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On the purity of the free boundary condition Potts measure on random trees

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Abstract

We consider the free boundary condition Gibbs measure of the Potts model on a random tree. We provide an explicit temperature interval below the ferromagnetic transition temperature for which this measure is extremal, improving older bounds of Mossel and Peres. In information theoretic language extremality of the Gibbs measure corresponds to non-reconstructability for symmetric q-ary channels. The bounds for the corresponding threshold value of the inverse temperature are optimal for the Ising model and differ from the Kesten Stigum bound by only 1.50% in the case $q = 3$ and 3.65% for $q = 4$, independently of d. Our proof uses an iteration of random boundary entropies from the outside of the tree to the inside, along with a symmetrization argument.
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1. Introduction

Interacting stochastic models on trees and lattices often differ in a fundamental way: where a lattice model has a single transition point (a critical value for a parameter of the model), the corresponding model on a tree might possess multiple transition points. Such phenomena happen more generally for non-amenable graphs (where surface terms are no smaller than volume terms), trees being major examples [1].

The main example of an interacting model is the usual ferromagnetic Ising model. Here the interesting property which gives rise to a new transition is the extremality of the free b.c. (boundary condition) state. In an Ising model on the lattice, below the ferromagnetic transition temperature the free boundary limiting measure will be a symmetric combination between the plus-state and the minus-state. On the tree, however, the open boundary state will still be extremal in a temperature interval strictly below the ferromagnetic transition temperature. It ceases to be extremal at even lower temperatures.

Ferromagnetic order on a tree is characterized by the fact that a plus-boundary condition at the leaves of a finite tree of depth n persists to have influence on the origin when n tends to infinity. For the tree it now happens in a range of temperatures that, even though an all plus-boundary condition will be felt at the origin, a *typical boundary condition* chosen from the free b.c. measure itself will not be felt at the origin for a range of temperatures below the ferromagnetic transition. The latter implies the extremality of the free b.c. state.

We write throughout the paper $\theta = \tanh \beta$ where β is the inverse temperature of the Ising (or Potts) model and denote by d the number of children on a regular rooted tree. Then the Ising ferromagnetic transition temperature is given by $d\theta = 1$, and the transition temperature where the free b.c. state ceases to be extremal is given by $d\theta^2 = 1$.

A proof of the latter fact is contained in [2]. A beautiful alternate proof of the extremality for $d\theta^2 \leq 1$ for regular trees was given by Ioffe [3]. The method used therein was elegant but very much dependent on the two-valuedness of the Ising spin variable. This was exploited for the control of conditional probabilities in terms of projections to products of spins. Some care is necessary to treat the marginal case where equality holds in the condition. Indeed, one needs to control quadratic terms in a recursion; this is difficult for a general tree where the degrees are not fixed. A second paper [4] proves an analogue of the condition for general trees with arbitrary degrees but leaves this case open. Finally, for a general tree which does not possess any symmetries, [5] give a sharp criterion for extremality in terms of capacities. It remains an open problem to determine the extremal measures and the measure in the extreme decomposition of the open b.c. state for $d\theta^2 > 1$.

Let us remark that the problem of extremality of the open b.c. state is equivalent to the so-called reconstruction problem: We send a signal (a plus or a minus) from the origin to the boundary, making a prescribed error probability (that is related to the temperature of the Ising model) at every edge of the tree. In this way one obtains a Markov chain indexed by the tree. The reconstruction problem on a tree is said to be solvable, if the measure, obtained on the boundary at distance n by sending an initial $+$, keeps a finite variational distance to the measure obtained by sending a $-$, as n tends to infinity. Non-solvability of reconstruction is equivalent to the extremality of the open b.c. state [6,7]. This is to say that there can be no transport of information along the tree between root and boundary, for typical signals.

1.1. The Potts model

We denote by T^N a finite tree rooted at 0 of depth N. Then the *free b.c. Potts measure on* T^N is the probability distribution $\mathbb{P}^N$ that assigns to a configuration $\eta_{T^N} = (\eta(v))_{v \in T^N} \in \{1, 2, \ldots, q\}^{T^N}$ the probability weights

$$\mathbb{P}^N(\eta_{T^N}) = \frac{\exp\left(2\beta \sum_{(v,w)} \delta_{\eta(v),\eta(w)}\right)}{Z_{\beta,T^N}}, \tag{1}$$

where the sum is over all edges (v, w) of the tree T^N and Z_{β, T^N} is the partition function that makes the r.h.s. a probability measure.

The *free b.c. Potts measure on an infinite tree T* is by definition the ak limit $\mathbb{P} = \lim_{N\uparrow\infty} \mathbb{P}_{T^N}$ when T^N is an exhaustion of T. $\mathbb{P}$ is identical to what is called *the symmetric chain on q symbols* in the context of the reconstruction problems in [6]. This chain has one parameter, namely the probability to change the symbol that is transmitted to any of the $q - 1$ others, which is given by $\frac{1}{e^{2\beta}+q-1}$.

Recalling the DLR equations, *a Potts Gibbs measure* on a graph with vertex set G is any measure $\mathbb{P}$ such that, for all finite subsets $\Lambda \subset G$, the corresponding conditional probabilities of $\mathbb{P}$ are given by

$$\mathbb{P}(\eta_\Lambda | \bar{\eta}_{G \setminus V}) = \frac{\exp\left(2\beta \sum_{\substack{(v,w) \\ v,w \in \Lambda}} \delta_{\eta(v),\eta(w)} + \sum_{\substack{(v,w) \\ v \in \Lambda, w \in G \setminus \Lambda}} \delta_{\eta(v),\bar{\eta}(w)}\right)}{Z_{\beta,\Lambda}^{\bar{\eta}}}, \tag{2}$$

where the sums are again along edges (v, w) of the graph.

Clearly the free b.c. measure on an infinite tree T is a Gibbs measure. Recall that a Gibbs measure is said to be extremal if it cannot be written as convex combination of other Gibbs measures.

1.2. Random trees

Consider a random tree T with vertices i and number of children at the site i given by d_i. We choose d_i to be independent random variables with the same distribution $\mathbb{Q}$. We use the symbol $\mathbb{Q}$ also to describe the expected value. As is well known these appear as local approximations of random graphs which has newly emphasized their interest [8]. Our results however are already interesting in the case of regular trees where every vertex i has precisely d children.

1.3. A criterion for extremality on random trees

In this situation our main result, formulated for a random tree, is the following. Write

$$P = \left\{ (p_i)_{i=1,\dots,q}, \; p_i \geq 0 \; \forall i, \; \sum_{i=1}^{q} p_i = 1 \right\}$$

for the simplex of Potts probability vectors.

Theorem 1.1. *The free boundary condition Gibbs measure $\mathbb{P}$ is extremal, for $\mathbb{Q}$-a.e. tree T when the condition $\mathbb{Q}(d_0) \frac{2\theta}{q-(q-2)\theta} \bar{c}(\beta, q) < 1$ is satisfied. Here,*

$$\bar{c}(\beta, q) := \sup_{p \in P} \frac{\sum_{i=1}^{q} (qp_i - 1) \log(1 + (e^{2\beta} - 1)p_i)}{\sum_{i=1}^{q} (qp_i - 1) \log qp_i}. \tag{3}$$

Remark. It appears that the supremum over P is achieved at the symmetric point $\frac{1}{q}(1, 1, \dots, 1)$ *only in the Ising model $q = 2$. This implies the sharpness of the bound in the Ising case, see also the discussion at the end of the paper. It is not surprising that the Potts model shows peculiarities*

in comparison with the Ising model. That Potts is more intricate is seen already on the level of the much simpler problem of determining the ferromagnetic transition temperature (where the Gibbs measure becomes unique). Due to the lack of concavity of the r.h.s. of the recursion relation the transition is first (instead of second) order.

Remark. The best bound which has been previously given appears in [9] on a d-ary tree. We recover it from our bounds when we use the estimate $\bar{c}(\beta, q) \leq \theta$ which will be discussed below. Moreover, numerically $\bar{c}(\beta, q)$ seems to decrease monotonically in q at fixed β.

Note also the bounds of Martinelli et al. [10] (see Theorem 9.3., Theorem 9.3.', Theorem 9.3'') who give a nice criterion for non-reconstruction involving a Dobrushin constant of the corresponding Markov specification which however give worse estimates in the Potts model.

Let us put our result in perspective. For the purpose of the discussion we specialize to the case of the regular tree with d children. Denote by $\mathbb{P}^{N,k}$ the measures on T^N obtained by putting the boundary condition k to all Potts-spins at the outer boundary, and denote by $\mathbb{P}^k$ the corresponding limiting measures on T.

Absence of ferromagnetic order (uniqueness of the Gibbs measure) can be detected by the fact that the distribution of the spin $\eta(0)$ at the origin under the infinite volume measure $\mathbb{P}^k$ is the equidistribution, independently of the boundary condition k. This condition is easy to obtain by considering a simple one-dimensional recursion of numbers (instead of measures). For more details see Section 2.2. Absence of ferromagnetic order in particular implies purity of the free b.c. state. In the language of the reconstruction problem this means non-solvability and as such the condition is mentioned as Proposition 4 in [6].

Let us compare with opposite results: It is known from the so-called Kesten–Stigum bound [11] that $d\lambda_2(\theta, q)^2 > 1$ implies reconstructability (i.e. non-extremality of the free b.c. measure). Here $\lambda_2(\theta, q)$ is the second eigenvalue of the transition matrix that produces the free b.c. Potts model by broadcasting from the origin to the boundary; it is decreasing in q at fixed θ, and increasing in θ at fixed q. This is intuitively clear: the bigger the number of states q and the smaller the inverse temperature, the easier it is to forget about the information put at the boundary. Moreover it is proved as Theorem 2 in [6] that when one fixes d and a value of $d\lambda_2(\theta, q) \equiv \lambda > 1$, for q large enough the reconstruction problem is solvable for the corresponding value of θ.

Now, our method of proof is based on controlling recursions for the probability distributions at roots of subtrees from the outside to the inside of a tree. These are recursions on log-likelihood ratios of Potts probability vectors for the root of subtrees, and these ratios are random w.r.t. the boundary condition (which is chosen according to the free b.c. measure).

Understanding recursions for probability distributions (needed to investigate the purity of the free b.c. state) is much less straightforward than controlling recursions for real numbers (needed for investigating the existence of ferromagnetic order). We prove convergence to a Dirac-distribution by controlling the boundary relative entropy, generalizing from the approach of [5] for the Ising model. Novelties appear for the Potts model, a key point being proper symmetrization to bring out the constant (39), beginning with Lemma 2.2.

2. Proof

To show the triviality of a measure μ on the tail sigma-algebra it suffices to show that, for any fixed cylinder event A we have

$$\lim_{N \uparrow \infty} \mu |\mu(A|\mathcal{T}_N) - \mu(A)| = 0, \tag{4}$$

where $\mathcal{T}_N$ is the sigma-algebra created by the spins that have at least distance N to the origin (see [12] Proposition 7.9).

We denote by T^N the tree rooted at 0 of depth N. The notation T_v^N indicates the subtree of T^N rooted at v obtained from "looking to the outside" on the tree T^N. We denote by $\mathbb{P}_v^{N,\xi}$ the corresponding Potts–Gibbs measure on T_v^N with boundary condition on ∂T_v^N given by $\xi = (\xi_i)_{i \in \partial T_v^N}$. We denote by $\mathbb{P}_v^N$ the corresponding Potts–Gibbs measure on T_v^N with free boundary conditions, as in (1).

We are going to show that the distribution of the probabilities to see a value s at the origin, obtained by putting a boundary condition ξ at distance N that is chosen according to the free measure $\mathbb{P}$ itself, converges to the equidistribution in probability. This reads

$$\lim_{N \uparrow \infty} \mathbb{P}\left(\xi : \left| \mathbb{P}^{N,\xi}(\eta(0) = s) - \frac{1}{q} \right| \geq \varepsilon \right) \to 0. \tag{5}$$

This then implies (4).

To achieve (5) it is more convenient to look at the probability distribution for the spin at the root v obtained with the boundary condition ξ in terms of the "log-likelihood ratios" defined by

$$X_k^j(v; \xi) := \log \frac{\mathbb{P}_v^{N,\xi}(\eta(v) = j)}{\mathbb{P}_v^{N,\xi}(\eta(v) = k)}, \tag{6}$$

where $1 \leq j \neq k \leq q$. Ultimately we are interested to show the convergence of these quantities at $v = 0$ to zero, for all pairs j, k, in $\mathbb{P}$-probability, as the depth N of the tree tends to infinity.

We denote the measure at the boundary at distance N from the root on the tree emerging from v, which is obtained by conditioning the spin in the site v to take the value to be j, by

$$Q_v^{N,j}(\xi) := \mathbb{P}_v^N(\eta : \eta_{|\partial T_v^N} = \xi | \eta(v) = j). \tag{7}$$

Definition 2.1. Denote the relative entropy of the boundary measures between the states obtained by conditioning the spin at v to be 1 respectively 2, by

$$m_v^{(N)} = S(Q_v^{N,2} | Q_v^{N,1}) = \int Q_v^{N,2}(\mathrm{d}\xi) \log \frac{Q_v^{N,2}(\xi)}{Q_v^{N,1}(\xi)}. \tag{8}$$

Here and in what follows denote by w the children of v, indicated by the symbol $v \to w$.

Lemma 2.2. *The boundary relative entropy can be written as an expected value w.r.t. the open boundary condition Gibbs measure $\mathbb{P}$ in the form*

$$S(Q_v^{N,2} | Q_v^{N,1}) = \frac{1}{q-1} \int \mathbb{P}(\mathrm{d}\xi) \sum_{i=1}^q \varphi\left(q \mathbb{P}_v^{N,\xi}(\eta(v) = i)\right), \tag{9}$$

with $\varphi(x) = (x - 1) \log x$.

Proof. In the first step we express the relative entropy as an expected value

$$S(Q_v^{N,2} | Q_v^{N,1}) = q \int \mathbb{P}(\mathrm{d}\xi) g\left(\mathbb{P}_v^{N,\xi}(\eta(v) = 2), \mathbb{P}_v^{N,\xi}(\eta(v) = 1)\right), \tag{10}$$

with

$$g(p_2, p_1) = p_2 \log \frac{p_2}{p_1}. \tag{11}$$

To see this, we use that

$$\frac{\mathrm{d}Q_v^{N,2}}{\mathrm{d}\mathbb{P}_v^N}(\xi) = q\mathbb{P}_v^{N,\xi}(\eta(v) = 2), \tag{12}$$

by the definition of the conditional probability and the fact that the marginal of $\mathbb{P}$ at any site is the equidistribution.

In the next step we use the invariance of $\mathbb{P}$ under permutation of the Potts-indices to write

$$S(Q_v^{N,2}|Q_v^{N,1}) = q \int \mathbb{P}(\mathrm{d}\xi)(Rg)\Big(\mathbb{P}_v^{N,\xi}(\eta(v) = 1), \mathbb{P}_v^{N,\xi}(\eta(v) = 2),$$
$$\ldots, \mathbb{P}_v^{N,\xi}(\eta(v) = q)\Big), \tag{13}$$

where R is the symmetrization operator acting on functions $f(p_1, \ldots, p_q)$ of Potts-probability vectors by

$$(Rf)(p_1, p_1, \ldots, p_q) = \frac{1}{q!} \sum_{\pi} f(p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(q)}), \tag{14}$$

where π runs over the permutations of $\{1, \ldots, q\}$.

One verifies that

$$(Rg)(p_1, p_1, \ldots, p_q) = \frac{1}{q(q-1)} \sum_{i=1}^{q} (qp_i - 1) \log qp_i, \tag{15}$$

which proves the lemma. $\quad\square$

2.1. Recursions for the boundary entropy for subtrees

Proposition 2.3. *The boundary relative entropy $m_v^{(N)}$ at the site v obeys the following linear recursive inequalities in terms of the values at the children w, given by*

$$m_v^{(N)} \le \frac{2\theta}{q - \theta(q-2)} \bar{c}(\beta, q) \sum_{w:v \to w} m_w^{(N)}. \tag{16}$$

Remark. Noting that $\frac{Q_v^{N,j}(\xi)}{Q_v^{N,k}(\xi)} = X_k^j(v; \xi)$ we may write

$$m_v^{(N)} = \int Q_v^{N,2}(\mathrm{d}\xi) X_1^2(v; \xi). \tag{17}$$

Remark. Suppose that we are considering a spherically symmetric tree. This means that the number of offspring depends only on the generation, e.g. $d_v = d_{|v|}$ where $|v|$ is the distance of v to the origin (that is the length of the unique path from the origin to v). Then $m_v^{(N)} = m_{|v|}^{(N)}$ and so

$$m_k^{(N)} \le \frac{2\theta}{q - \theta(q-2)} \bar{c}(\beta, q) d_k m_{k+1}^{(N)}. \tag{18}$$

So $\lim_{N \uparrow \infty} m_0^{(N)} = 0$ is implied by $\sum_{k=1}^{\infty} \log(cd_k) = -\infty$ with $c = \frac{2\theta}{q-\theta(q-2)} \bar{c}(\beta, q)$.

Proof of Theorem 1.1. Taking expectation w.r.t. the random tree we note that $\mathbb{E}m_v^{(N)} = \mathbb{E}m_{|v|}^{(N)}$. Now, using Wald's inequality we have

$$\mathbb{E}m_k^{(N)} \leq \frac{2\theta}{q - \theta(q - 2)} \bar{c}(\beta, q) \mathbb{E}d_0 \, \mathbb{E}(m_{k+1}^{(N)}). \tag{19}$$

From this follows that $\lim_{N\uparrow\infty} \mathbb{E}m_0^{(N)} = 0$ using the uniform boundedness in N, $\mathbb{E}m_{N-1}^{(N)} \leq C\mathbb{E}(d_0)$. This can be seen from Lemma 2.4 a few lines below. $\square$

To prove Proposition 2.3 at first a recursion for the log-likelihood ratios $X_k^j(v; \xi)$ has to be derived, for fixed finite tree of depth N from the outside to the inside. This iteration is standard, but we include its derivation for the convenience of the reader. In the following we omit the dependence on the fixed boundary condition ξ in the notation.

Lemma 2.4. *For all indices $1 \leq j, k \leq q$ we have*

$$X_k^j(v) = \sum_{\omega: v \to w} \log \frac{\sum_{i \neq k, j} \exp[X_k^i(w)] + 1 + \exp(2\beta) \exp[X_k^j(w)]}{\sum_{i \neq k, j} \exp[X_k^i(w)] + \exp(2\beta) + \exp[X_k^j(w)]}. \tag{20}$$

Proof. Note that the Potts-measure $\mathbb{P}_v^{N,\xi}$ is proportional to the weight

$$W(\eta) = \prod_{x \to y, x \geq v} \exp[2\beta\delta_{\eta(x),\eta(y)}],$$

where the product is taken over the neighboring vertices coming after v looking from the root of the tree. The normalization factor will be Z_v^{-1}.

We want to rewrite $X_k^j(v)$ as a function of $X_k^j(w)$ where w are the children of v. The key observation is that

$$W(\eta_v) = \prod_{w: v \to w} W(\eta_w) \exp[2\beta\delta_{\eta(v),\eta(w)}],$$

where we have written η_v for the restriction of η to the sub-tree T_v^N. Now,

$$\mathbb{P}_v^{N,\xi}(\eta(v) = j) = Z_v^{-1} \prod_{w: v \to w} \sum_{\eta_w} W(\eta_w) \exp[2\beta\delta_{j,\eta(w)}]$$

$$= Z_v^{-1} \prod_{w: v \to w} Z_w \sum_{i=1}^q Z_w^{-1} \exp[2\beta\delta_{j,i}] \sum_{\eta_w: \eta(w)=i} W(\eta_w)$$

$$= Z_v^{-1} \prod_{w: v \to w} Z_w \sum_{i=1}^q \exp[2\beta\delta_{j,i}] \mathbb{P}_w^{N,\xi}(\eta(w) = i). \tag{21}$$

The same computation can be done for $\mathbb{P}_v^{N,\xi}(\eta(v) = k)$ to obtain:

$$\mathbb{P}_v^{N,\xi}(\eta(v) = k) = Z_v^{-1} \prod_{w: v \to w} Z_w \sum_{i=1}^q \exp[2\beta\delta_{k,i}] \mathbb{P}_w^{N,\xi}(\eta(w) = i).$$

Now consider the ratio and then divide everything by $\mathbb{P}_w^{N,\xi}(\eta(w) = k)$:

$$
\frac{\mathbb{P}_v^{N,\xi}(\eta(v) = j)}{\mathbb{P}_v^{N,\xi}(\eta(v) = k)} = \prod_{w:v \to w} \frac{\sum_{i=1}^{q} \exp[2\beta \delta_{j,i}] \mathbb{P}_w^{N,\xi}(\eta(w) = i)}{\sum_{i=1}^{q} \exp[2\beta \delta_{k,i}] \mathbb{P}_w^{N,\xi}(\eta(w) = i)}
$$

$$
= \prod_{w:v \to w} \frac{\sum_{i \neq k,j} \frac{\mathbb{P}_w^{N,\xi}(\eta(w)=i)}{\mathbb{P}_w^{N,\xi}(\eta(w)=k)} + 1 + \exp(2\beta) \frac{\mathbb{P}_w^{N,\xi}(\eta(w)=j)}{\mathbb{P}_w^{N,\xi}(\eta(w)=k)}}{\sum_{i \neq k,j} \frac{\mathbb{P}_w^{N,\xi}(\eta(w)=i)}{\mathbb{P}_w^{N,\xi}(\eta(w)=k)} + \exp(2\beta) + \frac{\mathbb{P}_w^{N,\xi}(\eta(w)=j)}{\mathbb{P}_w^{N,\xi}(\eta(w)=k)}},
$$

which proves the result. $\square$

2.2. The ferromagnetic ordering

Let us quickly deviate from the proof of Proposition 2.3 and discuss the threshold value for the ferromagnetic ordering (where the infinite volume states with uniform boundary conditions cease to be different).

Observe that for a boundary condition ξ that is all q we have that $X_k^j(v) = 0$ for all $1 \le i, j \le q - 1$, and further that $X_i^q(v) = X_1^q(v)$ for all $i = 1, \dots, q - 1$. So the iteration runs on the one-dimensional quantity $X_1^q(v)$ and reads

$$
X_1^q(v) = \sum_{\omega:v \to w} \log \frac{q - 1 + \exp(2\beta) \exp[X_1^q(w)]}{q - 2 + \exp(2\beta) + \exp[X_1^q(w)]}
$$

$$
=: \sum_{\omega:v \to w} \psi(X_1^q(w)). \tag{22}
$$

For a regular tree with d children we have

$$
X_1^q(k) = d\psi(X_1^q(k+1)). \tag{23}
$$

We have to distinguish now the cases of $q = 2$ and $q \ge 3$. For $q = 2$ we see by computation of the second derivative that the function ψ is concave. This means that the critical value β for which a positive solution X ceases to exist is given by $1 = d\psi'(0)$.

The derivative at $X = 0$ (which we state now for general q) reads

$$
\frac{\partial}{\partial X} \psi(X) \Big|_{X=0} = \frac{e^{2\beta} - 1}{e^{2\beta} + q - 1} = \frac{2\theta}{q - (q-2)\theta}. \tag{24}
$$

Hence, the critical value in the Ising case is given by $d \tanh \beta = 1$, for a regular tree where every vertex has d children.

We note that this quantity equals λ_2, the second eigenvalue of the transition matrix associated to the model.

Let us now turn to the Potts model with $q \ge 3$. A computation shows that $\psi''(0) > 0$ for $\beta > 0$ and $q \ge 3$, and hence the function ψ is *not* concave. This reflects the fact that the transition at the critical point where a positive solution ceases is a first order transition, where the non-zero solution is bounded away from zero.

For a regular tree with d children we can derive the transition value $\beta(q, d)$ as follows: We must have $1 = d\psi'(X^*)$, meaning that the function ψ touches the line X with the same slope.

This equation translates into $\frac{1}{d} = \frac{ax}{q-1+ax} - \frac{x}{q-2+a+x}$ in the variables $a = e^{2\beta}$, $x = \exp[X^*]$. The fixed point equation itself reads $x^{\frac{1}{d}} = \frac{q-1+ax}{q-2+a+x}$.

From these two equations the critical values can be derived numerically for any d, q. We note moreover that, for the special case of a binary tree $d = 2$, the fixed point equation is cubic in the variable $y := x^{\frac{1}{2}}$. The fixed point equation is equivalent to $y(q-2+a+y^2)-((q-1)+ay^2) = 0$. We already know one root, it is $y = 1$, so we can produce a quadratic equation by polynomial division. Writing $y = 1 + u$ we get the solutions $u = \frac{1}{2}(-3 + a - \sqrt{5 - 2a + a^2 - 4q})$ and $u = \frac{1}{2}(-3 + a + \sqrt{5 - 2a + a^2 - 4q})$. The solution ceases to exist when the argument of the squareroot becomes negative which results in a critical value $a = 1 + 2\sqrt{q - 1}$, or $\beta(d = 2, q) = \frac{1}{2}\log(1+2\sqrt{q - 1})$. We note the numerical values $\beta(d = 2, q = 3) = 0.671227$, $\beta(d = 2, q = 4) = 0.748034$.

The same type of reasoning can be used for $d = 3$ where the fixed point equation requires the solution of a fourth order equation in $z = x^{\frac{1}{3}}$, which can be reduced to a third order equation by dividing out the root $z = 1$. We do not give details here.

2.3. Controlling the recursion relation for the boundary entropy

Lemma 2.5.

$$X_i^j(v) = \sum_{\omega:v \to w} \left[u\left(\mathbb{P}_v^{N,\xi}(\eta(v) = j) \right) - u\left(\mathbb{P}_v^{N,\xi}(\eta(v) = i) \right) \right], \tag{25}$$

where

$$u(p_1) = \log(1 + p_1(e^{2\beta} - 1)). \tag{26}$$

Proof. Remember the recursion given in Lemma 2.4. Now re-express the X's by the p-variables and use the fact that they form a probability vector. $\square$

Using this we may derive the following equality on the iteration of the boundary entropy.

Lemma 2.6.

$$Q_v^{N,2} X_1^2(v) = \frac{2\theta}{q - (q - 2)\theta} \sum_{\omega:v \to w} Q_w^{N,2} \left[u\left(\mathbb{P}_w^{N,\xi}(\eta(w) = 2) \right) \right.$$

$$\left. - u\left(\mathbb{P}_w^{N,\xi}(\eta(w) = 1) \right) \right]. \tag{27}$$

Proof. As the second piece of information next to Lemma 2.5 which is needed to understand the iteration for the boundary relative entropy $m_v^{(N)}$ we must see how the boundary measure $Q_v^{N,j}(\mathrm{d}\xi)$, obtained by conditioning at v, relates to the boundary measures obtained by conditioning at the children, denoted by w.

For the Potts model a computation shows that

$$Q_v^{N,j} = \prod_{v \to w} \left[\frac{\exp(2\beta)}{(q - 1) + \exp(2\beta)} Q_w^{N,j} + \frac{1}{(q - 1) + \exp(2\beta)} \sum_{i \neq j} Q_w^{N,i} \right]$$

$$= \prod_{v \to w} \left[\frac{1 + \theta}{q - (q - 2)\theta} Q_w^{N,j} + \frac{1 - \theta}{q - (q - 2)\theta} \sum_{i \neq j} Q_w^{N,i} \right]. \tag{28}$$

Thus, to control the iteration we must look at the terms

$$\left[\frac{1+\theta}{q-(q-2)\theta} Q_w^{N,2} + \frac{1-\theta}{q-(q-2)\theta} Q_w^{N,1} + \frac{1-\theta}{q-(q-2)\theta} \sum_{i\geq 3} Q_w^{N,i} \right]$$
$$\left[u\left(\mathbb{P}_w^{N,\xi}(\eta(w)=2) \right) - u\left(\mathbb{P}_w^{N,\xi}(\eta(w)=1) \right) \right]. \tag{29}$$

We first note that, by symmetry under the measure $Q_w^{N,i}$, for $i = 3, \ldots, q$, the corresponding terms in the sum vanish. Now we use the permutation symmetry of the Potts indices to see the proof. $\square$

Next we use the following representation.

Lemma 2.7.

$$Q_v^{N,2} X_1^2(v) = \frac{2\theta}{q-(q-2)\theta} \sum_{w:v\to w} \int \mathbb{P}(\mathrm{d}\xi) h\left(\mathbb{P}_w^{N,\xi}(\eta(w)=2), \mathbb{P}_w^{N,\xi}(\eta(w)=1) \right), \tag{30}$$

with

$$h(p_2, p_1) = qp_2(u(p_2) - u(p_1)). \tag{31}$$

Proof. This follows as in the Proof of Lemma 2.2 by plugging in the Radon–Nikodym derivative of $Q_w^{N,2}$ w.r.t. the open b.c. measure. $\square$

With these preparations we can now finish the proof of the main proposition.

Proof of Proposition 2.3. Recalling the definition of the symmetrization operator (14) we obtain

$$Q_v^{N,2} X_1^2(v) = \frac{2\theta}{q-(q-2)\theta} \sum_{w:v\to w} \int \mathbb{P}(\mathrm{d}\xi)(Rh)$$
$$\times \left(\mathbb{P}_w^{N,\xi}(\eta(w)=1), \ldots, \mathbb{P}_w^{N,\xi}(\eta(w)=q) \right), \tag{32}$$

where

$$(Rh)(p_1, \ldots, p_q) = \frac{1}{q-1} \sum_{i=1}^{q} (qp_i - 1)u(p_i). \tag{33}$$

From here follows that

$$Q_v^{N,2} X_1^2(v) = \frac{2\theta}{q-(q-2)\theta} \sum_{w:v\to w} \int \mathbb{P}(\mathrm{d}\xi) H\left(\mathbb{P}_w^{N,\xi}(\eta(w)=1), \ldots, \mathbb{P}_w^{N,\xi}(\eta(w)=q) \right), \tag{34}$$

where

$$H(p_1, \ldots, p_q) = \frac{1}{q-1} \sum_{i=1}^{q} (qp_i - 1)\tilde{u}(p_i), \tag{35}$$

with

$$\tilde{u}(p_1) = \log \frac{1 + p_1(e^{2\beta} - 1)}{1 + \frac{1}{q}(e^{2\beta} - 1)}. \tag{36}$$

From (34) we have the linear recursion relation

$$
\begin{aligned}
m^N(v) &= Q_v^{N,2} X_1^2(v) \\
&\leq \frac{2\theta}{q - (q-2)\theta} \bar{c}(\beta, q) \sum_{w:v \to w} \int \mathbb{P}(d\xi) Rg \\
&\quad \times \left(\mathbb{P}_w^{N,\xi}(\eta(w) = 1), \ldots, \mathbb{P}_w^{N,\xi}(\eta(w) = q) \right) \\
&\leq \frac{2\theta}{q - (q-2)\theta} \bar{c}(\beta, q) \sum_{w:v \to w} m^N(w)
\end{aligned}
\tag{37}
$$

and from here the result of the proposition follows. □

We could end the paper at this point, but let us comment on the constant appearing, and provide the following conjecture.

Define

$$
\hat{c}(\beta, q) := \sup_{p \in P, \, p_2 = \cdots = p_q} \frac{H(p_1, \ldots, p_q)}{Rg(p_1, \ldots, p_q)}.
\tag{38}
$$

Conjecture 2.8. *We believe that $\hat{c}(\beta, q) = \bar{c}(\beta, q)$.*

We checked this numerically for small values of q. If the previous conjecture is true, the two properties of $\bar{c}(\beta, q)$, namely, monotonicity in q and the bound $\bar{c}(\beta, q) \leq \theta$ carry over. These two properties are seen as follows.

Lemma 2.9.

$$
\bar{c}(\beta, q) = \sup_{x \in D_q} \bar{\varphi}(q, \lambda_q)(x),
\tag{39}
$$

with the function

$$
\bar{\varphi}(q, \lambda_q)(x) = \frac{\log\left(\frac{1 + \lambda_q x}{1 - \lambda_q (q-1)x} \right)}{\log\left(\frac{1 + qx}{1 - q(q-1)x} \right)},
\tag{40}
$$

with parameter $\lambda_q = \frac{e^{2\beta} - 1}{1 + \frac{1}{q}(e^{2\beta} - 1)}$ on the range $D_q = \left[-\frac{1}{q}, \frac{1}{q(q-1)} \right]$ with $D_{(q-1)} \supset D_q$.

Proof. Change to new coordinates on the simplex of probability vectors $(p_1, \ldots, p_q)$ given by

$$
x_i = p_i - \frac{1}{q} \quad \text{for } i = 1, \ldots, q-1,
\tag{41}
$$

take $x = x_i$ for $i = 1, \ldots, q-1$, and use Conjecture 2.8. □

Lemma 2.10. *For all $q \geq 3$ we have that*

$$
\hat{c}(\beta, q) < \hat{c}(\beta, q-1) \leq \frac{\lambda_2}{2} = \theta.
\tag{42}
$$

Proof. We use that

$$\frac{\partial \bar{\varphi}(q, \lambda_q)(x)}{\partial q} < 0, \tag{43}$$

for $x \in D_q$. This gives

$$\bar{\varphi}(q, \lambda_q)(x) < \bar{\varphi}(q - 1, \lambda_{q-1})(x) < \bar{\varphi}(2, \lambda_2)(x), \quad x \in D_q. \quad \square \tag{44}$$

This observation makes it very simple to compute $\hat{c}(\beta, q)$ numerically, for every q.

Next, what about the sharpness of the constant? Could it be possible that Theorem 1.1 in fact holds with the sharp value $\frac{e^{2\beta}-1}{q-1+e^{2\beta}}$ replacing the constant $\bar{c}(\beta, q)$? In our approach such a conjecture would be based on looking at the Hessian of the function

$$\partial_{x_i, x_j} \varphi_{\lambda, q}(x_1, \ldots, x_{q-1})|_{x_k=0} \, \forall k = 4\lambda 1_{i=j} + 2\lambda 1_{i \neq j}.$$

Indeed, heuristically it should suffice to look at the quadratic approximation around the equidistribution. This results in the rigorous lower bound $\bar{c}(\beta, q) \geq \frac{\lambda}{q} = \frac{e^{2\beta}-1}{q-1+e^{2\beta}}$ which we recognize as the Kesten–Stigum bound. For the Ising model we have equality, which is not true for $q = 3$. For the Ising model in the presence of weak asymmetry we refer to [15].

Let us compare with the recent literature. In their paper [13] Montanari and Mezard make the conjecture that the Kesten–Stigum bound is sharp for $q \leq 4$, or more precisely:

Conjecture 2.11 (*Mézard and Montanari [13]*). *Consider the Potts model with q symbols on a d-ary tree and let $\lambda_2 = \frac{e^{2\beta}-1}{e^{2\beta}+q-1} = \frac{2\theta}{q-(q-2)\theta}$, with $\theta = \tanh(\beta)$, then, if $q \leq 4$ and $d < d_{\max}$, there is reconstruction if and only if $d\lambda_2^2 > 1$.*

This conjecture is based on extensive numerical simulations of the random recursion. Moreover, the restriction on d comes from the limitation on the values of d they can treat numerically and they actually think that $d_{\max} = +\infty$. How close are the Kesten–Stigum bounds and our constants? We obtain numerically $\bar{c}(\beta, q) = \frac{e^{2\beta}-1}{q-1+e^{2\beta}}(1 + \varepsilon(q))$ with $\varepsilon(3) = 0.0150$ and $\varepsilon(4) = 0.0365$. If we specialize to a binary tree, and take advantage of the possible temperature dependence of ε we obtain $\beta_c := \sup\{\beta : 2\frac{2\theta}{3-\theta}\bar{c}(\beta, 3) < 1\} = 1.0434$ for $q = 3$ and $\beta_c := \sup\{\beta : 2\frac{2\theta}{4-2\theta}\bar{c}(\beta, 4) < 1\} = 1.1555$ in the case $q = 4$.

After completion of the first draft of our present work Sly's preprint [14] appeared where he proves the following.

Theorem 2.12 (*Sly [14]*). *When $q \leq 3$, and $d > d_{min}$, then the Kesten–Stigum bound is sharp, while the Kesten–Stigum bound is never sharp when $q \geq 5$.*

His method uses large degrees to justify quadratic expansions by means of central limit theorem approximation and makes no statements for small degrees where our estimates also apply. In view of these results it is natural for us to conjecture the following.

Conjecture 2.13. *For $q \leq 4$ there is $\mathbb{Q}$-a.s. no reconstruction if*

$$\mathbb{Q}(d_0) \left(\frac{2\theta}{q - (q-2)\theta} \right)^2 < 1. \tag{45}$$

Table 1
Simulation results for the exact reconstruction thresholds by Mezard and Montanari [13].

$q = 5$	ϵ_r	$\beta_r = -0.5\log\left(\frac{\epsilon_r}{(q-1)(1-\epsilon_r)}\right)$	$\lambda_r = 1 - \frac{q}{q-1}\epsilon_r$
$d = 2$	0.2348	1.2838	0.7065
$d = 3$	0.33881	1.0285	0.5765
$d = 4$	0.4008	0.8942	0.4990
$d = 7$	0.4986	0.6955	0.3767
$d = 15$	0.5955	0.4998	0.2556

Table 2
Our bounds deduced from Theorem 1.1.

$q = 5$	β_c	λ_c
$d = 2$	1.2425	0.6875
$d = 3$	0.98535	0.5526
$d = 4$	0.8520	0.47346
$d = 7$	0.65465	0.35095
$d = 15$	0.4640	0.2342

Finally, what can we say about $q \geq 5$? Montanari and Mézard [13] find in all the test-cases of $q \geq 5$ and d which they treat by simulations that the Kesten–Stigum bound is *not* sharp. Let us therefore conclude the paper by making a comparison of our and their values. Table 1 contains the simulation values from Montanari and Mézard for the critical error threshold ε_r (probability to switch to a new symbol), taken from [13], as well as the corresponding numerical values of the critical inverse temperature β_r and the second eigenvalue λ_r. (The three columns are equivalent but we give them for easy comparison.) Table 2 contains as β_c our lower bound on the presumed true inverse reconstruction temperature β_r, and the corresponding numerical value λ_c of the second eigenvalue. We remark that our bounds appear to be close also here.

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