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# Stochastic representation of subdiffusion processes with time-dependent drift

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**Abstract**

In statistical physics, subdiffusion processes are characterized by certain power-law deviations from the classical Brownian linear time dependence of the mean square displacement. For the mathematical description of subdiffusion, one uses fractional Fokker–Planck equations. In this paper we construct a stochastic process, whose probability density function is the solution of the fractional Fokker–Planck equation with time-dependent drift. We propose a strongly and uniformly convergent approximation scheme which allows us to approximate solutions of the fractional Fokker–Planck equation using Monte Carlo methods. The obtained results for moments of stochastic integrals driven by the inverse $\alpha$-stable subordinator play a crucial role in the proofs, but may be also of independent interest.


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## 1. Introduction

Over the past decades, starting with the pioneering papers of Montroll and his collaborators, see [21], the physical community has shown a growing interest in modelling anomalous diffusion processes. The term *anomalous diffusion* refers to a broad family of processes described by certain deviations from the classical Brownian linear time dependence of the centered second

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moment. The distinct subclass of anomalous diffusion processes forms *subdiffusion processes*. They are characterized through the power-law form of the variance $\mathrm{Var}[X(t)] \sim ct^{\alpha}$ as $t \to \infty$ with $0 < \alpha < 1$. The last formula compared with the expression for the second moment of standard Brownian motion explains why subdiffusion is also termed as *slow diffusion*. The empirically confirmed list of systems displaying a subdiffusive regime is very extensive. It encompasses, among others, charge carrier transport in amorphous semiconductors, nuclear magnetic resonance, diffusion in percolative and porous systems, transport on fractal geometries and dynamics of a bead in a polymeric network, as well as protein conformational dynamics; see [19] and references therein. Therefore, it is of great interest to develop new mathematical tools and methods, which can be used to investigate the properties of subdiffusive systems.

The usual mathematical description of subdiffusion is in terms the fractional Fokker–Planck equation [18,19]:

$$\frac{\partial w(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[K\frac{\partial^2}{\partial x^2}\right]w(x,t) \tag{1}$$

with the initial condition $w(x,0) = \delta(x)$. The operator

$${}_0D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s,$$

$0 < \alpha < 1$, $f \in C^1([0,\infty))$, is the fractional derivative of the Riemann–Liouville type [23]. Here, $\Gamma(\cdot)$ is the Gamma function. The constant $K$ denotes the anomalous diffusion coefficient. In Eq. (1), $w(x,t)$ denotes the probability density function (PDF) of a stochastic process $\{Z(t)\}_{t\geq 0}$. Properties of $\{Z(t)\}$ will be discussed in some detail in the next section. Note that for $\alpha \to 1$, Eq. (1) becomes the ordinary Fokker–Planck equation and its solution is the PDF of the Brownian Motion. It is easy to verify, [19], that $E[Z(t)] = 0$ and $E[Z^2(t)] = \frac{2K}{\Gamma(\alpha+1)}t^{\alpha}$, which confirms that the fractional Fokker–Planck equation (1) can be used to model subdiffusion.

To get some more insight into the structure of (1) let us discuss briefly its derivation. Consider the following continuous-time random walk (CTRW)

$$W(t) = \sum_{i=1}^{N_t} R_i.$$

Here, the counting process is given by $N_t = \max\{n \in \mathbb{N} : \sum_{i=1}^{n} T_i \leq t\}$ with the sequence $\{T_i\}_{i=1}^{\infty}$ of nonnegative i.i.d. random variables representing the waiting times between successive jumps of a particle. It is assumed here that the waiting times $T_i$ belong to the domain of attraction of a completely asymmetric stable distribution, $P(T_i > t) \sim ct^{-\alpha}$ as $t \to \infty$. The sequence $\{R_i\}_{i=1}^{\infty}$ of symmetric i.i.d. random variables with finite second moment represents the jumps of a particle. The sequence $\{R_i\}$ is assumed to be independent of $\{T_i\}$. Then, the scaled process $s^{-\alpha/2}W(st)$ converges weakly as $s \to \infty$ to the process $Z(t)$, whose PDF is the solution of the fractional Fokker–Planck equation (1), [7,14,15,22]. The subdiffusive behaviour of the process $\{Z(t)\}$ is the consequence of the heavy-tailed distributions of each $T_i$ in the underlying CTRW scenario. After every jump, the particle gets immobilized and has to wait a relatively long time for the next move. The heavy-tailed waiting times $T_i$ slow down the diffusion and determine the appearance of the fractional derivative ${}_0D_t^{1-\alpha}$ in (1). The typical trajectory of the subdiffusive process is presented in Fig. 1.

Many physical transport problems take place under the influence of an external force field. To model subdiffusion in the presence of an external force $F(x) \in C^1([0,\infty))$ one uses the

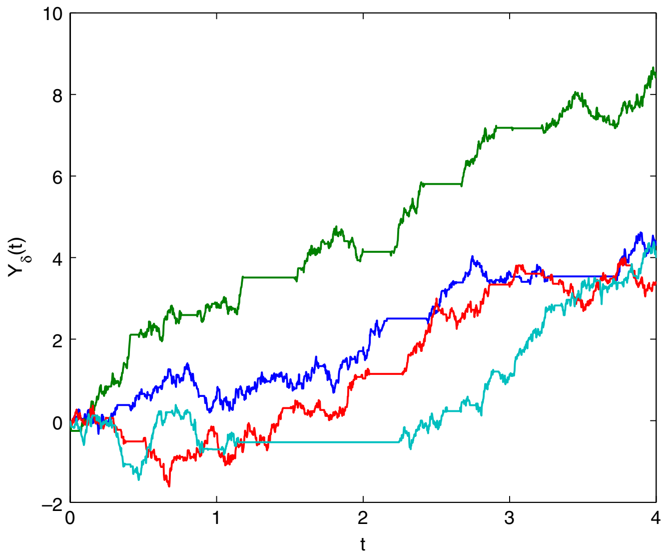


Fig. 1. Sample realizations of the process $\{Y_\delta(t)\}$ in the case of a constant force $F \equiv 1$ and $\alpha = 0.9$. The flat periods of the process are typical for subdiffusion and represent the heavy-tailed rests of the particle.

following version of the fractional Fokker–Planck equation [19]

$$\frac{\partial w(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[-\frac{\partial}{\partial x}F(x) + K\frac{\partial^2}{\partial x^2}\right]w(x,t), \tag{2}$$

$w(x,0) = \delta(x)$, which is a generalization of (1). Note that for $\alpha \to 1$ we recover the standard Fokker–Planck equation corresponding to the diffusion process with drift $F(x)$. For the case of time-dependent force $F(t) \in C([0,\infty))$, the recently derived version of the fractional Fokker–Planck equation [25] has the form

$$\frac{\partial w(x,t)}{\partial t} = \left[-F(t)\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right]{}_0D_t^{1-\alpha}\, w(x,t), \tag{3}$$

$w(x,0) = \delta(x)$. As compared to (2) the fractional operator in the last equation does not act on $F(t)$ and, therefore, it does not modify the time-dependent force.

Eqs. (2) and (3) are fundamental for statistical physicists in the modelling of subdiffusion. In the next section we discuss the question of stochastic representation of these equations, namely, how to define a stochastic process whose PDF is the solution of (2) or (3). The answer is known only for the case of Eq. (2); see [13]. In this paper we construct a stochastic process, whose PDF is the solution of the fractional Fokker–Planck equation (3). Our verification method is based on the analysis of the moments of the introduced process. However, the obtained results for moments of stochastic integrals driven by the inverse subordinator may be also of independent interest.

It should be noted that the physically correct extension of Eqs. (2) and (3) to the case of a space–time-dependent force $F(x,t)$ is still not known. One can obviously plug an arbitrary force in one of the above equations, however, it is not clear if such new mathematical object corresponds to any physical process. The detailed discussion of this problem can be found in [5,27]. Nevertheless, there is a considerable progress in generalizing fractional Fokker–Planck equations to some physically accepted models. Recently, the fractional Black–Scholes formula for subdiffusive geometric Brownian motion was derived in [11]. The case of space-and-time fractional Fokker–Planck equation and its stochastic representation can be found in [12]. Some advanced studies on fractional Feynman–Kac formula are still in progress.

Since the solution of Eq. (3) in the explicit form is not known, in the last section we make use of the obtained stochastic representation and introduce a strongly convergent approximation scheme. It allows us to approximate solutions of (3) using Monte Carlo techniques for arbitrary force $F(t)$. Finally, taking advantage of the proposed methods, we present some numerical results.

## 2. Stochastic representation of subdiffusion

In this section we discuss the question of stochastic representation of the fractional Fokker–Planck equations (2) and (3). Let us begin with recalling some basic facts concerning subordinators and their inverses. A Lévy process $\{U(t)\}_{t\geq 0}$ with nonnegative increments is called a subordinator. It is a well-known fact [24] that the Laplace transform of $\{U(t)\}$ has the form

$$E\left[\mathrm{e}^{-uU(t)}\right] = \mathrm{e}^{-t\Psi(u)},$$

where $\Psi(u)$ is the Lévy exponent. It can be written as

$$\Psi(u) = \lambda u + \int_0^\infty (1-\mathrm{e}^{-ux})\nu(\mathrm{d}x).$$

Here, $\lambda \geq 0$ is the drift and $\nu(\mathrm{d}x)$ is the appropriate Lévy measure.

Given a subordinator $\{U(t)\}$, the first-passage time process defined as

$$S(t) = \inf\{\tau > 0 : U(\tau) > t\}$$

is called the *inverse subordinator*. The inverse subordinators have found applications in many areas of probability theory. Their relations to local times of Markov processes are discussed in [1]. Some recent results on the connection between inverse subordinators and renewal processes can be found in [2,26,10]. Applications to finance and physics are presented in [28,17], respectively.

It turns out that in the context of modelling of subdiffusion, the *inverse $\alpha$-stable subordinator* $\{S_\alpha(t)\}_{t\geq 0}$, $0 < \alpha < 1$, is of special importance. It is defined as

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}, \tag{4}$$

where $\{U_\alpha(\tau)\}_{\tau\geq 0}$ is the $\alpha$-stable subordinator [24] with Laplace transform $E\left[\mathrm{e}^{-uU_\alpha(\tau)}\right] = \mathrm{e}^{-\tau u^\alpha}$, $0 < \alpha < 1$. Since $\{U_\alpha(\tau)\}$ is a pure-jump process with cadlag trajectories, the sample paths of $\{S_\alpha(t)\}$ are continuous and singular with respect to the Lebesgue measure. Additionally, every jump of $\{U_\alpha(\tau)\}$ corresponds to the flat period of its inverse. Notably, these heavy-tailed flat periods of $\{S_\alpha(t)\}$ are characteristic for the subdiffusive dynamics, as they represent long waiting times in which the particle is immobilized.

Using $1/\alpha$-self-similarity of $\{U_\alpha(\tau)\}$, we obtain $P(S_\alpha(t) \leq \tau) = P(U_\alpha(\tau) \geq t) = P((t/U_\alpha(1))^\alpha \leq \tau)$. Therefore, the distribution of $S_\alpha(t)$ is equal to the distribution of the random variable $(t/U_\alpha(1))^\alpha$. Computing the moments of the last random variable shows, [22,16], that the Laplace transform of $\{S_\alpha(t)\}$ equals

$$E\left[\mathrm{e}^{-uS_\alpha(t)}\right] = E_\alpha(-ut^\alpha). \tag{5}$$

Here, the function $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}$ is the Mittag-Leffler function, [23]. The above result will be used later.

Coming back to the problem of stochastic representation of fractional Fokker–Planck equations, the following result concerning Eq. (2) has been presented in [13]; see also [15,19]:

**Theorem** (*[13]*). *Let $\{S_\alpha(t)\}$ be the inverse $\alpha$-stable subordinator (4) and let $\{X(\tau)\}_{\tau\geq 0}$ be the solution of the following Itô stochastic differential equation*

$$\mathrm{d}X(\tau) = F(X(\tau))\mathrm{d}\tau + (2K)^{\frac{1}{2}}\mathrm{d}B(\tau), \quad X(0) = 0, \tag{6}$$

*with drift $F(x)$ and diffusion constant $K > 0$ driven by the standard Brownian motion $\{B(\tau)\}$. Assume that the processes $\{B(\tau)\}$ and $\{S_\alpha(t)\}$ are independent. Then, the PDF of the subordinated process*

$$Z(t) = X(S_\alpha(t)), \quad t \geq 0, \tag{7}$$

*is the solution of the fractional Fokker–Planck equation (2).*

The proof of the above theorem is based on the Laplace transform techniques. The crucial fact in the proof is that the processes $\{X(\tau)\}$ and $\{S_\alpha(t)\}$ are assumed independent. Denoting by $p(x, t)$ the PDF of $Z(t)$, one shows that the Laplace transform $\widehat{p}(x, k)$ of $p(x, t)$ satisfies

$$k\widehat{p}(x, k) - p(x, 0) = k^{1-\alpha}\left[-\frac{\partial}{\partial x}F(x)\widehat{p}(x, k) + K\frac{\partial^2}{\partial x^2}\widehat{p}(x, k)\right].$$

Inverting the Laplace transform one obtains

$$\frac{\partial p(x, t)}{\partial t} = {}_0D_t^{1-\alpha}\left[-\frac{\partial}{\partial x}F(x) + K\frac{\partial^2}{\partial x^2}\right]p(x, t).$$

The above theorem provides an insight into the structure of the subdiffusion process modelled by (2). The fractional Fokker–Planck equation (2) describes the PDF of the standard diffusion process $\{X(\tau)\}$ subordinated to $\{S_\alpha(t)\}$. The heavy-tailed flat periods of $\{S_\alpha(t)\}$, representing the temporary immobilization of the particle, are typical for subdiffusive dynamics. (See Fig. 1 for the typical trajectory of $\{X(S_\alpha(t))\}$.)

The next theorem, which is the main result of this paper, solves the problem of stochastic representation of the fractional Fokker–Planck equation (3) with time-dependent force $F(t)$.

**Theorem 1.** *Let $\{S_\alpha(t)\}$ be the inverse $\alpha$-stable subordinator (4) and let $\{B(\tau)\}$ be the standard Brownian motion. Assume that $\{S_\alpha(t)\}$ and $\{B(\tau)\}$ are independent. Then, the PDF of the process*

$$Y(t) = \int_0^t F(u)\mathrm{d}S_\alpha(u) + B(S_\alpha(t)), \quad t \geq 0, \tag{8}$$

*is the solution of the fractional Fokker–Planck equation (3) with time-dependent force $F(t)$.*

We postpone the proof of the above theorem till the end of this section. Since the process $\{S_\alpha(t)\}$ is nondecreasing, the integral on the right side of (8) is interpreted pathwise as the Lebesgue–Stieltjes integral. The similar stochastic integrals were obtained in [20] as limits of certain random sequences.

The following result allows us to compare the structure of processes (7) and (8). Additionally, it gives a physical interpretation to $\{Y(t)\}$.

**Lemma 1.** *Let $\{U_\alpha(\tau)\}$ be the $\alpha$-stable subordinator and let $\{S_\alpha(t)\}$ be its inverse. Assume that both processes are independent of the Brownian motion $\{B(\tau)\}$. Then, the process $\{Y(t)\}$ defined in (8) can be equivalently represented as*

$$Y(t) = \widehat{X}(S_\alpha(t)), \tag{9}$$

*where $\{\widehat{X}(\tau)\}_{\tau\geq 0}$ is given by*

$$\mathrm{d}\widehat{X}(\tau) = F(U_\alpha(\tau))\mathrm{d}\tau + \mathrm{d}B(\tau), \quad \widehat{X}(0) = 0. \tag{10}$$

**Proof.** From (10) we get that $\{\widehat{X}(\tau)\}$ is given by

$$\widehat{X}(\tau) = \int_0^\tau F(U_\alpha(u))\mathrm{d}u + B(\tau).$$

Therefore, it is enough to verify that

$$\int_0^{S_\alpha(t)} F(U_\alpha(u))\mathrm{d}u = \int_0^t F(u)\mathrm{d}S_\alpha(u). \tag{11}$$

Fix $t > 0$. Assume first that $F(t) = \mathbf{1}_{[t_1,t_2)}(t)$ with $0 < t_1 < t_2 \leq t$. Then, we have

$$\int_0^t F(u)\mathrm{d}S_\alpha(u) = S_\alpha(t_2) - S_\alpha(t_1).$$

Additionally, using the definition of $\{S_\alpha(t)\}$, we immediately obtain that $S_\alpha(t) > \tau \Leftrightarrow U_\alpha(\tau) < t$. Therefore,

$$\int_0^{S_\alpha(t)} F(U_\alpha(u))\mathrm{d}u = \int_0^{S_\alpha(t)} \mathbf{1}_{[S_\alpha(t_1),S_\alpha(t_2)]}(u)\mathrm{d}u = S_\alpha(t_2) - S_\alpha(t_1).$$

The intervals $[t_1, t_2)$ generate the Borel $\sigma$-field $\mathcal{B}([0, t])$; therefore (11) holds for every $F$ of the form $F(t) = \mathbf{1}_A(t)$, $A \in \mathcal{B}([0, t])$. Since $F$ is assumed continuous (thus measurable), it can be approximated by the step functions. Thus, using the standard arguments, we get (11) for any continuous $F$. □

Comparing the structures of (7) and (9), we see that the main difference between these processes lies in the underlying stochastic differential equations (6) and (10). The crucial factor is the process $\{U_\alpha(\tau)\}$, which turns up in (10). Its appearance fulfills the physical requirement that the time-dependent force $F$ should vary in the real time $t$. Indeed, the process $\{U_\alpha(\tau)\}$ reverses the time and cancels the effect of the subordinator $\{S_\alpha(t)\}$ on the force $F$. Therefore, $\{S_\alpha(t)\}$ subordinates the process $\{\widehat{X}(\tau)\}$ without subordinating the time-dependent force. Moreover, the processes $\{\widehat{X}(\tau)\}$ and $\{S_\alpha(t)\}$ are evidently dependent. Thus, one cannot repeat the methods used in the proof of the theorem for the case of space-dependent drift $F(x)$.

Since the inverse subordinator $\{S_\alpha(t)\}$ is non-Markovian (it is a local time of some Markov process, thus it has memory [1]), so is the process $\{Y(t)\}$. The fractional Fokker–Planck equation describes only the evolution in time of one-dimensional distributions; thus it is an incomplete mathematical description of the non-Markovian process. The stochastic representation obtained in Theorem 1 overcomes this problem by giving the complete mathematical picture of subdiffusion.

The fractional Fokker–Planck equation (3) and its solution $w(x,t)$ were constructed in [25] as a limit distribution in certain CTRW scheme. Its derivation was based on generalized master equation with two balance conditions: the probability conservation in a given state and under transition between different states. The considered physical system was assumed to be infinite and spatially homogeneous. In [25], authors showed that the moments

$$m_n(t) = \int_{-\infty}^{\infty} x^n w(x,t)\mathrm{d}x$$

of the distribution $w(x,t)$ satisfy the following recursive relation

$$m_n(t) = n\int_0^t F(t_1)\,{}_0D_{t_1}^{1-\alpha}\, m_{n-1}(t_1)\mathrm{d}t_1 + \frac{n(n-1)}{2}\int_0^t {}_0D_{t_1}^{1-\alpha}\, m_{n-2}(t_1)\mathrm{d}t_1 \tag{12}$$

with $m_0(t) = 1$ and $m_{-1}(t) = 0$. Recall that the operator ${}_0D_t^{1-\alpha}$ is the fractional Riemann–Liouville derivative. In order to verify the stochastic representation of the fractional Fokker–Planck equation (3), one cannot use the Laplace transform technique employed in the proof of Theorem in [13]. The main obstacle is the fact that the force in (3) is time-dependent. Therefore, no useful expression for the Laplace transform of Eq. (3) can be obtained. Thus, it is necessary to find a different way to prove Theorem 1. It turns out that the recursive formula (12) plays a crucial role in the proof. The next lemma allows us to calculate moments of stochastic integrals driven by the inverse $\alpha$-stable subordinator.

**Lemma 2.** *Let $\{S_\alpha(t)\}$ be the inverse $\alpha$-stable subordinator (4) and let $\{B(\tau)\}$ be the standard Brownian motion. Assume that $\{S_\alpha(t)\}$ and $\{B(\tau)\}$ are independent. Denote the moments*

$$a_n(t) = E[(B(S_\alpha(t)))^n],$$
$$b_n(t) = E\left[\left(\int_0^t F(t_1)\mathrm{d}S_\alpha(t_1)\right)^n\right],$$
$$c_{k,n}(t) = E\left[(B(S_\alpha(t)))^k\left(\int_0^t F(t_1)\mathrm{d}S_\alpha(t_1)\right)^n\right],$$

*with $k,n \in \mathbb{N}$, $F \in C([0,\infty))$. Then, the following relations are satisfied by $a_n(t)$, $b_n(t)$ and $c_{k,n}(t)$:*

(i) *If $n = 2m$, $m \in \mathbb{N}$, then*

$$a_n(t) = \frac{n(n-1)}{2}\int_0^t {}_0D_{t_1}^{1-\alpha}\, a_{n-2}(t_1)\mathrm{d}t_1,$$

*if $n = 2m-1$, $m \in \mathbb{N}$, then $a_n(t) = 0$.*

(ii) *For every $n \in \mathbb{N}$*

$$b_n(t) = n\int_0^t F(t_1)\,{}_0D_{t_1}^{1-\alpha}\, b_{n-1}(t_1)\mathrm{d}t_1. \tag{13}$$

(iii) *If $k = 2m$, $m \in \mathbb{N}$, then*

$$c_{k,n}(t) = n\int_0^t F(t_1)\,{}_0D_{t_1}^{1-\alpha}\, c_{k,n-1}(t_1)\mathrm{d}t_1 + \frac{k(k-1)}{2}\int_0^t {}_0D_{t_1}^{1-\alpha}\, c_{k-2,n}(t_1)\mathrm{d}t_1, \tag{14}$$

*if $k = 2m-1$, $m \in \mathbb{N}$, then $c_{k,n}(t) = 0$.*

**Proof.** (i) From (5) we immediately obtain that

$$E[S_\alpha^n(t)] = \frac{t^{n\alpha} n!}{\Gamma(n\alpha + 1)}.$$

Since $\{B(\tau)\}$ is 1/2-self-similar, conditioning on $S_\alpha(t)$, we obtain

$$a_n(t) = E[S_\alpha^{n/2}(t)]E[B^n(1)].$$

Thus, for $n = 2m - 1$ we have $a_n(t) = 0$. For $n = 2m$ the above expression yields

$$a_n(t) = E[S_\alpha^m(t)]E[B^{2m}(1)] = \frac{t^{m\alpha} m!}{\Gamma(m\alpha + 1)} 1 \cdot 3 \cdot \ldots \cdot (2m - 1).$$

Now, taking advantage of the last formula and the fact that ${}_0D_t^{1-\alpha}\, t^\mu = \frac{\Gamma(1+\mu)}{\Gamma(\alpha+\mu)} t^{\alpha+\mu-1}$, $\mu \geq 0$, we get the desired result.

(ii) By the change of variable formula (or by the Itô's lemma for semimartingales), we get

$$\left(\int_0^t F(t_1)\mathrm{d}S_\alpha(t_1)\right)^n = n \int_0^t \left(\int_0^{t_1} F(t_2)\mathrm{d}S_\alpha(t_2)\right)^{n-1} F(t_1)\mathrm{d}S_\alpha(t_1).$$

Thus, after $n$ iterations

$$\left(\int_0^t F(t_1)\mathrm{d}S_\alpha(t_1)\right)^n = n! \int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-1}} F(t_1)\ldots F(t_n)\mathrm{d}S_\alpha(t_n)\ldots\mathrm{d}S_\alpha(t_1). \tag{15}$$

Now, introduce the random measure on $[0, \infty)$ by $\Pi((s, t]) = S_\alpha(t) - S_\alpha(s)$, where $t > s \geq 0$. Let $\{C(t)\}_{t\geq 0}$ be the Cox process directed by $\Pi$, i.e., conditionally on $\Pi = \lambda$, $\{C(t)\}$ is equal in distribution to the inhomogeneous Poisson process with intensity $\lambda$. Note, [8], that $\{C(t)\}$ is the renewal process with the renewal function

$$u(t) = E[C(t)] = E[S_\alpha(t)] = \frac{t^\alpha}{\Gamma(\alpha + 1)}.$$

For the renewal process $\{C(t)\}$ we have (see [3] page 73)

$$E[\mathrm{d}C(t_1)\ldots\mathrm{d}C(t_n)] = \prod_{i=1}^n u'(t_i - t_{i+1})\mathrm{d}t_i,$$

where $t_1 > t_2 > \cdots > t_n > t_{n+1} = 0$. Since the factorial moments of the Cox process $\{C(t)\}$ are equal to the ordinary moments of its directing measure $\Pi$, see [3] page 170, we obtain

$$E[\mathrm{d}S_\alpha(t_1)\ldots\mathrm{d}S_\alpha(t_n)] = \prod_{i=1}^n u'(t_i - t_{i+1})\mathrm{d}t_i.$$

The above result in combination with (15) yields

$$\begin{aligned} b_n(t) &= E\left[\left(\int_0^t F(t_1)\mathrm{d}S_\alpha(t_1)\right)^n\right] \\ &= \frac{n!\alpha^n}{\Gamma^n(\alpha + 1)} \int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-1}} \prod_{i=1}^n F(t_i)(t_i - t_{i+1})^{\alpha-1}\mathrm{d}t_n\ldots\mathrm{d}t_1 \end{aligned}$$

with $t_{n+1} = 0$. Consequently,

$$
\begin{aligned}
b_n(t) &= \frac{n\alpha}{\Gamma(\alpha+1)} \int_0^t F(t_1) \int_0^{t_1} (t_1 - t_2)^{\alpha-1} \frac{\mathrm{d}}{\mathrm{d}t_2} b_{n-1}(t_2)\mathrm{d}t_2\mathrm{d}t_1 \\
&= \frac{n}{\Gamma(\alpha+1)} \int_0^t F(t_1) \frac{\mathrm{d}}{\mathrm{d}t_1} \int_0^{t_1} (t_1 - t_2)^{\alpha} \frac{\mathrm{d}}{\mathrm{d}t_2} b_{n-1}(t_2)\mathrm{d}t_2\mathrm{d}t_1 \\
&= n \int_0^t F(t_1)\, {}_0D_{t_1}^{1-\alpha}\, b_{n-1}(t_1)\mathrm{d}t_1,
\end{aligned}
$$

which ends the proof of part (ii).

(iii) Conditioning on $\mathcal{F}_t = \sigma(S_\alpha(\tau) : \tau \le t)$, we get

$$
c_{k,n}(t) = E[B^k(1)]E\left[S_\alpha^{k/2}(t)\left(\int_0^t F(t_1)\mathrm{d}S_\alpha(t_1)\right)^n\right].
$$

Thus, for $k = 2m-1$, $m \in \mathbb{N}$, we have $c_{k,n}(t) = 0$. For $k = 2m$ the above expression yields

$$
c_{k,n}(t) = p_k E\left[S_\alpha^{k/2}(t)\left(\int_0^t F(t_1)\mathrm{d}S_\alpha(t_1)\right)^n\right]
$$

with $p_k = E[B^k(1)] = 1 \cdot 3 \cdot \ldots \cdot (k-1)$. By integration by parts, we obtain

$$
\begin{aligned}
S_\alpha^{k/2}(t)\left(\int_0^t F(t_1)\mathrm{d}S_\alpha(t_1)\right)^n &= n\int_0^t \left(\int_0^{t_1} F(t_2)\mathrm{d}S_\alpha(t_2)\right)^{n-1} F(t_1)S_\alpha^{k/2}(t_1)\mathrm{d}S_\alpha(t_1) \\
&\quad + k/2 \int_0^t \left(\int_0^{t_1} F(t_2)\mathrm{d}S_\alpha(t_2)\right)^n S_\alpha^{k/2-1}(t_1)\mathrm{d}S_\alpha(t_1).
\end{aligned}
$$

Iterating the change of variable formula, we have

$$
\begin{aligned}
& n\int_0^t \left(\int_0^{t_1} F(t_2)\mathrm{d}S_\alpha(t_2)\right)^{n-1} F(t_1)S_\alpha^{k/2}(t_1)\mathrm{d}S_\alpha(t_1) \\
&\quad = n!\int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-1}} \prod_{i=1}^n F(t_i)S_\alpha^{k/2}(t_1)\mathrm{d}S_\alpha(t_n)\ldots\mathrm{d}S_\alpha(t_1);
\end{aligned}
$$

similarly

$$
\begin{aligned}
& k/2\int_0^t \left(\int_0^{t_1} F(t_2)\mathrm{d}S_\alpha(t_2)\right)^n S_\alpha^{k/2-1}(t_1)\mathrm{d}S_\alpha(t_1) \\
&\quad = n!k/2\int_0^t \int_0^{t_1} \ldots \int_0^{t_n} \prod_{i=2}^{n+1} F(t_i)S_\alpha^{k/2-1}(t_1)\mathrm{d}S_\alpha(t_{n+1})\ldots\mathrm{d}S_\alpha(t_1).
\end{aligned}
$$

Moreover, for every $q \in \mathbb{N}$, we have

$$
S_\alpha^q(t_1) = q!\int_0^{t_1} \ldots \int_0^{t_q} \mathrm{d}S_\alpha(t_{q+1})\ldots\mathrm{d}S_\alpha(t_2).
$$

Thus, repeating the reasoning from the proof of part (ii) of the lemma, we obtain

$$\begin{aligned}c_{k,n}(t) &= \frac{n\alpha}{\Gamma(\alpha+1)}\int_0^t F(t_1)\int_0^{t_1}(t_1-t_2)^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t_2}c_{k,n-1}(t_2)\mathrm{d}t_2\mathrm{d}t_1\\ &\quad+\frac{k(k-1)\alpha}{2\Gamma(\alpha+1)}\int_0^t\int_0^{t_1}(t_1-t_2)^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t_2}c_{k-2,n}(t_2)\mathrm{d}t_2\mathrm{d}t_1\\ &= n\int_0^t F(t_1)\,{}_0D_{t_1}^{1-\alpha}\,c_{k,n-1}(t_1)\mathrm{d}t_1+\frac{k(k-1)}{2}\int_0^t {}_0D_{t_1}^{1-\alpha}\,c_{k-2,n}(t_1)\mathrm{d}t_1,\end{aligned}$$

which yields (14) and ends the proof of the lemma. □

Note that for $F \equiv 1$, relation (14) follows immediately from (13) by conditioning. We finish the section with the proof of Theorem 1.

**Proof of Theorem 1.** Recall the definition (8) of the process $\{Y(t)\}$. We start with calculating the moments of $\{Y(t)\}$. Put

$$r_n(t) = E[Y^n(t)] = E\left[\left(\int_0^t F(u)\mathrm{d}S_\alpha(u)+B(S_\alpha(t))\right)^n\right],$$

$n \in \mathbb{N}$. We will show that the moments $r_n$ can be calculated from the recursive formula

$$r_n(t) = n\int_0^t F(t_1)\,{}_0D_{t_1}^{1-\alpha}\,r_{n-1}(t_1)\mathrm{d}t_1+\frac{n(n-1)}{2}\int_0^t {}_0D_{t_1}^{1-\alpha}\,r_{n-2}(t_1)\mathrm{d}t_1 \tag{16}$$

by formally putting $r_0(t) = 1$ and $r_{-1}(t) = 0$. Using Newton's binomial expansion, we get that equality (16) is equivalent to

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}c_{k,n-k}(t) &= n\sum_{k=0}^{n-1}\binom{n-1}{k}\int_0^t F(t_1)\,{}_0D_{t_1}^{1-\alpha}\,c_{k,n-k-1}(t_1)\mathrm{d}t_1\\ &\quad+\frac{n(n-1)}{2}\sum_{k=0}^{n-2}\binom{n-2}{k}\int_0^t {}_0D_{t_1}^{1-\alpha}\,c_{k,n-k-2}(t_1)\mathrm{d}t_1,\end{aligned}\tag{17}$$

where

$$c_{p,q}(t) = E\left[(B(S_\alpha(t)))^p\left(\int_0^t F(t_1)\mathrm{d}S_\alpha(t_1)\right)^q\right],$$

$p, q \in \mathbb{N}$. From (14) we get that for $k \geq 2$

$$\begin{aligned}\binom{n}{k}c_{k,n-k}(t) &= n\binom{n-1}{k}\int_0^t F(t_1)\,{}_0D_{t_1}^{1-\alpha}\,c_{k,n-k-1}(t_1)\mathrm{d}t_1\\ &\quad+\frac{n(n-1)}{2}\binom{n-2}{k-2}\int_0^t {}_0D_{t_1}^{1-\alpha}\,c_{k-2,n-k}(t_1)\mathrm{d}t_1.\end{aligned}$$

Therefore, by Lemma 2, we get that (17) and equivalently (16) hold.

Comparing expressions (12) and (16), we see that moments $m_n(t)$ and $r_n(t)$ coincide. Therefore, to prove that the PDF of $Y(t)$ is equal to the solution $w(x,t)$ of the fractional Fokker–Planck equation (3), it is enough to show that the characteristic function of $Y(t)$ is holomorphic in a neighborhood of zero. In such case the moments determine uniquely the distribution. Fix

$t_0 > 0$. We have

$$\begin{aligned} r_{2n}(t_0) &= E[Y^{2n}(t_0)] \le 4^n E\left[\left(\int_0^{t_0} F(u)\mathrm{d}S_\alpha(u)\right)^{2n}\right] + 4^n E\left[B^{2n}(S_\alpha(t_0))\right] \\ &\le 4^n M^{2n} E[S_\alpha^{2n}(t_0)] + 4^n E\left[S_\alpha^n(t_0)\right] E[B^{2n}(1)] \\ &= 4^n M^{2n} \frac{t_0^{2\alpha n}(2n)!}{\Gamma(2n\alpha+1)} + 4^n \frac{t_0^{\alpha n}(2n)!}{\Gamma(n\alpha+1)2^n}, \end{aligned}$$

where $M = \sup_{0\le s\le t_0} |F(s)|$. Consequently, the series $\sum_{n=1}^{\infty} r_{2n}(t_0)z^n/(2n)!$ is convergent for $|z| < \min\{(4M^2 t_0^{2\alpha})^{-1}; (4t_0^\alpha)^{-1}\}$. Therefore, the moments determine the distribution (see Chapter VII, Section 3 of [4]), and the PDF of $Y(t_0)$ is equal to the solution $w(x,t_0)$ of the fractional Fokker–Planck equation (3). □

## 3. Approximation scheme

In this section we introduce a strongly and uniformly convergent approximation scheme of the obtained stochastic representation process (8). Since the solution of Eq. (3) in the explicit and numerically treatable form is not known, the proposed scheme allows us to approximate solutions of (3) using Monte Carlo methods for arbitrary force $F(t)$.

In what follows, we make an additional weak assumption that the force $F$ is of bounded variation on every interval of the form $[0,t]$, $t > 0$. Then, the function

$$V_F(t) = \sup_P \sum_{i=1}^{n} |F(t_i) - F(t_{i-1})|,$$

where the supremum is taken over all the partitions of the interval $[0,t]$, is called the total variation of $F$.

Recall the definition (4) of the inverse $\alpha$-stable subordinator. Let us introduce the following straightforward approximation $\{S_{\alpha,\delta}(t)\}_{t\ge 0}$ of the process $\{S_\alpha(t)\}$:

$$S_{\alpha,\delta}(t) = (\min\{n \in \mathbb{N} : U_\alpha(\delta n) > t\} - 1)\,\delta, \tag{18}$$

where $\delta > 0$ is the step length. The '$-1$' term in the above expression comes from the fact that we want the process $\{S_{\alpha,\delta}(t)\}$ to start at the origin. Consequently, the proposed approximation of the stochastic representation process $\{Y(t)\}$ defined in (8) has the form

$$Y_\delta(t) = \int_0^t F(u)\mathrm{d}S_{\alpha,\delta}(u) + B(S_{\alpha,\delta}(t)), \quad t \ge 0. \tag{19}$$

We say, [9], that an approximation $Y_\delta$ *converges strongly* to $Y$ at time $T > 0$ if

$$\lim_{\delta\searrow 0} E\left[|Y(T) - Y_\delta(T)|\right] = 0.$$

Additionally, we say that an approximation $Y_\delta$ *converges strongly with order* $\gamma > 0$ to $Y$ at time $T > 0$ if there exists a positive constant $C$, which does not depend on $\delta$, such that

$$E\left[|Y(T) - Y_\delta(T)|\right] \le C\delta^\gamma.$$

The next theorem shows the uniform convergence and verifies the order of convergence of the approximation (19).

**Theorem 2.** *Let $\{S_{\alpha,\delta}(t)\}$ and $\{Y_\delta(t)\}$ be the introduced approximations of the processes $\{S_\alpha(t)\}$ and $\{Y(t)\}$, respectively. Then, for every $T > 0$, the following conditions are satisfied*

(i)
$$\sup_{0\le s\le T} |S_\alpha(s) - S_{\alpha,\delta}(s)| \le \delta \quad a.s.$$

(ii) *Let $0 < q < 1/2$. Then, for appropriately small $\delta > 0$*
$$\sup_{0\le s\le T} |Y(s) - Y_\delta(s)| \le C\delta + \delta^q \quad a.s.,$$
*where $C = \sup_{0\le s\le T} |F(s)| + 2V_F(T) - F(T) + F(0)$.*

(iii)
$$E[|Y(T) - Y_\delta(T)|] \le C_1\delta + C_2\delta^{1/2},$$
*where $C_1 = |F(T)| + 2V_F(T) - F(T) + F(0)$ and $C_2 = E[|B(1)|]$.*

*Consequently, the approximation $\{Y_\delta(t)\}$ is strongly convergent with order $\gamma = 1/2$.*

**Proof.** (i) We have
$$\begin{aligned} S_{\alpha,\delta}(t) &= (\min\{n \in \mathbb{N} : U_\alpha(\delta n) > t\} - 1)\,\delta \\ &\le \inf\{\tau > 0 : U_\alpha(\delta\tau) > t\}\delta \\ &= \inf\{\tau > 0 : U_\alpha(\tau) > t\} = S_\alpha(t). \end{aligned}$$

On the other hand
$$\begin{aligned} S_{\alpha,\delta}(t) &= (\min\{n \in \mathbb{N} : U_\alpha(\delta n) > t\} - 1)\,\delta \\ &\ge (\inf\{\tau > 0 : U_\alpha(\delta\tau) > t\} - 1)\delta \\ &= S_\alpha(t) - \delta. \end{aligned}$$

Thus, we get
$$S_\alpha(t) - \delta \le S_{\alpha,\delta}(t) \le S_\alpha(t),$$

which ends the proof of part (i).

(ii) We have
$$\begin{aligned} \sup_{0\le s\le T} |Y(s) - Y_\delta(s)| \le{}& \sup_{0\le s\le T} |B(S_{\alpha,\delta}(s)) - B(S_\alpha(s))| \\ &+ \sup_{0\le s\le T} \left| \int_0^s F(u)\mathrm{d}S_{\alpha,\delta}(u) - \int_0^s F(u)\mathrm{d}S_\alpha(u) \right|. \end{aligned} \tag{20}$$

To estimate the first term in the above formula, we use the fact that the paths of $B(t)$ are a.s. Hölder continuous on compacts. Namely, for every $0 < q < 1/2$ there exist $c > 0$ such that
$$|B(u) - B(v)| \le c|u - v|^q$$

with $u, v \le L$. If we further assume that $|u - v| \le \delta$ for appropriately small $\delta$, then the constant can be chosen as $c = 1$. Thus, using (i), for small $\delta$ we get
$$\sup_{0\le s\le T} |B(S_{\alpha,\delta}(s)) - B(S_\alpha(s))| \le \sup_{0\le s\le T} |S_{\alpha,\delta}(s) - S_\alpha(s)|^q \le \delta^q.$$

For the second term in (20), applying integration by parts, we obtain

$$\left|\int_0^s F(u)\mathrm{d}S_{\alpha,\delta}(u) - \int_0^s F(u)\mathrm{d}S_\alpha(u)\right| \le |F(s)||S_{\alpha,\delta}(s) - S_\alpha(s)| + \left|\int_0^s S_{\alpha,\delta}(u)\mathrm{d}F(u) - \int_0^s S_\alpha(u)F(u)\right|.$$

Since $F$ is of bounded variation, it can be written as a difference of two increasing functions $F = V_F - D$, where $V_F$ is the total variation of $F$ and $D = V_F - F$. Thus, we have

$$\begin{aligned}\left|\int_0^s F(u)\mathrm{d}S_{\alpha,\delta}(u) - \int_0^s F(u)\mathrm{d}S_\alpha(u)\right| &\le |F(s)|\delta \\ &+ \int_0^s |S_{\alpha,\delta}(u) - S_\alpha(u)|\mathrm{d}V_F(u) + \int_0^s |S_{\alpha,\delta}(u) - S_\alpha(u)|\mathrm{d}D(u) \\ &\le |F(s)|\delta + V_F(s)\delta + (V_F(s) - F(s) + F(0))\delta.\end{aligned}$$

Consequently,

$$\sup_{0\le s\le T}\left|\int_0^s F(u)\mathrm{d}S_{\alpha,\delta}(u) - \int_0^s F(u)\mathrm{d}S_\alpha(u)\right| \le \left(\sup_{0\le s\le T}|F(s)| + 2V_F(T) - F(T) + F(0)\right)\delta,$$

which ends part (ii).

(iii) We have

$$E\left[|Y(T) - Y_\delta(T)|\right] \le E[|B(S_{\alpha,\delta}(T)) - B(S_\alpha(T))|] + E\left[\left|\int_0^T F(u)\mathrm{d}S_{\alpha,\delta}(u) - \int_0^T F(u)\mathrm{d}S_\alpha(u)\right|\right].$$

For the first term on the right side of the above inequality, taking advantage of (i), we get

$$E[|B(S_{\alpha,\delta}(T)) - B(S_\alpha(T))|] \le E[|S_{\alpha,\delta}(T) - S_\alpha(T)|^{1/2}]E[|B(1)|] \le \delta^{1/2}E[|B(1)|].$$

For the second term, from the proof of (ii) we have

$$E\left[\left|\int_0^T F(u)\mathrm{d}S_{\alpha,\delta}(u) - \int_0^T F(u)\mathrm{d}S_\alpha(u)\right|\right] \le (|F(T)| + 2V_F(T) - F(T) + F(0))\delta,$$

and the proof of part (iii) is completed. □

We end the section with some remarks concerning the numerical simulation of the approximations $\{S_{\alpha,\delta}(t)\}$ and $\{Y_\delta(t)\}$. To evaluate the first process, one only needs to generate the values $U_\alpha(\delta n)$, $n = 1, 2, \ldots$. This can be done by the following method of summing up the increments of the subordinator $\{U_\alpha(\tau)\}$ (see [6]):

$$\begin{aligned}&U_\alpha(0) = 0,\\ &U_\alpha(\delta n) = U_\alpha(\delta(n-1)) + \delta^{1/\alpha}\xi_n,\end{aligned}$$

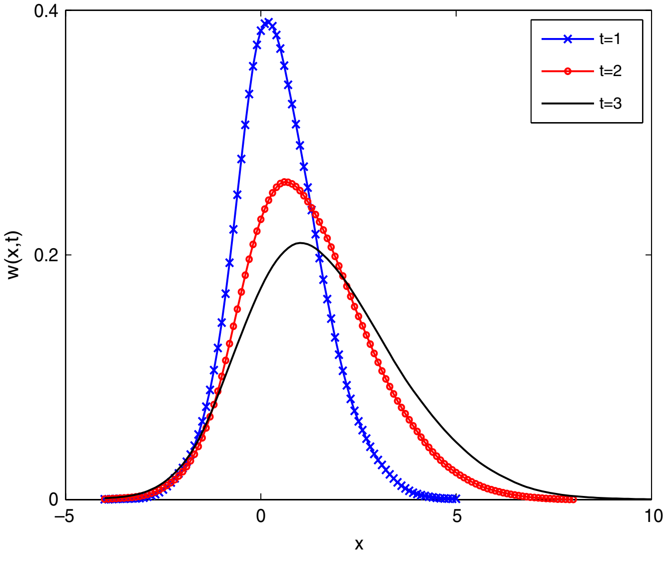


Fig. 2. The approximated solutions $w(x, t)$ of the fractional Fokker–Planck equation (3) with $F(t) = \sin t$ and $\alpha = 0.8$, obtained by the Monte Carlo method. The solutions were estimated from the sample of $10^4$ simulated trajectories of the process $\{Y_\delta(t)\}$.

where $\xi_n$, $n \in \mathbb{N}$, are the i.i.d. totally skewed positive $\alpha$-stable random variables. The procedure of generating realizations of $\xi_n$ is the following, [6]:

$$\xi_n = \frac{\sin(\alpha(V + c_1))}{(\cos(V))^{1/\alpha}} \left( \frac{\cos(V - \alpha(V + c_1))}{W} \right)^{(1-\alpha)/\alpha},$$

where $c_1 = \pi/2$, the random variable $V$ is uniformly distributed on $(-\pi/2, \pi/2)$ and W has exponential distribution with mean one.

As for the process $\{Y_\delta(t)\}$, since $\{S_{\alpha,\delta}(t)\}$ is a scaled renewal process, the integral in (19) can be written as

$$\int_0^t F(u) \mathrm{d}S_{\alpha,\delta}(u) = \delta \sum_{n=1}^{N} F(U_\alpha(\delta n)).$$

Here, $N$ is an integer number such that $U_\alpha(\delta N) < t \le U_\alpha(\delta(N + 1))$. Since the last sum can easily be calculated numerically, the above formula tells us how to evaluate the approximation $\{Y_\delta(t)\}$. Note that the numerical method of simulating the trajectories of Brownian motion $\{B(\tau)\}$ is well-known [9].

In Fig. 1 we present typical trajectories of the process $\{Y_\delta(t)\}$ for the case of a constant force $F \equiv 1$. Fig. 2 shows the approximated solutions of the fractional Fokker–Planck equation (3) with $F(t) = \sin t$, obtained by the Monte Carlo method. The solutions were estimated from the sample of $10^4$ simulated trajectories of the process $\{Y_\delta(t)\}$ with the help of the Rozenblatt–Parzen kernel estimator [6].

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