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Ergodicity of Spitzer's renewal model ☆

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Abstract

Answering a question raised in Andjel and Vares (1992), we prove the ergodicity of the infinite-dimensional renewal process whose coordinates are indexed by $\mathbb{Z}^d$ and whose failure rate at any given site is the average of the ages of its neighbors plus a positive constant c, for any $d \geq 1$, $c > 0$. The main point is to prove the convergence of zero boundary Gibbs measures as the volume tends to $\mathbb{Z}^d$. This also yields uniqueness of Gibbs measures.

Keywords: Multi-dimensional renewal process; Ergodicity; Attractiveness; Absence of phase transition

1. Introduction

In this article we study the higher-dimensional version of the renewal process introduced by Spitzer (1986) and studied by Andjel and Vares (1992) in the one-dimensional case. This is a Markovian evolution taking place on some suitable subset X of $Y = \mathbb{R}_+^{\mathbb{Z}^d}$, where $\eta(x)$ has the interpretation as the age of some renewing object sitting at site x. The renewal rate at site x is

$$\varphi_x(\eta) = c + \frac{1}{2d} \sum_{y:|y-x|=1} \eta(y), \tag{1.1}$$

where $c > 0$.

When $d = 1$, Andjel and Vares (1992) provided a suitable X so that a Markov process (η_t) could be constructed, satisfying the above description, for each $\eta_0 \in X$. They also show that when $c > 0$ this process is ergodic. Their proof strongly relies on the one-dimensional character and a natural question is: what happens when $d \geq 2$? Do we have ergodicity for every $c > 0$, or does there exist some critical c? Here we prove ergodicity for any $c > 0$, in any dimension d. Similarly to Andjel and Vares (1992), we make strong use of attractiveness properties, which reduce the problem to a question of convergence of zero boundary Gibbs measures on $\mathbb{R}_+^{\Lambda}$ as $\Lambda \uparrow \mathbb{Z}^d$. Making

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use of special properties of our Gibbs measures we can prove such convergence in a very simple way. Again, due to attractiveness properties this in fact implies uniqueness of Gibbs measure for the given interaction.

The restriction to a suitable subset X of Y comes in naturally if one wishes to define the infinite volume dynamics as the limit, for $n \to +\infty$, of those on $\mathbb{R}_+^{\Lambda_n}$ (zero boundary) where $\Lambda_n = \{x: |x| \le n\}$ and $|x| = \sum_{i=1}^d |x_i|$ for any $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$. As seen in Andjel and Vares (1992) one cannot hope this limit to exist if all configurations $\eta \in Y$ are allowed. The set X here obtained for any dimension d is more restrictive than the corresponding one in Andjel and Vares (1992) for $d = 1$, as seen in Proposition 6.1. Nevertheless, the investigation of ergodicity can be done in a more general way, avoiding this restriction, since the above mentioned finite volume dynamics on the whole Y are clearly seen to converge along the two subsequences Λ_n for n even, and for n odd (Section 3). In Theorem 4.4 we prove our main result: if $c > 0$ these two infinite volume semigroups (which are always ordered) are both ergodic with the *same* invariant measure μ, which is the limit as $n \to +\infty$ of the finite volume Gibbs measures μ_{Λ_n} on $\mathbb{R}_+^{\Lambda_n}$, whose density with respect to Lebesgue measure is

$$h_n(\eta) = \frac{1}{Z_n} \exp\left(- c \sum_{x \in \Lambda_n} \eta(x) - \frac{1}{2d} \sum_{\substack{|x-y|=1 \\ x, y \in \Lambda_n}} \eta(x)\eta(y) \right). \tag{1.2}$$

In Section 5 we prove the uniqueness of Gibbs measure on Y with the same prescriptions of μ, which follows from the results in Section 4.

2. Finite volume dynamics

If Λ is a finite volume in $\mathbb{Z}^d$ (to fix ideas $\Lambda = \Lambda_n \overset{\text{def}}{=} \{i: |i| \le n\}$) the construction of Section 2.1 of Andjel and Vares (1992) yields a Markov process on $Y_\Lambda = \mathbb{R}_+^\Lambda$ with transition probabilities $P_\Lambda(t, \eta, \cdot)$, and which corresponds to the formal generator L_Λ acting on $C^1(X_\Lambda)$ as

$$L_\Lambda f(\eta) = \sum_{i \in \Lambda} \{\partial_i f(\eta) + \varphi_i(\eta)(f(\eta^i) - f(\eta))\}, \tag{2.1}$$

where

$$\eta^i(j) = \begin{cases} \eta(j) & \text{if } j \ne i, \\ 0 & \text{if } j = i, \end{cases} \qquad \partial_i f(\eta) = \frac{\partial}{\partial \eta_i} f(\eta),$$

and

$$\varphi_i(\eta) = c + \frac{1}{2d} \sum_{\substack{|j-i|=1 \\ j \in \Lambda}} \eta(j) \tag{2.2}$$

(i.e. zero boundary on Λ^c).

For the construction of infinite volume dynamics we want to take limits of $P_\Lambda(t, \eta, \cdot)$ as $\Lambda \uparrow \mathbb{Z}^d$ in a suitable way. For this, it may be convenient to think of each $P_\Lambda(t, \eta, \cdot)$ as a probability measure on $Y = \mathbb{R}_+^{\mathbb{Z}^d}$, letting it concentrate on $\{\xi \in X: \xi(x) = 0\ \forall x \notin \Lambda\}$. We also need to fix some notations.

(a) When $\Lambda = \Lambda_n = \{i: |i| \le n\}$ we shall write $P_n(t, \eta, \cdot)$ instead of $P_{\Lambda_n}(t, \eta, \cdot)$, and Y_n for Y_{Λ_n}.

(b) If $\eta \in Y_\Lambda$ (or Y) and $t \ge 0$ we let $K_\eta = \{\xi \in Y_\Lambda$ (or Y): $\xi \le \eta\}$ where $\xi \le \eta$ iff $\xi(x) \le \eta(x)$ for all $x \in \Lambda$ (or $\mathbb{Z}^d$), and let $\eta + t$ be the configuration in Y_Λ (or Y) defined by $(\eta + t)(x) = \eta(x) + t$ for all $x \in \Lambda$ (or $\mathbb{Z}^d$).

(c) As in Andjel and Vares (1992) we shall make strong use of an "attractiveness" property of our dynamics. For this, recall the partial ordering $\prec$ on Y (or Y_Λ): we will say that $\eta \prec \xi$ iff $\eta(x) \le \xi(x)$ for all x even and $\eta(x) \ge \xi(x)$ for all x odd, where $x = (x_i, \ldots, x_d) \in \mathbb{Z}^d$ is said to be even or odd according to $\sum_{j=1}^d |x_j|$ being even or odd. We also use $\prec$ to denote the corresponding partial order on the set of probability measures on Y_Λ (or Y):

$$\mu \prec \mu' \text{ iff } \int f \mathrm{d}\mu \le \int f \mathrm{d}\mu'$$

for any bounded continuous function $f: Y_\Lambda$ (or Y) $\to \mathbb{R}$ which is nondecreasing with respect to $\prec$. As it is well known this is equivalent to the existence of a measure $\bar{\mu}$ on $Y_\Lambda \times Y_\Lambda$ (or $Y \times Y$) with first and second marginals μ and μ' respectively and concentrated on $\{(\eta, \eta'): \eta \prec \eta'\}$. (See Kamae et al. (1977).)

(d) If Z is a metric space $\boldsymbol{B}(Z)$ denotes its Borel σ-field.

The attractiveness property of P_Λ, which was proven in Andjel and Vares (1992) may be summarized in the following lemma.

Lemma 2.1 (Andjel and Vares, 1992). *Let $\eta, \xi \in Y_\Lambda$ with $\eta \prec \xi$. Then for any $t \ge 0$ we have $P_\Lambda(t, \eta, \cdot) \prec P_\Lambda(t, \xi, \cdot)$. If f is bounded, continuous and increasing for $\prec$, then so is $P_\Lambda(t, \cdot, f)$. Moreover, for each t, η $P_\Lambda(t, \eta, K_{\eta+t}) = 1$.*

Lemma 2.2 (Andjel and Vares 1992). *Let f be bounded, continuous and nondecreasing with respect to $\prec$. Assume f to depend only on the coordinates in Λ_k, and that $2n - 1 \ge k$, $2m \ge k$. Then, for any η we have*

$$P_{2n-1}(t, \eta, f) \le P_{2m}(t, \eta, f); \tag{2.3a}$$

$$P_{2n-1}(t, \eta, f) \le P_{2n+1}(t, \eta, f); \tag{2.3b}$$

$$P_{2m}(t, \eta, f) \ge P_{2m+2}(t, \eta, f). \tag{2.3c}$$

Finally, we also recall from Andjel and Vares (1992) the ergodic behavior of finite volume dynamics, when $c > 0$.

Theorem 2.3 (Andjel and Vares, 1992). *Let $c > 0$ and $n \in \mathbb{N}$. Then*

(a) *There exist constants $a_n, b_n \in (0, \infty)$ depending only on n and c so that for all $t \ge 0$:*

$$\sup_{n, \xi \in Y_n} \| P_n(t, \eta, \cdot) - P_n(t, \xi, \cdot) \| \le a_n \mathrm{e}^{-tb_n},$$

where $\|\cdot\|$ denotes the total variation norm.

(b) *Let μ_n be the unique probability on Y_n, which is absolutely continuous with respect to Lebesgue measure and whose density $h_n(\eta)$ is given by*

$$h_n(\eta) = \frac{1}{Z_n} \exp\left\{ - c \sum_{i \in \Lambda_n} \eta(i) - \frac{1}{2d} \sum_{\substack{i, j \in \Lambda \\ |i-j|=1}} \eta(i)\eta(j) \right\} \tag{2.3}$$

with Z_n a normalizing constant. Then μ_n is the unique invariant measure for P_n, i.e.

$$\int P_n(t, \eta, \cdot)\mu_n(\mathrm{d}\eta) = \mu_n(\cdot), \quad \forall t \geq 0.$$

(c) $\forall \eta \in Y_n$, $P_n(t, \eta, \cdot) \to \mu_n(\cdot)$ *in variational distance, as* $t \to +\infty$.

3. Infinite volume dynamics

The attractiveness properties of P_n were used in Andjel and Vares (1992) to bound the infinite dynamics $P(t, \eta, \cdot)$, both from above and below, by finite dynamics, if $\eta \in X$ as defined in Andjel and Vares (1992). Nevertheless, the restriction on η may be avoided. If $t \geq 0$ and $\eta \in \mathbb{R}_+^{\mathbb{Z}^d}$, the probabilities $P_\Lambda(t, \eta, \cdot)$ are all concentrated on the compact set $K_{\eta+t}$, so that $\{P_\Lambda(t, \eta, \cdot)\}_\Lambda$ is a tight family. Moreover, Lemma 2.2 tells us that if f is cylinder, bounded, continuous and nondecreasing for $\prec$, then $(P_{2m}(t, \eta, f))$ $(P_{2m+1}(t, \eta, f)$ resp.) decreases (increases resp.), for m large enough. This allows us to define probabilities $\bar{P}(t, \eta, \cdot)$ and $\underline{P}(t, \eta, \cdot)$ on Y via (limits are in w^*-topology)

$$\bar{P}(t, \eta, \cdot) = \lim P_{2n}(t, \eta, \cdot), \tag{3.1}$$

$$\underline{P}(t, \eta, \cdot) = \lim P_{2n+1}(t, \eta, \cdot). \tag{3.2}$$

(We are simply using that if M, $(M_n)_n$ are probability measures on $\mathbb{R}_+^{\mathbb{Z}^d}$ and $M_n(g) \to M(g)$ for all g bounded, continuous, cylinder and increasing for $\prec$, then $M_n \xrightarrow{w^*} M$, which is easily proven.)

Moreover, if f, k, m, n are as in Lemma 2.2, then

$$P_{2n-1}(t, \eta, f) \leq \underline{P}(t, \eta, f) \leq \bar{P}(t, \eta, f) \leq P_{2m}(t, \eta, f). \tag{3.3}$$

The next step is to show that $\underline{P}(t, \eta, \cdot)$ and $\bar{P}(t, \eta, \cdot)$ are Markov transition probabilities, which would allow the construction of infinite volume dynamics. As we know from one-dimensional case some condition on the growth of $\eta(x)$ as $|x| \to \infty$ is needed to have $\underline{P}(t, \eta, \cdot) = \bar{P}(t, \eta, \cdot)$. (See Section 6.) For the moment we do not assume this and look at two possibly different processes.

The basic property needed for construction is the following proposition.

Proposition 3.1. *The family of probability measures* $\{\bar{P}(t, \eta, \cdot)\colon t > 0, \eta \in Y\}$ $(\{\underline{P}(t, \eta, \cdot)\colon t > 0, \eta \in Y\})$ *satisfy the following:*

(a) $\bar{P}(t, \eta, K_{\eta+t}) = 1 = \underline{P}(t, \eta, K_{\eta+t})$,

(b) $\bar{P}(t, \eta, \cdot) \underset{t \to 0}{\xrightarrow{w^*}} \delta_\eta$ *and* $\underline{P}(t, \eta, \cdot) \underset{t \to 0}{\xrightarrow{w^*}} \delta_\eta$, *where δ_η is the Dirac measure at η,*

(c) $\eta \to \bar{P}(t,\eta,\Gamma)$ and $\eta \to \underline{P}(t,\eta,\Gamma)$ are *$\boldsymbol{B}(Y)$-measurable, for each* $\Gamma \in \boldsymbol{B}(Y)$.

(d) *Furthermore,*

$$\int \bar{P}(t,\eta,\mathrm{d}\xi)\bar{P}(s,\xi,f) = \bar{P}(t+s,\eta,f), \tag{3.4}$$

$$\int \underline{P}(t,\eta,\mathrm{d}\xi)\underline{P}(s,\xi,f) = \underline{P}(t+s,\eta,f), \tag{3.5}$$

for each f bounded continuous on Y, each $t > 0$, $s > 0$, and each $\eta \in Y$.

Proof. We consider only $\bar{P}(t,\eta,\cdot)$; $\underline{P}(t,\eta,\cdot)$ is analogous.

(a) Is immediate since $P_{2n}(t,\eta,K_{\eta+t}) = 1$ for each n.

(b) It is enough to show that if f is bounded, cylinder, continuous and increasing for $\prec$, then $\bar{P}(t,\eta,f) \to f(\eta)$ as $t \to 0$. Let $k \geq 1$ be such that f does not depend on the coordinates off Λ_k, and $2n \geq k$. We have

$$P_{2n+1}(t,\eta,f) \leq \bar{P}(t,\eta,f) \leq P_{2n}(t,\eta,f) \tag{3.6}$$

for all $t \geq 0$, all η. From the construction of $P_n(\cdot)$ (Proposition 2.4 of Andjel and Vares (1992)) both $P_{2n+1}(t,\eta,f)$ and $P_{2n}(t,\eta,f)$ tend pointwise to $f(\eta)$ and so (b) follows from (3.6). [Notice that if $f \in C_b(Y_n)$ the convergence of $P_n(t,\cdot,f)$ to $f(\cdot)$ as $t \to 0$ is *not* in general uniform in Y_m.]

(c) Is immediate from typical measure theoretic argument.

(d) It is enough to consider (3.4) with f bounded, continuous, cylinder and increasing for $\prec$ (usual monotone class argument). Let k be such that f does not depend on the coordinates off Λ_{2k}. We have if $2n \geq 2k$

$$\begin{aligned}\bar{P}(t+s,\eta,f) \leq P_{2n}(t+s,\eta,f) &= \int P_{2n}(t,\eta,\mathrm{d}\xi)P_{2n}(s,\xi,f) \\ &\leq \int P_{2n}(t,\eta,\mathrm{d}\xi)P_{2k}(s,\xi,f).\end{aligned} \tag{3.7}$$

Moreover, since $\xi \to P_{2k}(s,\xi,f)$ is bounded and continuous in Λ_{2k}, from the definition of $\bar{P}$ we have that the last expression on the r.h.s. of (3.7) tends to

$$\int \bar{P}(t,\eta,\mathrm{d}\xi)P_{2k}(s,\xi,f) \quad \text{as } n \to \infty.$$

Thus we get

$$\bar{P}(t+s,\eta,f) \leq \int \bar{P}(t,\eta,\mathrm{d}\xi)P_{2k}(s,\xi,f)$$

for all such f and k. Letting $k \to \infty$ we get by Dominated Convergence Theorem

$$\bar{P}(t+s,\eta,f) \leq \int \bar{P}(t,\eta,\mathrm{d}\xi)\bar{P}(s,\xi,f).$$

On the other side if f and k are as above

$$\int \bar{P}(t,\eta,\mathrm{d}\xi)\bar{P}(s,\xi,f) \le \int \bar{P}(t,\eta,\mathrm{d}\xi)P_{2k}(s,\xi,f)$$

$$\le \int P_{2k}(t,\eta,\mathrm{d}\xi)P_{2k}(s,\xi,f) = P_{2k}(t+s,\eta,f)$$

which tends to $\bar{P}(t+s,\eta,f)$ as $k \to \infty$, proving (3.4) for such f, and so (d). □

From Proposition 3.1 we immediately have the following theorem.

Theorem 3.2. *There exists a Markov process on Y whose transition probabilities are $\bar{P}(t,\eta,\cdot)$ (resp. $\underline{P}(t,\eta,\cdot)$).*

Proof. Analogous to Theorem 2.9 of Andjel and Vares (1992). If $\eta \in Y$, since $\bar{P}(t,\eta,\cdot)$ is tight, a classical result of Kolmogorov (Ethier and Kurtz (1986) p. 157) tells us that there exists a probability measure $\bar{\boldsymbol{P}}_\eta$ on $(Y^{[0,\infty)}, \boldsymbol{B}(Y^{[0,\infty)}))$ so that

$$\bar{\boldsymbol{P}}_\eta(\eta(t_i) \in \Gamma_i,\ 1 \le i \le m)$$

$$= \int_{\Gamma_1} \cdots \int_{\Gamma_m} \bar{P}(t_1,\eta,\mathrm{d}\eta_1)\bar{P}(t_2-t_1,\eta_1,\mathrm{d}\eta_2)\ldots\bar{P}(t_m-t_{m-1},\eta_{m-1},d\eta_m).$$

For $\underline{\boldsymbol{P}}_\eta$ the proof is exactly the same. □

4. Ergodicity of $\bar{\boldsymbol{P}}_\eta$ and $\underline{\boldsymbol{P}}_\eta$

Let us extend μ_n to Y by letting it concentrate on $\{\eta \in Y\colon \eta(x) = 0, \forall x \notin \Lambda_n\}$.

Let us consider again the finite volume dynamics $P_\Lambda(t,\eta,\cdot)$ of Section 2. Since $\varphi_i(\eta) \ge c$ for all $i \in \Lambda$, all $\eta \in Y_\Lambda$, it is easily seen that if $\tilde{P}_\Lambda(t,\eta,\cdot)$ corresponds to the process of independent renewals with rate c, then

$$P_\Lambda(t,\eta,\cdot) \le \tilde{P}_\Lambda(t,\eta,\cdot)$$

in the usual stochastic ordering for probability measures in Y_Λ (i.e. for $\eta \le \xi$ if $\eta(x) \le \xi(x)$, $\forall x$). From this, Theorem 2.3 and the ergodicity of $\tilde{P}_{\mathbb{Z}^d}$ (which has as unique invariant measure the product of exponentials with rate c denoted by ν_c, it follows that all $\{\mu_n\}_n$ are dominated in the usual stochastic order by ν_c. This immediately implies that the family $\{\mu_n\}_{n\ge 1}$ is tight on Y. On the other side, Lemma 2.2 implies that if f is cylinder, bounded, continuous and nondecreasing for $\prec$, then $\mu_{2n}(f)$ decreases and $\mu_{2n+1}(f)$ increases, for n large enough. It then follows that

$$\bar{\mu} = \lim_{n\to\infty} \mu_{2n}, \tag{4.1}$$

$$\underline{\mu} = \lim_{n\to\infty} \mu_{2n+1} \tag{4.2}$$

exist and are probabilities on Y. Their basic properties are summarized in the next lemma.

Notation. If $x \in \mathbb{Z}^d$ we let τ_x be the shift defined by $\tau_x\eta(y) = \eta(x+y)$ for all $\eta \in Y$. If μ is a measure on Y we let $\tau_{-x}\mu(A) = \mu(\tau_x A)$. Also $\tau_x g(\eta) = g(\tau_x \eta)$.

Lemma 4.1. *The following properties hold*:

(a) $\underline{\mu} \prec \bar{\mu}$,

(b) *If x is even then* $\underline{\mu} = \tau_x \underline{\mu}$, $\bar{\mu} = \tau_x \bar{\mu}$,

(c) *If x is odd then* $\underline{\mu} = \tau_x \bar{\mu}$, $\bar{\mu} = \tau_x \underline{\mu}$,

(d) *The conditional distribution of $\eta(x)$ given $\mathscr{F}_x$, the σ-field generated by coordinates $\eta(z)$, $z \neq x$, under each of the measures $\bar{\mu}$ or $\underline{\mu}$ is exponential with rate*

$$c + \frac{1}{2d} \sum_{y:|y-x|=1} \eta(y).$$

Proof. (a) It follows at once from (2.3a) and Theorem 2.3(c).

(d) It is a simple consequence of (4.1), (4.2) and Theorem 2.3(b). Indeed from simple integration it follows from Eq. (2.3) that the law of $\eta(x)$ given $(\eta(z), z \neq x, z \in \Lambda_n)$ under μ_n is the above exponential distribution. Since this depends only on $\eta(z)$, $|z-x| = 1$, letting $n \to \infty$ (4.1) and (4.2) yield (d).

(b) It is an immediate consequence of (c).

(c) It is enough to prove that $\tau_{e_j}\bar{\mu} = \underline{\mu}$ where $j = 1,\dots,d$ and $e_1,\dots,e_d$ are the canonical unit vectors in $\mathbb{Z}^d$. For simplicity take $j = 1$.

Fix a bounded, continuous, cylinder function f which is increasing for $\prec$, with support in Λ_k. Take n even so that $n-1 \geq k$. From the attractiveness properties of Section 2 we have

$$\mu_{\tilde{\Lambda}_{n+1}}(g) \prec \mu_{\Lambda_n}(g) \prec \mu_{\tilde{\Lambda}_{n-1}}(g), \tag{4.3}$$

where $\tilde{\Lambda}_m = \{x: |x - e_1| \leq m\}$ and g is bounded, continuous, increasing for $\prec$, with support in $\tilde{\Lambda}_{n-1}$. But $\tau_{e_1} f$ is decreasing for $\prec$, so

$$\mu_{\Lambda_{n-1}}(f) = \mu_{\tilde{\Lambda}_{n-1}}(\tau_{e_1} f) \leq \mu_{\Lambda_n}(\tau_{e_1} f) \leq \mu_{\tilde{\Lambda}_{n+1}}(\tau_{e_1} f) = \mu_{\Lambda_{n+1}}(f), \tag{4.4}$$

letting $n \to +\infty$ in (4.4) we get $\tau_{e_1}\bar{\mu} = \underline{\mu}$. A similar argument yields $\tau_{e_1}\underline{\mu} = \bar{\mu}$. □

Proposition 4.2. $\bar{\mu} = \underline{\mu}$.

Proof. We first show that $\bar{\mu}(\eta(0)) = \underline{\mu}(\eta(0))$. If $x \in \mathbb{Z}^d$, let $\mathscr{F}_x$ be as in Lemma 4.1(d) $\mathscr{F}_x = \sigma(\eta(y){:}\ y \neq x) \subseteq \mathscr{B}(Y)$. Then by Lemma 4.1(d) we have

$$\bar{\mu}(\eta(0)\eta(e_1)) = \bar{\mu}(\eta(0)\bar{\mu}(\eta(e_1)|\mathscr{F}_{e_1})) = \bar{\mu}\left(\eta(0)\Big/\left(c + \frac{1}{2d}\sum_{|x-e_1|=1}\eta(x)\right)\right) \tag{4.5}$$

and also

$$\begin{aligned}\bar{\mu}(\eta(0)\eta(e_1)) = \bar{\mu}(\eta(e_1)\bar{\mu}(\eta(0)|\mathscr{F}_0)) &= \bar{\mu}\left(\eta(e_1)\Big/\left(c + \frac{1}{2d}\sum_{|x|=1}\eta(x)\right)\right)\\ &= \underline{\mu}(\eta(2e_1))\Big/\left(c + \frac{1}{2d}\sum_{|y-e_1|=1}\eta(y)\right)\right),\end{aligned} \tag{4.6}$$

where we have used Lemma 4.1(c) in the last equality in (4.6). That is, setting

$$A(\eta) = \frac{1}{2d}\sum_{|y-e_1|=1}\eta(y),$$

we have proven that

$$\bar{\mu}\left(\frac{\eta(0)}{c+A}\right) = \underline{\mu}\left(\frac{\eta(2e_1)}{c+A}\right).$$

Letting $y_i = e_1 + e_i$ and $a(y_i) = e_1 - e_i$, $i = 1, 2, \ldots, d$, proceeding as in (4.5) and (4.6) we have

$$\bar{\mu}\left(\frac{\eta(y_i)}{c+A}\right) = \underline{\mu}\left(\frac{\eta(a(y_i))}{c+A}\right), \quad i = 1, \ldots, d. \tag{4.7}$$

Adding up we get

$$\bar{\mu}\left(\frac{A}{c+A}\right) = \underline{\mu}\left(\frac{A}{c+A}\right). \tag{4.8}$$

Let now $\tilde{\mu}$ be a measure on $Y \times Y$ such that:

(i) $\tilde{\mu}(B \times Y) = \underline{\mu}(B)$, $\tilde{\mu}(Y \times B) = \bar{\mu}(B)$, for all $B \in \boldsymbol{B}(Y)$;

(ii) $\tilde{\mu}\{(\eta, \eta') \,|\, \eta \prec \eta'\} = 1$. (This exists because $\underline{\mu} \prec \bar{\mu}$.) Since $\eta \to A(\eta)/(c + A(\eta))$ is increasing for $\prec$ we have

$$\tilde{\mu}\left\{(\eta, \eta') \,\middle|\, \frac{A(\eta)}{c + A(\eta)} \le \frac{A(\eta')}{c + A(\eta')}\right\} = 1$$

which together with (4.8) implies

$$\tilde{\mu}\left\{(\eta, \eta') \,\middle|\, \frac{A(\eta)}{c + A(\eta)} = \frac{A(\eta')}{c + A(\eta')}\right\} = 1,$$

and so

$$\tilde{\mu}\{(\eta, \eta') \,|\, A(\eta) = A(\eta')\} = 1.$$

From Lemma 4.1(b) we then have

$$\underline{\mu}(\eta(0)) = \frac{1}{2d}\underline{\mu}(A) = \frac{1}{2d}\bar{\mu}(A) = \bar{\mu}(\eta(0)).$$

for all such n. But $\mu_{2n} \to \mu$ and $\mu_{2n+1} \to \mu$ so that we have (remember $\underline{P}(t,\eta,\cdot) \prec \bar{P}(t,\eta,\cdot)$)

$$\lim_{t\to\infty} \underline{P}(t,\eta,f) = \mu(f) = \lim_{t\to\infty} \bar{P}(t,\eta,f),$$

for all such f, proving the theorem. □

5. Uniqueness of the Gibbs measure

Proposition 4.2 in fact implies the uniqueness of the Gibbs measure on Y with the same prescriptions as μ. In fact if Λ_n is as before and we take on Λ_n the Gibbs measure $\mu_n(\cdot\,|\bar{\eta})$ with arbitrary boundary condition $\bar{\eta}_i$ $i \in \partial\Lambda_n$ i.e. its Radon Nikodym derivative w.r.t. Lebesgue measure is

$$h_{\Lambda_n}(\eta\,|\,\bar{\eta}) = \frac{1}{Z_n}\exp\left(-c\sum_{j\in\Lambda_n}\eta_j - \frac{1}{2d}\sum_{\substack{|i-j|=1\\ i,j\in\Lambda_n}}\eta_i\eta_j - \frac{1}{2d}\sum_{\substack{i\in\Lambda_n\\ j\in\partial\Lambda_n\\ |i-j|=1}}\eta_i\bar{\eta}_j\right)$$

with Z_n a proper normalizing constant and $\partial\Lambda_n = \{x \notin \Lambda_n\colon \exists y \in \Lambda_n \text{ with } |x-y| = 1\}$, it is easily seen that for f bounded, continuous, increasing for $\prec$, and cylinder with support in Λ_{n-1}, we have

$$\begin{aligned} &\mu_{n-1}(f) \le \mu_n(f\,|\,\bar{\eta}) \le \mu_n(f) \quad \text{for } n \text{ even},\\ &\mu_n(f) \le \mu_n(f\,|\,\bar{\eta}) \le \mu_{n-1}(f) \quad \text{for } n \text{ odd}. \end{aligned} \tag{5.1}$$

Indeed (6.1) is an easy consequence of Lemmas 2.1, 2.2 and Theorem 2.3, since one can see that $\mu_n(\cdot\,|\bar{\eta})$ is the invariant measure of the process with properly modified boundary conditions, and the comparison between these and zero boundary process follows the same lines as Lemma 2.2. Taking limits in (5.1) it follows that $\mu_n(\cdot\,|\bar{\eta}) \xrightarrow{w^*} \mu$.

6. Infinite volume dynamics for "good" configurations

As observed in Andjel and Vares (1992), and easily noticed, there is no reason to have $\bar{P}(t,\eta,\cdot) = \underline{P}(t,\eta,\cdot)$ for a general η. In fact, if $\eta(x)$ grows very fast as $|x|$ grows, then it makes a lot of difference to look at $P_{2n}(t,\eta,\Gamma)$ or $P_{2n+1}(t,\eta,\Gamma)$, where Γ is a fixed cylinder set, no matter how large is n. Since we are considering $c > 0$, we have seen that as $t \to \infty$ this difference relaxes, and both tend to μ. Moreover, if we recall Lemma 4.3, as well as the fact that $\underline{P}(t,\eta,\cdot) \prec \bar{P}(t,\eta,\cdot)$ we immediately conclude that

$$\mu\{\eta\colon \underline{P}(t,\eta,\cdot) = \bar{P}(t,\eta,\cdot)\} = 1.$$

Nevertheless it is convenient to present a concrete example of a set $X \subseteq Y$, with $\mu(X) = 1$ and such that if $\eta \in X$ then $w^* - \lim_{n\to+\infty} P_n(t,\eta,\cdot)$ exists, i.e. $\underline{P}(t,\eta,\cdot) = \bar{P}(t,\eta,\cdot)$, for all $t > 0$. This is the content of the next proposition.

Proposition 6.1. *Let*

$$X = \{\eta \in Y: \exists A > 0, \exists \alpha < 1 \text{ so that } \eta(i) \le A|i|^\alpha \text{ for all } i \in \mathbb{Z}^d/\{0\}\}.$$

Then, for each $\eta \in X$ and $t > 0$

$$\underline{P}(t,\eta,\cdot) = \bar{P}(t,\eta,\cdot). \tag{6.1}$$

Proof. For each $m < n$ and $\eta \in X$ we couple the finite-dimensional processes on Λ_m and Λ_n in a suitable way. (It is enough to take $n = m + 1$, but this does not change what follows.) First, we use, as in Section 2.1 of Andjel and Vares (1992), the transition probabilities $P_{n,k}$, for which the renewals are eliminated if $\sum_{i\in\Lambda_n}\eta(i) > 2k$ i.e. φ_i is changed to $g_k(\sum_{j\in\Lambda_n}\eta(j))\varphi_i$ in (2.1) and (2.2), where

$$g_k(t) = \begin{cases} 1 & t \le k, \\ 2 - t/k & k < t < 2k, \\ 0 & t \ge 2k. \end{cases}$$

Notice that for $k \ge t + \sum_{j\in\Lambda_n}\eta(j)$ we have $P_n(t,\eta,\cdot) = P_{n,k}(t,\eta,\cdot)$.

Fix $\eta \in X$ and $k \ge 1$ and let $(\eta_s^1, \eta_s^2, \alpha_s)$ be a process on $Y_m \times Y_n \times \{0,1\}^{\Lambda_{m+1}}$ as in Proposition 2.5 of Andjel and Vares (1992) i.e.

(i) η^1 has transition probabilities $P_{m,k}$ and η_0^1 is η (restricted to Λ_m),

(ii) η^2 has transition probabilities $P_{n,k}$ and η_0^2 is η (restricted to Λ_n),

(iii) $\alpha_0(i) = 0$ if $i \in \Lambda_m$, $\alpha_0(i) = 1$ if $|i| = m + 1$. $\alpha_s(i)$ flips only from zero to one with rate

$$\frac{1}{2d}\sum_{|j-i|=1}\alpha_s(j)(\eta(j) + t),$$

(iv) $P(\eta_s^1(i) \ne \eta_s^2(i), \alpha_s(i) = 0) = 0$ if $i \in \Lambda_m$.

As in Andjel and Vares (1992), for the proof of the proposition it is enough to show that if $\eta \in X$, B is finite, and $t > 0$ then

$$\lim_{m\to\infty} P(\alpha_t(i) = 1 \text{ for some } i \in B) = 0. \tag{6.2}$$

Without loss of generality we consider $B = \{0\}$.

For each bond $\ell = \langle i,j\rangle$ with $i,j \in \mathbb{Z}^d$, $|i-j| = 1$, let $t(\ell)$ be an exponential random variable with rate $(\eta(i) + \eta(j) + 2t)/2d$, and take them independent. Consider the first passage percolation problem associated to these random times (as defined in Smythe and Wierman (1978) or Grimmett (1985)).

If q is a path, we set

$$t(q) = \sum_{\ell\in q} t(\ell)$$

and

$$A_t = \{i \in \Lambda_m: \exists \text{ path } q \text{ starting at } \partial\Lambda_n \text{ and ending at } i \text{ with } t(q) \le t\}$$

then by (iii) we easily see that

$$P(\alpha_t(0) = 1) \leq P(0 \in A_t). \tag{6.3}$$

To estimate the r.h.s. of (6.3) we say that the bond ℓ is open (closed) if $t(\ell) \leq (>) T/(|i|^\alpha \wedge |j|^\alpha + 1)$, where $0 < T$ is chosen small enough so that

$$P(\ell \text{ is open}) \leq p < p_c^d \tag{6.4}$$

for p_c^d the critical parameter for Bernoulli bond percolation on $\mathbb{Z}^d$.

If C_0 is the open cluster containing the origin then it is known that $E|C_0|^r < \infty$ for all r, where by $|C_0|$ we denote the cardinal of C_0 (see Grimmett (1989)).

Now if m is large enough and $0 < \beta < 1 - \alpha$ we have

$$[0 \in A_t] \subseteq [\exists \text{ open cluster } \subseteq \Lambda_m \text{ of size } \geq m^\beta]. \tag{6.5}$$

Indeed, if there are no open clusters of size at least m^β inside Λ_m, then each path q connecting 0 and $\partial\Lambda_m$ must have at least $[m^{1-\beta}]$ closed bonds $\ell_1, \ldots, \ell_{[m^{1-\beta}]}$, so that $\ell_k = \langle i_k, j_k \rangle$ with i_k or $j_k \in \Lambda_{km^\beta}$ $(k = 1, \ldots, [m^{1-\beta}])$. Thus

$$t(q) \geq \frac{T}{2} \sum_{k=1}^{[m^{1-\beta}]} \frac{1}{(km^\beta)^\alpha} = \frac{T}{2m^{\beta\alpha}} \sum_{k=1}^{[m^{1-\beta}]} \frac{1}{k^\alpha} > t,$$

if m is large enough.

Then, for m large enough

$$P(0 \in A_t) \leq (2m+1)^d P(|C_0| \geq m^\beta) \leq \frac{(2m+1)^d}{m^{\beta r}} E|C_0|^r,$$

which tends to zero if $\beta r > d$.

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