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A diffusion approximation for limit order book models☆

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Abstract

This paper derives a diffusion approximation for a sequence of discrete-time one-sided limit order book models with non-linear state dependent order arrival and cancellation dynamics. The discrete time sequences are specified in terms of an $\mathbb{R}_+$-valued best bid price process and an L^2_{loc}-valued volume process. It is shown that under suitable assumptions the sequence of interpolated discrete time models is relatively compact in a localized sense and that any limit point satisfies a certain infinite dimensional SDE. Under additional assumptions on the dependence structure we construct two classes of models, which fit in the general framework, such that the limiting SDE admits a unique solution and thus the discrete dynamics converge to a diffusion limit in a localized sense.

MSC: 60F17; 91G80

Keywords: Functional limit theorem; Diffusion limit; Scaling limit; Convergence of stochastic differential equations; Limit order book

1. Motivation and setup

In modern financial markets almost all transactions are settled through limit order books (LOBs). A LOB is a record of unexecuted orders awaiting execution. Stochastic analysis provides powerful tools for understanding the complex system of order aggregation and execution in limit order markets via the description of suitable scaling ("high-frequency") limits. Scaling limits

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allow for a tractable description of the macroscopic LOB dynamics (prices and standing volumes) from the underlying microscopic dynamics (individual order arrivals and cancellations). In this paper we prove a novel functional convergence result for a class of Markov chains arising in microstructure models of LOBs to an infinite dimensional diffusion.

Scaling limits for LOBs have recently attracted considerable attention in the probability and financial mathematics literature. Depending on the scaling assumptions either fluid limits (cf. [8–11]) or diffusion limits (cf. [2,6,12]) can be derived. Fluid limits for the full order book were first studied in [11] and afterwards in [10], where it was shown that under certain assumptions on the scaling parameters the sequence of discrete-time LOB models converges in probability to the solution of a deterministic differential equation. Although there is some work on probabilistic LOB models that assumes an SPDE or measure-valued dynamics for the volume process (cf. [13,17]), there is little work on the derivation of a measure valued diffusion limit starting from a microscopic ("event-by-event") description of the limit order book. Two exceptions are the particular models considered in [2,20]. The work [2] extends the models in [10,11] by introducing additional noise terms in the pre-limit in which case the dynamics can then be approximated by an SPDE in the scaling limit. The papers [2,10,11] rely on the same scaling assumptions. Our work is motivated by the question whether under *different* scaling assumptions the same event-by-event dynamics can be approximated by a diffusion process in the high frequency regime *without* adding additional noise terms in the pre-limit.

1.1. The LOB dynamics

The one-sided LOB models considered in this paper are specified by a sequence of discrete time $\mathbb{R}\times L^2(\mathbb{R}_+;\mathbb{R})$-valued processes $\widetilde{S}^{(n)} = \left(B^{(n)}, v^{(n)}\right)$, where for each $n \in \mathbb{N}$, the non-negative one dimensional process $B^{(n)}$ specifies the dynamics of the *best bid price*, and the $L^2(\mathbb{R}_+;\mathbb{R})$-valued process $v^{(n)}$ specifies the dynamics of the bid-side *volume density function*.

We fix some $T > 0$ and introduce the scaling parameters $\Delta x^{(n)}$, $\Delta v^{(n)}$, and $\Delta t^{(n)}$. They denote the tick-size, the impact of an individual order on the state of the book, and the time between two consecutive order arrivals, respectively. We put $T_n := \lfloor T/\Delta t^{(n)} \rfloor$, $x_j^{(n)} := j\Delta x^{(n)}$ and $t_j^{(n)} := j\Delta t^{(n)} \wedge T$ for all $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$ we define the interval $I^{(n)}(x)$ as

$$I^{(n)}(x) := \left[x_j^{(n)}, x_{j+1}^{(n)}\right) \quad \text{for} \quad x_j^{(n)} \le x < x_{j+1}^{(n)}.$$

The initial best bid price is given by $B_0^{(n)} = b_n \Delta x^{(n)}$ for some $b_n \in \mathbb{N}$. The initial volume density function is given by a non-negative deterministic step function $v_0^{(n)} \in L^2(\mathbb{R}_+;\mathbb{R})$ on the $\Delta x^{(n)}$-grid. Following [10] we assume that there are three events that change the state of the book: price increases (event A), price decreases (event B) and limit order placements, respectively cancellations (event C). In terms of the placement operator

$$M_k^{(n)}(\cdot) := \mathbb{1}_C\left(\phi_k^{(n)}\right)\frac{\omega_k^{(n)}}{\Delta x^{(n)}}\mathbb{1}_{I^{(n)}\left(\pi_k^{(n)}\right)}(\cdot) \tag{1}$$

the dynamics of the one-sided LOB models can then be described by the following point process: for each $n \in \mathbb{N}$ and all $k = 1, \ldots, T_n$,

$$\begin{aligned} B_k^{(n)} &= B_{k-1}^{(n)} + \Delta x^{(n)}\left[\mathbb{1}_B\left(\phi_k^{(n)}\right) - \mathbb{1}_A\left(\phi_k^{(n)}\right)\right] \\ v_k^{(n)} &= v_{k-1}^{(n)} + \Delta v^{(n)} M_k^{(n)} \end{aligned} \tag{2}$$

where the event indicator function $\phi_k^{(n)}$ is a random variable taking values in the set $\{A, B, C\}$, the $[-M, M]$-valued random variable $\omega_k^{(n)}$ specifies the size of a placement or cancellation ($M > 0$), and the non-negative random variable $\pi_k^{(n)}$ specifies the location of a placement or cancellation.

1.2. Fluid or diffusion approximation: empirical evidence

An array of scaling limits for LOBs have recently been suggested in the financial mathematics literature. These limits are (mostly) either of fluid or of diffusion type. Which type of limit is appropriate for a specific stock or class of stocks will depend on the stock characteristics and/or the considered time scales. In the sequel we provide preliminary empirical evidence that volume dynamics over intermediate time scales (one second) of liquidly traded stocks such as MSFT are best approximated by diffusion processes.

In [5,6] the authors argue that the volumes at the top of the book should be approximated by a diffusion rather than a fluid limit if the fluctuations dominate the mean in the long run. To this end, they compute the empirical distribution of the ratio

$$R := \frac{\sqrt{n} \cdot \text{mean}}{\text{std}},$$

where n denotes the average number of trades during a ten second interval and *mean* (resp. *std*) stands for the average size of a submitted order (resp. the standard deviation thereof). They provide empirical evidence that R concentrated around zero for certain stocks from the DowJones index during June 2008 when the data is grouped into ten second intervals, suggesting that volumes at the top should be modelled using a stochastic component on intermediate time horizons.

We complement their analysis using LOBSTER data of MSFT from June 5, 2018[1]: We consider volumes at several price levels, group the orders into price baskets of five ticks (i.e. five cents) and compute R separately for every price basket. As we are working with daily instead of monthly data, we grouped the data into one second intervals.[2] As the bid price took values between \$101,53 and \$102,32 on June 5, 2018, we consider price level between \$101,60 and \$101,24, grouped into intervals of five cents and let n be the average number of trades occurring at a particular price basket during a one second interval, given that at least one trade occurred in that interval.[3] Our results are documented in Table 1. As in [6], the ratio R takes very small values for our data sample across all price levels, suggesting that fluid approximations for the MSFT order book are not appropriate on one-second time scales.

The question is, then, whether the volumes are best modelled by a stochastic process with a diffusive component or a pure jump process. To this end, we apply a test proposed in [1]. For a one-dimensional Itô semimartingale, whose small jumps essentially behave like those of a stable process and whose volatility process itself is again an Itô semimartingale, that paper constructs a test statistic involving the p-variation of the observed process for some $p \in (0, 2)$, which should converge to some value strictly larger than 1 under the null hypothesis that the Itô semimartingale is driven by a Brownian motion (cf. the theoretical value reported in Table 2)

1 We considered all trades taking place on the bid side of the order book during that day except the execution of hidden orders, since they do not have a visible effect on the limit order book.

2 Since in [6] the authors considered ten second intervals, while we are working with one second intervals, our values should be multiplied by $\sqrt{10}$ to compare with the values in [6]. Nonetheless, this still yields a very comparable result.

3 We are working with absolute coordinates instead of relative coordinates for convenience. Since MSFT is a very liquid stock, we do not expect the results to differ much whether one looks at absolute or relative volumes.

Table 1
Ratio $R = \sqrt{n} \cdot$ mean/std for different price ticks of MSFT data from June 5, 2018.

Price tick	101,60–64	101,65–69	101,70–74	101,75–79	101,80–84	101,85–89	
Number of orders	11008	16299	28470	45503	29548	28176	
Ratio R	−0,0025	0,0076	0,0034	0,0011	0,0091	0,0031	
Price tick	101,90–94	101,95–99	102,00–04	102,05–09	102,10–14	102,15–19	102,20–24
Number of orders	37091	40849	39464	44045	44001	34720	46098
Ratio R	0,0086	0,0073	0,0042	−0,0187	−0,0087	0,0004	0,0019

Table 2
Test statistics for H_0 : *Brownian motion present*. For all values marked * (resp. **; ***; ****) the null hypothesis H_0 is not rejected at any chosen level of significance $\alpha \leq 0, 5$ (resp. 0,2; 0,1; 0,05).

	H_0 value	H_1 value	101,60–64	101,65–69	101,70–74	101,75–79	101,80–84	101,85–89	
$p = 1{,}25$	1,297	1	1,305*	1,257**	1,246***	1,323*	1,293**	1,226****	
$p = 1{,}5$	1,189	1	1,235*	1,195*	1,189**	1,272*	1,237*	1,169**	
$p = 1{,}75$	1,091	1	1,170*	1,140*	1,141*	1,231*	1,192*	1,122*	
	H_0 value	H_1 value	101,90–94	101,95–99	102,00–04	102,05–09	102,10–14	102,15–19	102,20–24
$p = 1{,}25$	1,297	1	1,319*	1,260**	1,278**	1,230****	1,264**	1,199***	1,281**
$p = 1{,}5$	1,189	1	1,275*	1,204*	1,230*	1,173**	1,212*	1,147**	1,232*
$p = 1{,}75$	1,091	1	1,240*	1,159*	1,194*	1,127*	1,170*	1,106*	1,194*

and to 1 otherwise. In [1] the test is applied to INL and MSFT high frequency data from 2006, yielding empirical evidence that on an intermediate time scale (any sampling interval between five seconds and 30 min) the prices of liquidly traded stocks in electronic markets should indeed be modelled with a Brownian component. We test if volumes should also be modelled with a Brownian component using the same data set as above, i.e. MSFT data from June 5, 2018. We aggregate the data on the tick level building baskets of five ticks and use sampling intervals of one second. The computed test statistics together with their theoretical values under the null hypothesis are reported in Table 2 for different values of p.[4] As one can see from our results, we will not reject the null hypothesis at any reasonable significance level. This suggests to consider diffusive limits for LOB dynamics of liquid stocks on intermediate time scales.

1.3. Preview of the main results

In deriving a diffusion limit for the sequence of LOB models (2), the first challenge is to define a suitable convergence concept. While for any $\pi \in \mathbb{R}_+$,

$$\left\| \mathbb{1}_{I^{(n)}(\pi)} \right\|_{L^2(\mathbb{R}_+)} = \left(\Delta x^{(n)}\right)^{1/2},$$

we have for any bounded $f \in L^2(\mathbb{R}_+)$,

$$\left\langle \mathbb{1}_{I^{(n)}(\pi)}, f \right\rangle_{L^2(\mathbb{R}_+)} = \int_{I^{(n)}(\pi)} f(x)dx = \mathcal{O}\left(\Delta x^{(n)}\right).$$

Hence, it seems impossible to formulate a scaling assumption with $\Delta x^{(n)} \to 0$, $\Delta t^{(n)} \to 0$, and $\Delta v^{(n)} \to 0$ that allows to prove convergence of the volume density functions to an

[4] To compute the values in Table 2 we used a cutoff level of $K = 2000$ for the order sizes. However, the test does not yield qualitatively different results if one varies the cutoff level within a reasonable range.

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$L^2(\mathbb{R}_+;\mathbb{R})$-valued diffusion process. However, observe that for any $m,\pi>0$ we have

$$\left\| \Delta x^{(n)} \sum_{j=0}^{\lfloor \cdot/\Delta x^{(n)} \rfloor} \mathbb{1}_{I^{(n)}(\pi)}\left(x_j^{(n)}\right) \mathbb{1}_{[0,m]}(\cdot) \right\|_{L^2} = \Delta x^{(n)} \left(m - \Delta x^{(n)} \left\lfloor \frac{\pi}{\Delta x^{(n)}} \right\rfloor \right)^{1/2} = \mathcal{O}\left(\Delta x^{(n)}\right)$$

and for any bounded $f\in L^2$ also

$$\left\langle \Delta x^{(n)} \sum_{j=0}^{\lfloor \cdot/\Delta x^{(n)} \rfloor} \mathbb{1}_{I^{(n)}(\pi)}\left(x_j^{(n)}\right), f\mathbb{1}_{[0,m]} \right\rangle_{L^2} = \Delta x^{(n)} \int_{\Delta x^{(n)}\left\lfloor \frac{\pi}{\Delta x^{(n)}} \right\rfloor}^{m} f(x)dx = \mathcal{O}\left(\Delta x^{(n)}\right).$$

This suggests to study the convergence of the cumulated volume processes $V^{(n)}=\left(V_k^{(n)}\right)_{k\le T_n}$ with

$$V_k^{(n)}(x) := \Delta x^{(n)} \sum_{j=0}^{\lfloor x/\Delta x^{(n)} \rfloor} v_k^{(n)}\left(x_j^{(n)}\right), \quad x\in\mathbb{R}_+, \tag{3}$$

instead of analysing directly the convergence of the volume density functions. To do this we will choose a localized convergence concept, since the functions $V^{(n)}$ are not square integrable on the whole line.

Our main contribution is to establish a convergence concept and a convergence result for the sequence $S^{(n)} := \left(B^{(n)}, V^{(n)}\right)$, $n\in\mathbb{N}$. In particular, we state sufficient conditions that guarantee that (i) this sequence is relatively compact; (ii) any limit point solves an infinite dimensional SDE driven by a standard Brownian and a cylindrical Brownian motion; (iii) the limiting SDE has a unique solution.

Having established a convergence concept, the second major challenge is that the dynamics of the process $S^{(n)}$, $n\in\mathbb{N}$, is not given in standard SDE form, due to the event-by-event dynamics, and that the system can only be controlled by specifying the conditional distribution of the random variables $\pi_k^{(n)}$, $\omega_k^{(n)}$, and $\phi_k^{(n)}$. Much of our work is, therefore, devoted to the identification of suitable integrands $G^{(n)}\left(S^{(n)}(t)\right)$ and semimartingale random measures $Y^{(n)}$ such that $S^{(n)}(t)$ can be represented as

$$S^{(n)}(t) = S_0^{(n)} + \int_0^t G^{(n)}\left(S^{(n)}(u)\right) dY^{(n)}(u), \quad t\in[0,T] \tag{4}$$

after continuous time-interpolation. Once the dynamics of the sequence $S^{(n)}$, $n\in\mathbb{N}$, has been brought into standard SDE form, it remains to study its convergence. The convergence of infinite dimensional stochastic integrals has been studied by several authors. Chao [4] and Walsh [22] consider semimartingale random measures as distribution valued processes in some nuclear space. Kallianpur and Xiong [16] prove diffusion approximations of nuclear space-valued SDEs. Their approach requires a dependence structure that is incompatible with our spatial pointwise dynamics, and is hence not applicable to our modelling framework. Jakubowski [15] provides convergence results for Hilbert space valued semimartingales under a uniform tightness condition. Kurtz and Protter [19] work with the same uniform tightness condition, but allow for a more general setting. Especially, they also study the convergence of solutions of stochastic differential equations in infinite dimension. The results are further extended by Ganguly [7] to study the convergence of infinite dimensional stochastic differential equations when the approximating sequence of integrators is not uniformly tight anymore.

Our proof relies on the results in [19]. We first establish sufficient conditions that guarantee that the sequence $Y^{(n)}$, $n\in\mathbb{N}$, converges to some $L^2(\mathbb{R}_+)^\#$-semimartingale Y. Subsequently we prove that the sequence $G^{(n)}$, $n \in \mathbb{N}$, satisfies a compactness property and converges in a localized sense to some function G. Finally, we show that the sequence of stochastic differential equations in (4) converges in law in a localized sense to a solution to an SDE of the form

$$S(t) = S_0 + \int_0^t G(S(u))\, dY(u), \quad t \in [0, T]. \tag{5}$$

The challenge in proving the converges of the SDEs is the verification of the conditions in [19] on the integrators and coefficient functions of the approximating sequence, and the fact that our convergence concept localizes in space, not time. Finally, we give sufficient conditions for the uniqueness of solutions to the above SDE. For instance, we show that uniqueness holds if only the drift but not the volatility is state-dependent.

1.4. Structure of the paper

The rest of the paper is structured as follows. In Section 2 we state conditions on the dynamics of the price processes that guarantee the converge of their normalized fluctuations to a standard Brownian motion. In Section 3 we state conditions on the dynamics of the order arrivals and cancellations that guarantee convergence of the standardized fluctuations of the volume processes to a cylindrical Brownian motion. While the analysis of the price is quite standard, deriving similar results for the volumes is much more tedious. First we show in Section 3.2 the convergence of the drift, volatility and correlation functions. Using an orthogonal decomposition of the covariance matrix we then establish in Section 3.3 a representation of the volume process as a discrete stochastic differential equation driven by "infinitely many discretized Brownian motions". In Section 3.5 we prove the convergence in law of the "infinitely many discretized Brownian motions" to a cylindrical Brownian motion. In Section 4 we define the stochastic integrals and stochastic differential equations that describe the LOB dynamics and verify that the conditions from [19] are satisfied. This allows us to derive our results on the characterization of the limiting LOB dynamics as solutions to an infinite dimensional SDE in Section 5. We then provide two specific examples in which the LOB dynamics converges weakly to the unique solution of an infinite dimensional SDE. In Section 5.3 we conclude with a short discussion on how our result can explain the noise term in macroscopic SPDE models for limit order books found in the literature.

1.5. Notation

For each $n \in \mathbb{N}$ we fix a probability space $\left(\Omega^{(n)}, \mathcal{F}^{(n)}, \mathbb{P}^{(n)}\right)$[5] with filtration

$$\left\{\emptyset, \Omega^{(n)}\right\} = \mathcal{F}_0^{(n)} \subset \mathcal{F}_1^{(n)} \subset \cdots \subset \mathcal{F}_k^{(n)} \subset \cdots \subset \mathcal{F}_{T_n}^{(n)} \subset \mathcal{F}^{(n)}.$$

We assume that the random vector $\left(\phi_k^{(n)}, \omega_k^{(n)}, \pi_k^{(n)}\right)$ is $\mathcal{F}_k^{(n)}$-measurable for all $n \in \mathbb{N}$ and $k \leq T_n$. We define the Hilbert space

$$E := \mathbb{R} \times L^2(\mathbb{R}_+; \mathbb{R}), \qquad \|(X_1, X_2)\|_E := |X_1| + \|X_2\|_{L^2}$$

and its localized version

$$E_{loc} := \mathbb{R} \times L^2_{loc}(\mathbb{R}_+; \mathbb{R})$$

[5] For ease of notation we will simply write $\mathbb{P}$ and $\mathbb{E}$ in the following instead of $\mathbb{P}^{(n)}$ and $\mathbb{E}^{(n)}$, since it is clear from the context on which probability space we work.

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with

$$L^2_{loc}(\mathbb{R}_+) := \left\{ f : \mathbb{R}_+ \to \mathbb{R} \;\middle|\; \int_0^m f^2(x)dx < \infty \; \forall\, m \in \mathbb{N} \right\}.$$

Moreover, we define for all $n \in \mathbb{N}$ the E_{loc}-valued stochastic process $S^{(n)} = \left(S^{(n)}_k\right)_{k=0,\dots,T_n}$ via

$$S^{(n)}_k := \left(B^{(n)}_k, V^{(n)}_k\right),$$

where $B^{(n)}_k$ and $V^{(n)}_k$ were defined in Eqs. (2) and (3). For all $n \in \mathbb{N}$ and $k = 1, \dots, T_n$ we set

$$\begin{aligned}
\delta V^{(n)}_k &:= V^{(n)}_k - V^{(n)}_{k-1}, & \delta B^{(n)}_k &:= B^{(n)}_k - B^{(n)}_{k-1},\\
\delta \hat{v}^{(n)}_k(x) &:= \mathbb{E}\left(\delta V^{(n)}_k(x) \,\middle|\, \mathcal{F}^{(n)}_{k-1}\right), & \delta \hat{B}^{(n)}_k &:= \mathbb{E}\left(\delta B^{(n)}_k \,\middle|\, \mathcal{F}^{(n)}_{k-1}\right),\\
\delta \overline{v}^{(n)}_k(x) &:= \delta V^{(n)}_k(x) - \delta \hat{v}^{(n)}_k(x), & \delta \overline{B}^{(n)}_k &:= \delta B^{(n)}_k - \delta \hat{B}^{(n)}_k.
\end{aligned}$$

W.l.o.g. we will assume that $\left(\Delta x^{(n)}\right)^{-1} \in \mathbb{N}$ for all $n \in \mathbb{N}$.

2. Fluctuations of the price process

In this section we analyse the fluctuations of the best bid price process $B^{(n)}$. To this end, we introduce a fourth scaling parameter $\Delta p^{(n)} = o(1)$ that controls the proportion of price changes among all events.[6] The scaling limits in [2,10,11] require two time scales, a fast time scale for limit order placements and cancellations and a comparably slow time scale for price changes. The scaling parameter $\Delta p^{(n)}$ introduces the "slow" time scale.

Assumption 2.1. For each $n \in \mathbb{N}$ there exist two functions $p^{(n)} : E_{loc} \to \mathbb{R}$ and $r^{(n)} : E_{loc} \to \mathbb{R}_+$ satisfying the boundary condition

$$\left(r^{(n)}(s)\right)^2 = \Delta x^{(n)} p^{(n)}(s) \quad \forall\, s = (0, v) \in E_{loc}, \tag{6}$$

such that for all $k = 1, \dots, T_n$,

$$\mathbb{P}\left(\phi^{(n)}_k \in \{A, B\} \,\middle|\, \mathcal{F}^{(n)}_{k-1}\right) = \Delta p^{(n)} \left(r^{(n)}\left(S^{(n)}_{k-1}\right)\right)^2 \quad \text{a.s.} \tag{7}$$

and

$$\mathbb{P}\left(\phi^{(n)}_k = B \,\middle|\, \mathcal{F}^{(n)}_{k-1}\right) - \mathbb{P}\left(\phi^{(n)}_k = A \,\middle|\, \mathcal{F}^{(n)}_{k-1}\right) = \Delta p^{(n)} \Delta x^{(n)} p^{(n)}\left(S^{(n)}_{k-1}\right) \quad \text{a.s.} \tag{8}$$

There exists $\eta > 0$ such that for all $n \in \mathbb{N}$ and $s \in E_{loc}$,

$$r^{(n)}(s) + \left(p^{(n)}(s)\right)^+ > \eta. \tag{9}$$

Note that the conditional distribution of the event variables is uniquely determined by Eqs. (7) and (8). Moreover, Eq. (6) guarantees that the price process $B^{(n)}$ will always stay positive.

Remark 2.2. The scaling parameter $\Delta p^{(n)}$ corresponds to the proportion of market orders and spread placements among all LOB events. It can easily be estimated from flow data; see [11]. Conditioning the proportion on selected characteristics of the book such as spreads, volumes at the top or volume imbalances yields the functions $p^{(n)}(\cdot)$ and $r^{(n)}(\cdot)$.

[6] The proportion of price changes among all orders book events is typically quite small for liquidly traded stocks such as APPL, MSFT or BAC; see [8,11] for empirical evidence.

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The next assumption controls the relative speed at which the different scaling parameters converge to zero. Since the discrete system dynamics are the same as in [10], we must use a different scaling to get a diffusion limit instead of a fluid limit. Intuitively, the average impact of all individual events must be of larger size to generate volatility. By comparing the scaling assumption from [10] with Assumption 2.3, we see that this is indeed the case.

Assumption 2.3. For all $n \in \mathbb{N}$,

$$\Delta t^{(n)} = \Delta p^{(n)} \left(\Delta x^{(n)}\right)^2 = \left(\Delta v^{(n)}\right)^2 = o(1).$$

Remark 2.4. The fact that the conditional distribution of the event variables is uniquely determined by Eqs. (7) and (8) is different from the corresponding assumption made in [10] to derive a large of large numbers in the high frequency regime. Indeed, while (7) can also be found in [10], (8) is the only important additional assumption – apart from the different scaling – which is needed to derive a diffusion dynamic for the price process in the high frequency limit. A similar assumption can also be found in [2].

Eqs. (7) and (8) of Assumption 2.1 yield together with Assumption 2.3 that for all $n \in \mathbb{N}$ and $k \leq T_n$ almost surely

$$\Delta t^{(n)} \left[r^{(n)}\left(S_{k-1}^{(n)}\right)\right]^2 = \mathbb{E}\left[\left(\delta B_k^{(n)}\right)^2 \middle| \mathcal{F}_{k-1}^{(n)}\right],$$

$$\Delta t^{(n)} p^{(n)}\left(S_{k-1}^{(n)}\right) = \mathbb{E}\left[\delta B_k^{(n)} \middle| \mathcal{F}_{k-1}^{(n)}\right] = \delta \hat{B}_k^{(n)}.$$

Let us define the process of the (nearly) normalized increments of $B^{(n)}$ as

$$\delta Z_k^{(n)} := \frac{\delta \overline{B}_k^{(n)}}{r^{(n)}\left(S_{k-1}^{(n)}\right)}, \quad Z_k^{(n)} := \sum_{j=1}^{k} \delta Z_j^{(n)} \quad \text{for all } k = 1, \ldots, T_n. \tag{10}$$

Then we may write for all $n \in \mathbb{N}$,

$$\begin{aligned} B^{(n)}(t) &= B_0^{(n)} + \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \delta B_k^{(n)} \\ &= B_0^{(n)} + \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \left[p^{(n)}\left(S_{k-1}^{(n)}\right)\Delta t^{(n)} + r^{(n)}\left(S_{k-1}^{(n)}\right)\delta Z_k^{(n)}\right] \end{aligned} \tag{11}$$

Through linear interpolation of the $Z_k^{(n)}$, $k = 1, \ldots, T_n$, we obtain the continuous time process

$$Z^{(n)}(t) := \sum_{k=0}^{T_n} Z_k^{(n)} \mathbb{1}_{\left[t_k^{(n)}, t_{k+1}^{(n)}\right)}(t), \quad t \in [0, T].$$

By construction the process $(Z^{(n)}(t))_{t\in[0,T]}$ is a càdlàg martingale for each $n \in \mathbb{N}$.

Theorem 2.5. *Under Assumptions 2.1 and 2.3,* $Z^{(n)} = \left(Z^{(n)}(t)\right)_{t\in[0,T]}$ *converges weakly in* $\mathcal{D}([0, T]; \mathbb{R})$ *to a standard Brownian motion* Z *as* $n \to \infty$.

Proof. By construction the field $\left(\delta Z_k^{(n)}\right)_{k\leq T_n,\, n\in\mathbb{N}}$ is a martingale difference array. Therefore, the claim will follow from the functional central limit theorem for martingale difference arrays, once we show that the Lindeberg condition is satisfied and that for every $t \in [0, T]$ the sum of the conditional second moments up to time t converges almost surely to t, which is the quadratic variation of Brownian motion.

For this we first note that Eqs. (7) and (8) imply that for all $n \in \mathbb{N}$ and $k \le T_n$,

$$-1 \le \frac{\Delta x^{(n)} p^{(n)}\left(S_{k-1}^{(n)}\right)}{\left(r^{(n)}\left(S_{k-1}^{(n)}\right)\right)^2} \le 1 \quad \text{a.s.} \tag{12}$$

Moreover, by definition

$$\Delta t^{(n)} \left| p^{(n)}\left(S_{k-1}^{(n)}\right)\right| \le \mathbb{E}\left(\left|\delta B_k^{(n)}\right| \Big| \mathcal{F}_{k-1}^{(n)}\right) \le \Delta x^{(n)} \overset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{13}$$

Hence, for any $t \in [0, T]$

$$\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \mathbb{E}\left(\left(\delta Z_k^{(n)}\right)^2 \Big| \mathcal{F}_{k-1}^{(n)}\right) = \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \frac{\Delta t^{(n)}\left(r^{(n)}\left(S_{k-1}^{(n)}\right)\right)^2 - \left(\Delta t^{(n)} p^{(n)}\left(S_{k-1}^{(n)}\right)\right)^2}{\left(r^{(n)}\left(S_{k-1}^{(n)}\right)\right)^2}$$
$$\to t \quad \text{a.s.}$$

Second, (9) and (12) imply that for all $n \in \mathbb{N}$ and $k \le T_n$,

$$\begin{aligned}
\left[r^{(n)}\left(S_{k-1}^{(n)}\right)\right]^{-2} &\le \left[r^{(n)}\left(S_{k-1}^{(n)}\right)\right]^{-2} \mathbb{1}_{\left\{r^{(n)}\left(S_{k-1}^{(n)}\right) > \frac{\eta}{2}\right\}} + \left[r^{(n)}\left(S_{k-1}^{(n)}\right)\right]^{-2} \mathbb{1}_{\left\{\left(p^{(n)}\left(S_{k-1}^{(n)}\right)\right)^+ > \frac{\eta}{2}\right\}} \\
&\le \frac{4}{\eta^2} + \left[r^{(n)}\left(S_{k-1}^{(n)}\right)\right]^{-2} \mathbb{1}_{\left\{\left[r^{(n)}\left(S_{k-1}^{(n)}\right)\right]^2 > \Delta x^{(n)} \frac{\eta}{2}\right\}} \\
&\le \frac{4}{\eta^2} + \frac{2}{\Delta x^{(n)}\eta} \quad \text{a.s.}
\end{aligned}$$

Therefore, there exists a deterministic sequence (c_n) converging to zero such that for all $k = 1, \ldots, T_n$,

$$\begin{aligned}
\left|\delta Z_k^{(n)}\right|^2 &= \frac{\left[\Delta x^{(n)}\left(\mathbb{1}_B\left(\phi_k^{(n)}\right) - \mathbb{1}_A\left(\phi_k^{(n)}\right)\right) - \Delta t^{(n)} p^{(n)}\left(S_{k-1}^{(n)}\right)\right]^2}{\left[r^{(n)}\left(S_{k-1}^{(n)}\right)\right]^2} \\
&\le 2\left[\left(\Delta x^{(n)}\right)^2 + \left(\Delta t^{(n)} p^{(n)}\left(S_{k-1}^{(n)}\right)\right)^2\right] \frac{2}{\eta}\left(\frac{2}{\eta} + \frac{1}{\Delta x^{(n)}}\right) \\
&\le c_n \quad \text{a.s.}
\end{aligned} \tag{14}$$

We conclude that for all $\varepsilon > 0$,

$$\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \mathbb{E}\left(\left|\delta Z_k^{(n)}\right|^2 \mathbb{1}_{\left\{\left|\delta Z_k^{(n)}\right| > \varepsilon\right\}}\right) \le \frac{c_n}{\varepsilon^2} \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \mathbb{E}\left|\delta Z_k^{(n)}\right|^2 \le \frac{t}{\varepsilon^2} \cdot c_n \to 0,$$

i.e. the Lindeberg condition is satisfied. Therefore, the functional central limit theorem for martingale difference arrays (cf. Theorem 18.2 in [3]) implies that $Z^{(n)}$ converges weakly to a standard Brownian motion. □

In order to obtain the convergence of the full price process in Section 5 we also have to assume that the drift and volatility functions $p^{(n)}$ and $r^{(n)}$, $n \in \mathbb{N}$, satisfy a continuity condition and that they converge to some functions p and r as $n \to \infty$.

Assumption 2.6.

(i) There exist functions $p : E_{loc} \to \mathbb{R}$, $r : E_{loc} \to \mathbb{R}_+$, and $C < \infty$ such that for all $s = (b, v), \widetilde{s} = (\widetilde{b}, \widetilde{v}) \in E_{loc}$,

$$|p(s)| + r(s) \le C(1 + |b|)$$

and for all $m \in \mathbb{N}$,

$$\sup_{s=(b,v)\in E_{loc}} \left| p^{(n)}((b \wedge m, v)) - p((b \wedge m, v)) \right| + \left| r^{(n)}((b \wedge m, v)) - r((b \wedge m, v)) \right| \to 0.$$

(ii) There exists $L < \infty$ such that for all $n \in \mathbb{N}$ and $s = (b, v), \widetilde{s} = (\widetilde{b}, \widetilde{v}) \in E_{loc}$,

$$\begin{aligned} \max & \left\{ \left| p^{(n)}(s) - p^{(n)}(\widetilde{s}) \right|, \left| r^{(n)}(s) - r^{(n)}(\widetilde{s}) \right| \right\} \\ & \le L \left(1 + |b| + |\widetilde{b}|\right) \left(1 + \left\| v \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} + \left\| \widetilde{v} \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2}\right) \left\{ \left| b - \widetilde{b} \right| \right. \\ & \quad \left. + \left\| (v - \widetilde{v}) \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} \right\}. \end{aligned}$$

Assumption 2.6(ii) is similar to a local Lipschitz assumption. It will play a key role in the proof of the main theorem later on. The following example illustrates the assumed dependence structure.

Example 2.7. In order to model dependence on standing volumes we can integrate a Lipschitz continuous function $h : \mathbb{R} \to \mathbb{R}$ against cumulated volumes standing to the left of the price process. If we suppose that h has compact support in $\mathbb{R}_-$, then for all $s = (b, v), \widetilde{s} = (\widetilde{b}, \widetilde{v}) \in E_{loc}$,

$$\begin{aligned} & \left| \left\langle v(\cdot + b) \mathbb{1}_{[-b,0]}, h \right\rangle - \left\langle \widetilde{v}\left(\cdot + \widetilde{b}\right) \mathbb{1}_{[-\widetilde{b},0]}, h \right\rangle \right| = \left| \left\langle v, h(\cdot - b) \mathbb{1}_{[0,b]} \right\rangle - \left\langle \widetilde{v}, h\left(\cdot - \widetilde{b}\right) \mathbb{1}_{[0,\widetilde{b}]} \right\rangle \right| \\ & \le \left| \left\langle v - \widetilde{v}, h\left(\cdot - \widetilde{b}\right) \mathbb{1}_{[0,\widetilde{b}]} \right\rangle \right| + \left| \left\langle v \mathbb{1}_{[0, b \vee \widetilde{b}]}, h(\cdot - b) - h\left(\cdot - \widetilde{b}\right) \right\rangle \right| \\ & \le \|h\|_{L^2} \cdot \left\| (v - \widetilde{v}) \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} + \left\| v \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} \cdot L \left\| \mathbb{1}_{[0, b \vee \widetilde{b}]} \left(b - \widetilde{b}\right) \right\|_{L^2} \\ & \le \|h\|_{L^2} \cdot \left\| (v - \widetilde{v}) \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} + L \left\| v \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} \left(1 + |b| + |\widetilde{b}|\right) \left| b - \widetilde{b} \right|. \end{aligned}$$

Now if P, R are Lipschitz continuous functions, we may define for all $s = (b, v) \in E_{loc}$,

$$p^{(n)}(s) := P\left(\left\langle v(\cdot + b) \mathbb{1}_{[-b,0]}, h \right\rangle \right), \qquad r^{(n)}(s) := R\left(\left\langle v(\cdot + b) \mathbb{1}_{[-b,0]}, h \right\rangle \right)$$

and the so defined functions $p^{(n)}$ and $r^{(n)}$ satisfy Assumption 2.6(ii).

3. Fluctuations of the volume process

In this section we analyse the fluctuation of the infinite dimensional volume process $V^{(n)}$. In a first step we compute its conditional moments and prove their convergence as $n \to \infty$. Subsequently, we represent it as the solution to a stochastic differential equations driven by infinite dimensional martingale that converges in distribution to a cylindrical Brownian motion as $n \to \infty$. Since $V^{(n)}$ is not an L^2-valued process, but only L^2_{loc}-valued, we need to localize the analysis.

We make the following assumption on the joint distribution of the random variables $\omega_k^{(n)}$ and $\pi_k^{(n)}$.

Assumption 3.1. There exists an $M > 0$ such that for all $n \in \mathbb{N}$ and $k \le T_n$,

$$\mathbb{P}\left(\omega_k^{(n)} \in [-M, M],\ \pi_k^{(n)} \in [0, \infty) \right) = 1. \tag{15}$$

For every $n \in \mathbb{N}$ there exist two measurable functions $g^{(n)},\ h^{(n)} : E_{loc} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $k = 1, \dots, T_n$ and all $D \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbb{E}\left(\left(\omega_k^{(n)}\right)^2 \mathbb{1}_C\left(\phi_k^{(n)}\right) \mathbb{1}_D\left(\pi_k^{(n)}\right) \,\middle|\, \mathcal{F}_{k-1}^{(n)} \right) = \int_D g^{(n)}\left(S_{k-1}^{(n)}; y\right) dy \quad \text{a.s.}$$

and

$$\mathbb{E}\left(\omega_k^{(n)} \mathbb{1}_C\left(\phi_k^{(n)}\right) \mathbb{1}_D\left(\pi_k^{(n)}\right) \,\middle|\, \mathcal{F}_{k-1}^{(n)} \right) = \Delta v^{(n)} \int_D h^{(n)}\left(S_{k-1}^{(n)}; y\right) dy \quad \text{a.s.}$$

Remark 3.2. The functions $\Delta v^{(n)} h^{(n)}(\cdot)$ describe the average net arrivals (placements minus cancellations) of limit orders at a particular price interval as a function of the state of the book. Estimating arrival intensities of limit orders and cancellations at different price levels conditionally on the spread, volumes at the top, etc. yields point estimates $h_i^{(n)}(\cdot)$ for different price levels. The function $h^{(n)}$ can then be derived from these estimates through any 'smoothing procedure' as e.g. in [11]. The same applies to the functions $g^{(n)}(\cdot)$ that describe the (state-dependent) second moments of volume changes across different price levels.

According to Assumptions 2.1 and 3.1 the process $\left(S_k^{(n)}\right)_{k=0,\dots,T_n}$ is a homogeneous Markov chain for each $n \in \mathbb{N}$. Furthermore, (15) and Assumption 2.3 imply that for all $m > 0$, $n \in \mathbb{N}$, and $k \le T_n$,

$$\begin{aligned}\left\| \delta V_k^{(n)} \mathbb{1}_{[0,m]} \right\|_{L^2}^2 &\le \left(\Delta v^{(n)}\right)^2 M^2 \left\| \sum_{j=0}^{\lfloor \cdot/\Delta x^{(n)} \rfloor} \mathbb{1}_{I^{(n)}\left(\pi_k^{(n)}\right)}\left(x_j^{(n)}\right) \mathbb{1}_{[0,m]} \right\|_{L^2}^2 \\ &\le \Delta t^{(n)} M^2 m \quad \text{a.s.}\end{aligned}$$

and therefore for all $m > 0$ also

$$\begin{aligned}\left\| \delta \overline{v}_k^{(n)} \mathbb{1}_{[0,m]} \right\|_{L^2}^2 &\le \left\| \delta \hat{v}_k^{(n)} \mathbb{1}_{[0,m]} \right\|_{L^2}^2 + \left\| \delta V_k^{(n)} \mathbb{1}_{[0,m]} \right\|_{L^2}^2 \\ &\le \mathbb{E}\left(\left\| \delta V_k^{(n)} \mathbb{1}_{[0,m]} \right\|_{L^2}^2 \,\middle|\, \mathcal{F}_{k-1}^{(n)} \right) + \left\| \delta V_k^{(n)} \mathbb{1}_{[0,m]} \right\|_{L^2}^2 \\ &\le 2M^2 m \Delta t^{(n)} \quad \text{a.s.}\end{aligned} \tag{16}$$

The next two assumptions deal with the convergence and continuity of $g^{(n)}$ and $h^{(n)}$.

Assumption 3.3.

(i) There exists a measurable function $g : E_{loc} \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\inf_{s \in E_{loc}} g(s; y) > 0 \quad \forall\, y \in \mathbb{R}_+$$

such that

$$\sup_{s \in E_{loc}} \int_0^\infty \left| g^{(n)}(s; y) - g(s; y) \right| dy \to 0.$$

(ii) There exists an $L < \infty$ such that for all $n \in \mathbb{N}$ and $s = (b, v), \widetilde{s} = (\widetilde{b}, \widetilde{v}) \in E_{loc}$,

$$\begin{aligned}&\int_0^\infty \left| g^{(n)}(s; y) - g^{(n)}(\widetilde{s}; y) \right| dy \\ &\quad \le L\left(1 + |b| + |\widetilde{b}|\right)\left(1 + \left\| v \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} + \left\| \widetilde{v} \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2}\right) \left\{ \left| b - \widetilde{b} \right| \right. \\ &\quad\quad \left. + \left\| \left(v - \widetilde{v}\right) \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} \right\}.\end{aligned}$$

The next assumption is key to the derivation of a diffusion limit for the L^2_{loc}-valued functions $V^{(n)}$. It states that order placements and cancellations are expected to be approximately of the same size and that the expected disbalance between both also scales in n. This guarantees that the cumulated volume process will not explode when passing to the scaling limit.

Assumption 3.4.

(i) There exists a measurable function $h : E_{loc} \times \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\sup_{s \in E_{loc}} \int_0^\infty |h(s; y)|^2 \, dy < \infty$$

such that

$$\sup_{s \in E_{loc}} \int_0^\infty \left| h^{(n)}(s; y) - h(s; y) \right|^2 dy \to 0.$$

(ii) There exists an $L < \infty$ such that for all $n \in \mathbb{N}$ and $s = (b, v), \widetilde{s} = \left(\widetilde{b}, \widetilde{v}\right) \in E_{loc}$,

$$\begin{aligned}\left(\int_0^\infty \left| h^{(n)}(s; y) - h^{(n)}(\widetilde{s}; y) \right|^2 dy \right)^{1/2} & \\ \leq L \left(1 + |b| + |\widetilde{b}|\right) & \left(1 + \left\| v \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} + \left\| \widetilde{v} \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2}\right) \left\{ \left| b - \widetilde{b} \right| \right. \\ & \left. + \left\| (v - \widetilde{v}) \, \mathbb{1}_{[0, b \vee \widetilde{b}]} \right\|_{L^2} \right\}.\end{aligned}$$

3.1. Basis functions

Our goal is to represent the volume function as a stochastic differential equation driven by an infinite dimensional martingale whose increments are orthogonal across different basis functions of $L^2(\mathbb{R}_+; \mathbb{R})$. We choose the Haar basis, i.e. we specify the basis functions (f_i) as follows: for each $k \in \mathbb{N}_0$ we set $g^k_{-1}(x) = \mathbb{1}_{[k,k+1)}(x)$. Moreover, we set for all $k, l \in \mathbb{N}_0$,

$$g^k_l(x) := \begin{cases} 2^{l/2} & : \; x \in \left[k 2^{-l}, \left(k + \frac{1}{2}\right) 2^{-l}\right) \\ -2^{l/2} & : \; x \in \left[\left(k + \frac{1}{2}\right) 2^{-l}, (k+1) 2^{-l}\right) \\ 0 & : \; \text{else.} \end{cases}$$

To define the (f_i) we now reorder the (g^k_l) in a diagonal procedure:

$$f_1 := g^0_{-1}, \; f_2 := g^1_{-1}, \; f_3 := g^0_0, \; f_4 := g^2_{-1}, \; f_5 := g^1_0, \; f_6 := g^0_1, \; \ldots$$

In the following we denote by $k(i) \in \mathbb{N}_0$ and $l(i) \in \mathbb{N}_{-1} := \mathbb{N}_0 \cup \{-1\}$ the indices such that $f_i \equiv g^{k(i)}_{l(i)}$.

Let us define for each $i \in \mathbb{N}$ the functions $F_i : \mathbb{R}_+ \to \mathbb{R}$ and $F^{(n)}_i : \mathbb{R}_+ \to \mathbb{R}, \; n \in \mathbb{N}$, via

$$F_i(y) := \int_y^\infty f_i(x) dx, \qquad F^{(n)}_i(y) := \int_{\Delta x^{(n)} \lfloor y / \Delta x^{(n)} \rfloor}^\infty f_i(x) dx.$$

We shall see that the drift and the volatility of the volume processes can be expressed in terms of the functions F_i and $F^{(n)}_i$. We notice that $|F_i(y)| \vee \left| F^{(n)}_i(y) \right| \leq 1$ for all $y \in \mathbb{R}_+$ and $i, n \in \mathbb{N}$. In addition, we will often use the fact that if $l(i) \geq 0$, then

$$supp(F_i) = \left[k(i) 2^{-l(i)}, (k(i) + 1) 2^{-l(i)}\right],$$

i.e. $|supp(F_i)| \leq 1$. Similarly, also $\left|supp\left(F_i^{(n)}\right)\right| \leq 1$ for all $i, n \in \mathbb{N}$ with $l(i) \geq 0$. We also notice that if $l(i) = -1$, then $supp(F_i) = supp\left(F_i^{(n)}\right) = [0, k(i) + 1]$. Moreover, we have the L^2_{loc}-representation

$$\mathbb{1}_{[y,\infty)}(x) = \sum_i F_i(y) f_i(x), \qquad \mathbb{1}_{\left[\Delta x^{(n)} \lfloor y/\Delta x^{(n)} \rfloor, \infty\right)}(x) = \sum_i F_i^{(n)}(y) f_i(x).$$

Finally, for all $m \in \mathbb{N}$ we define the index set

$$\mathcal{I}_m := \{i \in \mathbb{N} : \; supp(f_i) \cap (0, m) \neq \emptyset\}. \tag{17}$$

Note that for all $m \in \mathbb{N}$, $(f_i)_{i \in \mathcal{I}_m}$ is a basis of $L^2([0, m])$. Furthermore, for all $n, m \in \mathbb{N}$ and $y \in \mathbb{R}_+$,

$$\sum_{i \in \mathcal{I}_m} \left[F_i^{(n)}(y)\right]^2 \leq m \qquad \text{and} \qquad \sum_{i \in \mathcal{I}_m} [F_i(y)]^2 \leq m.$$

We shall repeatedly use the following technical lemma. It allows us to approximate the conditional moments of volume increments using finitely many basis functions after localization.

Lemma 3.5. *For each $\varepsilon > 0$ and $m \in \mathbb{N}$ there exists a finite subset $J \subset \mathcal{I}_m$ such that for all $y \in \mathbb{R}_+$,*

$$\sum_{i \in \mathcal{I}_m \setminus J} (F_i(y))^2 \leq \varepsilon \qquad \textit{and} \qquad \sum_{i \in \mathcal{I}_m \setminus J} \left(F_i^{(n)}(y)\right)^2 \leq \varepsilon \quad \forall\, n \in \mathbb{N}.$$

Proof. For fixed $\varepsilon > 0$ and $m \in \mathbb{N}$ set $l_0 := \min\left\{l \in \mathbb{N} : \; 2^{-l} \leq \varepsilon\right\}$ and $J := \{i \in \mathcal{I}_m : \; l(i) \leq l_0\}$. Now note that for all $i \in \mathbb{N}$,

$$|F_i(y)| = \left|\int_y^\infty f_i(x)dx\right| \leq 2^{-l(i)/2} \quad \forall\, y \in \mathbb{R}_+.$$

Furthermore for every $l \in \mathbb{N}$ and $y \in \mathbb{R}_+$ there exists exactly one $i \in \mathbb{N}$ with $l(i) = l$ such that $F_i(y) \neq 0$. Therefore,

$$\sum_{i \in \mathcal{I}_m \setminus J} (F_i(y))^2 \leq \sum_{l > l_0} 2^{-l(i)} = 2^{-l_0} \leq \varepsilon \quad \forall\, y \in \mathbb{R}_+.$$

Since this is true for all $y \in \mathbb{R}_+$, it is also true for all $\Delta x^{(n)} \lfloor y/\Delta x^{(n)} \rfloor$ with $n \in \mathbb{N}$ and $y \in \mathbb{R}_+$. Hence,

$$\sum_{i \in \mathcal{I}_m \setminus J} \left(F_i^{(n)}(y)\right)^2 \leq \varepsilon \quad \forall\, y \in \mathbb{R}_+,\; n \in \mathbb{N}. \quad \square$$

3.2. Convergence of drift, volatility and correlation functions

We are now going to analyse the convergence of the conditional expectations and variances of the volume increments. It will turn out that in the limit they can be described in terms of the functions $\mu_i : E_{loc} \to \mathbb{R}$ and $\sigma_i : E_{loc} \to \mathbb{R}_+$ ($i \in \mathbb{N}$) defined by:

$$\mu_i(s) := \int_0^\infty h(s; y) F_i(y) dy, \qquad (\sigma_i(s))^2 := \int_0^\infty g(s; y) \left[F_i(y)\right]^2 dy.$$

Lemma 3.6. *Given Assumption 3.3(i) we have for all $i \in \mathbb{N}$, $\inf_{s\in E_{loc}} \sigma_i(s) > 0$.*

Proof. By definition $F_i(y) \neq 0$ for all $y \in \left(k(i)2^{-l(i)}, (k(i)+1)2^{-l(i)}\right)$. Thus, the claim follows from the fact that $g(\cdot; y)$ is bounded away from zero for each $y \in \mathbb{R}_+$ according to Assumption 3.3(i). □

In view of the preceding lemma we can define for all $i, j \in \mathbb{N}$ the function $\rho_{ij} : E_{loc} \to [-1, 1]$ via

$$\sigma_i(s)\sigma_j(s)\rho_{ij}(s) := \int_0^\infty g(s; y)F_i(y)F_j(y)dy.$$

Moreover, we define for each $n, i, j \in \mathbb{N}$ the following functions from E_{loc} to $\mathbb{R}$,

$$\begin{aligned}
\mu_i^{(n)}(s) &:= \int_0^\infty h^{(n)}(s; y)F_i^{(n)}(y)dy,\\
\sigma_i^{(n)}(s) &:= \left(\int_0^\infty g^{(n)}(s; y)\left[F_i^{(n)}(y)\right]^2 dy - \Delta t^{(n)}\left(\mu_i^{(n)}(s)\right)^2\right)^{1/2},\\
\rho_{ij}^{(n)}(s) &:= \frac{\mathbb{1}_{(0,\infty)}\left(\sigma_i^{(n)}(s)\sigma_j^{(n)}(s)\right)}{\sigma_i^{(n)}(s)\sigma_j^{(n)}(s)}\\
&\quad\times\left(\int_0^\infty g^{(n)}(s; y)F_i^{(n)}(y)F_j^{(n)}(y)dy - \Delta t^{(n)}\mu_i^{(n)}(s)\mu_j^{(n)}(s)\right),
\end{aligned}$$

and the $L^2_{loc}(\mathbb{R}_+)$-valued functions

$$\mu^{(n)}(s; \cdot) := \sum_i \mu_i^{(n)}(s) f_i(\cdot) \qquad \text{and} \qquad \mu(s; \cdot) := \sum_i \mu_i(s) f_i(\cdot).$$

Note that with this notation we have for all $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$, making use of Assumption 2.3,

$$\begin{aligned}
\delta\hat{v}_k^{(n)}(x) &= \Delta v^{(n)}\mathbb{E}\left(\Delta x^{(n)} \sum_{j=0}^{\lfloor x/\Delta x^{(n)}\rfloor} M_k^{(n)}\left(x_j^{(n)}\right)\middle| \mathcal{F}_{k-1}^{(n)}\right)\\
&= \Delta v^{(n)} \int_0^\infty \Delta v^{(n)} h^{(n)}\left(S_{k-1}^{(n)}; y\right) \sum_{j=0}^{\lfloor x/\Delta x^{(n)}\rfloor} \mathbb{1}_{I^{(n)}(y)}\left(x_j^{(n)}\right) dy \qquad (18)\\
&= \Delta t^{(n)} \int_0^\infty h^{(n)}\left(S_{k-1}^{(n)}; y\right) \mathbb{1}_{[\Delta x^{(n)}\lfloor y/\Delta x^{(n)}\rfloor,\infty)}(x)dy = \Delta t^{(n)}\mu^{(n)}\left(S_{k-1}^{(n)}; x\right)
\end{aligned}$$

as well as

$$\begin{aligned}
&\mathbb{E}\left(\left\langle \delta V_k^{(n)}, f_i\right\rangle^2 \middle| \mathcal{F}_{k-1}^{(n)}\right)\\
&= \Delta t^{(n)}\mathbb{E}\left(\mathbb{1}_C\left(\phi_k^{(n)}\right)\left[\omega_k^{(n)}\int_{\mathbb{R}_+} f_i(x) \sum_{j=0}^{\lfloor x/\Delta x^{(n)}\rfloor} \mathbb{1}_{I^{(n)}\left(\pi_k^{(n)}\right)}\left(x_j^{(n)}\right) dx\right]^2 \middle| \mathcal{F}_{k-1}^{(n)}\right)
\end{aligned}$$

$$\begin{aligned}
&= \Delta t^{(n)} \mathbb{E}\left(\mathbb{1}_C \left(\phi_k^{(n)}\right) \left(\omega_k^{(n)}\right)^2 \left[F_i^{(n)} \left(\pi_k^{(n)}\right)\right]^2 \middle| \mathcal{F}_{k-1}^{(n)} \right) \\
&= \Delta t^{(n)} \int_0^\infty g^{(n)} \left(S_{k-1}^{(n)}; y\right) \left[F_i^{(n)}(y)\right]^2 dy \\
&= \Delta t^{(n)} \left[\left(\sigma_i^{(n)}\left(S_{k-1}^{(n)}\right)\right)^2 + \Delta t^{(n)} \left(\mu_i^{(n)}\left(S_{k-1}^{(n)}\right)\right)^2 \right],
\end{aligned}$$

i.e.

$$\mathbb{E}\left(\left\langle \delta \overline{v}_k^{(n)}, f_i \right\rangle^2 \middle| \mathcal{F}_{k-1}^{(n)} \right) = \Delta t^{(n)} \left(\sigma_i^{(n)}\left(S_{k-1}^{(n)}\right)\right)^2. \tag{19}$$

Similar calculations show that

$$\rho_{ij}^{(n)}\left(S_{k-1}^{(n)}\right) = \frac{\mathbb{E}\left(\left\langle \delta \overline{v}_k^{(n)}, f_i \right\rangle \left\langle \delta \overline{v}_k^{(n)}, f_j \right\rangle \middle| \mathcal{F}_{k-1}^{(n)} \right)}{\sigma_i^{(n)}\left(S_{k-1}^{(n)}\right) \sigma_j^{(n)}\left(S_{k-1}^{(n)}\right)} \mathbb{1}_{(0,\infty)} \left(\sigma_i^{(n)}\left(S_{k-1}^{(n)}\right) \sigma_j^{(n)}\left(S_{k-1}^{(n)}\right)\right). \tag{20}$$

The next three lemmata establish the convergence of the drift, the volatility and the covariance functions introduced above.

Lemma 3.7. *Given Assumption 3.4(i) we have for all $m \in \mathbb{N}$,*

$$\sup_{s \in E_{loc}} \left\| \mu(s) \mathbb{1}_{[0,m]} \right\|_{L^2} < \infty \qquad \textit{and} \qquad \sup_{s \in E_{loc}} \left\| \left(\mu^{(n)}(s) - \mu(s)\right) \mathbb{1}_{[0,m]} \right\|_{L^2} \to 0.$$

Proof. Since $supp(F_i) \subset [0, m]$ for all $i \in \mathcal{I}_m$,

$$\begin{aligned}
\sup_{s \in E_{loc}} \left\| \mu(s) \mathbb{1}_{[0,m]} \right\|_{L^2}^2 &= \sup_{s \in E_{loc}} \sum_{i \in \mathcal{I}_m} \left(\int_0^\infty h(s; y) F_i(y) dy \right)^2 \\
&\leq \sup_{s \in E_{loc}} m \sum_{i \in \mathcal{I}_m} \int_0^m (h(s; y))^2 (F_i(y))^2 dy \\
&\leq m^2 \cdot \sup_{s \in E_{loc}} \int_0^\infty (h(s; y))^2 dy < \infty.
\end{aligned}$$

By a similar reasoning we can estimate for all $m \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{s \in E_{loc}} \sum_{i \in \mathcal{I}_m} \left(\int_0^\infty h(s; y) \left(F_i^{(n)}(y) - F_i(y)\right) dy \right)^2 \\
&\leq \sup_{s \in E_{loc}} m \sum_{i \in \mathcal{I}_m} \int_0^\infty (h(s; y))^2 \left(F_i^{(n)}(y) - F_i(y)\right)^2 dy \\
&= m \cdot \sup_{s \in E_{loc}} \int_0^\infty (h(s; y))^2 \left\| \mathbb{1}_{[\Delta x^{(n)} \lfloor y/\Delta x^{(n)} \rfloor, y]}(\cdot) \right\|_{L^2}^2 dy \\
&\leq m \Delta x^{(n)} \sup_{s \in E_{loc}} \int_0^\infty (h(s; y))^2 dy \to 0
\end{aligned}$$

and by Assumption 3.4(i) also

$$\sup_{s\in E_{loc}}\sum_{i\in\mathcal{I}_m}\left(\int_0^\infty\left(h^{(n)}(s;y)-h(s;y)\right)F_i^{(n)}(y)dy\right)^2$$
$$\leq \sup_{s\in E_{loc}} m\sum_{i\in\mathcal{I}_m}\int_0^\infty\left(h^{(n)}(s;y)-h(s;y)\right)^2\left(F_i^{(n)}(y)\right)^2dy$$
$$\leq m^2\sup_{s\in E_{loc}}\int_0^\infty\left(h^{(n)}(s;y)-h(s;y)\right)^2dy\to 0.\quad\square$$

Lemma 3.8. *Given Assumptions 3.1, 3.3(i) and 3.4(i) we have for all $m\in\mathbb{N}$,*

$$\sup_{s\in E_{loc}}\sum_{i\in\mathcal{I}_m}\left(\sigma_i^{(n)}(s)\right)^2\leq mM^2\qquad\textit{and}\qquad\sup_{s\in E_{loc}}\sum_{i\in\mathcal{I}_m}\left|\sigma_i^{(n)}(s)-\sigma_i(s)\right|^2\to 0.$$

Proof. First, it follows from Assumption 3.1 and Eq. (19) that for all $m\in\mathbb{N}$ and $s\in E_{loc}$,

$$\sum_{i\in\mathcal{I}_m}(\sigma_i(s))^2\leq\sum_{i\in\mathcal{I}_m}\int_0^\infty g(s;y)\,[F_i(y)]^2\,dy\leq mM^2.$$

Second, by Assumption 3.3(i) for all $m\in\mathbb{N}$,

$$\sup_{s\in E_{loc}}\sum_{i\in\mathcal{I}_m}\left|\int_0^\infty\left(g^{(n)}(s;y)-g(s;y)\right)[F_i(y)]^2\,dy\right|$$
$$\leq m\cdot\sup_{s\in E_{loc}}\int_0^\infty\left|g^{(n)}(s;y)-g(s;y)\right|dy\to 0$$

and it follows from Lemma 3.7 that for all $m\in\mathbb{N}$,

$$\Delta t^{(n)}\sup_{s\in E}\sum_{i\in\mathcal{I}_m}\left(\mu_i^{(n)}(s)\right)^2\to 0.$$

Next fix $m\in\mathbb{N}$ and let $\varepsilon>0$. By Lemma 3.5 we find a finite subset $J\subset\mathcal{I}_m$ such that for all $n\in\mathbb{N}$ and $y\in\mathbb{R}_+$,

$$\sum_{i\in\mathcal{I}_m\setminus J}\left[F_i^{(n)}(y)\right]^2\leq\frac{\varepsilon}{4M^2}\qquad\text{and}\qquad\sum_{i\in\mathcal{I}_m\setminus J}[F_i(y)]^2\leq\frac{\varepsilon}{4M^2}.$$

Now we choose $n_0=n_0(\varepsilon,m)$ such that for all $i\in\mathbb{N}$, $y\in\mathbb{R}_+$, and $n\geq n_0$,

$$\left|\left[F_i^{(n)}(y)\right]^2-[F_i(y)]^2\right|\leq 2\left|F_i^{(n)}(y)-F_i(y)\right|$$
$$\leq 2\left\|\mathbb{1}_{[\Delta x^{(n)}\lfloor y/\Delta x^{(n)}\rfloor,y]}\right\|_{L^2}\leq 2\left(\Delta x^{(n)}\right)^{1/2}<\frac{\varepsilon}{2M^2|J|}.$$

We deduce that for all $n\geq n_0$ and $s\in E_{loc}$,

$$\sum_{i\in\mathcal{I}_m}\left|\int_0^\infty g^{(n)}(s;y)\left(\left[F_i^{(n)}(y)\right]^2-[F_i(y)]^2\right)dy\right|$$
$$\leq\frac{\varepsilon}{2M^2}\int_0^\infty g^{(n)}(s;y)dy+\int_0^\infty g^{(n)}(s;y)\sum_{i\in J}\left|\left[F_i^{(n)}(y)\right]^2-[F_i(y)]^2\right|dy$$
$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2M^2}\int_0^\infty g^{(n)}(s;y)dy\leq\varepsilon.$$

Therefore, we have,

$$\begin{aligned}&\lim_{n\to\infty}\sup_{s\in E_{loc}}\sum_{i\in\mathcal{I}_m}\left|\sigma_i^{(n)}(s)-\sigma_i(s)\right|^2\\ \le&\lim_{n\to\infty}\sup_{s\in E_{loc}}\sum_{i\in\mathcal{I}_m}\left|\left(\sigma_i^{(n)}(s)\right)^2-(\sigma_i(s))^2\right|\\ \le&\lim_{n\to\infty}\sup_{s\in E_{loc}}\sum_{i\in\mathcal{I}_m}\left|\left(\sigma_i^{(n)}(s)\right)^2+\Delta t^{(n)}\left(\mu_i^{(n)}(s)\right)^2-(\sigma_i(s))^2\right|\\ &+\lim_{n\to\infty}\Delta t^{(n)}\sup_{s\in E_{loc}}\sum_{i\in\mathcal{I}_m}\left(\mu_i^{(n)}(s)\right)^2=0.\quad\square\end{aligned}$$

Lemma 3.9. *Given Assumptions 2.3, 3.1, 3.3(i) and 3.4(i) we have for all $i, j \in \mathbb{N}$,*

$$\sup_{s\in E_{loc}}\left|\rho_{ij}^{(n)}(s)-\rho_{ij}(s)\right|\to 0.$$

Proof. One can show similarly to the proof of Lemma 3.8 that for every fixed $i, j \in \mathbb{N}$,

$$\rho_{ij}^{(n)}(s)\sigma_i^{(n)}(s)^{(n)}\sigma_j(s)=\int_0^\infty g^{(n)}(s;y)F_i^{(n)}(y)F_j^{(n)}(y)dy-\Delta t^{(n)}\mu_i^{(n)}(s)\mu_j^{(n)}(s)$$

converges to $\rho_{ij}(s)\sigma_i(s)\sigma_j(s)$ uniformly in $s \in E_{loc}$. Since $\sigma_i^{(n)}$ and $\sigma_j^{(n)}$ converge to σ_i respectively σ_j uniformly by Lemma 3.8 and since both, σ_i and σ_j are uniformly bounded from below by Lemma 3.6, the claim follows. $\square$

3.3. Orthogonal decomposition

In order to identify the volume as the solution of some stochastic differential equation we need to decorrelate the normalized volume increments. To this end, we introduce in this subsection an orthogonal decomposition of the increments using the algorithm from Appendix A. We assume that the probability spaces are rich enough to support i.i.d. Bernoulli random variables.

Assumption 3.10. For every $n \in \mathbb{N}$ there exists a field of i.i.d. random variables $\left(U_k^{(n),i}\right)_{k,i\in\mathbb{N}}$ on $\left(\Omega^{(n)},\mathcal{F}^{(n)},\mathbb{P}^{(n)}\right)$, which are independent of $S^{(n)}$, such that

$$\mathbb{P}\left(U_k^{(n),i}=-1\right)=\mathbb{P}\left(U_k^{(n),i}=1\right)=\frac{1}{2}.$$

In (20) we introduced the correlation coefficients $\rho_{ij}^{(n)}(\cdot)$, $n, i, j \in \mathbb{N}$, $j \le i$. Now if we define for any $n, i \in \mathbb{N}$ and $k \le T_n$ the normalized random variables

$$Z_k^{(n),i}:=\begin{cases}\frac{\left\langle\delta\overline{v}_k^{(n)},f_i\right\rangle}{\sigma_i^{(n)}\left(S_{k-1}^{(n)}\right)} & :\ \sigma_i^{(n)}\left(S_{k-1}^{(n)}\right)>0\\ \left(\Delta t^{(n)}\right)^{1/2}U_k^{(n),i} & :\ \sigma_i^{(n)}\left(S_{k-1}^{(n)}\right)=0,\end{cases}$$

then by construction the conditional covariance between $Z_k^{(n),i}$ and $Z_k^{(n),j}$ is precisely $\rho_{ij}\left(S_{k-1}^{(n)}\right)$. Next we have to decorrelate the $Z_k^{(n),i}$, $i \in \mathbb{N}$, for all $k \le T_n$, so that we can express the volume process $V^{(n)}$ as a discrete stochastic integral, cf. Eq. (22). This is achieved by the algorithm in Appendix A, which provides for each $n \in \mathbb{N}$ an array $\left(c_{ij}^{(n)}(\cdot)\right)_{j\le i}$ of measurable functions from E_{loc} to $[-1,1]$ together with an "inverse" array $\left(\alpha_{ij}^{(n)}(\cdot)\right)_{j\leq i}$ in terms of the Borel measurable correlation coefficients $\left(\rho_{ij}^{(n)}(\cdot)\right)_{j,i}$, such that the following result holds true – being an immediate corollary of Lemma A.1 in Appendix A – for the random variables $\delta W_k^{(n),i}$, $i\in\mathbb{N}$, given for all $k\leq T_n$, $n\in\mathbb{N}$, by

$$\delta W_k^{(n),1} := Z_k^{(n),1}\left(S_{k-1}^{(n)}\right)$$

and for $i>1$,

$$\delta W_k^{(n),i} := \begin{cases} \frac{1}{c_{ii}^{(n)}\left(S_{k-1}^{(n)}\right)}\left(Z_k^{(n),i}\left(S_{k-1}^{(n)}\right) - \sum_{j<i} c_{ij}^{(n)}\left(S_{k-1}^{(n)}\right)\delta W_k^{(n),j}\right) & : c_{ii}^{(n)}\left(S_{k-1}^{(n)}\right) > 0 \\ \left(\Delta t^{(n)}\right)^{1/2} U_k^{(n),i} & : c_{ii}^{(n)}\left(S_{k-1}^{(n)}\right) = 0. \end{cases} \tag{21}$$

Corollary 3.11. *Let Assumptions 2.3, 3.1 and 3.10 be satisfied. Then for all $n,i\in\mathbb{N}$ and $k=1,\ldots,T_n$,*

$$Z_k^{(n),i}\left(S_{k-1}^{(n)}\right) = \sum_{j\leq i} c_{ij}^{(n)}\left(S_{k-1}^{(n)}\right)\delta W_k^{(n),j}$$

as well as

$$\mathbb{E}\left(Z_k^{(n),i}\delta W_k^{(n),j}\,\middle|\,\mathcal{F}_{k-1}^{(n)}\right) = \Delta t^{(n)} c_{ij}^{(n)}\left(S_{k-1}^{(n)}\right) \quad \text{and}$$
$$\mathbb{E}\left(\delta W_k^{(n),i}\delta W_k^{(n),j}\,\middle|\,\mathcal{F}_{k-1}^{(n)}\right) = \Delta t^{(n)}\delta_{ij}.$$

In order to see that the random variables $\delta W_k^{(n),i}$, $i\in\mathbb{N}$, allow us to represent the volume process as a stochastic integral, we define for all $i,j,n\in\mathbb{N}$ a function $d_{ij}^{(n)}: E_{loc}\to[-M,M]$ via

$$d_{ij}^{(n)}(s) := \begin{cases} \sigma_i^{(n)}(s)c_{ij}^{(n)}(s) & : j\leq i \\ 0 & : j>i. \end{cases}$$

Note that for each $m\in\mathbb{N}$, the matrix $\left(d_{ij}^{(n)}(s)\right)_{i,j\leq m}$ is the triangular matrix that one obtains from the Cholesky factorization of the covariance matrix $\left(\sigma_i^{(n)}(s)\sigma_j^{(n)}(s)\rho_{ij}^{(n)}(s)\right)_{i,j\leq m}$. Therefore, the functions $\left(d_{ij}^{(n)}(s)\right)_{i,j\in\mathbb{N}}$ will serve as the volatility operator in the stochastic equation representing $V^{(n)}$. Indeed, Eqs. (18), (19) and Corollary 3.11 imply that almost surely

$$\begin{aligned} V^{(n)}(t,x) &= V_0(x) + \sum_i f_i(x)\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\left\langle \delta V_k^{(n)}, f_i\right\rangle \\ &= V_0(x) + \sum_i f_i(x)\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\left[\mu_i^{(n)}\left(S_{k-1}^{(n)}\right)\Delta t^{(n)} + \sigma_i^{(n)}\left(S_{k-1}^{(n)}\right)\delta Z_k^{(n),i}\right] \\ &= V_0(x) + \sum_i f_i(x)\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\Bigg[\mu_i^{(n)}\left(S_{k-1}^{(n)}\right)\Delta t^{(n)} \\ &\qquad + \sigma_i^{(n)}\left(S_{k-1}^{(n)}\right)\sum_{j\leq i} c_{ij}^{(n)}\left(S_{k-1}^{(n)}\right)\delta W_k^{(n),j}\Bigg] \end{aligned} \tag{22}$$

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The convergence of the drift has already been established. In the following two subsections we prove the convergence of the volatility operator and the martingale driving the SDE.

3.4. Convergence of the volatility operator

In this section we prove convergence of the functions $c_{ij}^{(n)}(\cdot)$ and $d_{ij}^{(n)}(\cdot)$. As a byproduct we obtain a key estimate for the functions $\alpha_{ij}^{(n)}(\cdot)$. This estimate allows, for instance, to verify that the random variables $\delta W_k^{(n),i}$, $k \in \mathbb{N}$, satisfy the Lindeberg condition in the proof of Theorem 3.16.

Lemma 3.12. *Suppose that Assumptions 2.3, 3.1, 3.3(i) and 3.4(i) are satisfied. Then there exist for every $i \in \mathbb{N}$ and $j \leq i$ functions $c_{ij}, \alpha_{ij} : E_{loc} \to \mathbb{R}$ such that*

$$\sup_{s \in E_{loc}} \left| c_{ij}^{(n)}(s) - c_{ij}(s) \right| \to 0 \quad \text{and} \quad \sup_{s \in E_{loc}} \left| \alpha_{ij}^{(n)}(s) - \alpha_{ij}(s) \right| \to 0.$$

Moreover, for all $i \in \mathbb{N}$ and $j \leq i$,

$$\inf_{s \in E_{loc}} c_{ii}(s) > 0 \quad \text{and} \quad \sup_{s \in E_{loc}} \left| \alpha_{ij}(s) \right| < \infty.$$

Proof. The claim is proven by induction on i. Clearly, for $i = 1$ we have $c_{11} \equiv 1 \equiv \alpha_{11}$. Now assume the claim is true for all functions $c_{jl}^{(n)}, \alpha_{jl}^{(n)}$ with $l \leq j \leq i - 1$. Especially, this implies that for all $j < i$ and for n large enough we have $\inf_{s \in E_{loc}} c_{jj}^{(n)}(s) > 0$ and hence

$$c_{ij}^{(n)}(s) = \frac{1}{c_{jj}^{(n)}(s)} \left(\rho_{ij}^{(n)}(s) - \sum_{l<j} c_{il}^{(n)}(s) c_{jl}^{(n)}(s) \right).$$

By iterative reasoning from $j = 1$ to $j = i - 1$ we see that this term converges uniformly in $s \in E_{loc}$ to some function c_{ij} (defined via a similar recursion scheme) due to the induction hypothesis and Lemma 3.9. The same is then true for

$$c_{ii}^{(n)}(s) = \left(1 - \sum_{j<i} \left(c_{ij}^{(n)}(s) \right)^2 \right)^{1/2}.$$

Next we have to show that the limit satisfies $\inf_{s \in E_{loc}} c_{ii}(s) > 0$. First, note that by the induction hypothesis for large enough n, $c_{jj}^{(n)}(s) > 0$ for all $j < i$ and hence by Eq. (A.2),

$$\begin{aligned} Z^{(n),i}(s) - \sum_{j<i} c_{ij}^{(n)}(s)\, W^{(n),j}(s) &= Z^{(n),i}(s) - \sum_{j<i} c_{ij}^{(n)}(s) \sum_{l \leq j} \alpha_{jl}^{(n)}(s)\, Z^{(n),l}(s) \\ &= Z^{(n),i}(s) - \sum_{l<i} Z^{(n),l}(s) \sum_{l \leq j < i} c_{ij}^{(n)}(s)\, \alpha_{jl}^{(n)}(s). \end{aligned}$$

We set for all $l < i$,

$$\beta_l^{(n)}(s) := \begin{cases} \frac{-1}{\sigma_l^{(n)}(s)} \sum_{l \leq j < i} c_{ij}^{(n)}(s) \alpha_{jl}^{(n)}(s) & : \text{ if } \sigma_l^{(n)}(s) > 0 \\ 0 & : \text{ else} \end{cases}$$

as well as

$$\beta_i^{(n)}(s) := \begin{cases} \frac{1}{\sigma_i^{(n)}(s)} & : \text{ if } \sigma_i^{(n)}(s) > 0 \\ 0 & : \text{ else.} \end{cases}$$

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By the induction hypothesis, Lemmas 3.6 and 3.8 we know that for every $j \leq i$ there exists a bounded function $\beta_j : E \to \mathbb{R}$ such that

$$\sup_{s \in E_{loc}} \left| \beta_j^{(n)}(s) - \beta_j(s) \right| \to 0. \tag{23}$$

But for n large enough we have by definition for all $s \in E_{loc}$,

$$\begin{aligned} & c_{ii}^{(n)}(s) W^{(n),i}(s) \\ &= Z^{(n),i}(s) - \sum_{j<i} c_{ij}^{(n)}(s)\, W^{(n),j}(s) \\ &= \Delta v^{(n)} \left\langle X_1^{(n)}(s) \sum_{j=0}^{\lfloor \cdot / \Delta x^{(n)} \rfloor} \mathbb{1}_{I^{(n)}\left(X_2^{(n)}(s)\right)}\left(x_j^{(n)}\right), \sum_{l \leq i} \beta_l^{(n)}(s)\, f_l \right\rangle - \Delta t^{(n)} \sum_{l \leq i} \beta_l^{(n)}(s)\, \mu_l^{(n)}(s) \end{aligned}$$

and then also

$$\begin{aligned} \Delta t^{(n)} \left(c_{ii}^{(n)}(s) \right)^2 &= \mathbb{E} \left(c_{ii}^{(n)}(s) W^{(n),i}(s) \right)^2 \\ &= \Delta t^{(n)} \int_0^\infty g^{(n)}(s; y) \left[\sum_{l \leq i} \beta_l^{(n)}(s) F_l^{(n)}(y) \right]^2 dy - \left[\Delta t^{(n)} \sum_{l \leq i} \beta_l^{(n)}(s) \mu_l^{(n)}(s) \right]^2. \end{aligned}$$

Clearly, (23) implies that $\sup_{n\in\mathbb{N}} \sup_{s \in E_{loc}} \left| \beta_l^{(n)}(s) \right| =: C < \infty$ for all $l \leq i$. Hence, the last term on the right hand side in the above equation converges to zero uniformly in $s \in E_{loc}$ using that $\sup_{n\in\mathbb{N}} \sup_{s\in E_{loc}} \left| \mu_l^{(n)}(s) \right| < \infty$ for all $l \leq i$ by Lemma 3.7. Moreover,

$$\begin{aligned} & \sup_{s \in E_{loc}} \left| \int_0^\infty \left(g^{(n)}(s; y) - g(s; y) \right) \left[\sum_{l \leq i} \beta_l^{(n)}(s) F_l^{(n)}(y) \right]^2 dy \right| \\ & \leq C^2 i^2 \cdot \sup_{s \in E_{loc}} \int_0^\infty \left| g^{(n)}(s; y) - g(s; y) \right| dy \to 0 \end{aligned}$$

and by dominated convergence we deduce that, uniformly in $s \in E_{loc}$,

$$\int_0^\infty g(s; y) \left[\sum_{l \leq i} \beta_l^{(n)}(s) F_l^{(n)}(y) \right]^2 dy \to \int_0^\infty g(s; y) \left[\sum_{l \leq i} \beta_l(s) F_l(y) \right]^2 dy.$$

Therefore,

$$c_{ii}(s) = \int_0^\infty g(s; y) \left[\sum_{l \leq i} \beta_l(s) F_l(y) \right]^2 dy. \tag{24}$$

Now suppose that $\inf_{s \in E_{loc}} c_{ii}(s) = 0$. Since $g(\cdot; y)$ is bounded away from zero for all $y \in \mathbb{R}_+$ by Assumption 3.3(i), we deduce from (24) that there must exist an E_{loc}-valued sequence (s_n) such that

$$\sum_{l \leq i} \beta_l(s_n) F_l(y) \to 0 \quad \text{for almost all } y \in \mathbb{R}_+.$$

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Since $\sup_{s\in E_{loc}} |\beta_l(s)| < \infty$ for all $l \leq i$, this implies that there exists some vector $b \in \mathbb{R}^i$ such that

$$\sum_{l\leq i} b_l F_l(y) = 0 \quad \text{for almost all } y \in \mathbb{R}_+$$

and thus also

$$H(y) := \sum_{l\leq i} b_l f_l(y) = 0 \quad \text{for almost all } y \in \mathbb{R}_+.$$

However,

$$0 = \|H\|_{L^2}^2 = \sum_{l\leq i} b_l^2$$

implies that $b_l = 0$ for all $l \leq i$ and hence we must have $\beta_l(s_n) \to 0$ for all $l \leq i$. But for $l = i$ this gives a contradiction, since

$$\sup_{s\in E_{loc}} (\sigma_i(s))^2 = \sup_{s\in E_{loc}} \int_0^\infty g(s;y)\,[F_i(y)]^2\,dy \leq M^2 < \infty.$$

Hence, σ_i is bounded and thus β_i is bounded away from 0. This proves that $\inf_{s\in E_{loc}} c_{ii}(s) > 0$.

Now the convergence of the $\alpha_{ij}^{(n)}$, $j \leq i$, to some α_{ij} satisfying $\sup_{s\in E_{loc}} \left|\alpha_{ij}(s)\right| < \infty$ follows from the definition of the $\alpha_{ij}^{(n)}$ by backwards iteration from $j = i$ to $j = 1$. $\square$

The following remark is key for our subsequent analysis.

Remark 3.13. If Assumptions 2.3, 3.1, 3.3(i) and 3.4(i) are satisfied, then there exists according to Lemmata 3.6, 3.8, and 3.12 for every $m \in \mathbb{N}$ a constant $q_m < \infty$ and an $n_m \in \mathbb{N}$ such that for all $n \geq n_m$ and $j \leq i \leq m$,

$$\sup_{s\in E_{loc}} \frac{\left|\alpha_{ij}^{(n)}(s)\right|}{\sigma_j^{(n)}(s)} < q_m.$$

Let us snow turn to the convergence of the volatility operator. Similarly, to the functions $d_{ij}^{(n)}$ we set for all $i, j \in \mathbb{N}$ and $s \in E_{loc}$,

$$d_{ij}(s) := \begin{cases} \sigma_i(s)c_{ij}(s) & : \ j \leq i \\ 0 & : \ j > i. \end{cases}$$

Lemma 3.14. *Given Assumptions 2.3, 3.1, 3.3(i) and 3.4(i) we have for all $m \in \mathbb{N}$,*

$$\sup_{s\in E_{loc}} \sum_{i\in\mathcal{I}_m} \sum_{j\leq i} \left(d_{ij}^{(n)}(s) - d_{ij}(s)\right)^2 \to 0.$$

Proof. Fix $m \in \mathbb{N}$ and let $\varepsilon > 0$. According to Lemma 3.5 we can a finite subset $J \subset \mathcal{I}_m$ such that for all $n \in \mathbb{N}$ and $y \in \mathbb{R}_+$,

$$\sum_{i\in\mathcal{I}_m\setminus J} \left(F_i^{(n)}(y)\right)^2 \leq \frac{\varepsilon}{8M^2}.$$

Hence for any $n \in \mathbb{N}$ and $s \in E_{loc}$,

$$\sum_{i\in\mathcal{I}_m\setminus J}\left(\sigma_i^{(n)}(s)\right)^2\sum_{j\le i}\left(c_{ij}^{(n)}(s)-c_{ij}(s)\right)^2$$

$$\le 2\sum_{i\in\mathcal{I}_m\setminus J}\left(\sigma_i^{(n)}(s)\right)^2\sum_{j\le i}\left[\left(c_{ij}^{(n)}(s)\right)^2+\left(c_{ij}(s)\right)^2\right]$$

$$= 4\sum_{i\in\mathcal{I}_m\setminus J}\left(\sigma_i^{(n)}(s)\right)^2$$

$$\le 4\sum_{i\in\mathcal{I}_m\setminus J}\int_0^\infty g^{(n)}(s;y)dx\left(F_i^{(n)}(y)\right)^2dy\le\frac{\varepsilon}{2}.$$

According to Lemma 3.12 there exists for all $i, j \in \mathbb{N}$ an $n_{ij} = n_{ij}(\varepsilon, m)$ such that for any $n \ge n_{ij}$,

$$\sup_{s\in E_{loc}}\left|c_{ij}^{(n)}(s)-c_{ij}(s)\right|^2<\frac{\varepsilon}{2|J|M^2m}.$$

Hence, for any $n \ge n_0 := \max\{n_{ij} : j \le i, i \in J\}$ and $s \in E_{loc}$,

$$\sum_{i\in\mathcal{I}_m}\left(\sigma_i^{(n)}(s)\right)^2\sum_{j\le i}\left(c_{ij}^{(n)}(s)-c_{ij}(s)\right)^2\le\frac{\varepsilon}{2}+\sum_{i\in J}\left(\sigma_i^{(n)}(s)\right)^2\sum_{j\le i}\left(c_{ij}^{(n)}(s)-c_{ij}(s)\right)^2$$

$$<\frac{\varepsilon}{2}+\sum_{i\in J}\left(\sigma_i^{(n)}(s)\right)^2\frac{\varepsilon}{2M^2m}\le\varepsilon.$$

Now the claim follows from the above and Lemma 3.8 because

$$\sum_{i\in\mathcal{I}_m}\sum_{j\le i}\left(d_{ij}^{(n)}(s)-d_{ij}(s)\right)^2\le 2\sum_{i\in\mathcal{I}_m}\left(\sigma_i^{(n)}(s)\right)^2\sum_{j\le i}\left(c_{ij}^{(n)}(s)-c_{ij}(s)\right)^2 + 2\sum_{i\in\mathcal{I}_m}\left(\sigma_i^{(n)}(s)-\sigma_i(s)\right)^2. \quad \square$$

3.5. *Convergence of the martingale to a Gaussian random measure*

We are now going to prove the convergence of the martingale driving the SDE in (22) to a cylindrical Brownian motion on $L^2(\mathbb{R}_+)$. We start with the following simple lemma.

Lemma 3.15. *Let Assumptions 2.3, 3.1 and 3.10 be satisfied. Then there exists for any $\varphi \in L^2(\mathbb{R}_+)$ and $\varepsilon > 0$ an $m_0 \in \mathbb{N}$ such that for all $m_2 \ge m_1 \ge m_0$, $n \in \mathbb{N}$, and $t \in [0, T]$,*

$$\mathbb{E}\left(\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\sum_{i=m_1+1}^{m_2}\delta W_k^{(n),i}\langle\varphi, f_i\rangle\right)^2<\varepsilon.$$

Proof. We choose

$$m_0 := \inf\left\{m\in\mathbb{N}:\sum_{i=m+1}^\infty\langle\varphi, f_i\rangle^2<\frac{\varepsilon}{T}\right\}.$$

Then due to Corollary 3.11 we have for all $n \in \mathbb{N}$ and $t \in [0, T]$,

$$\mathbb{E}\left(\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \sum_{i=m_1+1}^{m_2} \delta W_k^{(n),i}\langle \varphi, f_i\rangle\right)^2 = \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \sum_{i=m_1+1}^{m_2} \Delta t^{(n)}\langle \varphi, f_i\rangle^2$$
$$\leq T \sum_{i=m_1+1}^{m_2} \langle \varphi, f_i\rangle^2 < \varepsilon. \quad \square$$

The preceding lemma allows us to define for each $n \in \mathbb{N}$ a so called $L^2(\mathbb{R}_+)^\#$-semimartingale (for the definition see [19]): for any $t \in [0, T]$ and $\varphi \in L^2(\mathbb{R}_+)$ we set

$$W^{(n)}(\varphi, t) := \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \sum_{i} \delta W_k^{(n),i} \langle \varphi, f_i\rangle\,, \tag{25}$$

where the above series is defined as the $L^2\left(\mathbb{P}^{(n)}\right)$-limit.

Theorem 3.16. *Suppose that Assumptions 2.3, 3.1, 3.3(i), 3.4(i) and 3.10 are satisfied. Let $l \in \mathbb{N}$ and take any $\varphi_1, \dots, \varphi_l \in L^2(\mathbb{R}_+)$. Then as $n \to \infty$,*

$$\left(W^{(n)}(\varphi_1, \cdot), \dots, W^{(n)}(\varphi_l, \cdot)\right) \Rightarrow (W(\varphi_1, \cdot), \dots, W(\varphi_l, \cdot))$$

in $\mathcal{D}\left([0, T]; \mathbb{R}^l\right)$, where W is a cylindrical Brownian motion on $L^2(\mathbb{R}_+)$. Thus, in the terminology of [19], W is a centred Gaussian $L^2(\mathbb{R}_+)^\#$-semimartingale with covariance structure

$$\mathbb{E}\left[W(\varphi_1, t)W(\varphi_2, s)\right] = (t \wedge s)\langle \varphi_1, \varphi_2\rangle$$

for $\varphi_1, \varphi_2 \in L^2(\mathbb{R}_+)$ and $s, t \in [0, T]$.

Proof. For any $\varphi \in L^2(\mathbb{R}_+)$ we define the approximating sequence

$$\varphi^m := \sum_{i=1}^{m} \langle \varphi, f_i\rangle f_i.$$

Take $\varphi_1, \dots, \varphi_l \in L^2(\mathbb{R}_+)$ for some $l \in \mathbb{N}$. We will show that $\left(W^{(n)}(\varphi_1, \cdot), \dots, W^{(n)}(\varphi_l, \cdot)\right)$ converges to a centred Gaussian process with covariance function

$$\mathbb{E}\left[W(\varphi_i, t)W(\varphi_j, s)\right] = (t \wedge s)\langle \varphi_i, \varphi_j\rangle$$

for any $1 \leq i, j \leq l$ and $s, t \in [0, T]$. To this end, first note that for all $n \in \mathbb{N}$ and for all $k \leq T_n$,

$$\begin{aligned}
\mathbb{E}\left(W^{(n)}\left(\varphi_i, t_k^{(n)}\right)\Big| \mathcal{F}_{k-1}^{(n)}\right) &= \lim_{m\to\infty} \mathbb{E}\left(W^{(n)}\left(\varphi_i^m, t_k^{(n)}\right)\Big| \mathcal{F}_{k-1}^{(n)}\right) \\
&= \lim_{m\to\infty} W^{(n)}\left(\varphi_i^m, t_{k-1}^{(n)}\right) \\
&= W^{(n)}\left(\varphi, t_{k-1}^{(n)}\right).
\end{aligned}$$

Secondly, for all $n \in \mathbb{N}$ and $k_1, k_2 \in \{1, \dots, T_n\}$ denoting

$$\delta W^{(n)}\left(\varphi_i, t_k^{(n)}\right) := W^{(n)}\left(\varphi_i, t_k^{(n)}\right) - W^{(n)}\left(\varphi_i, t_{k-1}^{(n)}\right),$$

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we have

$$\begin{aligned}
&\mathbb{E}\left(\delta W^{(n)}\left(\varphi_i, t_k^{(n)}\right)\delta W^{(n)}\left(\varphi_j, t_k^{(n)}\right)\middle|\,\mathcal{F}_{k-1}^{(n)}\right)\\
&\quad=\lim_{m\to\infty}\mathbb{E}\left(\sum_{g,h=1}^{m}\delta W_k^{(n),g}\langle\varphi_i, f_g\rangle\delta W_k^{(n),h}\langle\varphi_j, f_h\rangle\middle|\,\mathcal{F}_{k-1}^{(n)}\right)\\
&\quad=\lim_{m\to\infty}\Delta t^{(n)}\sum_{h=1}^{m}\langle\varphi_i, f_h\rangle\langle\varphi_j, f_h\rangle=\Delta t^{(n)}\langle\varphi_i,\varphi_j\rangle
\end{aligned}$$

and therefore for all $1\le i, j\le l$ and $t\in[0,T]$,

$$\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\mathbb{E}\left(\delta W^{(n)}\left(\varphi_i, t_k^{(n)}\right)\delta W^{(n)}\left(\varphi_j, t_k^{(n)}\right)\middle|\,\mathcal{F}_{k-1}^{(n)}\right)\to t\langle\varphi_i,\varphi_j\rangle\quad\text{a.s.}$$

In order to apply the functional convergence theorem for martingale difference arrays it remains to check that the conditional Lindeberg condition is satisfied. For ease of notation we will assume that $l=2$ in the following, noting that the general case follows by similar arguments.

Let us fix some $\varepsilon>0$ and $t\in[0,T]$. We want to show that for any $\delta>0$ there exists an $n_0=n_0(\varepsilon,\delta)$ such that for all $n\ge n_0$,

$$\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\mathbb{E}\left(\left[\delta W^{(n)}\left(\varphi_1, t_k^{(n)}\right)\right]^2\mathbb{1}_{\left\{\left[\delta W^{(n)}\left(\varphi_1,t_k^{(n)}\right)\right]^2+\left[\delta W^{(n)}\left(\varphi_2,t_k^{(n)}\right)\right]^2>\varepsilon\right\}}\middle|\,\mathcal{F}_{k-1}^{(n)}\right)<\delta\quad\text{a.s.}$$

To this end we first apply Lemma 3.15 and choose $m=m(\delta)$ such that for all $n\in\mathbb{N}$,

$$\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\mathbb{E}\left(\left[\delta W^{(n)}\left(\varphi_1-\varphi_1^m, t_k^{(n)}\right)\right]^2\middle|\,\mathcal{F}_{k-1}^{(n)}\right)=\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\Delta t^{(n)}\sum_{i=m+1}^{\infty}\langle\varphi_1, f_i\rangle^2<\frac{\delta}{4}.$$

Hence,

$$\begin{aligned}
&\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\mathbb{E}\left(\left[\delta W^{(n)}\left(\varphi_1, t_k^{(n)}\right)\right]^2\mathbb{1}_{\left\{\left[\delta W^{(n)}\left(\varphi_1,t_k^{(n)}\right)\right]^2+\left[\delta W^{(n)}\left(\varphi_2,t_k^{(n)}\right)\right]^2>\varepsilon\right\}}\middle|\,\mathcal{F}_{k-1}^{(n)}\right)\\
&\quad<\frac{\delta}{2}+2\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\mathbb{E}\left(\left[\sum_{i=1}^{m}\delta W_k^{(n),i}\langle\varphi_1, f_i\rangle\right]^2\mathbb{1}_{\left\{\left[\delta W^{(n)}\left(\varphi_1,t_k^{(n)}\right)\right]^2+\left[\delta W^{(n)}\left(\varphi_2,t_k^{(n)}\right)\right]^2>\varepsilon\right\}}\middle|\,\mathcal{F}_{k-1}^{(n)}\right).
\end{aligned}$$

According to Remark 3.13 there exists an $n_m\in\mathbb{N}$ and a constant $q_m<\infty$ such that for all $n\ge n_m$,

$$\begin{aligned}
&\sum_{i=1}^{m}\left(\delta W_k^{(n),i}\right)^2\\
&\quad=\sum_{i=1}^{m}\left[\mathbb{1}_{\left\{c_{ii}^{(n)}\left(S_{k-1}^{(n)}\right)>0\right\}}\left(\sum_{j\le i}\alpha_{ij}^{(n)}\left(S_{k-1}^{(n)}\right)Z_k^{(n),j}\right)^2+\mathbb{1}_{\left\{c_{ii}^{(n)}\left(S_{k-1}^{(n)}\right)=0\right\}}\Delta t^{(n)}\left(U_k^{(n),i}\right)^2\right]\\
&\quad\le\sum_{i=1}^{m}\left[\mathbb{1}_{\left\{c_{ii}^{(n)}\left(S_{k-1}^{(n)}\right)>0\right\}}2^i\sum_{j\le i}\left(\alpha_{ij}^{(n)}\left(S_{k-1}^{(n)}\right)Z_k^{(n),j}\right)^2+\mathbb{1}_{\left\{c_{ii}^{(n)}\left(S_{k-1}^{(n)}\right)=0\right\}}\Delta t^{(n)}\right]
\end{aligned}$$

$$= \sum_{i=1}^{m} \left[\mathbb{1}_{\left\{c_{ii}^{(n)}\left(S_{k-1}^{(n)}\right)>0\right\}} 2^i \sum_{j \le i} \left(\frac{\alpha_{ij}^{(n)}\left(S_{k-1}^{(n)}\right)}{\sigma_j^{(n)}\left(S_{k-1}^{(n)}\right)} \right)^2 \left\langle \delta \overline{v}_k^{(n)}, f_j \right\rangle^2 + \mathbb{1}_{\left\{c_{ii}^{(n)}\left(S_{k-1}^{(n)}\right)=0\right\}} \Delta t^{(n)} \right]$$

$$\le \sum_{i=1}^{m} \left[2^m q_m^2 \left\| \delta \overline{v}_k^{(n)} \mathbb{1}_{[0,m]} \right\|_{L^2}^2 + \Delta t^{(n)} \right] \overset{(16)}{\le} \Delta t^{(n)} \left[m^2 q_m^2 2^{m+1} M^2 + m \right] \le d_n^m \quad a.s.$$

with $(d_n^m)_{n\in\mathbb{N}}$ being a deterministic sequence satisfying $d_n^m \to 0$ as $n \to \infty$. We choose

$$n_0 = n_0(\delta, \varepsilon) = n_0(m(\delta), \delta, \varepsilon) := \min \left\{ n \in \mathbb{N} : 8T \left\|\varphi_1\right\|_{L^2}^2 d_n^m \left(\left\|\varphi_1\right\|_{L^2}^2 + \left\|\varphi_2\right\|_{L^2}^2 \right) < \delta\varepsilon \right\}.$$

Then for all $n \ge n_m$ by the Cauchy–Schwarz inequality,

$$\mathbb{E} \left(\left[\sum_{i=1}^{m} \delta W_k^{(n),i} \langle \varphi_1, f_i \rangle \right]^2 \mathbb{1}_{\left\{ \left[\delta W^{(n)}\left(\varphi_1, t_k^{(n)}\right)\right]^2 + \left[\delta W^{(n)}\left(\varphi_2, t_k^{(n)}\right)\right]^2 > \varepsilon \right\}} \middle| \mathcal{F}_{k-1}^{(n)} \right)$$

$$\le \left\|\varphi_1\right\|_{L^2}^2 \cdot \mathbb{E} \left(\sum_{i=1}^{m} \left(\delta W_k^{(n),i} \right)^2 \left(\mathbb{1}_{\left\{ \left[\delta W^{(n)}\left(\varphi_1, t_k^{(n)}\right)\right]^2 > \frac{\varepsilon}{2} \right\}} + \mathbb{1}_{\left\{ \left[\delta W^{(n)}\left(\varphi_2, t_k^{(n)}\right)\right]^2 > \frac{\varepsilon}{2} \right\}} \right) \middle| \mathcal{F}_{k-1}^{(n)} \right)$$

$$\le \frac{2 \left\|\varphi_1\right\|_{L^2}^2 d_n^m}{\varepsilon} \cdot \mathbb{E} \left(\left[\delta W^{(n)}\left(\varphi_1, t_k^{(n)}\right) \right]^2 + \left[\delta W^{(n)}\left(\varphi_2, t_k^{(n)}\right) \right]^2 \middle| \mathcal{F}_{k-1}^{(n)} \right)$$

$$= \frac{2 \left\|\varphi_1\right\|_{L^2}^2 d_n^m}{\varepsilon} \Delta t^{(n)} \left(\left\|\varphi_1\right\|_{L^2}^2 + \left\|\varphi_2\right\|_{L^2}^2 \right) < \frac{\delta \Delta t^{(n)}}{4T} \quad \text{a.s.}$$

Hence, the conditional Lindeberg condition is satisfied and the functional central limit theorem for martingale difference arrays (cf. Theorem 3.33 in [14]) implies that

$$\left(W^{(n)}(\varphi_1, \cdot), \ldots, W^{(n)}(\varphi_l, \cdot) \right) \Rightarrow \left(W(\varphi_1, \cdot), \ldots, W(\varphi_l, \cdot) \right) \quad \text{in } \mathcal{D}([0, T]; \mathbb{R}^l),$$

where $(W(\varphi_1, \cdot), \ldots, W(\varphi_l, \cdot))$ is a centred Gaussian process with covariance function

$$\mathbb{E}\left[W(\varphi_i, t) W(\varphi_j, s) \right] = (t \wedge s) \langle \varphi_i, \varphi_j \rangle$$

for any $1 \le i, j \le l$ and $s, t \in [0, T]$. $\square$

Remark 3.17. The process W is not only an $L^2(\mathbb{R}_+)^\#$-semimartingale in the sense of [19], but can also be understood as a martingale random measure: If $\mathcal{A} := \{A \subset \mathcal{B}(\mathbb{R}_+) : A \text{ bounded}\}$, we can define for any $A \in \mathcal{A}$ and $t \in [0, T]$, $M(A, t) := W(\mathbb{1}_A, t)$. Then M is indeed a Gaussian martingale random measure indexed by $\mathcal{A} \times [0, T]$.

4. The state dynamics as an infinite dimensional SDE

In this section we show that the dynamics of $S^{(n)}$ can be written as an infinite dimensional SDE and prove the convergence of the integrands and integrators. Our concept of integration follows [19], to which we refer for any unknown terminology used in the following.

For each $n \in \mathbb{N}$ we define the E_{loc}-valued stochastic process $\left(S^{(n)}(t)\right)_{t\in[0,T]}$ as the piecewise constant interpolation of the $\left(S_k^{(n)}\right)_{k=0,\ldots,T_n}$, i.e.

$$S^{(n)}(t) := S_k^{(n)}, \quad \text{if } t \in \left[t_k^{(n)}, t_{k+1}^{(n)}\right).$$

Similarly, we set

$$B^{(n)}(t) := B_k^{(n)}, \qquad V^{(n)}(t, x) := V_k^{(n)}(x), \quad \text{if} \quad t_k^{(n)} \le t < t_{k+1}^{(n)}, \quad x \in \mathbb{R}_+.$$

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In view of Eqs. (11) and (22) we have that

$$B^{(n)}(t) = B_0^{(n)} + \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \left[p^{(n)}\left(S_{k-1}^{(n)}\right) \Delta t^{(n)} + r^{(n)}\left(S_{k-1}^{(n)}\right) \delta Z_k^{(n)} \right]$$

$$V^{(n)}(t,x) = V_0^{(n)}(x) + \sum_i f_i(x) \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \Bigg[\mu_i^{(n)}\left(S_{k-1}^{(n)}\right) \Delta t^{(n)} + \sigma_i^{(n)}\left(S_{k-1}^{(n)}\right) \sum_{j\le i} c_{ij}^{(n)}\left(S_{k-1}^{(n)}\right) \delta W_k^{(n),j} \Bigg]. \tag{26}$$

In terms of the processes $Z^{(n)}$ and $W^{(n)}$ introduced in (10) and (25), respectively, we can define a sequence of $L^2(\mathbb{R}_+)^\#$-semimartingales $Y^{(n)}$ by putting, for any $n \in \mathbb{N}$, $t \in [0,T]$, and $\varphi \in L^2(\mathbb{R}_+)$,

$$Y^{(n)}(\varphi,t) := \left(Z_k^{(n)}, W^{(n)}(\varphi,t), t_k^{(n)} \right), \quad \text{if } t \in \left[t_k^{(n)}, t_{k+1}^{(n)} \right).$$

The stochastic integral with respect to $Y^{(n)}$ is introduced in Appendix B. If we define, for any $n \in \mathbb{N}$, the coefficient functions $G^{(n)} : E_{loc} \to \hat{E}_{loc}$ (see Appendix B for the definition of the space $\hat{E}_{loc}$) via

$$G^{(n)} := \left(G^{(n),1},\ 0,\ G^{(n),3},\ 0,\ G^{(n),5},\ G^{(n),6} \right)$$

with

$$G^{(n),1}(s) := r^{(n)}(s), \qquad G^{(n),5}(s;x,y) := \sum_i \sum_{j\le i} d_{ij}^{(n)}(s) f_i(x) f_j(y),$$

$$G^{(n),3}(s) := p^{(n)}(s), \qquad G^{(n),6}(s;x) := \sum_i \mu_i^{(n)}(s) f_i(x) = \mu^{(n)}(s;x),$$

then the general integration theory guarantees that the integral

$$\int_0^t G^{(n)}\left(S^{(n)}(u-)\right) dY^{(n)}(u), \quad t \in [0,T],$$

is well-defined as an E_{loc}-valued stochastic process, and (26) yields the following representation of the state process:

$$S^{(n)}(t) = S_0^{(n)} + \int_0^t G^{(n)}\left(S^{(n)}(u-)\right) dY^{(n)}(u), \quad t \in [0,T]. \tag{27}$$

In the next subsection we are going to prove the convergence of the integrators and integrands.

4.1. Convergence of the integrator and integrand

The following theorem shows that the sequence $Y^{(n)}$ converges to the $L^2(\mathbb{R}_+)^\#$-semimartingale

$$Y(\varphi,t) := (Z(t), W(\varphi,t), t), \quad \varphi \in L^2(\mathbb{R}_+), \quad t \in [0,T], \tag{28}$$

where W is a cylindrical Brownian motion on $L^2(\mathbb{R}_+)$, and Z is an independent standard Brownian motion.

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Theorem 4.1. *Let Assumptions 2.1, 2.3, 3.1, 3.3(i), 3.4(i) and 3.10 be satisfied. Then, for every $k \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_k \in L^2(\mathbb{R}_+)$,*

$$\left(Y^{(n)}(\varphi_1, \cdot), \ldots, Y^{(n)}(\varphi_k, \cdot)\right) \Rightarrow (Y(\varphi_1, \cdot), \ldots, Y(\varphi_k, \cdot))$$

in $\mathcal{D}\left([0, T]; \mathbb{R}^{3k}\right)$, where Y is defined in (28).

Proof. The joint convergence follows directly from Theorems 2.5 and 3.16 because the processes $Z^{(n)}$, $n \in \mathbb{N}$, and $W^{(n)}(\varphi, \cdot)$, $n \in \mathbb{N}$, are C-tight for any $\varphi \in L^2(\mathbb{R})$. However, to derive the joint finite dimensional distributions (and especially to check the independence of the resulting cylindrical and standard Brownian motion), we have to show two more things: first, we will prove that for all $t \in [0, T]$ and $\varphi \in L^2(\mathbb{R}_+)$,

$$\sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \mathbb{E}\left(\delta W^{(n)}\left(\varphi, t_k^{(n)}\right) \delta Z_k^{(n)} \middle| \mathcal{F}_{k-1}^{(n)}\right) \to 0 \quad \text{a.s.}$$

and second, we will show that for all $\varepsilon > 0$, $t \in [0, T]$, and $\varphi \in L^2(\mathbb{R}_+)$,

$$\sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \mathbb{E}\left(\left(\left[\delta W^{(n)}\left(\varphi, t_k^{(n)}\right)\right]^2 + \left[\delta Z_k^{(n)}\right]^2\right) \mathbb{1}_{\left\{\left[\delta W^{(n)}\left(\varphi, t_k^{(n)}\right)\right]^2 + \left[\delta Z_k^{(n)}\right]^2 > \varepsilon\right\}} \middle| \mathcal{F}_{k-1}^{(n)}\right)$$
$$\to 0 \quad \text{a.s.}$$

To this end, observe that for any $n, i \in \mathbb{N}$ and $k \leq T_n$,

$$\mathbb{E}\left(\left\langle \delta \overline{v}_k^{(n)}, f_i \right\rangle \delta Z_k^{(n)} \middle| \mathcal{F}_{k-1}^{(n)}\right) = -\left(\Delta t^{(n)}\right)^2 \mu_i^{(n)}\left(S_{k-1}^{(n)}\right) p^{(n)}\left(S_{k-1}^{(n)}\right).$$

Let $\delta > 0$. We choose $m = m(\delta)$ such that for all $n \in \mathbb{N}$ and $t \in [0, T]$,

$$\sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \left| \mathbb{E}\left(\sum_{i=m+1}^{\infty} \langle \varphi, f_i \rangle \delta W_k^{(n),i} \delta Z_k^{(n)} \middle| \mathcal{F}_{k-1}^{(n)}\right) \right|$$
$$\leq \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \left(\mathbb{E}\left[\left(\delta Z_k^{(n)}\right)^2 \middle| \mathcal{F}_{k-1}^{(n)}\right] \mathbb{E}\left[\left(\sum_{i=m+1}^{\infty} \langle \varphi, f_i \rangle \delta W_k^{(n),i}\right)^2 \middle| \mathcal{F}_{k-1}^{(n)}\right]\right)^{1/2}$$
$$\leq \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \Delta t^{(n)} \left(\sum_{i=m+1}^{\infty} \langle \varphi, f_i \rangle^2\right)^{1/2} < \frac{\delta}{2} \quad \text{a.s.}$$

Moreover for large enough n and all $k \leq T_n$,

$$\mathbb{E}\left[\sum_{i=1}^{m} \langle \varphi, f_i \rangle \delta W_k^{(n),i} \delta Z_k^{(n)} \middle| \mathcal{F}_{k-1}^{(n)}\right]$$
$$= \sum_{i=1}^{m} \mathbb{E}\left[\langle \varphi, f_i \rangle \sum_{j \leq i} \frac{\alpha_{ij}^{(n)}\left(S_{k-1}^{(n)}\right)}{\sigma_j^{(n)}\left(S_{k-1}^{(n)}\right)} \left\langle \delta \overline{v}_k^{(n)}, f_j \right\rangle \delta Z_k^{(n)} \middle| \mathcal{F}_{k-1}^{(n)}\right]$$
$$= -\left(\Delta t^{(n)}\right)^2 p^{(n)}\left(S_{k-1}^{(n)}\right) \sum_{i=1}^{m} \langle \varphi, f_i \rangle \sum_{j \leq i} \frac{\alpha_{ij}^{(n)}\left(S_{k-1}^{(n)}\right)}{\sigma_j^{(n)}\left(S_{k-1}^{(n)}\right)} \mu_j^{(n)}\left(S_{k-1}^{(n)}\right).$$

According to Lemma 3.7 and Remark 3.13 there exist an $n_0 = n_0(m)$ and a constant $C_m < \infty$ such that for all $n \geq n_0$,

$$\sup_{s \in E_{loc}} \left| \sum_{i=1}^{m} \langle \varphi, f_i \rangle \sum_{j \leq i} \mu_j^{(n)}(s) \frac{\alpha_{ij}^{(n)}(s)}{\sigma_j^{(n)}(s)} \right| \leq C_m \sum_{i=1}^{m} |\langle \varphi, f_i \rangle| \leq m C_m \|\varphi\|_{L^2} < \infty.$$

Hence for all $n \geq n_0$,

$$\begin{aligned} \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \left| \mathbb{E} \left(\sum_{i=1}^{m} \langle \varphi, f_i \rangle \delta W_k^{(n),i} Z_k^{(n)} \middle| \mathcal{F}_{k-1}^{(n)} \right) \right| &\leq \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \left(\Delta t^{(n)} \right)^2 \left| p^{(n)} \left(S_{k-1}^{(n)} \right) \right| m C_m \|\varphi\|_{L^2} \\ &\overset{(13)}{\leq} T \Delta x^{(n)} m C_m \|\varphi\|_{L^2} < \frac{\delta}{2} \quad \text{a.s.} \end{aligned}$$

This proves that for any $\delta > 0$ there exists $n_0 = n_0(\delta)$ such that for all $n \geq n_0$,

$$\left| \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \mathbb{E} \left(\delta W^{(n)} \left(\varphi, t_k^{(n)} \right) \delta Z_k^{(n)} \middle| \mathcal{F}_{k-1}^{(n)} \right) \right| < \delta \quad \text{a.s.}$$

Next, using the estimate in Eq. (14) we have almost surely

$$\begin{aligned} &\sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \mathbb{E} \left(\left[\delta Z_k^{(n)} \right]^2 \mathbb{1}_{\left\{ \left[\delta W^{(n)} \left(\varphi, t_k^{(n)} \right) \right]^2 + \left[\delta Z_k^{(n)} \right]^2 > \varepsilon \right\}} \middle| \mathcal{F}_{k-1}^{(n)} \right) \\ &\leq c_n \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \mathbb{P} \left(\left[\delta W^{(n)} \left(\varphi, t_k^{(n)} \right) \right]^2 > \frac{\varepsilon}{2} \middle| \mathcal{F}_{k-1}^{(n)} \right) + \mathbb{P} \left(\left[\delta Z_k^{(n)} \right]^2 > \frac{\varepsilon}{2} \middle| \mathcal{F}_{k-1}^{(n)} \right) \\ &\leq \frac{2 c_n}{\varepsilon} \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \mathbb{E} \left(\left[\delta W^{(n)} \left(\varphi, t_k^{(n)} \right) \right]^2 + \left[\delta Z_k^{(n)} \right]^2 \middle| \mathcal{F}_{k-1}^{(n)} \right) \\ &\leq \frac{2 c_n}{\varepsilon} \sum_{k=1}^{\lfloor t/\Delta t^{(n)} \rfloor} \Delta t^{(n)} \left(\|\varphi\|_{L^2}^2 + 1 \right) \to 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\mathbb{E} \left(\left[\delta W^{(n)} \left(\varphi, t_k^{(n)} \right) \right]^2 \mathbb{1}_{\left\{ \left[\delta W^{(n)} \left(\varphi, t_k^{(n)} \right) \right]^2 + \left[\delta Z_k^{(n)} \right]^2 > \varepsilon \right\}} \middle| \mathcal{F}_{k-1}^{(n)} \right) \\ &\leq 2 \cdot \mathbb{E} \left(\left[\sum_{i=m+1}^{\infty} \delta W_k^{(n),i} \langle \varphi, f_i \rangle \right]^2 \right. \\ &\quad \left. + \left[\sum_{i=1}^{m} \delta W_k^{(n),i} \langle \varphi, f_i \rangle \right]^2 \mathbb{1}_{\left\{ \left[\delta W^{(n)} \left(\varphi, t_k^{(n)} \right) \right]^2 + \left[\delta Z_k^{(n)} \right]^2 > \varepsilon \right\}} \middle| \mathcal{F}_{k-1}^{(n)} \right) \\ &\leq 2 \Delta t^{(n)} \sum_{i=m+1}^{\infty} \langle \varphi, f_i \rangle^2 + 2 \|\varphi\|_{L^2}^2 \mathbb{E} \left(\left[\sum_{i=1}^{m} \delta W_k^{(n),i} \right]^2 \mathbb{1}_{\left\{ \left[\delta W^{(n)} \left(\varphi, t_k^{(n)} \right) \right]^2 + \left[\delta Z_k^{(n)} \right]^2 > \varepsilon \right\}} \middle| \mathcal{F}_{k-1}^{(n)} \right) \end{aligned}$$

and by a similar reasoning as above

$$\mathbb{E}\left(\left[\sum_{i=1}^{m}\delta W_k^{(n),i}\right]^2 \mathbb{1}_{\left\{\left[\delta W^{(n)}\left(\varphi,t_k^{(n)}\right)\right]^2+\left[\delta Z_k^{(n)}\right]^2>\varepsilon\right\}}\middle|\mathcal{F}_{k-1}^{(n)}\right) \leq \frac{2d_n^m}{\varepsilon}\Delta t^{(n)}\left(\|\varphi\|_{L^2}^2+1\right).$$

Now for any $\delta > 0$ we choose $m = m(\delta)$ and $n_0 = n_0(m, \delta, \varepsilon) = n_0(\delta, \varepsilon)$ such that for all $n \geq n_0$,

$$\sum_{i=m+1}^{\infty}\langle\varphi, f_i\rangle^2 < \frac{\delta}{4T} \qquad \text{and} \qquad \frac{2d_n^m}{\varepsilon}\|\varphi\|_{L^2}^2\left(\|\varphi\|_{L^2}^2+1\right) < \frac{\delta}{4T}$$

and therefore

$$\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\mathbb{E}\left(\left[\delta W^{(n)}\left(\varphi, t_k^{(n)}\right)\right]^2 \mathbb{1}_{\left\{\left[\delta W^{(n)}\left(\varphi,t_k^{(n)}\right)\right]^2+\left[\delta Z_k^{(n)}\right]^2>\varepsilon\right\}}\middle|\mathcal{F}_{k-1}^{(n)}\right) < \delta \quad \text{a.s.} \quad \square$$

Let us now turn to the integrands. The results of Section 3 suggest that the coefficient functions $G^{(n)}$ converge in a local sense to

$$G = \left(G^1,\ 0,\ G^3,\ 0,\ G^5,\ G^6\right) : E_{loc} \to \hat{E}_{loc}$$

with

$$G^1(s) := r(s), \qquad G^5(s; x, y) := \sum_i \sum_{j\leq i} d_{ij}(s)\, f_i(x) f_j(y),$$

$$G^3(s) := p(s), \qquad G^6(s; x) := \sum_i \mu_i(s)\, f_i(x) = \mu(s; x).$$

In order to formulate the convergence result we define for every $m \in \mathbb{N}$ the projections of G^5 and G^6 on $[0, m]$ as

$$G^{5,m}(s; x, y) := \sum_{i\in\mathcal{I}_m}\sum_{j\leq i} d_{ij}(s)\, f_i(x) f_j(y),$$

$$G^{6,m}(s; x) := \sum_{i\in\mathcal{I}_m}\mu_i(s)\, f_i(x) = \mu(s; x),$$

and set

$$G^m(s) := \left(G^1(s),\ 0,\ G^3(s),\ 0,\ G^{5,m}(s),\ G^{6,m}(s)\right), \quad s \in E_{loc}.$$

Moreover, for all $m \in \mathbb{N}$ we define the space

$$E_m := \left\{s = \left(b, v\mathbb{1}_{[0,m]}\right) : (b, v) \in E_{loc}\right\} \subset E_{loc}.$$

Next, we approximate $G^{(n)}$ by functions $G_m^{(n)} : E_m \to \hat{E},\ m \in \mathbb{N}$, given by

$$G_m^{(n)} := \left(G_m^{(n),1},\ 0,\ G_m^{(n),3},\ 0,\ G_m^{(n),5},\ G_m^{(n),6}\right),$$

where for all $s \in E_m \subset E_{loc}$ and $s_m := (s \wedge m, v)$,

$$G_m^{(n),1}(s) := p^{(n)}(s_m), \qquad G_m^{(n),5}(s; x, y) := \sum_{i\in\mathcal{I}_m}\sum_{j\leq i} d_{ij}^{(n)}(s)\, f_i(x) f_j(y),$$

$$G_m^{(n),3}(s) := r^{(n)}(s_m), \qquad G_m^{(n),6}(s; x, y) := \sum_{i\in\mathcal{I}_m}\mu_i^{(n)}(s)\, f_i(x).$$

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Analogously, we define for each $m \in \mathbb{N}$ a function $G_m : E_m \to \hat{E}$ via a similar modification of G, i.e. we have

$$G_m(s) := G^m(s_m), \quad s \in E_m.$$

We note that for $s = (b, v) \in E_m$ with $b \le m$, $G_m(s) = G^m(s)$, due to Assumptions 2.6(ii), 3.3(ii) and 3.4(ii).

Theorem 4.2. *Let Assumptions 2.3, 2.6, 3.1, 3.3 and 3.4 hold. Then for any $m \in \mathbb{N}$,*

$$\sup_{s \in E_m} \left\| G_m^{(n)}(s) - G_m(s) \right\|_{\hat{E}} \to 0.$$

Proof. By Assumption 2.6, Lemmas 3.7 and 3.14 we have for all $s = (b, v) \in E_m$,

$$\begin{aligned}\left\| G_m^{(n)}(s) - G_m(s) \right\|_{\hat{E}} &= \left| r^{(n)}(s_m) - r(s_m) \right| + \left| p^{(n)}(s_m) - p(s_m) \right| \\ &\quad + \left(\sum_{i \in \mathcal{I}_m} \left(\mu_i^{(n)}(s) - \mu_i(s) \right)^2 \right)^{1/2} \\ &\quad + \left(\sum_{i \in \mathcal{I}_m} \sum_{j \le i} \left(d_{ij}^{(n)}(s) - d_{ij}(s) \right)^2 \right)^{1/2}. \quad \square\end{aligned}$$

4.2. Compactness of the integrands

In this section it is shown that for each $m \in \mathbb{N}$ the $G_m^{(n)}$, $n \in \mathbb{N}$, satisfy a uniform compactness condition from which we shall later deduce relative compactness of the price-volume process and hence the existence of accumulation points.

Theorem 4.3. *Given Assumptions 2.3, 3.1 and 3.4(i), there exists for every $m \in \mathbb{N}$ a compact set $K_m \subset \hat{E}$ such that for all $n \in \mathbb{N}$ and $s \in E_m$,*

$$G_m^{(n)}(s) \in K_m.$$

Since $G_m^{(n),1}$ and $G_m^{(n),3}$ are uniformly bounded by Assumption 2.6(i), we only have to care about the last two components of $G_m^{(n)}$. Thus, Theorem 4.3 will directly follow from Lemmata 4.4 and 4.5.

Lemma 4.4. *Let Assumptions 2.3 and 3.1 be satisfied. Then for each $m \in \mathbb{N}$ the set*

$$K_m^5 := \left\{ G_m^{(n),5}(s) : \ s \in E_m, \ n \in \mathbb{N} \right\} \subset L^2\left(\mathbb{R}_+^2\right)$$

t is relatively compact.

Proof. First note that for all $s, \tilde{s} \in E_m$ we have

$$\left\| G_m^{(n),5}(s) - G_m^{(n),5}(\tilde{s}) \right\|_{L^2\left(\mathbb{R}_+^2\right)}^2 = \sum_{i \in \mathcal{I}_m} \sum_{j \le i} \left(d_{ij}^{(n)}(s) - d_{ij}^{(n)}(\tilde{s}) \right)^2.$$

Now consider a sequence $\left(G_m^{(n_k),5}(s_k) \right)_{k \in \mathbb{N}} \subset K_m^5$ and set $a_k := 2^{-k}$, $k \in \mathbb{N}$. W.l.o.g. we may assume that $s_k \in E_m$ for all $k \in \mathbb{N}$. As in the proof of Lemma 3.14 one can show that there exists a finite index set $J \subset \mathcal{I}_m$ such that for all $n \in \mathbb{N}$ and $s \in E_{loc}$,

$$\sum_{i \in \mathcal{I}_m \setminus J} \left(\sigma_i^{(n)}(s) \right)^2 < \frac{a_1}{8}.$$

For any $(i, j) \in \mathbb{N}^2$, $\left(d_{ij}^{(n_k)}(s_k) \right)_{k \in \mathbb{N}}$ is a real-valued sequence, bounded by M. Since J is a finite set, there exists a subsequence $(k_q) \subset \mathbb{N}$ and a $q_0 = q_0(a_1) \in \mathbb{N}$ such that for each pair (i, j) with $i \in J$ and $j \leq i$,

$$\left(d_{ij}^{(n_{k_q})}(s_{k_q}) - d_{ij}^{(n_{k_{q'}})}(s_{k_{q'}}) \right)^2 \leq \frac{a_1}{|J|(|J|+1)} \quad \text{for all } q, q' \geq q_0.$$

Hence, for all $q, q' \geq q_0$ we have

$$\begin{aligned} &\sum_{i \in \mathcal{I}_m} \sum_{j \leq i} \left(d_{ij}^{(n_{k_q})}(s_{k_q}) - d_{ij}^{(n_{k_{q'}})}(s_{k_{q'}}) \right)^2 \\ &\quad \leq \frac{a_1}{2} + 2 \sum_{i \in \mathcal{I}_{m_0} \setminus J} \sum_{j \leq i} \left\{ \left(d_{ij}^{(n_{k_q})}(s_{k_q}) \right)^2 + \left(d_{ij}^{(n_{k_{q'}})}(s_{k_{q'}}) \right)^2 \right\} \\ &\quad = \frac{a_1}{2} + 2 \sum_{i \in \mathcal{I}_m \setminus J} \left\{ \left(\sigma_i^{(n_{k_q})}(s_{k_q}) \right)^2 + \left(\sigma_i^{(n_{k_{q'}})}(s_{k_{q'}}) \right)^2 \right\} < a_1. \end{aligned}$$

Next, we consider the sequence $\left(G_m^{(n_{k_q}),5}(s_{k_q}) \right)_{q \in \mathbb{N}} \subset K_m^5$ and construct in a similar way as above – with a_1 being replaced by a_2 – a further subsequence. This will be done iteratively for all a_k, $k \in \mathbb{N}$. Finally, we choose the diagonal sequence of all these subsequences, which will be a Cauchy sequence and hence convergent in $L^2(\mathbb{R}_+^2)$. This shows that K_m^5 is relatively compact. □

Lemma 4.5. *Let Assumption 3.4(i) be satisfied. Then for each $m \in \mathbb{N}$ the set*

$$K_m^6 := \left\{ G_m^{(n),6}(s) : \ s \in E_m, \ n \in \mathbb{N} \right\} \subset L^2(\mathbb{R}_+)$$

is relatively compact.

Proof. Consider some sequence $\left(G_m^{(n_k),6}(s_k) \right)_{k \in \mathbb{N}} \subset K_m^6$ and set again $a_k := 2^{-k}$, $k \in \mathbb{N}$. As in the proof of Lemma 4.4 we may assume that $s_k \in E_m$ for all $k \in \mathbb{N}$. By Assumption 3.4(i) there exists $K > 0$ such that for all $n \in \mathbb{N}$ and $s \in E_{loc}$,

$$\int_0^\infty \left| h^{(n)}(s; y) \right|^2 dy < K.$$

We apply Lemma 3.5 to find a finite subset $J \subset \mathcal{I}_m$ such that for all $n \in \mathbb{N}$ and $y \in \mathbb{R}_+$,

$$\sum_{i \in \mathcal{I}_m \setminus J} \left(F_i^{(n)}(y) \right)^2 \leq \frac{a_1}{8Km}.$$

Hence for all $n \in \mathbb{N}$ and $s \in E_{loc}$,

$$\sum_{i \in \mathcal{I}_m \setminus J} \left(\mu_i^{(n)}(s)\right)^2 = \sum_{i \in \mathcal{I}_m \setminus J} \left(\int_0^\infty h^{(n)}(s;y) F_i^{(n)}(y) dy\right)^2$$
$$\leq \sum_{i \in \mathcal{I}_m \setminus J} m \int_0^m \left(h^{(n)}(s;y) F_i^{(n)}(y)\right)^2 dy$$
$$\leq \frac{a_1}{8K} \int_0^\infty \left|h^{(n)}(s;y)\right|^2 dy < \frac{a_1}{8}.$$

The rest of the proof follows as in the proof of Lemma 4.4. □

4.3. *Continuity of the integrand*

In this subsection we will prove for all $m \in \mathbb{N}$ the continuity of G_m. First note that by Assumption 2.6 there exists some $L > 0$ such that for all $s = (b, v)$, $\widetilde{s} = \left(\widetilde{b}, \widetilde{v}\right) \in E_m$,

$$\left|G_m^1(s) - G_m^1(\widetilde{s})\right| \leq L\left(1 + |b| + |\widetilde{b}|\right)\left(1 + \|v\|_{L^2} + \|\widetilde{v}\|_{L^2}\right)\left(\left|b - \widetilde{b}\right| + \|v - \widetilde{v}\|_{L^2}\right).$$

Hence, for any $c > 0$ there exists $L_c < \infty$ such that for all $s, \widetilde{s} \in E_m$ with $\|s\|_E \leq c$, $\|\widetilde{s}\|_E \leq c$,

$$\left|G_m^1(s) - G_m^1(\widetilde{s})\right| \leq L_c \|s - \widetilde{s}\|_E .$$

A similar result holds for G_m^3. It remains to show the continuity of G_m^5 and G_m^6.

Lemma 4.6. *Under Assumption 3.4 there exists for every $m \in \mathbb{N}$ and $c > 0$ a constant L_c^m such that for all $s, \widetilde{s} \in E_m$ with $\|s\|_E \leq c$, $\|\widetilde{s}\|_E \leq c$ we have*

$$\left\|G_m^6(s) - G_m^6(\widetilde{s})\right\|_{L^2(\mathbb{R}_+)} \leq L_c^m \|s - \widetilde{s}\|_E .$$

Proof. Due to Assumption 3.4 we have for all $s, \widetilde{s} \in E_m$,

$$\left\|G_m^6(s) - G_m^6(\widetilde{s})\right\|_{L^2}^2 = \sum_{i \in \mathcal{I}_m} (\mu_i(s) - \mu_i(\widetilde{s}))^2$$
$$= \sum_{i \in \mathcal{I}_m} \left(\int_0^\infty [h(s;y) - h(\widetilde{s};y)] F_i(y) dy\right)^2$$
$$\leq \sum_{i \in \mathcal{I}_m} m \int_0^m [h(s;y) - h(\widetilde{s};y)]^2 (F_i(y))^2 dy$$
$$\leq m^2 \int_0^\infty [h(s;y) - h(\widetilde{s};y)]^2 dy$$
$$\leq m^2 L^2 \left(1 + |b| + |\widetilde{b}|\right)^2 \left(1 + \|v\|_{L^2} + \|\widetilde{v}\|_{L^2}\right)^2$$
$$\times \left(\left|b - \widetilde{b}\right| + \|v - \widetilde{v}\|_{L^2}\right)^2. \quad \square$$

Lemma 4.7. *Suppose that Assumptions 2.3, 3.1, 3.3 and 3.4 are satisfied. Then there exists for all $c > 0$ and $m, i, j \in \mathbb{N}$ with $j \leq i$ a constant $L_{ij}^{m,c} > 0$ such that for all $s, \widetilde{s} \in E_m$ with $\|s\|_E \leq c$, $\|\widetilde{s}\|_E \leq c$,*

$$|d_{ij}(s) - d_{ij}(\widetilde{s})| \leq L_{ij}^{m,c} \|s - \widetilde{s}\|_E .$$

Proof. Since $d_{ij} = \sigma_i c_{ij}$ for all $j \le i$ and $|\sigma_i| \le M$, $|c_{ij}| \le 1$, it is sufficient to show the inequality for σ_i and c_{ij} separately. For all $i, j \in \mathbb{N}$ and $s, \widetilde{s} \in E_m$ by Assumption 3.3,

$$\begin{aligned}
&\left|\sigma_i(s)\sigma_j(s)\rho_{ij}(s) - \sigma_i(\widetilde{s})\sigma_j(\widetilde{s})\rho_{ij}(\widetilde{s})\right| \\
&\le \int_0^\infty |g(s; y) - g(\widetilde{s}; y)||F_i(y)F_j(y)|dy \\
&\le L\left(1 + |b| + |\widetilde{b}|\right)\left(1 + \|v\|_{L^2} + \|\widetilde{v}\|_{L^2}\right)\left(\left|b - \widetilde{b}\right| + \|v - \widetilde{v}\|_{L^2}\right).
\end{aligned}$$

In the case $i = j$, using the fact that $\inf_{s\in E_{loc}} \sigma_i(s) > 0$ by Lemma 3.6, we can thus find $L_i^{m,c} > 0$ for each $i \in \mathbb{N}$ such that for all $s, \widetilde{s} \in E_m$ with $\|s\|_E \le c$, $\|\widetilde{s}\|_E \le c$,

$$|\sigma_i(s) - \sigma_i(\widetilde{s})| = \frac{\left|\sigma_i^2(s) - \sigma_i^2(\widetilde{s})\right|}{\sigma_i(s) + \sigma_i(\widetilde{s})} \le L_i^{m,c} \|s - \widetilde{s}\|_E .$$

Using again the boundedness away from zero of σ_i and σ_j, we may also find $K_{ij}^{m,c} > 0$ for each (i, j) such that for all $s, \widetilde{s} \in E_m$ with $\|s\|_E \le c$, $\|\widetilde{s}\|_E \le c$,

$$\begin{aligned}
|\rho_{ij}(s) - \rho_{ij}(\widetilde{s})| &\le \frac{\left|\sigma_i(s)\sigma_j(s)\rho_{ij}(s) - \sigma_i(\widetilde{s})\sigma_j(\widetilde{s})\rho_{ij}(\widetilde{s})\right| + \left|\sigma_i(s)\sigma_j(s) - \sigma_i(\widetilde{s})\sigma_j(\widetilde{s})\right|}{\sigma_i(s)\sigma_j(s)} \\
&\le K_{ij}^{m,c} \|s - \widetilde{s}\|_E .
\end{aligned}$$

Because of the recursive definition of the c_{ij}, $j \le i$, as functions of the ρ_{ij}, $j \le i$, the same inequality (with a different constant) follows for each c_{ij} from the fact that all the c_{ij}, $j \le i$, are bounded by 1 and $\inf_{s\in E_{loc}} c_{ii}(s) > 0$ for all $i \in \mathbb{N}$ by Lemma 3.12. □

Lemma 4.8. *Let Assumptions 2.3, 3.1, 3.3 and 3.4 be satisfied. If $(s_n) \subset D(E_m; [0, T])$ is a sequence with $\sup_{u\le t} \|s_n(u) - s(u)\|_E \to 0$ for $t \in [0, T]$, then also*

$$\sup_{u\le t} \left\| G_m^5(s_n(u)) - G_m^5(s(u))\right\|_{L^2(\mathbb{R}_+^2)} \to 0.$$

Proof. Fix $\varepsilon > 0$ and let $(s_n) \subset D(E_m; [0, T])$ be any sequence satisfying $\sup_{u\le t} \|s_n(u) - s(u)\|_E \to 0$. Then there exists $c > 0$ such that $\|s(u)\|_E \le c$ and $\|s_n(u)\|_E \le c$ for all $n \in \mathbb{N}$ and $u \in [0, t]$. Similarly to the proof of Lemma 3.14 we can find a finite index set $J \subset \mathcal{I}_m$ such that for all $\widetilde{s} \in E_{loc}$,

$$\sum_{i\in\mathcal{I}_m\setminus J} \sigma_i^2(\widetilde{s}) < \frac{\varepsilon}{8}.$$

Moreover, by Lemma 4.7 we can find an $n_0 = n_0(\varepsilon, c)$ such that for all $n \ge n_0$ and $u \le t$,

$$\left(d_{ij}(s_n(u)) - d_{ij}(s(u))\right)^2 \le \frac{\varepsilon}{|J|(|J| + 1)} \quad \forall\, i \in J,\ j \le i.$$

Thus for all $n \ge n_0$,

$$\begin{aligned}
\sup_{u\le t} \left\| G_m^3(s_n(u)) - G_m^3(s(u))\right\|^2_{L^2\left(\mathbb{R}_+^2\right)} &= \sup_{u\le t} \sum_{i\in\mathcal{I}_m} \sum_{j\le i} \left(d_{ij}(s_n(u)) - d_{ij}(s(u))\right)^2 \\
&\le \frac{\varepsilon}{2} + 2\sup_{u\le t} \sum_{i\in\mathcal{I}_m\setminus J} \sum_{j\le i} \left\{\left(d_{ij}(s_n(u))\right)^2 + \left(d_{ij}(s(u))\right)^2\right\} \\
&= \frac{\varepsilon}{2} + 2\sup_{u\le t} \sum_{i\in\mathcal{I}_m\setminus J} \left\{(\sigma_i(s_n(u)))^2 + (\sigma_i(s(u)))^2\right\} < \varepsilon. \quad \square
\end{aligned}$$

The preceding results immediately yield the following theorem.

Theorem 4.9. *Let Assumptions 2.3, 2.6, 3.1, 3.3 and 3.4 be satisfied. If* $(s_n) \subset D(E_m; [0,T])$ *is a sequence such that* $\sup_{u\le t} \|s_n(u) - s(u)\|_E \to 0$ *for* $t \in [0,T]$*, then also*

$$\sup_{u\le t} \|G_m(s_n(u)) - G_m(s(u))\|_{\hat{E}} \to 0 \qquad \forall\, m \in \mathbb{N}.$$

5. Convergence of the stochastic integrals

Before stating our main result, we need one more assumption on the convergence of the initial values.

Assumption 5.1. There exists $S_0 = (B_0, V_0) \in E_{loc}$ such that for all $m \in \mathbb{N}$,

$$\left|B_0^{(n)} - B_0\right| + \left\|\left(V_0^{(n)} - V_0\right)\mathbb{1}_{[0,m]}\right\|_{L^2} \to 0.$$

For all $n, m \in \mathbb{N}$ we set

$$S_0^{(n),m} := \left(B_0^{(n)}, V_0^{(n)}\mathbb{1}_{[0,m]}\right), \quad S_0^m := \left(B_0, V_0\mathbb{1}_{[0,m]}\right)$$

and denote by $\widetilde{S}^{(n),m}$ the solution of

$$\widetilde{S}^{(n),m}(t) = S_0^{(n),m} + \int_0^t G_m^{(n)}\left(\widetilde{S}^{(n),m}(u-)\right) dY^{(n)}(u), \quad t \in [0,T].$$

Furthermore, we define for all $m, n \in \mathbb{N}$ the stopping time

$$\tau_m^{(n)} := \inf\left\{t \ge 0 : \ B^{(n)}(t) \ge m\right\} \wedge T$$

and the process

$$S^{(n),m}(t) := \left(B^{(n)}\left(t \wedge \tau_m^{(n)}\right), V^{(n)}\left(t \wedge \tau_m^{(n)}\right)\mathbb{1}_{[0,m]}\right), \quad t \in [0,T].$$

Note that, due to Assumptions 2.6(ii), 3.3(ii) and 3.4(ii) for all $n, m \in \mathbb{N}$ the process $\widetilde{S}^{(n),m}$ equals $S^{(n),m}$ on $\left[0, \tau_m^{(n)}\right]$ and

$$\tau_m^{(n)} = \inf\left\{t \ge 0 : \ \widetilde{B}^{(n),m}(t) \ge m\right\} \wedge T \quad \text{a.s.}$$

Definition 5.2. We say that S is a (global) solution of the infinite dimensional SDE

$$S(t) = S_0 + \int_0^t G(S(u))dY(u), \quad t \in [0,T], \tag{29}$$

if there exists a filtration $(\mathcal{F}_t)$ to which $S = (B, V)$ and Y are adapted and for **all** $m \in \mathbb{N}$,

$$\left(B(t), V(t)\mathbb{1}_{[0,m]}\right) = S_0^m + \int_0^t G^m(S(u))dY(u), \quad t \in [0,T].$$

We say that (S, τ, m) is a local solution of (29) if there exists a filtration $(\mathcal{F}_t)$ to which $S = (B, V)$ and Y are adapted, τ is an $(\mathcal{F}_t)$-stopping time, and $S = (B, V)$ satisfies the SDE

$$\left(B(t\wedge\tau), V(t\wedge\tau)\mathbb{1}_{[0,m]}\right) = S_0^m + \int_0^{t\wedge\tau} G^m(S(u))dY(u), \quad t \in [0,T].$$

5.1. Local relative compactness of the state process

Our main result states that the sequence of LOB models is relatively compact after localization and that any accumulation point is the solution to a certain infinite dimensional SDE driven by a pair consisting of a Brownian motion and a cylindrical Brownian motion.

Theorem 5.3. *Under Assumptions 2.1, 2.3, 2.6, 3.1, 3.3, 3.4, 3.10 and 5.1 the sequence $\left(S^{(n),m}\right)_{n\in\mathbb{N}}$ is relatively compact for all $m \in \mathbb{N}$ and any limit point $S^m = (B^m, V^m)$ gives a local solution (S^m, τ_m, m) of (29), i.e. for $(t, x) \in [0, T] \times [0, m]$,*

$$
\begin{aligned}
B^m(t\wedge\tau_m) &= B_0^m + \int_0^{t\wedge\tau_m} p\left(S^m(u)\right)du + \int_0^{t\wedge\tau_m} r\left(S^m(u)\right)dZ(u),\\
V^m(t\wedge\tau_m, x) &= V_0^m(x) + \int_0^{t\wedge\tau_m} \mu\left(S^m(u); x\right)du \\
&\quad + \sum_{i\in\mathcal{I}_m} f_i(x)\sum_{j\le i}\int_0^{t\wedge\tau_m} d_{ij}\left(S^m(u)\right)dW^j(u),
\end{aligned}
\tag{30}
$$

where W^j, $j \in \mathbb{N}$, and Z are independent Brownian motions and $\tau_m := \inf\{t \ge 0 : B^m(t) \ge m\} \wedge T$.

For the proof we will apply Theorem 7.6 of [19] and also partially follow the idea of the proof of Theorem 5.4 in [18]. However, note that there is a crucial difference between our Theorem 5.3 and Theorem 5.4 in [18]: while in [18] a local convergence result is derived by stopping the process appropriately and thereby localizing it in time, we do not only localize in time, but in fact have to localize in space as well.

Proof. Let us fix $m \in \mathbb{N}$. First, we will show that the sequence $\left(S_0^{(n),m}, \widetilde{S}^{(n),m}, Y^{(n)}\right)_{n\in\mathbb{N}}$ is relatively compact. To do this we will apply Theorem 7.6 in [19]. Let us verify the conditions of Theorem 7.6 in [19]: Theorems 4.1 and B.1 show that $\left(Y^{(n)}\right)_{n\in\mathbb{N}}$ is uniformly tight and converges weakly to Y in terms of finite dimensional distributions. Moreover, by Assumption 5.1 there exists $S_0^m \in E_m$ such that $S_0^{(n),m} \to S_0^m$. Hence, $\left(S_0^{(n),m}, Y^{(n)}\right) \Rightarrow (S_0^m, Y)$. Theorems 4.2 and 4.9 imply that $G_m^{(n)}$, $n \in \mathbb{N}$, and G_m satisfy Condition C.2 of [19]. Moreover, the compactness condition follows from Theorem 4.3 and we clearly have $\sup_n \sup_{s\in E_m} \left\|G_m^{(n)}(s)\right\|_{\hat{E}} < \infty$, due to Assumption 2.6(i), Lemmas 3.7 and 3.8. Hence, the requirements of Theorem 7.6 in [19] are satisfied and we may conclude that the sequence $\left(S_0^{(n),m}, \widetilde{S}^{(n),m}, Y^{(n)}\right)_{n\in\mathbb{N}}$ is relatively compact.

Next note that $\tau_m^{(n)}$ is a measurable function of $\widetilde{S}^{(n),m}$ for all $n \in \mathbb{N}$, say $\tau_m^{(n)} = h_m\left(\widetilde{S}^{(n),m}\right)$. We denote by D_{h_m} the set of discontinuities of h_m. Then Eq. (9) of Assumption 2.1 ensures that $\mathbb{P}(S^m \in D_{h_m}) = 0$ for any limit point S^m of $\widetilde{S}^{(n),m}$ and we may conclude by the continuous mapping theorem that the sequence $\left(S_0^{(n),m}, \widetilde{S}^{(n),m}\left(\cdot\wedge\tau_m^{(n)}\right), \tau_m^{(n)}, Y^{(n)}\right)_{n\in\mathbb{N}}$ is also relatively compact. Let $\left(S_0^m, \hat{S}^m, \tau_m^0, Y\right)$ denote a weak limit point of that sequence. Then Condition C.2 together with Theorem 5.5 in [19] yields that along a subsequence,

$$
S_0^{(n),m} + \int_0^{\cdot} G_m^{(n)}\left(\widetilde{S}^{(n),m}\left(u\wedge\tau_m^{(n)}\right)\right)dY^{(n)}(u) \Rightarrow S_0^m + \int_0^{\cdot} G_m\left(\hat{S}^m(u)\right)dY(u).
$$

Furthermore, as remarked earlier $\widetilde{S}^{(n),m}$ and $S^{(n),m}$ agree on $\left[0, \tau_m^{(n)}\right]$. Thus, by definition

$$S^{(n),m}(t) = \widetilde{S}^{(n),m}\left(t \wedge \tau_m^{(n)}\right) = S_0^{(n),m} + \int_0^{t\wedge\tau_m^{(n)}} G_m^{(n)}\left(\widetilde{S}^{(n),m}(u)\right) dY^{(n)}(u), \quad t \in [0,T].$$

Since $\hat{\tau}_m := h_m\left(\hat{S}^m\right) \le \tau_m^0$ a.s. and since $G_m\left(\hat{S}_m(u)\right) = G^m\left(\hat{S}_m(u)\right)$ for $u \le \hat{\tau}_m$, we conclude that $\left(S^{(n),m}\right)_{n\in\mathbb{N}}$ is relatively compact and that any limit point $\hat{S}^m$ of $\left(S^{(n),m}\right)_{n\in\mathbb{N}}$ gives a local solution of (29). □

5.2. Local weak convergence

So far we have shown that the sequence of our LOB model dynamics is relatively compact in a localized sense and that any accumulation point solves a certain infinite dimensional SDE. If the limiting SDE admits a unique strong solution, then the LOB dynamics converges to a unique limit as shown by the following theorem.

Theorem 5.4. *Suppose that all the assumptions of Theorem 5.3 are satisfied and that for all $m \in \mathbb{N}$ there exists a unique strong solution $\hat{S}^m = (\hat{B}^m, \hat{V}^m)$ of*

$$\begin{aligned} &\hat{S}^m(t) = S_0^m + \int_0^{t\wedge\tau_{m,m}} G_m\left(\hat{S}^m(u)\right) dY(u), \quad t \in [0,T], \\ &\tau_{m,l} := \inf\{t \ge 0 : \hat{B}^m(t) \ge l\} \wedge T. \end{aligned} \tag{31}$$

Then there exists a unique global solution $S = (B, V)$ of (29) and for all $m \in \mathbb{N}$,

$$S^{(n),m} \Rightarrow S^m \quad in \quad \mathcal{D}([0,T]; E),$$

where $S^m(t) := \left(B(t\wedge\tau_m), V(t\wedge\tau_m)\mathbb{1}_{[0,m]}\right)$, $t \in [0,T]$, and $\tau_m := \inf\{t \ge 0 : B(t) \ge m\} \wedge T$.

Proof. Strong uniqueness implies together with Assumptions 2.6, 3.3 and 3.4 that for all $m, k \in \mathbb{N}$, $\hat{S}^m$ equals $\left(\hat{B}^{m+k}, \hat{V}^{m+k}\mathbb{1}_{[0,m]}\right)$ almost surely on the interval $\left[0, \tau_{m,m} \wedge \tau_{m+k,m}\right]$. Thus for all $m, k \in \mathbb{N}$, $\tau_{m,m} = \tau_{m+k,m}$ and hence $\tau_{m,m} \le \tau_{m+k,m+k}$ a.s. Setting $\tau_0^0 := 0$ we define

$$B(t) := \sum_{m=1}^{\infty} \mathbb{1}_{[\tau_{m-1,m-1}, \tau_{m,m})}(t)\hat{B}^m(t), \quad t \in [0,T],$$

and for all $m \in \mathbb{N}$ and $x \in [m-1, m)$,

$$V(t,x) := \mathbb{1}_{[0,\tau_{m,m})}(t)\hat{V}^m(t,x) + \sum_{k=1}^{\infty} \mathbb{1}_{[\tau_{m+k-1,m+k-1}, \tau_{m+k,m+k})}(t)\hat{V}^{m+k}(t,x), \quad t \in [0,T].$$

Let $\tau_m := \inf\{t \ge 0 : B(t) \ge m\}$ and $\tau_\infty := \lim \tau_m$. Then by the linear growth condition of Assumption 2.6(i) we have for all $m \in \mathbb{N}$ and $t \in [0,T]$,

$$\begin{aligned} \mathbb{E}\left[B(t\wedge\tau_m)\right]^2 &\le 4\left[B_0^2 + \mathbb{E}\left(\int_0^t |p(S(u\wedge\tau_m))|\, du\right)^2 + \mathbb{E}\left(\int_0^t r(S(u\wedge\tau_m))\, dZ_u\right)^2\right] \\ &\le 4\left[B_0^2 + T\int_0^t \mathbb{E}|p(S(u\wedge\tau_m))|^2 du + \int_0^t \mathbb{E}|r(S(u\wedge\tau_m))|^2 du\right] \\ &\le 4B_0^2 + 2(4T+1)K^2 \int_0^t \left(1 + \mathbb{E}[B(u\wedge\tau_m)]^2\right) du, \end{aligned}$$

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which implies by Gronwall's inequality that

$$\mathbb{E}\left[B\left(T\wedge\tau_\infty\right)\right]^2\leq\liminf_m\mathbb{E}\left[B\left(T\wedge\tau_m\right)\right]^2<\infty.$$

Therefore $\tau_\infty = T$ a.s. and, since $G^m(S(u)) = G_m(S(u))$ on $\{u \leq \tau_m\}$ for all $m \in \mathbb{N}$, $S := (B, V)$ defines a global solution of (29), which must be unique as well. Now the weak convergence result follows from Theorem 5.3. □

5.2.1. Uniqueness

We are now going to analyse two classes of models which fit in the framework developed so far and which satisfy the assumptions of Theorem 5.4, i.e. they converge – in a local sense – in the scaling limit to the unique solution of the infinite dimensional SDE (29). For this it is sufficient to establish the local Lipschitz continuity of the coefficient function G_m on E_m for all $m \in \mathbb{N}$, so that (31) will have a unique solution for all $m \in \mathbb{N}$. Note that G_m^1, G_m^3, and G_m^6 are locally Lipschitz continuous and uniformly bounded on E_m by Assumption 2.6 and Lemmata 4.6 and 3.7. Hence, it remains to establish the local Lipschitz continuity of G_m^5.

Lemma 5.5. *Suppose in addition to the assumptions of Theorem 5.3 that $g(s; y)$ is independent of the state of the book for all $y \in \mathbb{R}_+$. Then each G_m is locally Lipschitz continuous and the conditions of Theorem 5.4 are satisfied.*

Proof. Since in this case G_m^5 does not depend on s, G_m^5 is trivially Lipschitz continuous and uniformly bounded on E_m^*. Therefore there exists a unique strong solution of

$$\widetilde{S}^m(t) = S_0^m + \int_0^t G_m\left(\widetilde{S}^m(u)\right)dY(u), \quad t \in [0,T], \tag{32}$$

by Corollary 7.8 in [19]. Hence, there exists a unique strong solution of (31) as well. □

The next lemma allows the volatility of the cumulated volume process to be state dependent. However, we require the dynamics of the system to only depend on current volumes through some approximation of the cumulated volume function.

For all $l_0, m \in \mathbb{N}$ we define the index sets $\mathcal{I}_m(l_0) := \{i \in \mathcal{I}_m : l(i) < l_0\}$ and $\mathcal{I}(l_0) := \{i \in \mathbb{N} : l(i) < l_0\}$.

Assumption 5.6. There is $l_0 \in \mathbb{N}$ such that for all pairs $s = (b, v), \widetilde{s} = (\widetilde{b}, \widetilde{v}) \in E_{loc}$ satisfying $b = \widetilde{b}$ and $\langle v, f_i\rangle = \langle \widetilde{v}, f_i\rangle \; \forall\, i \in \mathcal{I}(l_0)$, we have the equalities $p^{(n)}(s) = p^{(n)}(\widetilde{s})$, $q^{(n)}(s) = q^{(n)}(\widetilde{s})$, $h^{(n)}(s; y) = h^{(n)}(\widetilde{s}; y)$, $g^{(n)}(s; y) = g^{(n)}(\widetilde{s}; y)$ for all $n \in \mathbb{N}$, $y \in \mathbb{R}_+$.

Lemma 5.7. *Let the assumptions of Theorem 5.3 and Assumption 5.6 be satisfied. Then there exists a unique strong solution of (31) for each $m \in \mathbb{N}$.*

Proof. Fix $m \in \mathbb{N}$. We first show that there exists a unique strong solution to (32). Note that p and r are Lipschitz continuous on E_m by Assumption 2.6 and it follows from Lemmata 4.6 and 4.7 that d_{ij} and μ_i are also locally Lipschitz continuous on E_m. Moreover, each μ_i resp. d_{ij} is uniformly bounded and p and r satisfy a linear growth condition by Assumption 2.6(i). Therefore, the *finite* dimensional SDE

$$\overline{B}^m(t) = B_0 + \int_0^t p\left(\overline{S}^m(u)\right)du + \int_0^t r\left(\overline{S}^m(u)\right)dZ_u,$$

$$\overline{V}_i^m(t) = \langle V_0^m, f_i\rangle + \int_0^t \mu_i\left(\overline{S}^m(u)\right)du + \sum_{j\le i}\int_0^t d_{ij}\left(\overline{S}^m(u)\right)dW_u^j, \quad i\in\mathcal{I}_m(l_0),$$

$$\text{with} \quad \overline{V}^m := \sum_{i\in\mathcal{I}_m(l_0)} \overline{V}_i^m f_i \quad \text{and} \quad \overline{S}^m = \left(\overline{B}^m, \overline{V}^m\right)$$

has a unique strong solution $\overline{S}^m$. Given this solution let us define

$$\underline{V}^m(t) := V_0^m + \sum_{i\in\mathcal{I}_m} f_i \int_0^t \mu_i\left(\overline{S}^m(u)\right)du + \sum_{i\in\mathcal{I}_m} f_i \sum_{j\le i}\int_0^t d_{ij}\left(\overline{S}^m(u)\right)dW_u^j, \quad t\in[0,T].$$

Clearly, $\left(\overline{B}^m, \underline{V}^m\right)$ is a solution of (32) due to Assumption 5.6 and by construction it must be unique. It follows that $S^m := \left(\overline{B}^m(\cdot\wedge\tau_m), \underline{V}^m(\cdot\wedge\tau_m)\right)$ is the unique strong solution of (31). □

Remark 5.8. It is not clear to us how to establish the (strong or weak) uniqueness of a solution to the general infinite dimensional SDE of Theorem 5.3 apart from the two cases considered in this section. Indeed, even though the Cholesky factorization in finite dimensions is a Lipschitz continuous operation, it is known that the Lipschitz constant grows dramatically when the dimension is increased. This makes the search for conditions on g and h that yield strong uniqueness very difficult. Of course, one could alternatively look for weak uniqueness of a solution to the infinite dimensional SDE by considering the associated martingale problem. However, to the best of our knowledge also in this case the problem is still unsolved and requires further research.

5.2.2. *Examples*

We close this section with two examples where uniqueness of solutions to the limiting SDE can indeed be established.

Example 5.9. Let $\alpha, \delta, K, \eta, q > 0$ and suppose that there exist $c_1, c_2 \in (0,\infty)$ such that for all $n\in\mathbb{N}$ and $s=(b,v)\in E_{loc}$ with $0\le b\le c_1$ and $\left\|v\mathbb{1}_{[0,c_1]}\right\|_\infty < c_2$, the functions $p^{(n)}$ and $r^{(n)}$ are given by

$$p^{(n)}(s) = \frac{b}{20}\int_{(b-q)^+}^{b}(\alpha y - v(y))\,dy + \eta$$

$$\left(r^{(n)}(s)\right)^2 = \Delta x^{(n)}\eta + \delta^2 b^2.$$

This specifies uniquely the conditional distribution of the process $B^{(n)}$ (as long as $S^{(n)}$ does not exit the c_1-c_2-interval defined above). We have chosen $r^{(n)}$ and $p^{(n)}$ such that the volatility as well as the absolute value of the drift of the price process are increasing in the price itself. Moreover, high volumes at the top of the book (compared to some reference level specified by α) lead to a negative drift for the price process, while low volumes at the top of the book lead to a positive drift. In the scaling limit the price follows the volume-dependent, "generalized Black–Scholes" dynamics

$$dB(t) = \left(\frac{B(t)}{20}\int_{(B(t)-q)^+}^{B(t)}(\alpha y - V(t,y))dy + \eta\right)dt + \delta B(t)dZ(t).$$

Order placements/cancellations outside the spread are assumed to be of size ± 100, i.e. $\mathbb{P}\Big(\omega_k^{(n)} = \pm 100\Big) = 1$ for all $n \in \mathbb{N}$, $k \leq T_n$. Furthermore, we suppose that there exist two functions $f_{\pm}^{(n)} : E_{loc} \times \mathbb{R}_+ \to \mathbb{R}_+$ for every $n \in \mathbb{N}$ such that for all $B \in \mathcal{B}(\mathbb{R}_+)$ and $k = 1, \dots, T_n$,

$$\mathbb{P}\left(\phi_k^{(n)} = C,\ \omega_k^{(n)} = \pm 1,\ \pi_k^{(n)} \in B \,\Big|\, \mathcal{F}_{k-1}^{(n)}\right) = \int_B f_{\pm}^{(n)}\left(S_{k-1}^{(n)}; y\right) dy \quad \text{a.s.}$$

Let $\xi : \mathbb{R} \to \mathbb{R}_+$ be continuously differentiable with bounded derivative and suppose that ξ has compact support in $\mathbb{R}_-$. Let $D > 0$ and suppose that the $f_{\pm}^{(n)}$ are for all $y \in \mathbb{R}_+$ and $s = (b, v) \in E_{loc}$ with $0 \leq b \leq c_1$ and $\left\| v \mathbb{1}_{[0,c_1]} \right\|_\infty < c_2$ given by

$$\begin{aligned} f_+^{(n)}(s; y) &= \left(1 - \Delta p^{(n)} \left(r^{(n)}(s)\right)^2\right) \frac{\exp(-y/10)}{10\left(2 + \Delta v^{(n)}\right)} \\ &\quad \times \left(1 - \frac{\Delta v^{(n)}}{10} \left\langle v(\cdot + b)\mathbb{1}_{[-b,0]}, \xi \right\rangle + \frac{\Delta v^{(n)}}{10(1 + |y - b|)}\right), \\ f_-^{(n)}(s; y) &= \left(1 - \Delta p^{(n)} \left(r^{(n)}(s)\right)^2\right) \frac{\exp(-y/10)}{10\left(2 + \Delta v^{(n)}\right)} \\ &\quad \times \left(1 + \frac{\Delta v^{(n)}}{10} \left\langle v(\cdot + b)\mathbb{1}_{[-b,0]}, \xi \right\rangle + \frac{\Delta v^{(n)}|y - b|}{10(1 + |y - b|)}\right). \end{aligned}$$

This means that the location at which order placements and cancellations take place is exponentially distributed. Order cancellations are more likely to happen further away from the current best bid price or if cumulated volumes are quite high. On the other hand, order placements occur more frequently in the proximity of the current best bid price or if cumulated volumes are low. The above specification of $f^{(n)}$ yields for s as above

$$g(s; y) = 5 \exp\left(-\frac{y}{10}\right)$$

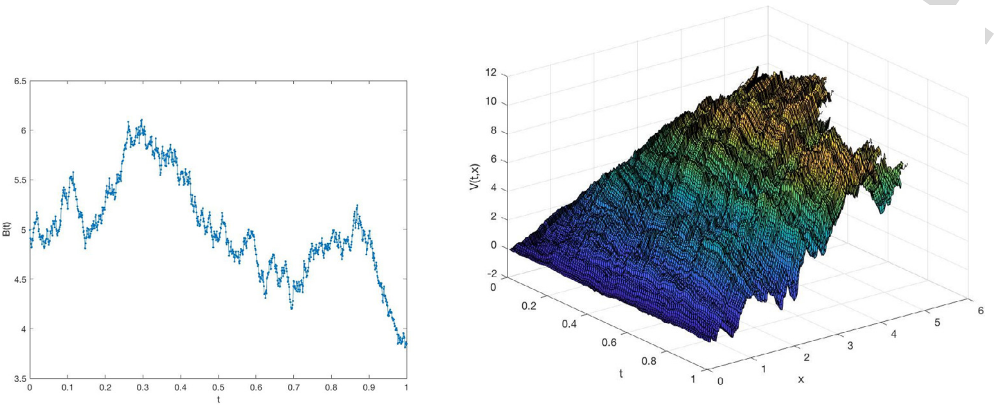
and

$$h(s; y) = \frac{1}{2} \exp\left(-\frac{y}{10}\right)\left(-2\left\langle v(\cdot + b)\mathbb{1}_{[-b,0]}, \xi \right\rangle + \frac{1 - |y - b|}{1 + |y - b|}\right).$$

Therefore, the covariance structure does not depend on s, which implies that $d_{ij}(s) = d_{ij}(\widetilde{s})$ for all $s, \widetilde{s}$ as above and $i, j \in \mathbb{N}$. However, note that h and hence also μ depend on s. Moreover, one can check that for n large enough all assumptions of Theorem 5.3 are satisfied. Hence, Theorem 5.4 and Lemma 5.7 imply that the limiting SDE has a unique solution in this case and that $S^{(n)}$ converges weakly to this solution in a localized sense (see Fig. 1).

While the above example shows that even with constant G^5 we can already model many interesting dependencies, one disadvantage is that the conditional distribution of the location variables $\pi_k^{(n)}$, $n \in \mathbb{N}$, $k \leq T_n$, of order placements resp. cancellations cannot be taken to be relative to the current best bid price, which would be reasonable from a microeconomic point of view. Another disadvantage is that for constant G^5 the $L^2(\mathbb{R}_+)$-valued process V is not necessarily positive respectively increasing in $x \in \mathbb{R}_+$. Therefore, it is more reasonable to think of log volumes instead of real volumes.

The next example allows to model the location of order placements being distributed relative to the current best bid price.



(a) Bid price process B (b) Bid volumes $V(t,x),\ x \leq B(t)$

Fig. 1. Realization of the process $S = (B, V)$ for $T = 1$, $\delta = 0.3$, $q = 0.1$, $\alpha = 2$, $\eta = 10^{-10}$, $\xi(x) = 10(x+1)^2 x^2 \exp(10x)$ and initial values $B_0 = 5$, $V_0(x) = x \mathbb{1}_{[0,10]}(x)$.

Example 5.10. For given $s = (b, v) \in E_{loc}$ we set

$$v_{l_0}(y) := \sum_{i \in \mathcal{I}(l_0)} \langle v, f_i \rangle f_i(y), \qquad y \in \mathbb{R}_+.$$

Note that v_{l_0} is the projection of v on the subspace spanned by $\{f_i : i \in \mathcal{I}(l_0)\}$, which consists of all step functions on the grid $k2^{-l_0}$, $k \in \mathbb{N}$. Hence, v_{l_0} has the alternative representation

$$v_{l_0}(y) = \sum_{k \in \mathbb{N}_0} a_k \mathbb{1}_{[k2^{-l_0},(k+1)2^{-l_0})}(y) \quad \text{with} \quad a_k := 2^{-l_0} \int_{k2^{-l_0}}^{(k+1)2^{-l_0}} v(x)dx.$$

Therefore, $\left\{v_{l_0}(y) : y \leq 2^{-l_0} \lfloor b2^{l_0} \rfloor\right\}$ only depends on $\{v(y) : y \leq b\}$ for any $s = (b, v) \in E_{loc}$. Similarly to Example 5.9 we specify the price dynamics as follows : let $\alpha, K, \eta, c_1, c_2 > 0$, $q \geq 2^{-l_0}$ and suppose that for all $n \in \mathbb{N}$ and $s = (b, v) \in E_{loc}$ with $0 \leq b \leq c_1$ and $\left\| v_{l_0} \mathbb{1}_{[0,c_1]} \right\|_\infty < c_2$,

$$p^{(n)}(s) = \frac{b}{10} \int_{(b-q)^+}^{\lfloor b2^{l_0} \rfloor 2^{-l_0}} \left(\alpha y - v_{l_0}(y)\right) dy + \eta$$

$$\left(r^{(n)}(s)\right)^2 = \Delta x^{(n)} \eta + \delta^2 b^2.$$

As in Example 5.9 we suppose that there exists a function $f^{(n)}$ for every $n \in \mathbb{N}$ such that for all $A \in \mathcal{B}([-M, M])$, $B \in \mathcal{B}(\mathbb{R}_+)$, and $k = 1, \ldots, T_n$,

$$\mathbb{P}\left(\phi_k^{(n)} = C,\ \omega_k^{(n)} \in A,\ \pi_k^{(n)} \in B \,\middle|\, \mathcal{F}_{k-1}^{(n)}\right) = \int_B \int_A f^{(n)}\left(S_{k-1}^{(n)}; x, y\right) dxdy \quad \text{a.s.}$$

For $s = (b, v) \in E_{loc}$ with $0 \leq b \leq c_1$ and $\left\| v_{l_0} \mathbb{1}_{[0,c_1]} \right\|_\infty < c_2$ let

$$f^{(n)}(s; x, y) := C_n(s) f^{(n),1}(v; x) \exp\left(-\frac{1}{2}(y-b)^2\right),$$

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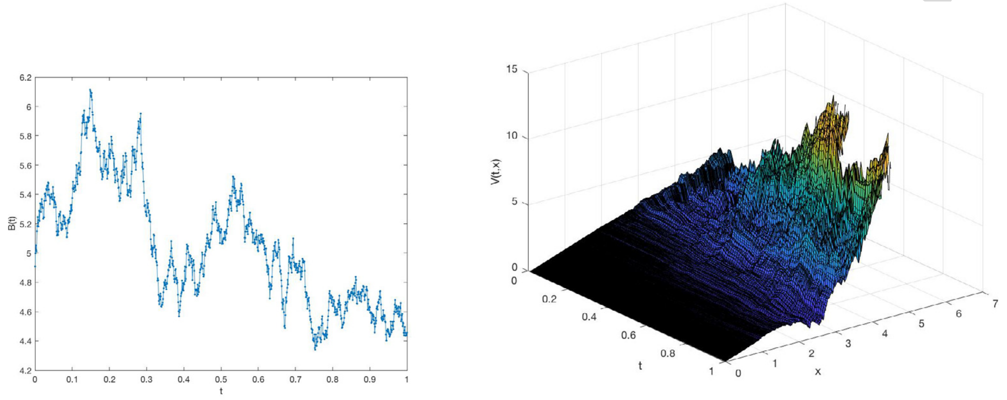

(a) Bid price process B

(b) Bid volumes $V(t,x),\ x \leq B(t)$

Fig. 2. Realization of the process $S = (B, V)$ for $T = 1$, $\delta = 0.3$, $q = 0.1$, $\alpha = 2$, $M = 10$, $\eta = 10^{-10}$, $\xi(x) = 6(x+1)^2 x^2 \exp(6x)$ and initial values $B_0 = 5$, $V_0(x) = x\mathbb{1}_{[0,10]}(x)$.

where $C_n(s)$ is chosen such that $\int_0^\infty \int_{-M}^M f^{(n)}(s; x, y)dxdy = 1 - \Delta p^{(n)} \left(r^{(n)}(s)\right)^2$. It can be shown that as $n \to \infty$, $C_n(s)$ converges to $1/\Phi(b)$. The function $f^{(n),1}$ specifies the conditional distribution of the order size and is given by

$$f^{(n),1}(v; x) := \frac{1 - a_n(v)}{M} \mathbb{1}_{[0,M]}(x) + \frac{a_n(v)}{M} \mathbb{1}_{[-M,0]}(x)$$

with

$$a_n(v) := \frac{1}{2} - \Delta v^{(n)} \left\langle v_{l_0}\left(\cdot + \lfloor b 2^{l_0} \rfloor 2^{-l_0}\right) \mathbb{1}_{\left[-\lfloor b2^{l_0}\rfloor 2^{-l_0}, 0\right]}, \xi \right\rangle,$$

for some function ξ with compact support in $\mathbb{R}_-$. In this case

$$g(s; y) = \frac{M^2}{3\Phi(b)} \exp\left(-\frac{1}{2}(y-b)^2\right)$$

as well as

$$h(s; y) = \frac{M}{\Phi(b)} \exp\left(-\frac{1}{2}(y-b)^2\right) \left\langle v_{l_0}\left(\cdot + \lfloor b2^{l_0} \rfloor 2^{-l_0}\right) \mathbb{1}_{[-\lfloor b2^{l_0}\rfloor 2^{-l_0}, 0]}, \xi \right\rangle$$

both depend on s. Hence, also the $d_{ij},\ i, j \in \mathbb{N}$, will vary with s. Still, it can be easily checked that all assumptions of Lemma 5.7 and Theorem 5.4 are satisfied.

In Fig. 2 we plot a realization of the process $S = (B, V)$ for certain parameter values. Since the volatility function g depends on b in this example, the volumes turn out to be much more volatile near the best bid price than deeper in the book, which is very reasonable from an empirical point of view.

5.3. *From microscopic to macroscopic order book models*

In this paper we start from a microscopic description of the limit order book dynamics and derive its high frequency limit when the number of orders goes to infinity, while each individual one is of negligible size. The resulting diffusion approximation can then be seen as a macroscopic description of the limit order book dynamics when viewed at intermediate time intervals (e.g. a

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few seconds for MSFT, cf. Section 1.2). In the literature macroscopic limit order book models have been suggested and analysed e.g. in [17,23]. In [23] the author suggests to model the volume density function of the bid side of the limit order book[7] as the solution to the following SPDE with $\alpha, \rho \in \mathbb{R}$ and a regular scaling function $\sigma : \mathbb{R}_+ \to \mathbb{R}$, driven by space–time white noise:

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= \alpha \frac{\partial^2 v}{\partial x^2}(t,x) + \sigma(x-P(t))\frac{\partial^2 W}{\partial x \partial t}, \quad && x < P(t), \\ v(t,x) &= 0, && x \geq P(t), \end{aligned} \tag{33}$$

with moving boundary p (called mid-price) satisfying

$$\rho P'(t) = -\frac{\partial v}{\partial x}(t, P(t)-).$$

This model was generalized in several aspects in [17], replacing the white noise process by a more general stochastic noise process and allowing for a larger class of integrands. In [23] the dynamics of v as in (33) are motivated as follows: the drift term stems from the assumption that ”limit orders are placed, cancelled, and executed in a manner where jitters tend to be rapidly smoothed out”, thereby not having a first order, but a second order impact on the evolution of volumes. The stochastic integral is supposed to model the randomness of the limit order flow, whose intensity varies across different price levels and tends to vanish at price levels far below the mid price. Finally, the mid price changes relative to the “strength” of the limit orders placed around it. Let us suppose that $\alpha = 0$, i.e. there are no jitters. In this case, if we are now looking at the integrated bid volume V, then

$$V(t,x) := \int_0^x v(t,y)dy = V_0(x) + \int_0^t \int_0^x \sigma(y-P(u))W(dy,du), \quad x < P(t). \tag{34}$$

In §1 of [21] it is shown that a cylindrical Brownian motion $\widetilde{W}$ on $L^2(\mathbb{R})$ can be constructed from space–time white noise as follows: for any $k \in L^2$ one sets

$$\widetilde{W}_t(k) := \int_0^t \int_{\mathbb{R}} k(x)W(dx,du).$$

Using the representation $\widetilde{W}_t = \sum_i f_i \widetilde{W}_t^i$ with independent Brownian motions $(\widetilde{W}^i)_{i\in\mathbb{N}}$, we see that the stochastic integral is of the form

$$\begin{aligned} \int_0^t \int_0^x \sigma(y-P(u))W(du,dy) &= \sum_i f_i(x) \int_0^t \int_{\mathbb{R}} F_i(y)\sigma(y-P(u))W(dy,du) \\ &= \sum_i f_i(x) \sum_j \int_0^t \langle \sigma(\cdot - P(u))F_i, f_j \rangle d\widetilde{W}_u^j. \end{aligned}$$

Replacing the bid price B by the mid price P in the state process $S = (P, V)$ and defining for any $s = (p, v) \in E_{loc}$ and $x \in \mathbb{R}$ the function $g(s;x) := \sigma^2(x-p)$, we see that for all $i_1, i_2 \in \mathbb{N}$.

$$\begin{aligned} \left[\langle V, f_{i_1}\rangle, \langle V, f_{i_2}\rangle\right]_t &= \int_0^t \sum_j \langle \sigma(\cdot - P(u))F_{i_1}, f_j\rangle \langle \sigma(\cdot - P(u))F_{i_2}, f_j\rangle du \\ &= \int_0^t \int_{\mathbb{R}} g(S(u);y)F_{i_1}(y)F_{i_2}(y)dydu. \end{aligned}$$

[7] In [23] the author studies the one-sided stochastic Stefan problem as given above and then extends it to propose a model for the two-sided limit order book.

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This shows that the stochastic integral in (34) coincides with the stochastic integral term in Eq. (30) of Theorem 5.3 for this particular choice of g.[8] Hence, one can conclude that in this paper we have provided a justification for the white noise term appearing in the macroscopic limit order book dynamics (33).

Appendix A. Orthogonal decomposition of sequences of random variables

In this appendix we derive an orthogonal decomposition result for sequences of random variables. Specifically, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a sequence of normalized random variables Z^i and denote by ρ_{ij} the correlation between the variables Z^i and Z^j $(i, j \in \mathbb{N}, j \leq i)$. In terms of these quantities we define an array of real numbers $c_{ij},\ j \leq i$, as well as a sequence of random variables $W^i,\ i \in \mathbb{N}$, via the following algorithm:

Put $c_{11} := 1$ and $W^1 := Z^1$.
For $i = 2, 3, 4, \cdots$:
For $j = 1, 2, \ldots, i-1$:
If $c_{jj} = 0$,
then $c_{ij} := 0$.
Else

$$c_{ij} := \frac{1}{c_{jj}} \left(\rho_{ij} - \sum_{l<j} c_{il} c_{jl} \right).$$

Next j.

$$c_{ii} := \left(1 - \sum_{j<i} (c_{ij})^2 \right)^{1/2}$$

$$W^i := \begin{cases} \frac{1}{c_{ii}} \left(Z^i - \sum_{j<i} c_{ij} W^j \right) & : c_{ii}\ 0 \\ U^i & : c_{ii} = 0. \end{cases}$$

Next i.

Lemma A.1. *For all $n, i \in \mathbb{N}$, $j \leq i$, the following holds:*

1. $\mathbb{E}\left(Z^i W^j\right) = c_{ij}$, *2.* $\sum_{j<i} c_{ij}^2 \leq 1$, *3.* $\mathbb{E}\left(W^i W^j\right) = \delta_{ij}$,

4. $Z^i = \sum_{j \leq i} c_{ij}\, W^j$.

Proof. We proceed by induction over i. For $i = 1$ we have $c_{11} \equiv 1$ and $W^1 = Z^1$, which trivially gives 1–4.

Now assume that 1–4 are true up to index $i - 1$; in particular c_{jj} and W^j are well defined for $j < i$. We first show 1 for indices i and $j < i$. This will be done by induction over j. For $j = 1$, we have by definition

$$\mathbb{E}\left(Z^i W^1\right) = \mathbb{E}\left(Z^i Z^1\right) = \rho_{i1} = c_{i1}.$$

[8] We note that the assumption on σ made in [23] does not align with our Assumption 3.3(i), which is made for technical convenience only to avoid further localization arguments.

Now consider an arbitrary $j < i$ and suppose that the claim is true up to $j-1$. If $c_{jj} = 0$, then

$$\mathbb{E}\left(Z^i W^j\right) = \mathbb{E}\left(Z^i U^j\right) = 0 = c_{ij}.$$

If $c_{jj} > 0$, then by definition and the induction hypothesis

$$\mathbb{E}\left(Z^i W^j\right) = \frac{1}{c_{jj}}\mathbb{E}\left(Z^i\left(Z^j - \sum_{l\le j-1} c_{jl}W^l\right)\right) = \frac{1}{c_{jj}}\left(\rho_{ij} - \sum_{l\le j-1} c_{il}c_{jl}\right) = c_{ij}.$$

This implies that

$$0 \le \mathbb{E}\left(Z^i - \sum_{j<i} c_{ij}W^j\right)^2 = \left(1 - 2\sum_{j<i} c_{ij}^2\right) + \mathbb{E}\left(\sum_{j<i} c_{ij}W^j\right)^2 = 1 - \sum_{j<i} c_{ij}^2,$$

where the last equality follows from part 3 of the induction hypothesis. This proves 2. Moreover, $\mathbb{E}\left(Z^i W^i\right) = c_{ij}$ for $j = i$ follows now in the same way as above for $j < i$. This completes the proof of 1.

Next we show 3. If $c_{ii} = 0$, the claim is trivial because U^i is independent of everything else. If $c_{ii} > 0$, then for all $j < i$ by definition and the induction hypothesis,

$$\mathbb{E}\left(W^i W^j\right) = \frac{1}{c_{ii}}\mathbb{E}\left(\left(Z^i - \sum_{l<i} c_{il}W^l\right)W^j\right) = \frac{1}{c_{ii}}\left[\mathbb{E}\left(Z^i W^j\right) - c_{ij}\right] \overset{1}{=} 0$$

as well as

$$\mathbb{E}\left(W^i\right)^2 = \frac{1}{c_{ii}^2}\cdot\mathbb{E}\left(Z^i - \sum_{j<i} c_{ij}W^j\right)^2 = \frac{1}{c_{ii}^2}\left(1 - \sum_{j<i} c_{ij}^2\right) = 1.$$

Thus, 3 is proven. It remains to show 4. If $c_{ii} \neq 0$, 4. is trivial. Hence, suppose that $c_{ii} = 0$. Then,

$$\mathbb{E}\left(Z^i - \sum_{j<i} c_{ij}W^i\right)^2 = \left(1 - \sum_{j<i} c_{ij}^2\right) = c_{ii}^2 = 0,$$

which shows that 4. is also true in this case. □

Next we define for all $i \in \mathbb{N}$, $j \le i$ numbers α_{ij} iteratively as follows:

For $i = 1, 2, 3, 4, \cdots$:

$$\alpha_{ii} := \begin{cases} \frac{1}{c_{ii}} & : \ c_{ii} > 0 \\ 1 & : \ c_{ii} = 0. \end{cases}$$

For $j = i-1, i-2, \ldots, 1$:

If $c_{jj} = 0$,

then $\alpha_{ij} := 0$.

Else

$$\alpha_{ij} := -\frac{1}{c_{jj}}\left(\sum_{j<l\le i} \alpha_{il}c_{lj}\right). \tag{A.1}$$

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Next j.

Next i.

Note that $(\alpha_{ij})_{i\in\mathbb{N},\ j\le i}$ can be regarded as the "inverse" of $(c_{ij})_{i\in\mathbb{N},\ j\le i}$ in the following sense: for fixed $i, j \in \mathbb{N}$ with $j \le i$ one has

$$\sum_{j\le l\le i} \alpha_{il} c_{lj} \overset{(A.1)}{=} \mathbb{1}_{\{c_{jj}=0\}} \sum_{j\le l\le i} \alpha_{il} c_{lj} + \mathbb{1}_{\left\{c^{(n)}_{jj}>0\right\}} \delta_{ij} = \mathbb{1}_{\{c_{ii}>0\}} \delta_{ij},$$

where the last equality follows from the fact that $c_{lj} = 0$ for all $l > j$ if $c_{jj} = 0$. Hence, if $c_{ii} > 0$, then

$$\sum_{j\le i} \alpha_{ij} Z^j = \sum_{j\le i} \alpha_{ij} \sum_{l\le j} c_{jl} W^l = \sum_{l\le i} W^l \sum_{l\le j\le i} \alpha_{ij} c_{jl} = W^i. \tag{A.2}$$

Appendix B. Integration with respect to $Y^{(n)}$ and Y

In this appendix we introduce the stochastic integrals with respect to $Y^{(n)}$ and Y. The concept of integration follows [19]. We recall the definition of the random variables $\delta W^{(n),i}_k$ in (21) and put ($i \in \mathbb{N},\ t \in [0, T]$),

$$W^{(n),i}(t) := W^{(n)}(f_i, t) = \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \delta W^{(n),i}_k \qquad \text{and} \qquad W^i := W(f_i, \cdot)$$

where W is a cylindrical Brownian motion. Thus, the random variables the W^i, $i \in \mathbb{N}$, are independent Brownian motions and each $Y^{(n)} = \left(Y^{(n)}_t\right)_{t\in[0,T]}$ is adapted to the filtration $\left(\hat{\mathcal{F}}^{(n)}_t\right)_{t\in[0,T]}$ defined via

$$\hat{\mathcal{F}}^{(n)}_t := \mathcal{F}^{(n)}_k, \quad t^{(n)}_k \le t < t^{(n)}_{k+1}.$$

As integrands for $Y^{(n)}$ we consider càdlàg, $\left(\hat{\mathcal{F}}^{(n)}_t\right)_{t\in[0,T]}$-adapted processes which take their values in the space

$$\hat{E} := \mathbb{R} \times L^2(\mathbb{R}_+; \mathbb{R}) \times \mathbb{R} \times L^2(\mathbb{R}_+; \mathbb{R}) \times L^2\left(\mathbb{R}^2_+; \mathbb{R}\right) \times L^2\left(\mathbb{R}_+; \mathbb{R}\right),$$

endowed with the norm

$$\begin{aligned}&\|(a_1, a_2, a_3, a_4, a_5, a_6)\|_{\hat{E}} \\ &\quad := |a_1| + \|a_2\|_{L^2(\mathbb{R}_+)} + |a_3| + \|a_4\|_{L^2(\mathbb{R}_+)} + \|a_5\|_{L^2\left(\mathbb{R}^2_+\right)} + \|a_6\|_{L^2(\mathbb{R}_+)}.\end{aligned}$$

We define $\mathcal{S}^{(n)}_{\hat{E}}$ as the set of processes $a^{(n)} : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \hat{E}$ that are of the form

$$\begin{aligned}a^{(n)}(t; x, y) := \Bigg(&a^{1,(n)}(t), \sum_j a^{2,(n)}_j(t) f_j(y), a^{3,(n)}(t), \sum_i a^{4,(n)}_i(t) f_i(x), \\ &\sum_{ij} a^{5,(n)}_{ij}(t) f_i(x) f_j(y), \sum_i a^{6,(n)}_i(t) f_i(x)\Bigg)\end{aligned} \tag{B.3}$$

for càdlàg and $\left(\hat{\mathcal{F}}^{(n)}_t\right)$-adapted processes $a^{1,(n)}, a^{2,(n)}_j, a^{3,(n)}, a^{4,(n)}_i, a^{5,(n)}_{ij}, a^{6,(n)}_i$, $i, j \in \mathbb{N}$, of which all but finitely many are zero. For $a^{(n)} \in \mathcal{S}^{(n)}_{\hat{E}}$ with the representation as above, the integral

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with respect to $Y^{(n)}$ is defined as

$$\int_0^t a^{(n)}(u-)dY^{(n)}(u) :=$$

$$\Bigg(\int_0^t a^{1,(n)}(u-)dZ^{(n)}(u) + \sum_j \int_0^t a_j^{2,(n)}(u-)dW^{(n),j}(u) + \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} a^{3,(n)}\left(t_k^{(n)}-\right)\Delta t^{(n)},$$

$$\sum_i f_i \int_0^t a_i^{4,(n)}(u-)dZ^{(n)}(u) + \sum_{ij} f_i \int_0^t a_{ij}^{5,(n)}(u-)dW^{(n),j}(u)$$

$$+ \sum_i f_i \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} a_i^{6,(n)}\left(t_k^{(n)}-\right)\Delta t^{(n)}\Bigg).$$

Theorem B.1. *Suppose that Assumptions 2.1, 2.3, 3.1, 3.3(i), 3.4(i) and 3.10 hold. Then the sequence $Y^{(n)}$ is uniformly tight, i.e.*

$$\mathcal{H}_t := \bigcup_n \left\{ \left\| \int_0^t a^{(n)}(u-)dY^{(n)}(u) \right\|_E \ : \ a^{(n)} \in \mathcal{S}_{\hat{E}}^{(n)},\ \sup_{u\le t} \left\|a^{(n)}(u)\right\|_{\hat{E}} \le 1 \text{ a.s.} \right\}$$

is stochastically bounded for all $t \in [0, T]$.

Proof. It is sufficient to show that for any $t \in [0, T]$ there exists a constant $C(t)$ such that for all $n \in \mathbb{N}$ and $a^{(n)} \in \mathcal{S}_{\hat{E}}^{(n)}$ with $\sup_{u\le t} \left\|a^{(n)}(u)\right\|_{\hat{E}} \le 1$,

$$\mathbb{E} \left\| \int_0^t a^{(n)}(u-)dY^{(n)}(u) \right\|_E \le C(t).$$

Let $a^{(n)} \in \mathcal{S}_{\hat{E}}^{(n)}$ satisfy $\sup_{u\le t} \left\|a^{(n)}(u)\right\|_{\hat{E}} \le 1$. Thus for all $u \le t$,

$$\max\Bigg\{ \left|a^{1,(n)}(u)\right|, \sum_j \left(a_j^{2,(n)}(u)\right)^2, \left|a^{3,(n)}(u)\right|, \sum_i \left(a_i^{4,(n)}(u)\right)^2,$$

$$\sum_{ij} \left(a_{ij}^{5,(n)}(u)\right)^2, \sum_i \left(a_i^{6,(n)}(u)\right)^2 \Bigg\} \le 1 \quad \text{a.s.}$$

This implies that

$$\mathbb{E}\left| \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} a^{3,(n)}\left(t_k^{(n)}-\right)\Delta t^{(n)} \right| \le \Delta t^{(n)} \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \mathbb{E}\left|a^{3,(n)}\left(t_k^{(n)}-\right)\right| \le t$$

and, since only finitely many of the $a_i^{6,(n)}$ are assumed to be unequal zero,

$$\mathbb{E}\left\| \sum_i f_i \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} a_i^{6,(n)}\left(t_k^{(n)}-\right)\Delta t^{(n)} \right\|_{L^2}$$

$$\le \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \Delta t^{(n)} \mathbb{E}\left(\sum_i \left(a_i^{6,(n)}\left(t_k^{(n)}-\right)\right)^2 \right)^{1/2} \le t.$$

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For the other four terms recall that $a^{(n)}$ is $\left(\hat{\mathcal{F}}_t^{(n)}\right)$-adapted. Thus $a^{(n)}\left(t_k^{(n)}-\right) \in \mathcal{F}_{k-1}^{(n)}$ for $k = 1, \dots, T_n$. So,

$$\mathbb{E}\left(\int_0^t a^{1,(n)}(s-)dZ^{(n)}(s)\right)^2 = \mathbb{E}\left(\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} a^{1,(n)}\left(t_k^{(n)}-\right)\delta Z_k^{(n)}\right)^2 \leq \Delta t^{(n)} \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \mathbb{E}\left(a^{1,(n)}\left(t_k^{(n)}-\right)\right)^2 \leq t$$

and, since only finitely many of the $a_{ij}^{4,(n)}$ are assumed to be unequal zero,

$$\mathbb{E}\left\|\sum_i f_i \int_0^t a_i^{4,(n)}(s-)dZ^{(n)}(s)\right\|_{L^2}^2 = \sum_i \mathbb{E}\left(\sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} a_i^{4,(n)}\left(t_k^{(n)}-\right)\delta Z_k^{(n)}\right)^2 \leq \Delta t^{(n)} \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\sum_i \mathbb{E}\left(a_i^{4,(n)}\left(t_k^{(n)}-\right)\right)^2 \leq t.$$

Similarly, since only finitely many of the $a_j^{2,(n)}$ and $a_{ij}^{5,(n)}$ are assumed to be unequal zero,

$$\begin{aligned}\mathbb{E}\left|\sum_j \int_0^t a_j^{2,(n)}(u-)dW^{(n),j}(u)\right| &\leq \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \mathbb{E}\left|\sum_j a_j^{2,(n)}\left(t_k^{(n)}-\right)\delta W_k^{(n),j}\right| \\ &\leq \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor}\left(\mathbb{E}\left[\sum_j a_j^{2,(n)}\left(t_k^{(n)}-\right)\delta W_k^{(n),j}\right]^2\right)^{1/2} \\ &\leq \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \Delta t^{(n)}\mathbb{E}\sum_j\left(a_j^{2,(n)}\left(t_k^{(n)}-\right)\right)^2 \leq t\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left\|\sum_{ij} f_i \int_0^t a_{ij}^{5,(n)}(u-)dW^{(n),j}(u)\right\|_{L^2}^2 &= \sum_i \mathbb{E}\left(\sum_j \int_0^t a_{ij}^{5,(n)}(u-)dW^{(n),j}(u)\right)^2 \\ &= \sum_i \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \mathbb{E}\left(\sum_j a_{ij}^{5,(n)}\left(t_k^{(n)}-\right)\delta W_k^{(n),j}\right)^2 \\ &= \sum_{ij}\Delta t^{(n)} \sum_{k=1}^{\lfloor t/\Delta t^{(n)}\rfloor} \mathbb{E}\left(a_{ij}^{5,(n)}\left(t_k^{(n)}-\right)\right)^2 \leq t. \quad \square\end{aligned}$$

The preceding theorem implies that $Y^{(n)} = \left(Y_t^{(n)}\right)_{t\in[0,T]}$ is a standard $(E, \hat{E})$-semimartingale in the sense of [19]. Therefore, the definition of the stochastic integral $\int a_-^{(n)} dY^{(n)}$ extends to all càdlàg, adapted, uniformly bounded, $\hat{E}$-valued processes $a^{(n)}$, where the resulting infinite sums can be shown to exist as limits in probability.

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Similarly, if $(\mathcal{F}_t)_{t\in[0,T]}$ is any filtration to which $Y=(Y_t)_{t\in[0,T]}$ is adapted, we will denote by $\mathcal{S}_{\hat{E}}$ the set of $\hat{E}$-valued processes a of the form (B.3), for which all $a^1, a^2_j, a^3, a^4_i, a^5_{ij}, a^6_i$, $i,j\in\mathbb{N}$, are càdlàg, $(\mathcal{F}_t)$-adapted processes, of which all but finitely many are zero. The integral of $a\in\mathcal{S}_{\hat{E}}$ with respect to Y is then defined by,

$$\begin{aligned}\int_0^t a(u-)dY(u) := \Bigg(&\int_0^t a^1(u-)dZ(u)+\sum_j\int_0^t a^2(u-)dW^j(u)+\int_0^t a^3(u)du,\\ &\sum_i\int_0^t a^4_i(u-)dZ(u)+\sum_{ij} f_i\int_0^t a^3_{ij}(u-)dW^j(u)\\ &+\sum_i f_i\int_0^t a^6_i(u)du\Bigg).\end{aligned}$$

Analogously as above, one can show that Y is also an $(E,\hat{E})$-semimartingale and thus we can again extend the definition of the integral $\int a\_dY$ to all càdlàg, $(\mathcal{F}_t)$-adapted $\hat{E}_{loc}$-valued processes a.

In view of (26) we only need to consider integrands of the form (B.3), for which $a^{2,(n)}_j\equiv 0$ and $a^{4,(n)}_i\equiv 0$ for all $i,j\in\mathbb{N}$. Moreover, we will further extend the definition of the integral $\int a^{(n)}(u-)dY^{(n)}(u)$ allowing as integrands all càdlàg, adapted processes $a^{(n)}$ which take their values in the set

$$\hat{E}_{loc} := \mathbb{R}\times\{0\}\times\mathbb{R}\times\{0\}\times L^2_{loc,diag}(\mathbb{R}^2_+;\mathbb{R})\times L^2_{loc}(\mathbb{R}_+;\mathbb{R}),$$

where

$$L^2_{loc,diag}(\mathbb{R}_+;\mathbb{R}) := \left\{ h(x,y)=\sum_i\sum_{j\le i} h_{ij} f_i(x) f_j(y) \;\middle|\; \sum_{i\in\mathcal{I}_m}\sum_{j\le i} h^2_{ij}<\infty\ \forall\, m\in\mathbb{N}\right\}.$$

The definition of the integral will be extended as follows: for any $a^{(n)}\in\hat{E}_{loc}$, the process

$$\int_0^t a^{(n)}(u-)dY^{(n)}(u),\ t\in[0,T],$$

is defined as the unique càdlàg E_{loc}-valued process $X=(X^1,X^2)$ (up to indistinguishability) such that for all $m\in\mathbb{N}$ and rational $t\in[0,T]$,

$$\left(X^1_t, \mathbb{1}_{[0,m]}X^2_t\right) = \int_0^t a^{(n),m}(u-)dY^{(n)}(u), \tag{B.4}$$

where the $\hat{E}$-valued processes $a^{(n),m}$, $m\in\mathbb{N}$, are defined as the projections of $a^{(n)}$ on the subspace $\{f_i : i\in\mathcal{I}_m\}$ with $\mathcal{I}_m$ defined in (17):

$$\begin{aligned} &a^{(n),m}(t;x,y)\\ &:= \left(a^{1,(n)}(t),\ 0,\ a^{3,(n)}(t),\ 0,\ \sum_{i\in\mathcal{I}_m}\sum_{j\le i} a^{5,(n)}_{ij}(t) f_i(x) f_j(y),\ \sum_{i\in\mathcal{I}_m} a^{6,(n)}_i(t) f_i(x)\right).\end{aligned}$$

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