambda>0$,

$$\int e^{\lambda|x|}\,\mu_V(dx)<\infty.$$

Furthermore, for each $r>0$, define

$$F(r):=\int_1^\infty e^{r\lambda}h(\lambda)\,d\lambda,$$

where for every $\lambda>1$,

$$h(\lambda):=\exp\left\{-(1+c_0)\lambda-\lambda\int_1^\lambda\frac{1}{s^2}\log\left[2\beta\left(\frac{1}{c_1s^2e^{2s}}\right)\right]ds\right\},$$

and

$$c_0:=\log\left(\int e^{|x|}\,\mu_V(dx)\right),$$

$$c_1:=\int_{\{|z|\leqslant1\}}\frac{dz}{|z|^{d+\alpha-2}}.$$

Then

$$\int F(|x|)\mu_V(dx)<\infty.$$

Proof. (1) For any $n \geqslant 1$, define $g_n(x) := e^{\lambda(|x|\wedge n)}$, where $\lambda > 0$ is a constant to be determined later. Clearly, g_n is a Lipschitz continuous bounded function. By the approximation procedure in the proof of Theorem 1.1, we can apply the function g_n into the Poincaré inequality. Thus,

$$\int g_n^2(x)\,\mu_V(dx) \leqslant \frac{C_1}{2}\iint_{\{|x-y|\leqslant 1\}} \frac{(g_n(x)-g_n(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_V(dx) + \left(\int g_n(x)\,\mu_V(dx)\right)^2.$$

By the mean value theorem and the fact that for any $x, y \in \mathbb{R}^d$, $n \geqslant 1$,

$$||x|\wedge n - |y|\wedge n| \leqslant |x-y|,$$

we know that for any $x \in \mathbb{R}^d$,

$$\begin{aligned}\int_{\{|x-y|\leqslant 1\}} \frac{(g_n(x)-g_n(y))^2}{|x-y|^{d+\alpha}}\,dy &= \int_{\{|x-y|\leqslant 1\}} \frac{(e^{\lambda(|x|\wedge n)}-e^{\lambda(|y|\wedge n)})^2}{|x-y|^{d+\alpha}}\,dy \\ &\leqslant \lambda^2 e^{2(\lambda(|x|\wedge n)+\lambda)}\int_{\{|x-y|\leqslant 1\}} \frac{|x-y|^2}{|x-y|^{d+\alpha}}\,dy \\ &\leqslant c_1\lambda^2 e^{2(\lambda(|x|\wedge n)+\lambda)} \\ &= c_1\lambda^2 e^{2\lambda}\,e^{2\lambda(|x|\wedge n)},\end{aligned}$$

where

$$c_1 := \int_{\{|z|\leqslant 1\}} \frac{dz}{|z|^{d+\alpha-2}} = \frac{d\pi^{d/2}}{(2-\alpha)\Gamma(d/2+1)}.$$

Therefore,

$$\iint_{\{|x-y|\leqslant 1\}} \frac{(g_n(x)-g_n(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_V(dx) \leqslant c_1\lambda^2 e^{2\lambda}\int e^{2\lambda(|x|\wedge n)}\,\mu_V(dx). \tag{3.30}$$

For any $n \geqslant 1$ and $\lambda > 0$, set

$$l_n(\lambda) := \int g_n^2(x)\,\mu_V(dx) = \int e^{2\lambda(|x|\wedge n)}\,\mu_V(dx).$$

Then, combining all the estimates above, for each $\lambda > 0$,

$$l_n(\lambda) \leqslant \frac{C_1}{2}c_1\lambda^2 e^{2\lambda} l_n(\lambda) + l_n^2(\lambda/2).$$

Furthermore, using the Cauchy–Schwarz inequality, for any $R > 0$, we have

$$l_n^2(\lambda/2) \leqslant \left(e^{\lambda R} + \int_{\{|x|>R\}} e^{\lambda(|x|\wedge n)}\,\mu_V(dx)\right)^2 \leqslant 2e^{2\lambda R} + 2p(R)\,l_n(\lambda),$$

where $p(R) := \mu_V(|x| > R)$. Therefore, for each $R > 0$ and $\lambda > 0$,

$$l_n(\lambda) \leqslant \left(\frac{C_1}{2}c_1\lambda^2 e^{2\lambda} + 2p(R)\right) l_n(\lambda) + 2e^{2\lambda R}.$$

Now, we fix $R_0 > 0$ large enough such that $p(R_0) < 1/8$, and then take $\lambda_0 > 0$ small enough such that $C_1 c_1 \lambda_0^2 e^{2\lambda_0} < 1/2$. Then, we arrive at

$$l_n(\lambda_0) \leqslant 4e^{2\lambda_0 R_0}.$$

Letting $n \to \infty$, we obtain the first desired assertion.

(2) We still use the same test function g_n as that in part (1). By applying this test function g_n into the super Poincaré inequality and by using (3.30), we have

$$\int g_n^2(x)\,\mu_V(dx) \leqslant \frac{c_1}{2}\lambda^2 e^{2\lambda} s \int g_n^2(x)\,\mu_V(dx) + \beta(s)\left(\int g_n(x)\,\mu_V(dx)\right)^2, \quad s > 0. \tag{3.31}$$

Following the argument in the proof of part (1), we can get that for any λ, s and $R > 0$,

$$l_n(\lambda) \leqslant \frac{c_1}{2} s\lambda^2 e^{2\lambda} l_n(\lambda) + \beta(s)\left[2e^{2\lambda R} + 2p(R) l_n(\lambda)\right],$$

where $l_n(\lambda)$ and $p(R)$ are the same functions defined in the proof of part (1).

Now, for any fixed $\lambda > 0$, choose $s_0 > 0$ small enough such that $c_1 s_0 \lambda^2 e^{2\lambda} < 1/2$, and then take R_0 large enough such that $\beta(s_0)p(R_0) < 1/8$, we get

$$l_n(\lambda) \leqslant 8\beta(s_0) e^{2\lambda R_0}.$$

Letting $n \to \infty$, we show $\int e^{\lambda|x|}\,\mu_V(dx) < \infty$ for any $\lambda > 0$.

In the remaining part, we will follow the method adopted in the proof of [10, Theorem 3.3.20], see also [12, Theorem 6.1]. For every $\lambda > 0$, set $l(\lambda) := \mu_V(e^{\lambda|x|})$. For any $\varepsilon > 0$, it holds that

$$\begin{aligned}
l'(\lambda) &= \mu_V(|x|e^{\lambda|x|}) \\
&= \mu_V\left[\left(\frac{1}{\lambda}(\lambda|x| + \log\varepsilon) - \frac{\log\varepsilon}{\lambda}\right)e^{\lambda|x|}\right] \\
&= \mu_V\left(\frac{1}{\lambda}(\lambda|x| + \log\varepsilon)e^{\lambda|x|}\right) - \frac{\log\varepsilon}{\lambda}\mu_V(e^{\lambda|x|}) \\
&\leqslant \varepsilon\mu_V(e^{2\lambda|x|}) - \frac{\log(\varepsilon\lambda e)}{\lambda}\mu_V(e^{\lambda|x|}) \\
&= \varepsilon l(2\lambda) - \frac{\log(\varepsilon\lambda e)}{\lambda} l(\lambda),
\end{aligned}$$

where in the inequality above we have applied the Young inequality

$$st \leqslant s\log s - s + e^t, \quad s \in \mathbb{R}_+, t \in \mathbb{R}$$

with $s = \frac{1}{\lambda}$ and $t = \lambda|x| + \log\varepsilon$.

On the other hand, according to (3.31) and letting $n \to \infty$,

$$l(2\lambda) \leqslant \frac{c_1}{2}\lambda^2 e^{2\lambda} s l(2\lambda) + \beta(s) l(\lambda)^2, \quad s > 0.$$

Taking $s = \left(c_1\lambda^2 e^{2\lambda}\right)^{-1}$, we obtain that

$$l(2\lambda) \leqslant 2\beta\left(\frac{1}{c_1\lambda^2 e^{2\lambda}}\right) l(\lambda)^2.$$

Combining all the estimates above,

$$l'(\lambda) \leqslant 2\varepsilon\beta\left(\frac{1}{c_1\lambda^2 e^{2\lambda}}\right) l(\lambda)^2 - \frac{\log(\varepsilon\lambda e)}{\lambda} l(\lambda).$$

Choosing $\varepsilon = \left(2\lambda l(\lambda)\beta\left(\frac{1}{c_1\lambda^2 e^{2\lambda}}\right)\right)^{-1}$, we derive

$$l'(\lambda) \leqslant \frac{l(\lambda)}{\lambda}\left[\log l(\lambda) + \log\left(2\beta\left(\frac{1}{c_1\lambda^2 e^{2\lambda}}\right)\right)\right],$$

hence

$$\frac{d}{d\lambda}\left(\frac{\log l(\lambda)}{\lambda}\right) \leqslant \frac{1}{\lambda^2}\log\left(2\beta\left(\frac{1}{c_1\lambda^2 e^{2\lambda}}\right)\right),$$

which implies that for any $\lambda \geqslant 1$,

$$l(\lambda) \leqslant \exp\left(\lambda \log l(1) + \lambda\int_1^\lambda \frac{1}{s^2}\log\left(2\beta\left(\frac{1}{c_1 s^2 e^{2s}}\right)\right) ds\right).$$

Then, by the Fubini theorem, we have

$$\int F(|x|)\,\mu_V(dx) = \int_1^{+\infty}\int e^{\lambda|x|}\,\mu_V(dx)h(\lambda)\,d\lambda \leqslant \int_1^\infty e^{-\lambda}\,d\lambda < \infty.$$

This finishes the proof. $\square$

Now, we turn to the proof of Example 1.2.

Proof of Example 1.2. (1) Let $\mu_{V_\lambda}(dx) = C_\lambda e^{-\lambda|x|}\,dx =: C_\lambda e^{-V_\lambda(x)}\,dx$ with $\lambda > 0$, we have

$$\frac{\inf\limits_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V_\lambda(z)}}{\sup\limits_{|x|\leqslant|z|\leqslant|x|+1} e^{-V_\lambda(z)}} \geqslant e^{\lambda/2}, \quad x \geqslant 1.$$

Then, for λ_0 defined in Example 1.2(1), if $\lambda > \lambda_0$,

$$\liminf_{|x|\to\infty}\frac{\inf\limits_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-V_\lambda(z)}}{\sup\limits_{|x|\leqslant|z|\leqslant|x|+1} e^{-V_\lambda(z)}} > \frac{1}{\alpha}2^{2d+1}(e+e^{1/2})(2^\alpha - 1).$$

According to Theorem 1.1(1), the Poincaré inequality (1.4) holds for $\mu_{V_\lambda}(dx)$ with $\lambda > \lambda_0$.

(2) If the super Poincaré inequality (1.6) holds for $\mu_{V_\delta}(dx) = C_\delta e^{-(1+|x|^\delta)}\,dx =: C_\delta e^{-V_\delta(x)}\,dx$, then, by Proposition 3.5(2), $\int e^{\lambda|x|}\mu_{V_\delta}(dx) < \infty$ for any $\lambda > 0$, which implies that the super Poincaré inequality (1.6) holds only if $\delta > 1$.

On the other hand, for every $\delta > 1$ and for $|x|$ large enough,

$$e^{V_\delta(x)}\inf_{|z|\leqslant|x|-1/2} e^{-V_\delta(z)} \geqslant C_1 e^{C_2|x|^{\delta-1}},$$

where C_1 and C_2 are two positive constants independent of x. Hence, for r large enough, $\Phi(r) \geqslant C_1 e^{C_2 r^{\delta-1}}$. Therefore, according to Theorem 1.1(2), we know that the super Poincaré inequality (1.6) holds with the rate function β given by (1.8).

According to [10, Theorem 3.3.14] (also see [12, Theorem 5.1]), if the rate function $\beta(s)$ satisfies that

$$\Psi(t) := \int_t^\infty \frac{\beta^{-1}(r)}{r}\,dr < \infty \quad \text{for any } t > \inf_{s>0}\beta(s), \tag{3.32}$$

then

$$\|P_t^{\alpha,V_\delta}\|_{L^1(\mu_{V_\delta})\to L^\infty(\mu_{V_\delta})} \leqslant 2\Psi^{-1}(t), \quad t > 0, \tag{3.33}$$

where

$$\Psi^{-1}(t) := \inf\left\{r \geqslant \inf_{s>0}\beta(s) : \Psi(r) \geqslant t\right\}.$$

It follows from (1.8) that

$$\beta^{-1}(r) \leqslant \exp\{-C_3(\log r + C_4)^{\frac{\delta-1}{\delta}}\}$$

holds for r large enough and some positive constants C_3 and C_4. Hence, for t large enough,

$$\begin{aligned}\Psi(t) &\leqslant \int_t^\infty \frac{1}{r\exp\{C_3(\log r + C_4)^{\frac{\delta-1}{\delta}}\}}\,dr\\ &\leqslant \int_t^\infty \frac{1}{r(\log r + C_4)^{\frac{1}{\delta}}\exp\{C_5(\log r + C_4)^{\frac{\delta-1}{\delta}}\}}\,dr\\ &= \frac{C_6}{\exp\{C_5(\log t + C_4)^{\frac{\delta-1}{\delta}}\}}.\end{aligned}$$

This along with (3.33) gives us the desired estimate for the associated semigroup P_t^{α,V_δ}.

Furthermore, assume that the super Poincaré inequality (1.6) holds for μ_{V_δ} with the rate function $\beta(s)$ satisfying (1.9). Then for any $\varepsilon > 0$ small enough, there is a $s_0 := s_0(\varepsilon) > 0$ such that for any $s \leqslant s_0$,

$$\log\beta(s) \leqslant \varepsilon\log^{\frac{\delta}{\delta-1}}(1 + s^{-1}).$$

Hence, there is a constant $C_7 > 0$ (independent of ε) such that for every $\varepsilon > 0$ and $s \geqslant 1$,

$$\log\left(2\beta\left(\frac{1}{c_1 s^2 e^{2s}}\right)\right) \leqslant C_7\varepsilon s^{\frac{\delta}{\delta-1}} + C_8(\varepsilon),$$

where $C_8(\varepsilon) > 0$ may depend on ε. Let $F(r)$ be the function defined in Proposition 3.5(2). Therefore, for every $r > 0$ large enough and $\varepsilon > 0$ small enough,

$$\begin{aligned}F(r) &\geqslant \int_1^\infty \exp\left\{r\lambda - (c_0+1)\lambda - \lambda\int_1^\lambda \frac{1}{s^2}\left(C_7\varepsilon s^{\frac{\delta}{\delta-1}} + C_8(\varepsilon)\right)ds\right\}d\lambda\\ &\geqslant \int_1^\infty \exp\left\{-C_9\varepsilon\lambda^{\frac{\delta}{\delta-1}} + (r - C_{10}(\varepsilon))\lambda\right\}d\lambda\\ &\geqslant \int_1^{\left(\frac{r}{2C_9\varepsilon}\right)^{\delta-1}} e^{(\frac{r}{2} - C_{11}(\varepsilon))\lambda}\,d\lambda,\end{aligned}$$

where in the last inequality we have used the fact that if $\lambda \leqslant (\frac{r}{2C_9\varepsilon})^{\delta-1}$, then $C_9\varepsilon\lambda^{\frac{1}{\delta-1}} \leqslant r/2$. The inequality above shows that, for any $\varepsilon > 0$ small enough there are two constants $C_{12} > 0$

(independent of ε and r) and $C_{13}(\varepsilon) > 0$ (independent of r) such that for $r > 0$ large enough,

$$F(r) \geqslant \frac{C_{13}(\varepsilon)}{r} \exp\left[\left(\frac{C_{12}}{\varepsilon}\right)^{\delta-1} r^{\delta}\right].$$

This, along with Proposition 3.5(2), yields that for any $\kappa > 0$,

$$\int e^{\kappa|x|^{\delta}} \mu_{V_\delta}(dx) < \infty.$$

However, the statement above cannot be true since $\mu_{V_\delta}(dx) = C_\delta e^{-(1+|x|^\delta)}\,dx$. That is, there is a contradiction, so the super Poincaré inequality (1.6) does not hold for μ_{V_δ} with the rate function $\beta(s)$ satisfying (1.9).

(3) Let $\mu_{V_\theta}(dx) = C_\theta e^{-|x|\log^\theta(1+|x|)}\,dx =: C_\theta e^{-V_\theta(x)}\,dx$. Suppose that in this case the super Poincaré inequality (1.6) holds. Then, according to Proposition 3.5, $\int e^{\lambda|x|}\mu_{V_\theta}(dx) < \infty$ for any $\lambda > 0$, which implies the super Poincaré inequality (1.6) holds for μ_{V_θ} only with $\theta > 0$.

On the other hand, for every $\theta > 0$, there exist two positive constants C_1 and C_2 such that for $|x|$ large enough,

$$e^{V_\theta(x)} \inf_{|z|\leqslant|x|-1/2} e^{-V_\theta(z)} \geqslant C_1 e^{C_2\log^\theta(1+|x|)}.$$

Then, for r large enough, we have $\Phi(r) \geqslant C_1 e^{C_2\log^\theta(1+r)}$. Therefore, by Theorem 1.1(2), we can get that the super Poincaré (1.6) holds for μ_{V_θ} with the rate function $\beta(s)$ given by (1.10).

On the other hand, according to (1.10), we have

$$\beta^{-1}(r) \leqslant \exp\left\{-C_3\log^\theta\left(C_4(\log r + C_5)\right)\right\}$$

for r large enough and some positive constants C_i ($i = 3, 4, 5$). Let $\Psi(t)$ be the function defined by (3.32). Then, for t large enough, we have

$$\begin{aligned}
\Psi(t) &\leqslant \int_t^\infty \frac{1}{r\exp\left\{C_3\log^\theta\left(C_4(\log r + C_5)\right)\right\}}\,dr \\
&\leqslant \int_t^\infty \frac{\log^{\theta-1}\left(C_4(\log r + C_5)\right)}{r(\log r + C_5)\exp\left\{C_6\log^\theta\left(C_4(\log r + C_5)\right)\right\}}\,dr \\
&= \frac{C_7}{\exp\left\{C_6\log^\theta\left(C_4(\log t + C_5)\right)\right\}},
\end{aligned}$$

where in the second inequality we have used the fact that if $\theta > 1$, then

$$\exp\left\{C_3\log^\theta\left(C_4(\log r + C_5)\right)\right\} \geqslant (\log r + C_5)\exp\left\{C_6\log^\theta\left(C_4(\log r + C_5)\right)\right\}$$

holds for r large enough and some positive constants $C_3 > C_6$. Combining the estimate above with (3.33), we get the desired estimate for the associated semigroup P_t^{α,V_θ}.

Next, we assume that (1.6) holds for μ_{V_θ} with the rate function $\beta(s)$ satisfying (1.11). Then for any $\varepsilon > 0$ small enough, there is a constant $s_0 := s_0(\varepsilon) > 0$ such that for any $s \leqslant s_0$,

$$\log\beta(s) \leqslant \exp\left\{\varepsilon\log^{\frac{1}{\theta}}(1 + s^{-1})\right\}.$$

Hence for every $s \geqslant 1$ and $\varepsilon > 0$ small enough,

$$\log\left(2\beta\left(\frac{1}{c_1 s^2 e^{2s}}\right)\right) \leqslant \exp\left\{C_8\varepsilon s^{\frac{1}{\theta}} + C_9(\varepsilon)\right\},$$

where $C_8 > 0$ is independent of ε, and $C_9(\varepsilon) > 0$ may depend on ε. Therefore, by a similar argument in the proof of part (2), for $r > 0$ large enough and $\varepsilon > 0$ small enough,

$$\begin{aligned} F(r) &\geqslant \int_1^\infty \exp\left\{r\lambda - (c_0+1)\lambda - \lambda\int_1^\lambda \frac{1}{s^2}\exp\left\{C_8\varepsilon s^{\frac{1}{\theta}} + C_9(\varepsilon)\right\} ds\right\} d\lambda \\ &\geqslant \int_1^\infty \exp\left\{-C_{10}(\varepsilon)\lambda e^{\varepsilon C_8 \lambda^{\frac{1}{\theta}}} + (r - C_{11}(\varepsilon))\lambda\right\} d\lambda \\ &\geqslant \int_1^{\left(\frac{\log r - \log(2C_{10}(\varepsilon))}{C_8\varepsilon}\right)^\theta} e^{\left(\frac{r}{2} - C_{11}(\varepsilon)\right)\lambda}\, d\lambda \\ &\geqslant \frac{C_{13}(\varepsilon)}{r}\exp\left\{\frac{C_{12}}{\varepsilon^\theta} r \log^\theta r\right\}, \end{aligned}$$

where $C_{12} > 0$ is independent of ε, r, and $C_{13}(\varepsilon) > 0$ is independent of r. Thus, according to Proposition 3.5(2), for any $\kappa > 0$,

$$\int e^{\kappa|x|\log^\theta(1+|x|)}\,\mu_{V_\theta}(dx) < \infty,$$

which however cannot be true, since $\mu_{V_\theta}(dx) = C_\theta e^{-|x|\log^\theta(1+|x|)}\,dx$. This contradiction shows that the super Poincaré inequality (1.6) does not hold for μ_{V_θ} with the rate function $\beta(s)$ satisfying (1.11). $\square$

3.3. *Comparison of the functional inequalities for $\mathscr{E}_{\alpha,V}$ and $D_{\rho,V}$*

In this subsection, we aim to compare the criteria for the Poincaré inequality and the super Poincaré inequality between $\mathscr{E}_{\alpha,V}$ and $D_{\rho,V}$. First, we take $\rho(r) = r^{-d-\alpha}e^{-\delta r}$ with $\alpha \in (0,2)$ and $\delta \geqslant 0$ in (1.1), and set

$$D_{\alpha,\delta,V}(f,f) := \frac{1}{2}\iint \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}e^{-\delta|x-y|}\,dy\,\mu_V(dx).$$

We denote $D_{\alpha,0,V}$ by $D_{\alpha,V}$ for simplicity. Theorem 1.1 yields the following.

Corollary 3.6. *Let $\alpha \in (0,2)$ and $\delta \in [0,\infty)$. For any $a > 0$, set $\tilde V_a(x) := V(ax)$.*

(1) *If there is a constant $a > 0$ such that*

$$\liminf_{|x|\to\infty} \frac{\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-\tilde V_a(z)}}{\sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-\tilde V_a(z)}} > \frac{1}{\alpha}2^{2d+1}(e+e^{1/2})(2^\alpha - 1), \tag{3.34}$$

then there is a constant $c_1 > 0$ such that for any $f \in C_b^\infty(\mathbb{R}^d)$,

$$\mu_V\left((f-\mu_V(f))^2\right) \leqslant c_1 D_{\alpha,\delta,V}(f,f).$$

(2) *Suppose there is a constant $a > 0$ such that*

$$\liminf_{|x|\to\infty} \frac{\inf_{|x|-1\leqslant|z|\leqslant|x|-1/2} e^{-\tilde V_a(z)}}{\sup_{|x|\leqslant|z|\leqslant|x|+1} e^{-\tilde V_a(z)}} = \infty. \tag{3.35}$$

Let $\tilde{\beta}_a(s)$ be the rate function defined by (1.7) with $\tilde{V}_a(x) := V(ax)$ in place of $V(x)$. If moreover there is a constant $c_2 > 0$ such that

$$\tilde{\beta}_a(s) \leqslant \exp\left(c_2\left(1+s^{-1}\right)\right), \quad s>0, \tag{3.36}$$

then the following log-Sobolev inequality holds

$$\mu_V(f^2\log f^2) - \mu_V(f^2)\log\mu_V(f^2) \leqslant c_3 D_{\alpha,\delta,V}(f,f), \quad f\in C_b^\infty(\mathbb{R}^d).$$

Proof. (a) For any function $f \in C_b^\infty(\mathbb{R}^d)$ with $\int f\,d\mu_V = 0$, define $\tilde{f}(x) := f(ax)$ for all $x\in\mathbb{R}^d$. By changing the variable, it is easy to check that $\int \tilde{f}\,d\mu_{\tilde{V}_a} = 0$. According to (3.34) and Theorem 1.1(1), we know that

$$\int \tilde{f}^2(x)\,\mu_{\tilde{V}_a}(dx) \leqslant \frac{C_0}{2}\iint_{\{|x-y|\leqslant 1\}} \frac{(\tilde{f}(x)-\tilde{f}(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_{\tilde{V}_a}(dx)$$

holds for some constant $C_0>0$ independent of f. Then, by changing the variable again, we arrive at

$$\int f^2(x)\,\mu_V(dx) \leqslant \frac{a^\alpha C_0}{2}\iint_{\{|x-y|\leqslant a\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_V(dx).$$

Combining this inequality with the fact that

$$\frac{1}{2}\iint_{\{|x-y|\leqslant a\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_V(dx) \leqslant e^{a\delta} D_{\alpha,\delta,V}(f,f), \tag{3.37}$$

we can get the first required conclusion.

(b) Suppose that (3.35) holds and the rate function $\tilde{\beta}_a(s)$ defined by (1.7) with respect to $\tilde{V}_a(x)$ satisfies (3.36). By Theorem 1.1(2) and [10, Corollary 3.3.4] (see also [12, Corollary 3.3]), the following defective log-Sobolev inequality holds for any $g\in C_b^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\mu_{\tilde{V}_a}(g^2\log g^2) - \mu_{\tilde{V}_a}(g^2)\log\mu_{\tilde{V}_a}(g^2) \leqslant{}& C_1\iint_{\{|x-y|\leqslant 1\}} \frac{(g(x)-g(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_{\tilde{V}_a}(dx)\\ &+C_2\mu_{\tilde{V}_a}(g^2),\end{aligned} \tag{3.38}$$

where C_1 and C_2 are two positive constants. Hence, for any $f\in C_b^\infty(\mathbb{R}^d)$, by applying $\tilde{f}(x) := f(ax)$ into (3.38) and by the change of variable and (3.37), we get that

$$\begin{aligned}\mu_V(f^2\log f^2) - \mu_V(f^2)\log\mu_V(f^2) \leqslant{}& 2a^\alpha e^{a\delta} C_1 D_{\alpha,\delta,V}(f,f)\\ &+(C_2 - d\log a)\mu_V(f^2).\end{aligned} \tag{3.39}$$

If $C_2 - d\log a \leqslant 0$, then, by (3.39), we get the second required conclusion. If $C_2 - d\log a > 0$, then (3.39) indeed is a defective log-Sobolev inequality. On the other hand, according to (3.35) and (1), we know that the Poincaré inequality holds for $D_{\alpha,\delta,V}(f,f)$, which along with (3.39) yields the real log-Sobolev inequality, e.g. see [10, Theorem 5.1.8]. □

Corollary 3.6 improves [4, Theorem 1.1] for $D_{\alpha,\delta,V}$ when $\delta > 0$ large enough. The detail also can be seen from the following example.

Example 3.7. (1) Let $\mu_V(dx) := \mu_\lambda(dx) = C_\lambda e^{-\lambda|x|}\,dx$ with $\lambda > 0$. Then, (3.34) is satisfied for such μ_V, and hence the Poincaré inequality holds for $D_{\alpha,\delta,V}$ with any $\delta \geqslant 0$, while [4, Theorem 1.1] only yields that the Poincaré inequality holds for $D_{\alpha,\delta,V}$ with $\delta \in [0, \lambda]$.

(2) Let $\mu_V(dx) := C_\lambda e^{-\lambda|x|\log(1+|x|)}\,dx$ with $\lambda > 0$. Then, (3.35) and (3.36) hold for such μ_V, and hence the log-Sobolev inequality holds for $D_{\alpha,\delta,V}$ with any $\delta \geqslant 0$.

Remark 3.8. Indeed, according to the arguments of Example 1.2 and Corollary 3.6, we can find the following two statements. (i) Let $\mu_{V_\lambda}(dx) := C_\lambda e^{-\lambda|x|}\,dx$ with $\lambda > 0$. Then, there are two positive constants a_1 and C_1 (may depend on λ) such that

$$\mu_{V_\lambda}(f^2) - \mu_{V_\lambda}(f)^2 \leqslant C_1 \iint_{\{|x-y|\leqslant a_1\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_{V_\lambda}(dx), \quad f \in C_b^\infty(\mathbb{R}^d).$$

(ii) Let $\mu_{\hat{V}_\lambda}(dx) := C_\lambda e^{-\lambda|x|\log(1+|x|)}\,dx$ with $\lambda > 0$. Then, there are two positive constants a_2 and C_2 (may depend on λ) such that

$$\begin{aligned}&\mu_{\hat{V}_\lambda}\left(f^2\log f^2\right) - \mu_{\hat{V}_\lambda}(f^2)\log\mu_{\hat{V}_\lambda}(f^2)\\ &\quad\leqslant C_2 \iint_{\{|x-y|\leqslant a_2\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dy\,\mu_{\hat{V}_\lambda}(dx), \quad f \in C_b^\infty(\mathbb{R}^d).\end{aligned}$$

In particular, a close inspection of the computation in Example 1.2 shows that, if λ is large enough then one can take both the jump sizes a_1 and a_2 in two inequalities above to be less than 1; however, for small λ we cannot expect the jump sizes a_1 and a_2 to be less than 1.

To compare the different properties of the functional inequalities for $D_{\alpha,\delta,V}$ and $\mathscr{E}_{\alpha,V}$, we will take the following three examples.

Example 3.9 (*Poincaré Inequalities and Super Poincaré Inequalities hold for $D_{\alpha,V}$ but not for $D_{\alpha,\delta,V}$ with $\delta > 0$ and $\mathscr{E}_{\alpha,V}$*)**.** Let $\mu_{V_\varepsilon}(dx) := C_\varepsilon(1+|x|)^{d+\varepsilon}\,dx$ with $\varepsilon \geqslant \alpha$. Then, according to [13, Corollary 1.2], the Poincaré inequality holds for D_{α,V_ε} with $\varepsilon \geqslant \alpha$, and the super Poincaré inequality holds for D_{α,V_ε} with $\varepsilon > \alpha$. However, by Chen and Wang [4, Proposition 1.3], the Poincaré inequality and so the super Poincaré inequality do not hold for $D_{\alpha,\delta,V_\varepsilon}$ with any $\delta > 0$. On the other hand, according to Proposition 3.5, the Poincaré inequality and the super Poincaré inequality either do not hold for $\mathscr{E}_{\alpha,V_\varepsilon}$.

Example 3.10 (*Super Poincaré Inequalities hold for $D_{\alpha,\delta,V}$ with $\delta > 0$ but not $\mathscr{E}_{\alpha,V}$*)**.** Let $\mu_{V_\lambda}(dx) := C_\lambda e^{-\lambda|x|}\,dx$ with $\lambda > 0$. For every $0 < \delta < \lambda$, according to [4, Lemma 4.3] and the argument of [13, Theorem 1.1(2)], the super Poincaré inequality holds for $D_{\alpha,\delta,V_\lambda}$ with the rate function $\beta(s) = c_1(1+s^{-p_1})$ for some positive constants c_1 and p_1. However, by Proposition 3.5, the super Poincaré inequality does not hold for $\mathscr{E}_{\alpha,V_\lambda}$.

Example 3.11 (*Super Poincaré Inequalities hold for both $D_{\alpha,\delta,V}$ and $\mathscr{E}_{\alpha,V}$, but with Different Rate Function*)**.** Let $\mu_{V_\kappa}(dx) := C_\kappa e^{-(1+|x|^\kappa)}\,dx$ with $\kappa > 1$. Also according to [4, Lemma 4.3] and the argument of [13, Theorem 1.1(2)], the super Poincaré inequality holds for $D_{\alpha,\delta,V_\kappa}$ with the rate function $\beta(s) = c_2(1+s^{-p_2})$ for some positive constants c_2 and p_2. On the other hand, according to Example 1.2 (2), the super Poincaré holds for $\mathscr{E}_{\alpha,V_\kappa}$ with the rate function

$$\beta(s) = c_3\exp\left(c_4\left(1+\log^{\kappa/(\kappa-1)}(1+s^{-1})\right)\right).$$

4. Functional inequalities for non-local Dirichlet forms with large jumps

Proof of Theorem 1.4. (1) The proof of (1.13) is almost the same as that of [4, Theorem 3.6]. For reader's convenience, here we write it in detail. Let $L_{\mathscr{D}_{\alpha,V}}$ be the generator associated with $\mathscr{D}_{\alpha,V}$. Then, according to [4, Lemma 4.2], we know that for any $f \in C_c^\infty(\mathbb{R}^d)$,

$$L_{\mathscr{D}_{\alpha,V}} f(x) = \frac{1}{2}\int_{\{|z|>1\}} (f(x+z) - f(x)) \left(e^{V(x)-V(x+z)} + 1\right) \frac{dz}{|z|^{d+\alpha}}.$$

For $\alpha_0 \in (0,1)$, let $\phi \in C^\infty(\mathbb{R}^d)$ such that $\phi \geqslant 1$ and $\phi(x) = 1 + |x|^{\alpha_0}$ for $|x| > 1$. By (1.12) and [4, Lemma 4.3], $L_{\mathscr{D}_{\alpha,V}}\phi$ is well defined, and there exist r_0, c_1 and $c_2 > 0$ such that

$$L_{\mathscr{D}_{\alpha,V}}\phi(x) \leqslant -c_1 \frac{e^{V(x)}}{(1+|x|)^{d+\alpha}}\phi(x) + c_2 \mathbb{1}_{B(0,r_0)}(x).$$

This, along with [4, Proposition 3.2], yields that there are $c_3, c_4 > 0$ such that for any $f \in C_b^\infty(\mathbb{R}^d)$,

$$\int f(x)^2 \frac{e^{V(x)}}{(1+|x|)^{d+\alpha}}\, \mu_V(dx) \leqslant c_3 \mathscr{D}_{\alpha,V}(f,f) + c_4 \int_{B(0,r_0)} f^2 \phi^{-1}\, d\mu_V.$$

In particular, for any $f \in C_b^\infty(\mathbb{R}^d)$ with $\mu_V(f) = 0$,

$$\int f(x)^2 \frac{e^{V(x)}}{(1+|x|)^{d+\alpha}}\, \mu_V(dx) \leqslant c_3 \mathscr{D}_{\alpha,V}(f,f) + c_4 \int_{B(0,r_0)} f^2 \phi^{-1}\, d\mu_V.$$

On the other hand, since $\phi \geqslant 1$, by the local Poincaré inequality (2.23), there is a constant $c_5 > 0$ such that for any $r > r_0 \vee 3$,

$$\begin{aligned}\int_{B(0,r_0)} f^2\phi^{-1}\, d\mu_V &\leqslant \int_{B(0,r_0)} f^2\, d\mu_V \\ &\leqslant \int_{B(0,r)} f^2\, d\mu_V \\ &\leqslant \frac{c_5 K(r) r^{2d+\alpha}}{k(r)} \mathscr{D}_{\alpha,V}(f,f) + \frac{1}{\mu_V(B(0,r))}\left(\int_{B(0,r)} f\, d\mu_V\right)^2 \\ &= \frac{c_5 K(r) r^{2d+\alpha}}{k(r)} \mathscr{D}_{\alpha,V}(f,f) + \frac{1}{\mu_V(B(0,r))}\left(\int_{B(0,r)^c} f\, d\mu_V\right)^2,\end{aligned}$$

where in the equality above we have used the fact that

$$\int_{B(0,r)} f\, d\mu_V = -\int_{B(0,r)^c} f\, d\mu_V.$$

Using the Cauchy–Schwarz inequality, we find

$$\left(\int_{B(0,r)^c} f\, d\mu_V\right)^2 \leqslant \left(\int_{B(0,r)^c} \frac{(1+|x|)^{d+\alpha}}{e^{V(x)}}\, \mu_V(dx)\right) \times \int_{B(0,r)^c} f(x)^2 \frac{e^{V(x)}}{(1+|x|)^{d+\alpha}}\, \mu_V(dx).$$

Therefore, for any $f \in C_b^\infty(\mathbb{R}^d)$ with $\int f\, d\mu_V = 0$ and any $r \geqslant r_0 \vee 3$,

$$\begin{aligned}\int f(x)^2 \frac{e^{V(x)}}{(1+|x|)^{d+\alpha}}\, \mu_V(dx) \leqslant \left(c_3 + \frac{c_6 K(r) r^{2d+\alpha}}{k(r)}\right) \mathscr{D}_{\alpha,V}(f,f) \\ + \frac{c_6 \int_{B(0,r)^c} \frac{(1+|x|)^{d+\alpha}}{e^{V(x)}}\, \mu_V(dx)}{\mu_V(B(0,r))} \\ \times \int f(x)^2 \frac{e^{V(x)}}{(1+|x|)^{d+\alpha}}\, \mu_V(dx).\end{aligned}$$

Due to (1.12), $\int \frac{(1+|x|)^{d+\alpha}}{e^{V(x)}}\, \mu_V(dx) < \infty$, and so we can choose $r_1 \geqslant r_0 \vee 3$ large enough such that

$$\frac{c_6 \int_{B(0,r_1)^c} \frac{(1+|x|)^{d+\alpha}}{e^{V(x)}}\, \mu_V(dx)}{\mu_V(B(0,r_1))} \leqslant 1/2,$$

which gives us the inequality (1.13) with $C_1 = 2\left(c_3 + \frac{c_6 K(r_1) r_1^{2d+\alpha}}{k(r_1)}\right)$.

(2) Let D be a bounded compact subset of $\mathbb{R}^d$. For any $f \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} f \subset D$, we find

$$\begin{aligned}\mathscr{D}_{\alpha,V}(f,f) &= \frac{1}{2}\iint_{\{D\times D, |x-y|>1\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\, dy\, \mu_V(dx) \\ &\quad + \frac{1}{2}\iint_{\{D\times D^c, |x-y|>1\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\, dy\, \mu_V(dx) \\ &\quad + \frac{1}{2}\iint_{\{D^c\times D, |x-y|>1\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\, dy\, \mu_V(dx) \\ &\leqslant \iint_{\{D\times D, |x-y|>1\}} \frac{f^2(x)}{|x-y|^{d+\alpha}}\, dy\, \mu_V(dx) \\ &\quad + \iint_{\{D\times D, |x-y|>1\}} \frac{f^2(y)}{|x-y|^{d+\alpha}}\, dy\, \mu_V(dx) \\ &\quad + \frac{1}{2}\iint_{\{D\times D^c, |x-y|>1\}} \frac{f^2(x)}{|x-y|^{d+\alpha}}\, dy\, \mu_V(dx) \\ &\quad + \frac{1}{2}\iint_{\{D^c\times D, |x-y|>1\}} \frac{f^2(y)}{|x-y|^{d+\alpha}}\, dy\, \mu_V(dx) \\ &=: \sum_{i=1}^4 J_i.\end{aligned}$$

Note that

$$\begin{aligned}J_1 &\leqslant \int_D \left(\int_{\{|x-y|>1\}} \frac{1}{|x-y|^{d+\alpha}}\, dy\right) f^2(x)\, \mu_V(dx) \\ &\leqslant \left(\int_{\{|z|>1\}} \frac{1}{|z|^{d+\alpha}}\, dz\right) \int_D f^2(x)\, \mu_V(dx).\end{aligned}$$

Since

$$\int_{\{|x-y|>1\}} \frac{1}{|x-y|^{d+\alpha}}\,\mu_V(dx) \leqslant \int \mu_V(dx) = 1,$$

we have

$$\begin{aligned} J_2 &\leqslant \int_D \left(\int_{\{|x-y|>1\}} \frac{1}{|x-y|^{d+\alpha}}\,\mu_V(dx) \right) f^2(y)\,dy \\ &\leqslant \left(C_V^{-1} \sup_{y\in D} e^{V(y)} \right) \int_D f^2(y)\,\mu_V(dy). \end{aligned}$$

Following the same way as above, we can get the similar estimates for J_3 and J_4, respectively. Therefore, for every $f \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f \subset D$,

$$\mathscr{D}_{\alpha,V}(f,f) \leqslant C_{V,D}\mu_V(f^2),$$

where $C_{V,D}$ is a positive constant independent of f.

Thus, according to (1.15), for every $f \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f \subset D$,

$$\mu_V(f^2) \leqslant sC_{V,D}\mu_V(f^2) + \beta(s)\mu(|f|)^2.$$

By taking $s = \frac{1}{2C_{V,D}}$, we derive that

$$\mu_V(f^2) \leqslant 2\beta\left(\frac{1}{2C_{V,D}}\right)\mu_V(|f|)^2. \tag{4.40}$$

On the other hand, since the function V is locally bounded, there exist a point $x_0 \in D$ and a constant $r_0 > 0$ such that $B(x_0,r_0) \subset D$, and

$$\int_{B(x_0,r_0)} \mu_V(dx) \leqslant \left[4\beta\left(\frac{1}{2C_{V,D}}\right) \right]^{-1}.$$

Let $f_0 \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} f_0 \subset B(x_0,r_0)$ and $f_0(x) > 0$ for every $x \in B(x_0,r_0/2)$. Hence, by the Cauchy–Schwartz inequality,

$$\mu_V(|f_0|)^2 = \mu_V(|f_0|\mathbb{1}_{B(x_0,r_0)})^2 \leqslant \mu_V(f_0^2)\mu_V(B(x_0,r_0)) \leqslant \frac{\mu_V(f_0^2)}{4\beta\left(\frac{1}{2C_{V,D}}\right)}.$$

This along with (4.40) yields that

$$\mu_V(f_0^2) \leqslant \frac{1}{2}\mu_V(f_0^2).$$

However, due to the fact that $f_0(x) > 0$ for $x \in B(x_0,r_0/2)$, $\mu_V(f_0^2) \neq 0$, which is a contradiction, and so the super Poincaré inequality (1.15) does not hold for $\mathscr{D}_{\alpha,V}$. $\square$

Remark 4.1. (1) As the same way, we can also prove that the super Poincaré inequality does not hold for the following Dirichlet form

$$\mathscr{D}_{\rho,V}(f,f) := \frac{1}{2}\iint (f(x)-f(y))^2\rho(|x-y|)\,dy\,\mu_V(dx),$$

where ρ is a positive measurable function on $\mathbb{R}_+$ such that $\int_{(0,\infty)} \rho(r)r^{d-1}\,dr < \infty$ and $\sup \rho(r) < \infty$.

(2) As shown in Theorem 1.4(1), if (1.12) holds, then we can get the weighted Poincaré inequality for $\mathscr{D}_{\alpha,V}$. However, different from the case for $D_{\alpha,V}$ (see [4, Proposition 1.6]) and due to the lack of local super Poincaré inequality for $\mathscr{D}_{\alpha,V}$, the global super Poincaré inequality fails for $\mathscr{D}_{\alpha,V}$, which reveals that in some situations, to derive the global super Poincaré inequality, the local super Poincaré inequality is inevitable.

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Note added in Proof. After the submission of this paper, we know the recent work [6] by P. T. Gressman about the fractional L^p Poincaré inequality and the generalized log-Sobolev inequality on general metric measure space. Although our findings on the Poincaré inequality for a class of non-local Dirichlet forms with finite range jumps partially overlap, the methods used here and in [6] are essentially different. The generalized log-Sobolev inequality in [6] is used to characterize some embedding properties and it is different from the standard log-Sobolev inequality in our paper.

References

[1] P. Cattiaux, A. Guillin, F.-Y. Wang, L. Wu, Lyapunov conditions for Super Poincaré inequalties, J. Funct. Anal. 256 (2009) 1821–1841.
[2] Z.-Q. Chen, P. Kim, T. Kumagai, Weighted Poincaré inequality and heat kernel estimates for finite range jump processes, Math. Ann. 342 (2008) 833–883.
[3] Z.-Q. Chen, T. Kumagai, Heat kernel estimates for stable-like processes on d-sets, Stoch. Proc. Appl. 108 (2003) 27–62.
[4] X. Chen, J. Wang, Weighted Poincaré inequalities for non-local Dirichlet forms, Preprint. Also see arXiv:1207.7140v1.
[5] A. Eberle, Spectral gaps on discretized loop spaces, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 6 (2003) 265–300.
[6] P.T. Gressman, Fractional Poincaré and logarithmic Sobolev inequalities for measure spaces, J. Funct. Anal. 265 (2013) 867–889.
[7] A. Holley, S. Kusuoka, D.W. Stroock, Asymptotics of the spectral gap with applications to the theory of simulated annealing, J. Funct. Anal. 83 (1989) 333–347.
[8] C. Mouhot, E. Russ, Y. Sire, Fractional Poincaré inequalities for general measures, J. Math. Pures Appl. 95 (2011) 72–84.
[9] M. Röckner, F.-Y. Wang, Weak Poincaré inequalities and L^2 convergence rates of Markov semigroups, J. Funct. Anal. 185 (2001) 564–603.
[10] F.-Y. Wang, Functional Inequalities, Markov Processes and Spectral Theory, Science Press, Beijing, 2005.
[11] F.-Y. Wang, Functional inequalities, semigroup properties and spectrum estimates, Infin. Dimens. Anal. Quant. Probab. Relat. Top. 3 (2000) 263–295.
[12] F.-Y. Wang, Functional inequalities for empty essential spectrum, J. Funct. Anal. 170 (2000) 219–245.
[13] F.-Y. Wang, J. Wang, Functional inequalities for stable-like Dirichlet forms, J. Theor. Probab. (2013) http://dx.doi.org/10.1007/s10959-013-0500-5. in press. Also see arXiv:1205.4508v2.