

Quantile inference for near-integrated autoregressive time series under infinite variance and strong dependence

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Abstract

Consider a near-integrated time series driven by a heavy-tailed and long-memory noise $\varepsilon_t = \sum_{j=0}^{\infty} c_j \eta_{t-j}$, where $\{\eta_j\}$ is a sequence of *i.i.d* random variables belonging to the domain of attraction of a stable law with index α . The limit distribution of the quantile estimate and the semi-parametric estimate of the autoregressive parameters with long- and short-range dependent innovations are established in this paper. Under certain regularity conditions, it is shown that when the noise is short-memory, the quantile estimate converges weakly to a mixture of a Gaussian process and a stable Ornstein–Uhlenbeck (O–U) process while the semi-parametric estimate converges weakly to a normal distribution. But when the noise is long-memory, the limit distribution of the quantile estimate becomes substantially different. Depending on the range of the stable index α , the limit distribution is shown to be either a functional of a fractional stable O–U process or a mixture of a stable process and a stable O–U process. These results indicate that although the quantile estimate tends to be more efficient for infinite variance time series, extreme caution should be exercised in the long-memory situation.

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1. Introduction

Consider a near-integrated first-order autoregressive (AR(1)) model

$$Y_i = \gamma_n Y_{i-1} + \varepsilon_i, \quad (1.1)$$

where $\gamma_n = 1 - \gamma/n$ and γ is a real number. The asymptotic theory of autoregressive time series with roots on or near the unit circle has been actively pursued by statisticians and econometricians alike. As of today, a relatively complete theory has been established under the finite variance situation. For a concise review on the recent developments of this topic, see Chan [5] and the references therein.

A large number of empirical studies ranging from signal processing, network traffic to insurance, however, indicates that time series with heavy tails offer a viable alternative. For background information on heavy-tailed time series and their applications, readers are referred to the *Séminaire Européen de Statistique* edited by Finkenstädt and Rootzén [13], where exemplary theories and applications of extreme values in finance, insurance, the environment and telecommunications are surveyed. In financial econometrics, there has also been increasing interest in modeling financial phenomena by time series driven by heavy-tailed innovations. For example, Fama [12] and Mandelbrot [23,24] argued that distributions of commodity and stock returns are often heavy-tailed with possible infinite variance. Rachev and Mitnik [28] considered stable paretian models in finance, [22] studied agent-based models in with heavy tails, and [2] studied financial market model where order flows follow heavy-tailed and long-memory durations.

Due to the intricacy of the asymptotic theory involved in the infinite variance model, much less is known when both long-range dependence and infinite variance structure are exhibited in the time series. Since the least squares procedure is known to be less robust and less effective when the time series is heavy-tailed, one of the main purposes of this paper is to establish a more robust estimate of α_n for nearly nonstationary AR(1) models (1.1) driven by strongly dependent and infinite variance innovations. For more information and applications concerning strong dependent and infinite variance processes, we refer the readers to [11,30] and the references therein. It should also be pointed out that an alternate way to describe long-range dependence is by means of aggregating short-memory processes with random coefficients, see for example [9,3] and the references therein. In this paper, we follow the traditional method of describing long-range dependence through linear processes.

Specifically, Chan and Zhang [7] considered the least squares inference for a nearly nonstationary time series with errors defined by a heavy-tailed and long-memory noise

$$\varepsilon_i = \sum_{j=0}^{\infty} c_j \eta_{i-j},$$

where $c_0 = 0$ and $c_j = j^{-\beta} l(j)$ when $j \geq 1$, $\beta > 1/\alpha$, $l(\cdot)$ is a slowly varying function and $\eta_i, i \in \mathcal{Z}$ are *i.i.d* variables and in the domain of attraction of a stable law. That is, there exists some sequence $a_n = \inf\{x : P(|\eta_0| > x) \leq 1/n\} = n^{1/\alpha} L(n)$, $L(x)$ which is a slowly varying function such that

$$a_n^{-1} \sum_{i=1}^{[ns]} \eta_i \Rightarrow^{J_1} Z_\alpha(s), \quad (1.2)$$

where $Z_\alpha(s)$ is a stable random variable with index $\alpha \in (0, 2)$ and \Rightarrow^{J_1} denotes weak convergence in the J_1 topology, see [4].

It is well known that when $E\varepsilon_t^2 = \infty$, the least squares estimate is not efficient. An important method used to deal with this problem is the so-called quantile regression, which has been receiving considerable attention since the seminal work of Koenker and Bassett [20], see also [19] for more discussion on this topic. Knight [17,18] established the limit distribution for least absolute deviations estimate for $\gamma_n = 1$ when the infinite variance errors $\{\varepsilon_i\}$ are independent or weakly dependent. Chan et al. [8] considered quantile inference for a nearly non-stationary time series (i.e., $\gamma_n = 1 - \gamma/n$) when $\{\varepsilon_t\}$ are independent with infinite variance.

In this paper, we generalize the results to the case when $\{\varepsilon_t\}$ are long- and short-memory processes with heavy tails. Limit distributions of quantile regression estimate are established under different scenarios. As the limit distributions for long- and short-memory errors are substantially different, these results indicate that when applying quantile regression to infinite variance time series, extreme caution should be exercised. In particular, for short-memory noise ($\beta > 2/\alpha$), the process $n^{-1/2} \sum_{i=1}^{\lfloor nt \rfloor} \varphi_\tau(\varepsilon_i - \beta_0(\tau))$ converges weakly to a Gaussian process, where $\varphi_\tau(x) = \tau - I(x < 0)$. In this case, standard arguments together with the continuous mapping theorem can then be used to show that the limit distribution of the quantile estimate of γ_n is a functional of a stable process and a Brownian motion. For the long-memory case ($\beta < 2/\alpha$), instead of converging weakly to a Gaussian process, the partial sum process $n^{-1/2} \sum_{i=1}^{\lfloor nt \rfloor} \varphi_\tau(\varepsilon_i - \beta_0(\tau))$ converges weakly to a stable process. The crux of the difficulty lies in establishing the limit of the process $\sum_{i=1}^n Y_i \varphi_\tau(\varepsilon_i - \beta_0(\tau))$. We show that when $1/\alpha < \beta < (\alpha + 2)/(3\alpha)$ and $\alpha > 1$, $\sum_{i=1}^n Y_i \varphi_\tau(\varepsilon_i - \beta_0(\tau))$ can be approximated by $\sum_{i=1}^n Y_i \varepsilon_i$ and as a result, the limit distribution of the quantile estimate of γ_n can be deduced from [7] as a functional of a fractional Ornstein–Uhlenbeck (O–U) stable process. On the other hand, when $1 < \beta < 2/\alpha$ and $\alpha > 1$, applying a result of [21] shows that the partial sum $\sum_{i=1}^n Y_i \varphi_\tau(\varepsilon_i - \beta_0(\tau))$ converges weakly to a stable process and as a result, the limit distribution of the quantile estimate of γ_n is a functional of two different stable processes.

The paper is organized as follows. Section 2 provides the asymptotic distribution of quantile regression estimate. As the limit process depends on unknown parameters of the density of ε at the quantile and the variance in the weakly dependent case, estimation of these parameters and their corresponding limit distributions are given in Section 3. Proofs of the main results are given in Section 4 and technical lemmas are relegated to the Appendix.

2. Quantile regression

Given $\tau \in (0, 1)$, let $\gamma(\tau) = \gamma_n$ and denote the τ -th quantile of ε_t by $\beta(\tau)$. Define $\rho_\tau(\mu) = \mu(\tau - I(\mu < 0))$, $\theta(\tau) = (\beta(\tau), \gamma(\tau))^T$ and $X_t = (1, Y_{t-1})^T$. Let $Q_t(\tau|t-1)$ be the τ -th conditional quantile of Y_t conditional on Y_{t-1} . Then $Q_t(\tau|t-1) = X_t^T \theta(\tau)$. According to [20], the quantile regression estimate is defined as

$$\hat{\theta}(\tau) = \operatorname{argmin}_{\theta(\tau)} \sum_{i=1}^n \rho_\tau(Y_i - X_i^T \theta(\tau)). \quad (2.1)$$

We impose the following conditions throughout the entire paper.

H₁. Let $\{Y_t\}$ follow model (1.1) with $\{\eta_j\}$ satisfying (1.2).

H₂. The density $p(x)$ of η_1 satisfies $|p'(x)| \leq C_1(1 + |x|)^{-(1+\delta)}$ for some $\delta > \max\{0, \alpha - 1\}$ and for all $x \in \mathbf{R}$ and $|p'(x) - p'(y)| \leq C_2|x - y|(1 + |x|)^{-(1+\delta)}$ for all $x, y \in \mathbf{R}$ with $|x - y| < 1$.

Let $\lambda = \sum_{j=0}^{\infty} c_j$, $f(x)$ be the density of ε and $\theta_0(\tau) = (\beta_0(\tau), \gamma_n)^T$ be the true value of $\theta(\tau)$. Define $A(x) = \int_0^1 (1, x(s))^T (1, x(s)) ds$. We have the following theorems.

Theorem 2.1. Assume conditions H₁ and H₂. If $\beta > 2/\alpha$, then

$$D_n(\hat{\theta}(\tau) - \theta_0(\tau)) \longrightarrow^d \frac{\sigma}{f(\beta_0(\tau))} (A(S))^{-1} \left(W(\tau, 1), \int_0^1 S(s) dW(\tau, s) \right)^T. \quad (2.2)$$

In particular,

$$a_n \sqrt{n}(\hat{\alpha}(\tau) - \gamma_n) \longrightarrow^d \frac{\sigma}{f(\beta_0(\tau))} \frac{\int_0^1 S(s) dW(\tau, s) - W(\tau, 1) \int_0^1 S(s) ds}{\int_0^1 S^2(s) ds - \left(\int_0^1 S(s) ds \right)^2} \quad (2.3)$$

and

$$\left(\sum_{i=1}^n Y_{t-1}^2 - \left(\sum_{i=1}^n Y_{t-1} \right)^2 \right)^{1/2} (\hat{\alpha}(\tau) - \gamma_n) \longrightarrow^d N\left(0, \frac{\sigma^2}{f^2(\beta_0(\tau))}\right), \quad (2.4)$$

where $D_n = \text{diag}(\sqrt{n}, a_n \sqrt{n})$, $\sigma^2 = E\varphi_\tau^2(\varepsilon_0) + 2 \sum_{j=1}^{\infty} E[\varphi_\tau(\varepsilon_0)\varphi_\tau(\varepsilon_j)]$ and $W(\tau, \cdot)$ is a standard Brownian motion independent of $S(s) = \lambda(Z_\alpha(s) - \gamma \int_0^s e^{-\gamma(s-t)} dZ_\alpha(t))$, with $Z_\alpha(t)$ being defined in (1.2).

Theorem 2.2. Assume conditions H₁ and H₂. If $c_j \sim b_0 j^{-\beta}$, $1/\alpha < \beta < (\alpha + 2)/(3\alpha)$ and $\alpha > 1$, then

$$D_n(\hat{\theta}(\tau) - \theta_0(\tau)) \longrightarrow^d (A(Z_{\alpha,\beta,\gamma}))^{-1} \left(Z_{\alpha,\beta}(1), \gamma \int_0^1 Z_{\alpha,\beta,\gamma}(s) ds + \frac{1}{2} Z_{\alpha,\beta,\gamma}^2(1) \right)^T, \quad (2.5)$$

where $D_n = \text{diag}(a_n^{-1} n^\beta, n)$, $Z_{\alpha,\beta}(t) = \int_{-\infty}^{\infty} \int_0^t (u - s)_+^{-\beta} du dZ_\alpha(s)$ and

$$Z_{\alpha,\beta,\gamma}(t) = Z_{\alpha,\beta}(t) - \gamma \int_0^t e^{-\gamma(t-s)} Z_{\alpha,\beta}(s) ds, \quad Z_{\alpha,\beta,\gamma}(0) = 0.$$

Theorem 2.3. Assume conditions H₁ and H₂. Suppose that $c_j \sim b_0 j^{-\beta}$ and $\lim_{x \rightarrow \infty} P(\eta_0 > x)/P(|\eta_0| > x) = 1/2$. Then for $1 + \sqrt{1 - 1/\alpha} < \beta < 2/\alpha$ and $\alpha > 1$,

$$D_n(\hat{\theta}(\tau) - \theta_0(\tau)) \longrightarrow^d \frac{1}{f(\beta_0(\tau))} (A(S))^{-1} \left(-L_{\alpha\beta}(1), -\int_0^1 S(s) dL_{\alpha\beta}(s) \right)^T, \quad (2.6)$$

where $D_n = \text{diag}(na_n^{-1/\beta}, na_n^{1-1/\beta})$, $L_{\alpha\beta}(s)$ is a stable process with index $\alpha\beta$ defined by

$$L_{\alpha\beta} = c^+ Z_{\alpha\beta}^+ + c^- Z_{\alpha\beta}^-,$$

$$c^\pm = \Lambda \int_0^\infty (F(\beta_0(\tau) \pm t) - \tau) t^{-1-1/\beta} dt,$$

$$\Lambda = \left(\frac{b_0^\alpha (\alpha\beta - 1)}{\Gamma(2 - \alpha\beta) \cos(\pi\alpha\beta/2) \beta^{\alpha\beta}} \right)^{1/(\alpha\beta)} \quad (2.7)$$

and $Z_{\alpha\beta}^-(s)$ is an independent copy of $Z_{\alpha\beta}^+(s)$ with characteristic function

$$\mathbb{E} e^{irZ_{\alpha\beta}^+(s)} = \exp\{-s|t|^{\alpha\beta}[1 - \text{isgn}(t) \tan(\pi\alpha\beta/2)]\}.$$

Note that [Theorems 2.1–2.3](#) point out the subtle differences and difficulties in quantile estimation of the near-integrated model (1.1). In the short-memory case ($\beta > 2/\alpha$), [Theorem 2.1](#) shows that the limit distribution of the quantile estimate converges weakly to a functional of a mixture of a Brownian motion and a stable O–U process. On the other hand, in the long-memory case with $1/\alpha < \beta < (\alpha + 2)/(3\alpha)$, [Theorem 2.2](#) shows that the limit distribution of the quantile estimate converges weakly to a functional of a different fractional stable O–U process. But for the long-memory case with $1 < \beta < 2/\alpha$, [Theorem 2.3](#) gives a completely different characterization of the limit distribution of the quantile estimate of α_n as a functional of a mixture of a stable process and a stable O–U process. Consequently, one needs to be extremely cautious in applying the quantile regression procedure in the near-integrated model as there is an abrupt change in the behavior of the limit distributions.

3. Semi-parametric estimates of α_n

Although [Theorems 2.1–2.3](#) give the limit distributions of the quantile estimate of γ_n , they involve the unknown parameters $f(\beta_0(\tau))$ and σ^2 . Likewise, $f(\beta_0(\tau))$ and $F(\beta_0(\tau) + t)$ in [Theorem 2.3](#) are also unknown a priori. In this section, we propose a semi-parametric estimate $\tilde{\alpha}_n$ to tackle this problem. Note that according to [Theorem 2.1](#), we have

$$\hat{t}_{\gamma_n} = \left(\sum_{t=1}^n Y_{t-1}^2 - \left(\sum_{t=1}^n Y_{t-1} \right)^2 \right)^{1/2} (\hat{\alpha}(\tau) - \alpha_n) \longrightarrow^d N\left(0, \frac{\sigma^2}{f^2(\beta_0(\tau))}\right).$$

This implies that

$$\frac{f(\beta_0(\tau))}{\sigma} \hat{t}_{\gamma_n} \longrightarrow^d N(0, 1).$$

Therefore, if we can construct consistent estimators $\hat{\sigma}$ and $\hat{f}(\beta_0(\tau))$ to estimate σ and $f(\beta_0(\tau))$ respectively, then

$$\tilde{t}_{\alpha_n} = \frac{\hat{f}(\beta_0(\tau))}{\hat{\sigma}} \hat{t}_{\gamma_n} \longrightarrow^d N(0, 1).$$

Similarly, to apply [Theorem 2.3](#), we need to construct a consistent estimator $\hat{F}(\beta_0(\tau) + t)$ of $F(\beta_0(\tau) + t)$. Since

$$\sigma^2 = \mathbb{E} \psi_\tau^2(\varepsilon_0 - \beta_0(\tau)) + 2 \sum_{k=1}^{\infty} \mathbb{E} \psi_\tau(\varepsilon_0 - \beta_0(\tau)) \psi_\tau(\varepsilon_k - \beta_0(\tau)) = 2\pi f_{\varepsilon\varepsilon}(0),$$

where $f_{\varepsilon\varepsilon}(\cdot)$ is the spectral density of $\{\psi_\tau(\varepsilon_t - \beta_0(\tau))\}$, we can estimate σ^2 by

$$\hat{\sigma}^2 = 2\pi \hat{f}_{\varepsilon\varepsilon}(0) = \sum_{j=-M}^M (1 - j/M) \hat{r}(j),$$

where $\hat{r}(j) = \frac{1}{n} \sum_{t=1}^n \varphi_\tau(\hat{\varepsilon}_t - \hat{\beta}_0(\tau)) \varphi_\tau(\hat{\varepsilon}_{t+j} - \hat{\beta}_0(\tau))$ and $M = o(n^{1/2})$, $M \rightarrow \infty$.

To estimate $f(\beta_0(\tau))$ in [Theorems 2.1](#) and [2.3](#), we use a kernel density estimate method. Let $\varepsilon_i = Y_i - \hat{\alpha}_n Y_{i-1}$ be the residuals and let $K(\cdot)$ be a symmetric and monotone kernel function with a bounded derivative, a compact support, $[-1, 1]$, say and $\int_{-1}^1 K(x)dx = 1$. Since $f(\cdot)$ is unknown, $\beta_0(\tau)$ is also unknown. We estimate $f(\beta_0(\tau))$ by

$$\hat{f}(\beta_0(\tau)) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{\hat{\varepsilon}_i - \hat{\beta}_0(\tau)}{h}\right).$$

To estimate the distribution $F(\beta_0(\tau) + t)$ of ε , we use an empirical process defined by

$$\tilde{F}_n(\beta_0(\tau) + t) = \frac{1}{n} \sum_{i=1}^n I(\hat{\varepsilon}_i - \hat{\beta}_0(\tau) \leq t).$$

We have the following theorems.

Theorem 3.1. *Under the conditions of [Theorem 2.1](#),*

$$\hat{f}(\beta_0(\tau)) - f(\beta_0(\tau)) = O_p\left((nh)^{-1/2} + h^2\right) \quad (3.1)$$

and

$$\hat{\sigma} - \sigma = o_p(1). \quad (3.2)$$

Theorem 3.2. *Under the conditions of [Theorem 2.3](#),*

$$\hat{f}(\beta_0(\tau)) - f(\beta_0(\tau)) = O_p((nh)^{-1} a_{nh}^{1/\beta} + h^2) \quad (3.3)$$

$$\tilde{F}_n(\beta_0(\tau) + t) - F(\beta_0(\tau) + t) = O_p(n^{-1} a_n^{1/\beta}), \quad (3.4)$$

and

$$na_n^{-1/\beta} \tilde{F}_n(\beta_0(\tau) + t) - F(\beta_0(\tau) + t) \Rightarrow^{J_1} Z_{\alpha\beta}^*(t), \quad (3.5)$$

where $a_n = n^{1/\alpha} l(n)$ is defined in [Section 1](#) and $Z_{\alpha\beta}^*(t)$ is a stable process with index $\alpha\beta$ defined in [Appendix](#).

[Theorems 3.1](#) and [3.2](#) show that the proposed semi-parametric estimates of f , σ^2 and F are consistent and as a result, they can be used in conjunction with [Theorems 2.1–2.3](#) to construct confidence intervals for the quantile estimates of the near-integrated process [\(1.1\)](#) in the long-memory and heavy-tailed situations.

4. Proofs

Proof of Theorem 2.1. Put $v = (v_1, v_2)^T = \sqrt{n}(\beta(\tau) - \beta_0(\tau), a_n(\gamma(\tau) - \gamma_n))^T$ and

$$Z_n(v) = \sum_{t=1}^n \rho_\tau(\varepsilon_t - \beta_0(\tau) - v^T D_n^{-1} X_t) - \rho_\tau(\varepsilon_t - \beta_0(\tau)).$$

Then

$$Z_n(v) = - \sum_{t=1}^n v^T D_n^{-1} X_t \varphi_\tau(\varepsilon_t - \beta_0(\tau))$$

$$\begin{aligned}
& + \sum_{t=1}^n (\varepsilon_t - \beta_0(\tau) - v^T D_n^{-1} X_t) I(0 < \varepsilon_t - \beta_0(\tau) < v^T D_n^{-1} X_t) \\
& - \sum_{t=1}^n (\varepsilon_t - \beta_0(\tau) - v^T D_n^{-1} X_t) I(0 > \varepsilon_t - \beta_0(\tau) > v^T D_n^{-1} X_t) \\
& =: II_1 + II_2 + II_3.
\end{aligned} \tag{4.1}$$

From Lemma A.3, it follows that

$$II_1 \longrightarrow^d -v^T \sigma(W(\tau, 1), \int_0^1 S(t) dW(\tau, t))^T \tag{4.2}$$

and for all $|v| \leq C$ for some $C > 0$,

$$\max_{1 \leq t \leq n} |\sqrt{n} v^T D_n^{-1} X_t| \leq |v_1| + |v_2| \sup_{0 \leq t \leq 1} |Y_{[nt]}/a_n| = O_p(1).$$

Let

$$\begin{aligned}
Z_{tn}(v) &= (v^T D_n^{-1} X_t - \varepsilon_t + \beta_0(\tau)) I(v^T D_n^{-1} X_t > \varepsilon_t - \beta_0(\tau) > 0) \\
&\quad \times I(0 < \sqrt{n} v^T D_n^{-1} X_t \leq \log n), \\
\mathcal{F}_t &= \sigma(\varepsilon_s, s \leq t), \quad \mu_{tn} = E(Z_{tn}(v) | \mathcal{F}_{t-1}), \\
A_{tn}(v) &= v^T D_n^{-1} X_t I(0 < \sqrt{n} v^T D_n^{-1} X_t \leq \log n).
\end{aligned} \tag{4.3}$$

Then

$$\begin{aligned}
\sum_{t=1}^n \mu_{tn} &= \sum_{t=1}^n \int_{\beta_0(\tau)}^{\beta_0(\tau) + A_{tn}(v)} (A_{tn}(v) + \beta_0(\tau) - x) f_{t-1}(x) dx \\
&= \sum_{t=1}^n \int_{\beta_0(\tau)}^{\beta_0(\tau) + A_{tn}(v)} \int_x^{A_{tn}(v) + \beta_0(\tau)} ds f_{t-1}(x) dx \\
&= \sum_{t=1}^n \int_{\beta_0(\tau)}^{\beta_0(\tau) + A_{tn}(v)} \int_{\beta_0(\tau)}^s f_{t-1}(x) dx ds \\
&= \sum_{t=1}^n \int_{\beta_0(\tau)}^{\beta_0(\tau) + A_{tn}(v)} (s - \beta_0(\tau)) f_{t-1}(\beta_0(\tau)) (1 + o_p(1)) ds \\
&= \frac{1}{2} \sum_{t=1}^n f_{t-1}(\beta_0(\tau)) A_{tn}(v)^2 + o_p(1).
\end{aligned} \tag{4.4}$$

Note that under H_2 , $E|f_{t-1}(\beta_0(\tau))|^\vartheta < \infty$ for some $\vartheta > 1$. Combining with the stationarity of $\{f_{t-1}(\beta_0(\tau))\}$ yields for some $\delta > 0$

$$\max_{1 \leq k \leq n} \frac{1}{n^{1-\delta}} \left| \sum_{t=1}^k (f_{t-1}(\beta_0(\tau)) - f(\beta_0(\tau))) \right| = o_p(1). \tag{4.5}$$

This implies that

$$\sum_{t=1}^n \mu_{tn} =^p f(\beta_0(\tau)) A_{tn}(v)^2.$$

Furthermore, by Lemma A.3, we have $\max_{1 \leq t \leq n} A_{tn}(v) = o_p(1)$, which implies

$$\sum_{t=1}^n \mathbb{E}(Z_{tn}^2(v) | \mathcal{F}_{t-1}) \leq (\max_{1 \leq t \leq n} A_{tn}(v)) \sum_{t=1}^n \mu_{tn} \xrightarrow{p} 0. \quad (4.6)$$

Therefore,

$$H_3 =^p \frac{1}{2} f(\beta_0(\tau)) \sum_{t=1}^n (v^T D_n^{-1} X_t)^2 I(v^T D_n^{-1} X_t > 0). \quad (4.7)$$

Similarly,

$$H_2 =^p \frac{1}{2} f(\beta_0(\tau)) \sum_{t=1}^n (v^T D_n^{-1} X_t)^2 I(v^T D_n^{-1} X_t < 0). \quad (4.8)$$

By Lemma A.3 and (4.2),

$$Z_n(v) \xrightarrow{d} -v^T \sigma \left(W(\tau, 1), \int_0^1 S(t) dW(\tau, t) \right)^T + \frac{1}{2} f(\beta_0(\tau)) v^T A(S) v.$$

Since $Z_n(v)$ has a convex sample path, Theorem 2.1 follows from Lemma 2.2 of [10]. \square

Proof of Theorem 2.2. Let $b_n = a_n n^{1-\beta}$, $v = (nb_n^{-1}(\beta(\tau) - \beta_0(\tau)), n(\gamma(\tau) - \gamma_n))^T$, $D_n = \text{diag}(n/b_n, n)$ and $Z_n(v) = \sum_{t=1}^n \rho_\tau(\varepsilon_t - \beta_0(\tau) - v^T D_n^{-1} X_t) - \rho_\tau(\varepsilon_t - \beta_0(\tau))$. Similar to (4.4) and (4.5), we have

$$\begin{aligned} nb_n^{-2} Z_n(v) &= -nb_n^{-2} \sum_{t=1}^n v^T D_n^{-1} X_t \varphi_\tau(\varepsilon_t - \beta_0(\tau)) + \frac{1}{2} nb_n^{-2} f(\beta_0(\tau)) \\ &\quad \times \sum_{t=1}^n (v^T D_n^{-1} X_t)^2 I(nb_n^{-1} |v^T D_n^{-1} X_t| \leq \log n) + o_p(1). \end{aligned} \quad (4.9)$$

Under the condition that $1/\alpha < \beta < (\alpha + 2)/(3\alpha)$, it can be shown after tedious calculations that for any $x \in R$,

$$\begin{aligned} \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{[nt]} \varphi_\tau(\varepsilon_i - \beta_0(\tau)) - \sum_{i=1}^{[nt]} f(\beta_0(\tau)) \varepsilon_i - \sum_{i=1}^{[nt]} \sum_{j=0}^{\infty} \mathbb{E}[\varphi_\tau(\varepsilon_i - \beta_0(\tau)) \right. \\ \left. - f(\beta_0(\tau)) \varepsilon_i | \eta_{i-j}] + \sum_{i=1}^{[nt]} \sum_{0 \leq j_1 < j_2} c_{j_1} c_{j_2} \eta_{i-j_1} \eta_{i-j_2} \right| = o_p(n^{1-2\beta} a_n^2). \end{aligned} \quad (4.10)$$

It follows from Theorem 3.3 of [31] that

$$\begin{aligned} F_n(t) &=: \frac{1}{n^{1-2\beta} a_n^2} \sum_{i=1}^{[nt]} \sum_{0 \leq j_1 < j_2} c_{j_1} c_{j_2} \eta_{i-j_1} \eta_{i-j_2} \\ &\Rightarrow^{J_1} C \int_{-\infty}^t \int_{u_2}^t \int_0^1 (x - u_1)_+^{-\beta} (x - u_2)_+^{-\beta} dx dZ_\alpha(u_1) dZ_\alpha(u_2), \end{aligned} \quad (4.11)$$

for some constant C . By (4.11) and the weak convergence of $S_n(t) =: \sum_{i=1}^{[nt]} \varepsilon_i / b_n \Rightarrow^{J_1} Z_{\alpha, \beta}(t)$, we have for δ small enough (see [4]),

- (a) $\sup_{0 \leq t \leq 1} |S_n(t)| = O_p(1)$ and $\sup_{0 \leq t \leq 1} |F_n(t)| = O_p(1)$;
- (b) $\sup_{|s-t| \leq \delta} |S_n(s) - S_n(t)| = o_p(1)$ and $\sup_{|s-t| \leq \delta} |F_n(s) - F_n(t)| = o_p(1)$.

Partition $[0, 1]$ into sub-intervals each with length δ , say $A_i = [(i-1)\delta, i\delta]$, $i = 1, 2, \dots, [1/\delta]$ and $A^* = [[1/\delta]\delta, 1]$. Then

$$\begin{aligned}
 & \frac{1}{b_n^2} \sum_{i=1}^n \sum_{l=1}^i \sum_{0 \leq j_1 < j_2} c_{j_1} c_{j_2} \eta_{l-j_1} \eta_{l-j_2} \varepsilon_i \\
 &= \frac{1}{n} \sum_{i=1}^n \left(\sum_{l=1}^i \sum_{0 \leq j_1 < j_2} c_{j_1} c_{j_2} \eta_{l-j_1} \eta_{l-j_2} / (n^{1-2\beta} a_n^2) \right) \varepsilon_i \\
 &= \frac{1}{n} \sum_{i=1}^{[1/\delta]} \sum_{l=[n(i-1)\delta]+1}^{[ni\delta]} \left[F_n\left(\frac{l}{n}\right) - F_n\left(\frac{[n(i-1)\delta]+1}{n}\right) \right] \varepsilon_l \\
 &\quad + \frac{1}{n} \sum_{i=1}^{[1/\delta]} \sum_{l=[n(i-1)\delta]+1}^{[ni\delta]} \varepsilon_l F_n\left(\frac{[n(i-1)\delta]+1}{n}\right) + \frac{1}{n} \sum_{l=[n[1/\delta]\delta]+1}^n F_n\left(\frac{l}{n}\right) \varepsilon_l \\
 &\leq \sup_{|s-t| \leq \delta} |F_n(s) - F_n(t)| \left(\frac{1}{n} \sum_{i=1}^n |\varepsilon_i| \right) \\
 &\quad + a_n n^{-\beta} [1/\delta] \sup_{|s-t| \leq \delta} |S_n(s) - S_n(t)| \left(\sup_{0 \leq t \leq 1} |F_n(t)| \right) \\
 &\quad + \sup_{0 \leq t \leq 1} |F_n(t)| \left(\frac{1}{n} \sum_{l=[n[1/\delta]\delta]+1}^n |\varepsilon_l| \right) = o_p(1), \tag{4.12}
 \end{aligned}$$

by letting $n \rightarrow \infty$ and then $\delta \rightarrow 0$.

Let $H(\eta_i) = \sum_{j=0}^{\infty} \{F(\beta_0(\tau) - c_j \eta_i) - E[F(\beta_0(\tau) - c_j \eta_i)] + f(\beta_0(\tau) c_j \eta_i)\}$ and $H'(\eta_i) = \sum_{j=0}^{\infty} E[\varphi_{\tau}(\varepsilon_i - \beta_0(\tau)) - f(\beta_0(\tau)) \varepsilon_i | \eta_{i-j}]$. Then

$$\begin{aligned}
 & \frac{1}{b_n^2} \sum_{i=1}^n \left(Y_{i-1} \sum_{j=0}^{\infty} E[\varphi_{\tau}(\varepsilon_i - \beta_0(\tau)) - f(\beta_0(\tau)) \varepsilon_i | \eta_{i-j}] \right) \\
 &= \frac{Y_{n-1}}{b_n^2} \sum_{i=1}^n H'(\eta_i) - \frac{1}{b_n^2} \sum_{i=1}^{n-1} \sum_{l=1}^i (H'(\eta_l) + H(\eta_l)) (Y_i - Y_{i-1}) \\
 &\quad + \frac{1}{b_n^2} \sum_{i=1}^{n-1} \sum_{l=1}^i H(\eta_l) (Y_i - Y_{i-1}) \\
 &= \frac{Y_{n-1}}{b_n^2} \left[2 \sum_{i=1}^{n-1} H'(\eta_i) + H'(\eta_n) \right] - \frac{1}{b_n^2} \left[\sum_{i=1}^{n-1} Y_{i-1} H(\eta_i) \right] \\
 &\quad - \frac{1}{b_n^2} \sum_{i=1}^{n-1} \sum_{l=1}^i (H'(\eta_l) + H(\eta_l)) \left(\varepsilon_i - \frac{\gamma Y_{i-1}}{n} \right). \tag{4.13}
 \end{aligned}$$

Similar to Lemma 3.2 of [33], we have $a_n^{-1/\beta} \sum_{i=1}^n H(\eta_i) \xrightarrow{d} Z'_{\alpha\beta}$, where $Z'_{\alpha\beta}$ is a stable process with index $\alpha\beta$. For any $0 \leq t \leq 1$,

$$\sum_{i=1}^{[nt]} (H'(\eta_i) + H(\eta_i)) = - \sum_{i \leq 0} \sum_{j=1-i}^{[nt]-i} \{F_j(\beta_0(\tau) - c_j \eta_i) - \tau + f(\beta_0(\tau) c_j \eta_i)\}$$

$$\begin{aligned}
 & + \sum_{i=1}^{[nt]} \sum_{j=[nt]-i+1}^{\infty} \{F(\beta_0(\tau) - c_j \eta_i) - E[F(\beta_0(\tau) - c_j \eta_i)] + f(\beta_0(\tau) c_j \eta_i)\} \\
 & - \sum_{i=1}^{[nt]} \sum_{j=1}^{[nt]-i} \{[F_j(\beta_0(\tau) - c_j \eta_i) - F(\beta_0(\tau) - c_j \eta_i)] - E[F_j(\beta_0(\tau) - c_j \eta_i) \\
 & - F(\beta_0(\tau) - c_j \eta_i)]\} \\
 & =: V_1(t) + V_2(t) + V_3(t),
 \end{aligned} \tag{4.14}$$

with $E|V_l(t)|^r \leq Cn^{1+r-\alpha\beta+\kappa}$ for any $\kappa > 0$, $1 < r < \alpha\beta$, $l = 1, 2$ and $E|V_3(t)|^2 \leq Cn$. Therefore, the first term of the right-hand side in (4.13) is equal to zero in probability and by (3.9) and (3.10) of [33], we can show the third term is

$$\begin{aligned}
 & -\frac{1}{b_n^2} \sum_{i=1}^{n-1} \sum_{l=1}^i (H'(\eta_l) + H(\eta_l)) \varepsilon_i + o_p(1) \\
 & = -\frac{1}{b_n^2} \sum_{i=1}^{n-1} \sum_{l=1}^i (H'(\eta_l) + H(\eta_l)) \varepsilon_i [I(|\varepsilon_i| > a_n \log n) + I(|\varepsilon_i| \leq a_n \log n)] + o_p(1) \\
 & = -\frac{1}{b_n^2} \sum_{i=1}^{n-1} [(V_1(i/n) + V_2(i/n)) \varepsilon_i + V_3 \varepsilon_i (i/n) I(|\varepsilon_i| \leq a_n \log n)] + o_p(1) \\
 & = -\frac{1}{b_n^2} \sum_{i=1}^{n-1} [V_3(i/n) \varepsilon_i I(|\varepsilon_i| \leq a_n \log n)] + o_p(1).
 \end{aligned}$$

Let $\varepsilon_{i,n} = \varepsilon_i I(|\varepsilon_i| \leq a_n \log n)$, by the Hölder inequality, we have

$$\begin{aligned}
 & E \left| \frac{1}{b_n^2} \sum_{i=1}^{n-1} [V_3(i/n) \varepsilon_i I(|\varepsilon_i| \leq a_n \log n)] \right| \\
 & \leq \frac{1}{b_n^2} \sum_{i=1}^{n-1} [E(|V_3(i/n)|^2)]^{1/2} [E(|\varepsilon_i|^2)]^{1/2} \\
 & = o(1).
 \end{aligned}$$

By Theorem 2.7 of [21], we have the second term of the right hand side of (4.13) is equal to zero in probability. Thus, by (4.9), (4.10), (4.12) and (4.13),

$$nb_n^{-2} Z_n(v) =^p f(\beta_0(\tau)) \left(- \sum_{t=1}^n v^T nb_n^{-2} D_n^{-1} X_t \varepsilon_t + \frac{1}{2} nb_n^{-2} \sum_{t=1}^n (v^T D_n^{-1} X_t)^2 \right). \tag{4.15}$$

Since $Z_n(v)$ is convex, it follows that

$$nb_n^{-1} (\widehat{\beta}(\tau) - \beta_0(\tau), b_n(\widehat{\alpha}(\tau) - \gamma_n)) =^p \Sigma_n^{-1} \left(\frac{1}{b_n} \sum_{i=1}^n \varepsilon_i, \frac{1}{b_n^2} \sum_{i=1}^n Y_{i-1} \varepsilon_i \right),$$

where

$$\Sigma_n^{-1} = \begin{pmatrix} 1 & \frac{1}{n} \sum_{i=1}^n \frac{Y_{i-1}}{b_n} \\ \frac{1}{n} \sum_{i=1}^n \frac{Y_{i-1}}{b_n} & \frac{1}{n} \sum_{i=1}^n \frac{Y_{i-1}^2}{b_n^2} \end{pmatrix}.$$

By Theorem 2.3 of [7], we have Theorem 2.2. \square

Proof of Theorem 2.3. Let $v = (na_n^{-1/\beta}(\beta(\tau) - \beta_0(\tau)), na_n^{1-1/\beta}(\gamma(\tau) - \gamma_n(\tau)))^T$, $D_n = \text{diag}(na_n^{-1/\beta}, na_n^{1-1/\beta})$ and $Z_n(v)$ defined as above. Using a similar argument of Theorem 2.2, we have

$$\begin{aligned} na_n^{-2/\beta} Z_n(v) &= na_n^{-2/\beta} \sum_{t=1}^n v^T D_n^{-1} X_t \varphi_\tau(\varepsilon_t - \beta_0(\tau)) \\ &\quad + \frac{1}{2} f(\beta_0(\tau)) na_n^{-2/\beta} \sum_{t=1}^n (v^T D_n^{-1} X_t)^2 + o_p(1). \end{aligned}$$

By Lemma A.4 and the convexity of $Z_n(v)$, we have Theorem 2.3. \square

Proofs of Theorem 3.1. To prove (3.1) and (3.2), it is enough to show that for some $\sigma_1 < \infty$,

$$(\sqrt{nh}) \left(\widehat{f}(\beta_0(\tau)) - f(\beta_0(\tau)) - \frac{1}{2} f''(\beta_0(\tau)) h^2 \right) \longrightarrow^d N(0, \sigma_1^2) \quad (4.16)$$

and that

$$\widehat{r}(j) - r(j) = O_p(n^{-1/2}). \quad (4.17)$$

Note that

$$\widehat{\varepsilon}_i - \widehat{\beta}_0(\tau) = \varepsilon_i - \beta_0(\tau) + (\widehat{\gamma}_n - \gamma_n) Y_{i-1} + \widehat{\beta}_0(\tau) - \beta_0(\tau) =: \varepsilon_i - \beta_0(\tau) + \widehat{\mu}_n. \quad (4.18)$$

Put $g(\beta_0(\tau), u) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{\varepsilon_i - \beta_0(\tau) + u/\sqrt{n}}{h}\right)$. Then by H_2 and the monotone property of $K(\cdot)$, for any $0 < C < \infty$,

$$\begin{aligned} &E \sup_{0 \leq |u| \leq C} |g(\beta_0(\tau), u) - g(\beta_0(\tau), 0)| \\ &\leq \frac{1}{nh} \sum_{i=1}^n E \sup_{0 \leq |u| \leq C} \left| K\left(\frac{\varepsilon_i - \beta_0(\tau) + u/\sqrt{n}}{h}\right) - K\left(\frac{\varepsilon_i - \beta_0(\tau)}{h}\right) \right| \\ &\leq \frac{1}{nh} \sum_{i=1}^n \int_{-\infty}^{\infty} |K((y - \beta_0(\tau))/h)| \sup_{0 \leq |u| \leq C} |f(y - u/\sqrt{n}) - f(y)| dy \\ &\leq 2(nh)^{-1} \sum_{i=1}^n h |f'(\beta_0(\tau))| C n^{-1/2} = O(n^{-1/2}). \end{aligned} \quad (4.19)$$

Thus,

$$\sup_{0 \leq |u| \leq C} |g(\beta_0(\tau), u) - g(\beta_0(\tau), 0)| = O_p(n^{-1/2}). \quad (4.20)$$

It follows from Theorem 2.1 that

$$\widehat{\mu}_n = (\widehat{\gamma}_n - \gamma_n) Y_{i-1} + \widehat{\beta}_0(\tau) - \beta_0(\tau) = O_p(n^{-1/2}). \quad (4.21)$$

Combining (4.20) and (4.21) yields

$$\begin{aligned} & \frac{1}{nh} \sum_{i=1}^n K((\widehat{\varepsilon}_i - \widehat{\beta}_0(\tau))/h) - \frac{1}{nh} \sum_{i=1}^n K((\varepsilon_i - \beta_0(\tau))/h) \\ & = g(\beta_0(\tau), \sqrt{n}\widehat{\mu}_n) - g(\beta_0(\tau), 0) = O_p(n^{-1/2}). \end{aligned} \quad (4.22)$$

Similar to the proof of (A.4) (see Lemma A.1), we have

$$\sqrt{nh} \left(g(\beta_0(\tau), 0) - f(\beta_0(\tau)) - f''(\beta_0(\tau))h^2/2 \right) \longrightarrow^d N(0, \sigma_1^2), \quad (4.23)$$

where $\sigma_1^2 = E[(nh)^{-1/2} \sum_{i=1}^n K((\varepsilon_i - \beta_0(\tau))/h)] < \infty$. Combining (4.22) and (4.23) gives (4.16).

For (4.17), let $\xi_t = \varepsilon_t - \beta_0(\tau)$ and

$$L_n(u) = \frac{1}{n} \sum_{t=1}^n \varphi_\tau(\xi_t + u/\sqrt{n}) \varphi_\tau(\xi_{t+j} + u/\sqrt{n}).$$

Then

$$\begin{aligned} L_n(u) - EL_n(0) &= L_n(u) - L_n(0) + L_n(0) - EL_n(0) \\ &= \frac{1}{n} \sum_{t=1}^n \varphi_\tau(\xi_t + u/\sqrt{n}) \left(I(\xi_{t+j} < 0) - I(\xi_{t+j} + u/\sqrt{n} < 0) \right) \\ &\quad \times \frac{1}{n} \sum_{t=1}^n \varphi_\tau(\xi_{t+j}) \left(I(\xi_t < 0) - I(\xi_t + u/\sqrt{n} < 0) \right) \\ &\quad + L_n(0) - EL_n(0) \\ &=: L_{n1}(u) + L_{n2}(u) + L_n(0) - EL_n(0). \end{aligned} \quad (4.24)$$

Observe that

$$\sup_{|u| \leq C} |L_{n1}(u)| \leq \frac{(\tau+1)}{n} \sum_{t=1}^n \left| I\left(-\frac{C}{\sqrt{n}} \leq \xi_{t+j} \leq \frac{C}{\sqrt{n}}\right) \right|. \quad (4.25)$$

Thus,

$$\begin{aligned} E \sup_{|u| \leq C} |L_{n1}(u)| &\leq \frac{(\tau+1)}{n} \sum_{t=1}^n P(-C/\sqrt{n} \leq \varepsilon_{t+j} - \beta_0(\tau) \leq C/\sqrt{n}) \\ &= (\tau+1) \int_{\beta_0(\tau)-C/\sqrt{n}}^{\beta_0(\tau)+C/\sqrt{n}} f(x) dx \\ &\leq 3C(\tau+1)f(\beta_0(\tau))/\sqrt{n}. \end{aligned} \quad (4.26)$$

This implies that $\sup_{|u| \leq C} |L_{n1}(u)| = O_p(n^{-1/2})$. Similarly, we have $\sup_{|u| \leq C} |L_{n2}(u)| = O_p(n^{-1/2})$. In the following, we will apply the method of Woodroffe for the central limit theorem for functions of Markov chains to show that

$$\sqrt{n}(L_n(0) - EL_n(0)) \longrightarrow^d N(0, \sigma_2^2), \quad (4.27)$$

where $\sigma_2^2 = \text{Var}(\sqrt{n}L_n(0)) < \infty$. Put $g(\xi_t, \xi_{t+j}) = \varphi_\tau(\xi_t)\varphi_\tau(\xi_{t+j})$. Since $\sup_{x \in R} |f(x)| \leq C$, it follows that

$$\begin{aligned}
 & \sum_{i=1}^{\infty} \|\mathbb{E}(g(\xi_t, \xi_{t+j})|\mathcal{F}_1) - \mathbb{E}(g(\xi_t, \xi_{t+j})|\mathcal{F}_0)\| \\
 &= \sum_{t=1}^{\infty} \|\tau\{\mathbb{E}[(I(\xi_t < 0) + I(\xi_{t+j} < 0))|\mathcal{F}_1] - \mathbb{E}[(I(\xi_t < 0) + I(\xi_{t+j} < 0))|\mathcal{F}_0]\} \\
 &\quad + \mathbb{E}[I(\xi_t < 0)I(\xi_{t+j} < 0)|\mathcal{F}_1] - \mathbb{E}[I(\xi_t < 0)I(\xi_{t+j} < 0)|\mathcal{F}_0]\| \\
 &\leq \tau \sum_{t=1}^{\infty} \|\mathbb{E}(I(\xi_t < 0)|\mathcal{F}_1) - \mathbb{E}(I(\xi_t < 0)|\mathcal{F}_0)\| \\
 &\quad + \tau \sum_{t=1}^{\infty} \|\mathbb{E}(I(\xi_{t+j} < 0)|\mathcal{F}_1) - \mathbb{E}(I(\xi_{t+j} < 0)|\mathcal{F}_0)\| \\
 &\quad + \sum_{t=1}^{\infty} \|\mathbb{E}[I(\xi_t < 0)I(\xi_{t+j} < 0)|\mathcal{F}_1] - \mathbb{E}[I(\xi_t < 0)I(\xi_{t+j} < 0)|\mathcal{F}_0]\| \\
 &=: \Gamma_1 + \Gamma_2 + \Gamma_3.
 \end{aligned} \tag{4.28}$$

Let $\underline{\varepsilon}_{t0}, \bar{\varepsilon}_{t2}$ be defined as that in [Lemma A.1](#). By [\(A.2\)](#), we have

$$\begin{aligned}
 \Gamma_1 &= \tau \sum_{t=1}^{\infty} \|\mathbb{E}(I(\underline{\varepsilon}_{t0} + c_{t-1}\eta_1 + \bar{\varepsilon}_{t2} < \beta_0(\tau))|\mathcal{F}_1) \\
 &\quad - \mathbb{E}(I(\underline{\varepsilon}_{t0} + c_{t-1}\eta_1 + \bar{\varepsilon}_{t2} < \beta_0(\tau))|\mathcal{F}_0)\| \\
 &= \tau \sum_{t=1}^{\infty} \|\mathbb{E}(G_t(\beta_0(\tau) - \underline{\varepsilon}_{t0} - c_{t-1}\eta_1) - G_t(\beta_0(\tau) - \underline{\varepsilon}_{t0} - c_{t-1}\eta'_1)|\mathcal{F}_1)\| \\
 &< \infty.
 \end{aligned} \tag{4.29}$$

Similarly, $\Gamma_2 < \infty$. To show that $\Gamma_3 < \infty$, let F_{tj} be the distribution of $\sum_{i=0}^{j-1} c_i \eta_{t+j-i}$. Note that

$$\begin{aligned}
 & \mathbb{E}(I(\xi_t < 0)I(\xi_{t+j} < 0)|\mathcal{F}_1) - \mathbb{E}(I(\xi_t < 0)I(\xi_{t+j} < 0)|\mathcal{F}_0) \\
 &= \mathbb{E}\left[I(\xi_t < 0)F_{tj}\left(\beta_0(\tau) - \sum_{i=j}^{\infty} c_i \eta_{t+j-i}\right)|\mathcal{F}_1\right] \\
 &\quad - \mathbb{E}\left[I(\xi_t < 0)F_{tj}\left(\beta_0(\tau) - \sum_{i=j}^{\infty} c_i \eta_{t+j-i}\right)|\mathcal{F}_0\right] \\
 &= \mathbb{E}\left[(I(\underline{\varepsilon}_{t0} + c_{t-1}\eta_1 + \bar{\varepsilon}_{t2} < \beta_0(\tau)) - I(\underline{\varepsilon}_{t0} + c_{t-1}\eta'_1 + \bar{\varepsilon}_{t2} < \beta_0(\tau))) \right. \\
 &\quad \left. F_{tj}(\beta_0(\tau) - \underline{\varepsilon}_{t+j,0} - c_{t+j-1}\eta_1 - \sum_{i=2}^t c_{t+j-i}\eta_i)|\mathcal{F}_1\right] \\
 &\quad - \mathbb{E}\left[I(\underline{\varepsilon}_{t0} + c_{t-1}\eta_1 + \bar{\varepsilon}_{t2} < \beta_0(\tau))F_{tj}\left(\beta_0(\tau) \right.\right.
 \end{aligned}$$

$$\begin{aligned}
& -\varepsilon_{t+j,0} - c_{t+j-1}\eta_1 - \sum_{i=2}^t c_{t+j-i}\eta_i \Bigg) \\
& - F_{tj} \left(\beta_0(\tau) - \varepsilon_{t+j,0} - c_{t+j-1}\eta'_1 - \sum_{i=2}^t c_{t+j-i}\eta_i \right) | \mathcal{F}_1 \Bigg] \\
& =: \Gamma_{31t} + \Gamma_{32t}.
\end{aligned} \tag{4.30}$$

Let $p_t(x)$, $p_1^*(y)$ be the densities of η_t and η'_1 . Then

$$\begin{aligned}
\Gamma_{31t} &= \mathbb{E}[\mathbb{E}(\Gamma_{31t} | \mathcal{F}_{t-1}) | \mathcal{F}_1] \\
&= \mathbb{E} \left(\int_R \int_R [I(\varepsilon_{t,t-1} + c_0x < \beta_0(\tau)) - I(\varepsilon_{t,t-1} + c_0x + c_{t-1}(y - \eta_1) < \beta_0(\tau))] \right. \\
&\quad \left. F_{tj}(\beta_0(\tau) - \varepsilon_{t,t-1} - c_0x) p_t(x) p_1^*(y) dx dy | \mathcal{F}_1 \right) \\
&= \mathbb{E} \left[\int_R \int_{\frac{1}{c_0}(\beta_0(\tau) - \varepsilon_{t,t-1})}^{\frac{1}{c_0}(\beta_0(\tau) - \varepsilon_{t,t-1} - c_{t-1}(y - \eta_1))} F_{tj}(\beta_0(\tau) - \varepsilon_{t,t-1} - c_0x) p_t(x) dx p_1^*(y) dy | \mathcal{F}_1 \right] \\
&\leq CE \left[\int_R \min \left\{ \frac{1}{c_0} |c_{t-1}(y - \eta_1)|, 1 \right\} p_1^*(y) dy | \mathcal{F}_1 \right] \\
&= CE(\min\{c_0^{-1}c_{t-1}(\eta'_1 - \eta_1), 1\} | \mathcal{F}_1).
\end{aligned} \tag{4.31}$$

Thus,

$$\begin{aligned}
\sum_{t=1}^{\infty} \|\Gamma_{31t}\| &\leq \sum_{t=1}^{\infty} C \left\| \mathbb{E} \left[\int_R \min \left\{ c_0^{-1} |c_{t-1}(y - \eta_1)|, 1 \right\} p'_1(y) dy | \mathcal{F}_1 \right] \right\| \\
&\leq \sum_{t=1}^{\infty} C \left[\mathbb{E} \min \left(1, c_0^{-1} c_{t-1} (\eta'_1 - \eta_1) \right)^2 \right]^{1/2} < \infty.
\end{aligned} \tag{4.32}$$

Furthermore, by (A.2), we have

$$\begin{aligned}
\sum_{t=1}^{\infty} \|\Gamma_{32t}\| &\leq \sum_{t=1}^{\infty} \left\| \mathbb{E} \left[F_{tj} \left(\beta_0(\tau) - \varepsilon_{t+j,0} - c_{t+j-1}\eta_1 - \sum_{i=2}^t c_{t+j-i}\eta_i \right) \right. \right. \\
&\quad \left. \left. - F_{tj} \left(\beta_0(\tau) - \varepsilon_{t+j,0} - c_{t+j-1}\eta'_1 - \sum_{i=2}^t c_{t+j-i}\eta_i \right) \right] | \mathcal{F}_1 \right\| \\
&\leq \sum_{t=1}^{\infty} \|\min\{1, c_{t+j-1}|\eta_1 - \eta'_1|\}\| \\
&= O \left(\sum_{t=1}^{\infty} |c_{t+j-1}|^{\min\{\alpha'/2, 1\}} \right) < \infty.
\end{aligned} \tag{4.33}$$

From (4.32) and (4.33), it follows that $|\Gamma_3| < \infty$. Arguing along the same line as in the proof of Lemma A.1, we have (4.27). Thus, $\sup_{|u| \leq C} |L_n(u) - \mathbb{E}L_n(0)| = O_p(n^{-1/2})$. Combining this with (4.21) implies (4.17). The proof of Theorem 3.1 is complete. \square

Proof of Theorem 3.2. Let $\hat{\mu}_n$ be defined in (4.18) and

$$\tilde{g}(\beta_0(\tau), \mu) = \frac{1}{nh} \sum_{i=1}^n K((\varepsilon_i - \beta_0(\tau) + \mu a_n^{-1/\beta})/h).$$

Similar to the proof of (4.22), we have

$$\begin{aligned} & \frac{1}{nh} \sum_{i=1}^n K((\hat{\varepsilon}_i - \hat{\beta}_0(\tau))/h) - \frac{1}{nh} \sum_{i=1}^n K((\varepsilon_i - \beta_0(\tau))/h) \\ &= \tilde{g}(\beta_0(\tau), a_n^{1/\beta} \hat{\mu}_n) - \tilde{g}(\beta_0(\tau), 0) = O_p(a_n^{-1/\beta}). \end{aligned} \quad (4.34)$$

Let $\zeta'(\eta_i, x) = \sum_{j=1}^{\infty} (F(x - c_j \eta_{i-j}) - EF(x - c_j \eta_{i-j}))$. Then by revising Lemma 5.2 of [32], we have

$$\begin{aligned} & \sup_{y \in [-1, 1]} \sum_{i=1}^n |\zeta'(\eta_i, \beta_0(\tau) + yh) - \zeta'(\eta_i, \beta_0(\tau))| \\ & \quad - [\zeta(\eta_i, \beta_0(\tau) + yh) - \zeta(\eta_i, \beta_0(\tau))] = O_p(a_{nh}^{1/\beta}). \end{aligned} \quad (4.35)$$

Thus, by Lemmas 5.3 and 5.4 of [32], we have

$$\begin{aligned} & n h a_{nh}^{-1/\beta} (\tilde{g}(\beta_0(\tau), 0) - E\tilde{g}(\beta_0(\tau), 0)) \\ &= -a_{nh}^{-1/\beta} \int_R \sum_{i=1}^n [I(\varepsilon_i < x) - F(x)] dK((x - \beta_0(\tau))/h) \\ &= -a_{nh}^{-1/\beta} \int_{-1}^1 \sum_{i=1}^n [I(\varepsilon_i < \beta_0(\tau) + yh) - F(\beta_0(\tau) + yh) \\ & \quad - I(\varepsilon_i < \beta_0(\tau)) - F(\beta_0(\tau))] dK(y) \\ &= - \int_{-1}^1 a_{nh}^{-1/\beta} \sum_{i=1}^n (\zeta(\eta_i, \beta_0(\tau) + yh) - \zeta(\eta_i, \beta_0(\tau))) dK(y) + O_p(1). \end{aligned} \quad (4.36)$$

By (3.9) of [32], there exists $1 < \gamma < \alpha$ such that for all $y \in [-1, 1]$,

$$|\zeta(\eta_i, \beta_0(\tau) + yh) - \zeta(\eta_i, \beta_0(\tau))| \leq C \max\{|\eta_i|^{1/\beta} h^{1/\gamma\beta}, 1\}.$$

This yields that for any $w > 0$ and a large enough n ,

$$\begin{aligned} & P \left(\sup_{y \in [-1, 1]} \left| \sum_{i=1}^n \zeta(\eta_i, \beta_0(\tau) + yh) - \zeta(\eta_i, \beta_0(\tau)) \right| \geq C w a_{nh}^{1/\beta} \right) \\ & \leq n P \{ \max\{|\eta_i|^{1/\beta} h^{1/\gamma\beta}, 1\} > w a_{nh}^{1/\beta} \} \leq C w^{-\alpha\beta}, \end{aligned} \quad (4.37)$$

which combines with (4.36) implies

$$n h a_{nh}^{-1/\beta} (\tilde{g}(\beta_0(\tau), 0) - E\tilde{g}(\beta_0(\tau), 0)) = O_p(1). \quad (4.38)$$

Note that $E\tilde{g}(\beta_0(\tau), 0) - f(\beta_0(\tau)) = -\frac{1}{2} f''(\beta_0(\tau)) h^2 \int_{-1}^1 y^2 K(y) dy + o(h^2)$ and $a_n^{-1/\beta} = o(n^{-1} a_n^{1/\beta})$. Thus (3.3) follows from (4.34) and (4.38).

Set $\bar{F}_n(\beta_0(\tau) + t, \mu) = \sum_{i=1}^n I(\varepsilon_i - \beta_0(\tau) + \mu a_n^{-1/\beta} \leq t)$. It is easy to see that for any $x, y \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{E} \left| \sup_{|\mu| \leq C} [\bar{F}_n(\beta_0(\tau) + x, \mu) - F_n(\beta_0(\tau) + x)] - [\bar{F}_n(\beta_0(\tau) + y, \mu) - F_n(\beta_0(\tau) + y)] \right| \\ & \leq 2 \int_0^{C/a_n^{1/\beta}} \int_0^{|x-y|} f'(\beta_0(\tau) + a + b) da db \leq C|x-y|/a_n^{1/\beta}. \end{aligned}$$

This implies that

$$\sup_{t \in \mathbb{R}} |\tilde{F}_n(\beta_0(\tau) + t) - F_n(\beta_0(\tau) + t)| = O_p(a_n^{-1/\beta}). \quad (4.39)$$

From Theorem 2.1 of [32], it follows that

$$\begin{aligned} & na_n^{-1/\beta} (F_n(\beta_0(\tau) + t) - F(\beta_0(\tau) + t)) \\ & \Rightarrow \Lambda Z_{\alpha\beta}^+ \int_0^\infty [F(\beta_0(\tau) + t - s) - F(\beta_0(\tau) + t)] s^{-1-1/\beta} ds \\ & \quad + \Lambda Z_{\alpha\beta}^- \int_0^\infty [F(\beta_0(\tau) + t + s) - F(\beta_0(\tau) + t)] s^{-1-1/\beta} ds =: Z_{\alpha\beta}^*(t). \end{aligned} \quad (4.40)$$

By (4.39) and (4.40), we have (3.4) and (3.5). This completes the proof of Theorem 3.2. \square

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Appendix

Let

$$S_n(s) = \frac{1}{a_n} \sum_{t=1}^{[ns]} \eta_t, \quad T_n(s) = \frac{1}{a_n} Y_{[ns]}, \quad W_n(\tau, s) = \frac{1}{\sqrt{n}} \sum_{t=1}^{[ns]} \varphi_\tau(\varepsilon_t - \beta_0(\tau))$$

where $\varphi_\tau(x) = \tau - I(x < 0)$. To prove Theorems 2.1–2.3 and Theorems 3.1 and 3.2, we need the following lemmas.

Lemma A.1. Under conditions H_1 and H_2 , for $\beta > 2/\alpha$, we have

$$\begin{pmatrix} S_n(s_1) \\ W_n(\tau, s_2) \end{pmatrix} \Rightarrow^{J_1} \begin{pmatrix} Z_\alpha(s_1) \\ \sigma W(\tau, s_2) \end{pmatrix} \quad \text{on } D(0, 1) \times D(0, 1). \quad (\text{A.1})$$

Proof. Let $\|X\|$ be the norm $(E|X|^2)^{1/2}$, $\underline{\varepsilon}_{t0} = \sum_{j=-\infty}^0 c_{t-j}\eta_j$, $\bar{\varepsilon}_{t2} = \sum_{j=2}^t c_{t-j}\eta_j$ and $\mathcal{F}_t = \sigma\{\varepsilon_s, s \leq t\}$. Let $\{\eta'_j\}$ be an independent copy of $\{\eta_j\}$ and G_t be the distribution of $\bar{\varepsilon}_{t2}$. Since $\sup_{x \in R} |f(x)| < C < \infty$, it follows that $g_t(x) = G'_t(x)$ is also bounded by C . This gives

$$\begin{aligned} & \sum_{t=1}^{\infty} \|E(\varphi_{\tau}(\varepsilon_t - \beta_0(\tau))|\mathcal{F}_1) - E(\varphi_{\tau}(\varepsilon_t - \beta_0(\tau))|\mathcal{F}_0)\| \\ &= \sum_{t=1}^{\infty} \|E(G_t(\beta_0(\tau) - \underline{\varepsilon}_{t0} - c_{t-1}\eta_1) - G_t(\beta_0(\tau) - \underline{\varepsilon}_{t0} - c_{t-1}\eta'_1)|\mathcal{F}_1)\| \\ &\leq \sum_{t=1}^{\infty} \|G_t(\beta_0(\tau) - \underline{\varepsilon}_{t0} - c_{t-1}\eta_1) - G_t(\beta_0(\tau) - \underline{\varepsilon}_{t0} - c_{t-1}\eta'_1)\| \\ &\leq \sum_{t=1}^{\infty} \|\min\{C|c_{t-1}(\eta_1 - \eta'_1)|, 1\}\| \\ &\leq \sum_{t=1}^{\infty} \left[E\left(C|c_{t-1}(\eta_1 - \eta'_1)|\right)^{\min\{\alpha', 2\}} \right]^{1/2} \\ &= O\left(\sum_{t=0}^{\infty} \{|c_t|^{\alpha'/2}\}\right) < \infty \end{aligned} \quad (\text{A.2})$$

for some $\alpha' < \alpha$, where we have used the fact that $[\min(1, |a|)]^2 \leq |a|^{\min(2, \alpha')}$. It follows from (A.2) that

$$\begin{aligned} \|E(W_n(\tau, 1)|\mathcal{F}_0)\|^2 &= \left\| \sum_{k=-\infty}^0 E(W_n(\tau, 1)|\mathcal{F}_k) - E(W_n(\tau, 1)|\mathcal{F}_{k-1}) \right\|^2 \\ &\leq \sum_{k=-\infty}^0 \|E(W_n(\tau, 1)|\mathcal{F}_k) - E(W_n(\tau, 1)|\mathcal{F}_{k-1})\|^2 \\ &\leq \frac{1}{n} \sum_{k=-\infty}^0 \left(\sum_{t=1}^n \|E(\varphi_{\tau}(\varepsilon_t - \beta_0(\tau))|\mathcal{F}_k) - E(\varphi_{\tau}(\varepsilon_t - \beta_0(\tau))|\mathcal{F}_{k-1})\| \right)^2 \\ &= \left(\frac{1}{n}\right) O\left[\sum_{k=0}^{\infty} \sum_{t=1}^n \|E(\varphi_{\tau}(\varepsilon_{t+k+1} - \beta_0(\tau))|\mathcal{F}_1) - E(\varphi_{\tau}(\varepsilon_{t+k+1} - \beta_0(\tau))|\mathcal{F}_0)\| \right] \\ &= o(1). \end{aligned} \quad (\text{A.3})$$

Since $\{\varphi_{\tau}(\varepsilon_t - \beta_0(\tau))\}$ is a stationary process with $E\varphi_{\tau}^2(\varepsilon_t - \beta_0(\tau)) < \infty$, from (A.2) and (A.3) and Theorem 1 of [34] (see also [35]), it follows that

$$W_n(\tau, 1) \longrightarrow^d N(0, \sigma^2). \quad (\text{A.4})$$

By (A.4) and a standard argument, we have

$$W_n(\tau, s) \longrightarrow^d \sigma W(\tau, s) \quad \text{in } D(0, 1).$$

Thus, the marginal distributions of $S_n(\cdot)$ and $W_n(\cdot)$ are $Z_{\alpha}(\cdot)$ and $\sigma W(\tau, \cdot)$ respectively. Following the argument of [29], we have the conclusion as desired. \square

Lemma A.2. *If $1 < \beta < 2/\alpha$ and $1 < \alpha < 2$, there exists a $\nu > 0$ such that for any $\mu > 0$ and for any $0 \leq t_1 < t_2 \leq 1$,*

$$P\left\{\frac{1}{n^{1/(\alpha\beta)}}\left|\sum_{i=[nt_1]}^{[nt_2]} v_i\left(\varphi_\tau(\varepsilon_i - \beta_0(\tau)) + \zeta'(\eta_i, \beta_0(\tau))\right)\right|\geq \mu\right\} \leq C_{12}(t_2 - t_1)n^{-\nu}, \quad (\text{A.5})$$

and for any $0 \leq t \leq 1$,

$$P\left\{a_n^{-1/\beta}\left|\sum_{i=1}^{[nt]}\left(\zeta'(\eta_i, \beta_0(\tau)) - \zeta(\eta_i, \beta_0(\tau))\right)\right|\geq \mu\right\} \leq C_{12}\mu^{-\alpha}n^{-\alpha(\beta-1)+1-1/\beta}, \quad (\text{A.6})$$

where $F(x)$ is the distribution of ε_0 , $\{v_i, 1 \leq i \leq n\}$ is a non-random real-valued sequence with $\max_{1 \leq i \leq n} |v_i| = O(\log n)$ and

$$\begin{aligned} \zeta(\eta_i, \beta_0(\tau)) &= \sum_{j=1}^{\infty} (F(\beta_0(\tau) - c_j \eta_i) - EF(\beta_0(\tau) - c_j \eta_i)), \\ \zeta'(\eta_i, \beta_0(\tau)) &= \sum_{j=1}^{\infty} (F(\beta_0(\tau) - c_j \eta_{i-j}) - EF(\beta_0(\tau) - c_j \eta_{i-j})). \end{aligned}$$

Proof. Let $X_{i,j} = \sum_{l=0}^j c_l \eta_{i-l}$, $\bar{X}_{i,j} = \sum_{l=j+1}^{\infty} c_l \eta_{i-l}$, $F_j(\beta_0(\tau)) = P(X_{i,j} \leq \beta_0(\tau))$ and $U_{i,j}(\beta_0(\tau)) = F_{j-1}(\beta_0(\tau) - \bar{X}_{i,j-1}) - F_j(\beta_0(\tau) - \bar{X}_{i,j}) - F(\beta_0(\tau) - c_j \eta_{i-j}) + EF(\beta_0(\tau) - c_j \eta_{i-j})$. Then

$$\begin{aligned} &\sum_{i=[nt_1]}^{[nt_2]} v_i \left(I(\varepsilon_i \leq \beta_0(\tau)) - \tau - \eta(\xi_i, \beta_0(\tau)) \right) \\ &= \sum_{i=[nt_1]}^{[nt_2]} \sum_{j=1}^{\infty} v_i U_{i,j}(\beta_0(\tau)) = \sum_{j=-\infty}^{[nt_2]-1} \sum_{i=[nt_1] \vee (j+1)}^{[nt_2]} v_i U_{i,i-j}(\beta_0(\tau)). \end{aligned}$$

Let $M_{[nt_2],j} = \{\sum_{i=[nt_1] \vee (j+1)}^{[nt_2]} U_{i,i-j}(\beta_0(\tau))\}$. Then $\{M_{[nt_2],j}\}$ is a martingale difference. By Bahr–Essen’s inequality for martingales, we have that for any $1 \leq \nu \leq 2$,

$$\begin{aligned} E\left(\left|\sum_{j=-\infty}^{[nt_2]-1} M_{[nt_2],j}\right|^{\nu}\right) &\leq 2 \sum_{j=-\infty}^{[nt_2]-1} E|M_{[nt_2],j}|^{\nu} \\ &\leq 2 \sum_{j=-\infty}^{[nt_2]} \left(\sum_{i=[nt_1] \vee (j+1)}^{[nt_2]} E^{1/\nu}(U_{i,i-j} |^{\nu})\right)^{\nu}. \end{aligned} \quad (\text{A.7})$$

Using (A.7) and a similar argument to that of Lemma 5.3 in [32] (see also Lemma 7 of [6]), we have (A.5) as desired. (A.6) can be shown by a similar argument of Lemma 5.2 in [32]. \square

Lemma A.3. *Assume conditions H_1 and H_2 hold. Then,*

$$\int_0^1 T_n(s)ds \xrightarrow{d} \int_0^1 S(s)ds, \quad \int_0^1 T_n^2(s)ds \xrightarrow{d} \int_0^1 S^2(s)ds, \quad (\text{A.8})$$

and

$$\int_0^1 T_n(s-) dW_n(\tau, s) \longrightarrow^d \sigma \int_0^1 S(s) dW(\tau, s), \quad (\text{A.9})$$

where

$$S(s) = \lambda \int_0^s e^{-\gamma(s-t)} dZ_\alpha(t) = \lambda \left(Z_\alpha(s) - \gamma \int_0^s e^{-\gamma(s-t)} Z_\alpha(t) dt \right)$$

and the convergence in (A.8) and (A.9) holds jointly.

Proof. The proof of $\int_0^1 T_n(s) ds \longrightarrow^d \int_0^1 S(s) ds$ is similar to that of $\int_0^1 T_n^2(s) ds \longrightarrow^d \int_0^1 S^2(s) ds$, so we only give the proof of the latter. Note that

$$T_n(s) = Y_{[ns]}/a_n = \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} \varepsilon_j / a_n, \quad (\text{A.10})$$

and

$$\varepsilon_t = \lambda \eta_t + (\varepsilon_t - \lambda \eta_t). \quad (\text{A.11})$$

In view of (A.10) and (A.11), we have

$$T_n(s) = \frac{\lambda}{a_n} \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} \eta_j + \frac{1}{a_n} \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} (\varepsilon_j - \lambda \eta_j). \quad (\text{A.12})$$

This yields

$$\begin{aligned} \int_0^1 T_n^2(s) ds &= \int_0^1 \left(\frac{\lambda}{a_n} \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} \eta_j + \frac{1}{a_n} \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} (\varepsilon_j - \lambda \eta_j) \right)^2 ds \\ &= \lambda^2 \int_0^1 \left(\frac{1}{a_n} \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} \eta_j \right)^2 ds + \int_0^1 \left(\frac{1}{a_n} \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} (\varepsilon_j - \lambda \eta_j) \right)^2 ds \\ &\quad + 2\lambda \int_0^1 \left(\frac{1}{a_n} \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} \eta_j \right) \left(\frac{1}{a_n} \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} (\varepsilon_j - \lambda \eta_j) \right) ds \\ &=: I_1 + I_2 + I_3. \end{aligned} \quad (\text{A.13})$$

By Lemma 2 of [8], we have

$$\frac{1}{a_n} \sum_{j=1}^{[nt]} \gamma_n^{[ns]-j} \eta_j \Rightarrow_{J_1} Z_\alpha(s) - \gamma \int_0^s e^{-\gamma(s-t)} Z_\alpha(t) dt \quad \text{in } D(0, 1). \quad (\text{A.14})$$

By means of the continuous mapping theorem, $I_1 \longrightarrow^d \int_0^1 S^2(t) dt$. Note that $I_3 \leq 2(I_1 I_2)^{1/2}$. It is enough to show $I_2 \longrightarrow^p 0$. Observe that

$$\begin{aligned} \frac{1}{a_n} \sum_{j=1}^t \gamma_n^{t-j} (\varepsilon_j - \lambda \eta_j) &= \frac{1}{a_n} \sum_{j=1}^t (\varepsilon_j - \lambda \eta_j) - \frac{\gamma}{na_n} \sum_{j=1}^t \sum_{k=1}^j \gamma_n^{t-j} (\varepsilon_k - \lambda \eta_k) \\ &= \frac{1}{a_n} \left(\sum_{j=1}^t \sum_{i=t-j+1}^\infty c_i \eta_j - \sum_{j=-\infty}^0 \sum_{i=1-j}^{t-j} c_i \eta_j \right) - \frac{\gamma}{na_n} \end{aligned}$$

$$\times \sum_{j=1}^t \gamma_n^{t-j} \left(\sum_{k=1}^j \sum_{l=j-k+1}^{\infty} c_l \eta_k - \sum_{k=-\infty}^0 \sum_{l=1-k}^{j-k} c_l \eta_k \right). \quad (\text{A.15})$$

When $\alpha > 1$, since

$$\sup_{1 \leq m \leq n} \mathbb{E} \left\{ \frac{1}{a_n} \left| \sum_{j=1}^m \sum_{i=m-j+1}^{\infty} c_i \eta_j - \sum_{j=-\infty}^0 \sum_{i=1-j}^{m-j} c_i \eta_j \right| \right\} \rightarrow 0, \quad (\text{A.16})$$

it follows that

$$\frac{1}{n} \sum_{i=1}^n \left| \frac{1}{a_n} \sum_{j=1}^i \gamma_n^{i-j} (\varepsilon_j - \lambda \eta_j) \right| \xrightarrow{p} 0. \quad (\text{A.17})$$

On the other hand, by the continuous mapping and Theorem 2 of [1],

$$\begin{aligned} & \sup_{0 \leq i \leq n} \left| \frac{1}{a_n} \sum_{j=1}^i \gamma_n^{i-j} (\varepsilon_j - \lambda \eta_j) \right| \\ & \leq \sup_{1 \leq i \leq n} \left| \frac{1}{a_n} \sum_{j=1}^i \gamma_n^{i-j} \varepsilon_j \right| + \sup_{1 \leq i \leq n} \left| \frac{1}{a_n} \sum_{j=1}^i \gamma_n^{i-j} \lambda \eta_j \right| = O_p(1). \end{aligned}$$

Thus,

$$\begin{aligned} I_2 &= \int_0^1 \left(\frac{1}{a_n} \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} (\varepsilon_j - \lambda \eta_j) \right)^2 ds \\ &= \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{a_n} \sum_{j=1}^i \gamma_n^{i-j} (\varepsilon_j - \lambda \eta_j) \right)^2 \xrightarrow{p} 0. \end{aligned} \quad (\text{A.18})$$

When $\alpha \leq 1$, by the ‘Beveridge–Nelson’ decomposition of $\{\varepsilon_t\}$, we have

$$\varepsilon_t - \lambda \eta_t = \bar{\varepsilon}_t - \bar{\varepsilon}_t \quad (\text{A.19})$$

where $\bar{\varepsilon}_t = \sum_{j=0}^{\infty} \bar{c}_j \varepsilon_{t-j}$ and $\bar{c}_j = \sum_{i=j+1}^{\infty} c_i$. Eq. (A.8) can be shown as in Theorem 2.1 of [27]. The proof of (A.8) is complete.

Next, we adopt an idea of [18] to show (A.9). Let $\{\xi_t\}$ be stationary ergodic martingale differences with respect to σ -fields generated by $\{\eta_k, k \leq t\}$ for $t = 1, 2, \dots, n$ with $\mathbb{E} \xi_t^2 < \infty$ and Z_t be a stationary process with $\mathbb{E} Z_t^2 < \infty$ such that

$$\varphi_\tau(\eta_t) = \xi_t + Z_t - Z_{t+1}, \quad t = 1, 2, \dots, n.$$

Let $W_n^*(\tau, s) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[ns]} \xi_j$. Then

$$\begin{aligned} & \left| \int_0^1 T_n(s-) dW_n(\tau, s) - \int_0^1 T_n(s-) dW_n^*(\tau, s) \right| \\ &= |T_n(1)(W_n^*(\tau, 1) - W_n(\tau, 1)) - \int_0^1 (W_n^*(\tau, s) - W_n(\tau, s)) dT_n(s)| \\ &\leq |T_n(1)| \sup_{0 \leq s \leq 1} |W_n^*(\tau, s) - W_n(\tau, s)| + \left| \frac{1}{a_n \sqrt{n}} \sum_{t=1}^n (Z_{t+1} - Z_1) \varepsilon_t \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \frac{1}{a_n \sqrt{n}} \sum_{i=1}^n (Z_{t+1} - Z_1) \sum_{j=1}^{t-1} \gamma_n^{t-j-1} (\gamma_n - 1) \varepsilon_j \right| \\
& =: I_4 + I_5 + I_6.
\end{aligned} \tag{A.20}$$

Note that $EZ_t^2 < \infty$. It follows that

$$\sup_{0 \leq s \leq 1} |W_n^*(\tau, s) - W_n(\tau, s)| \leq \frac{2}{\sqrt{n}} \sup_{1 \leq t \leq n+1} |Z_t| = o_p(1). \tag{A.21}$$

Thus $I_4 = o_p(1)$. For any $\zeta > 0$, define

$$\begin{aligned}
I_{51} &= P \left\{ \left| \sum_{i=1}^n \left(\frac{1}{\sqrt{n}} (Z_{t+1} - Z_1) I(|Z_{t+1} - Z_1| > \sqrt{n}\delta) \right) \left(\frac{\varepsilon_t}{a_n} \right) \right| > \zeta \right\}, \\
I_{52} &= P \left\{ \left| \sum_{i=1}^n \left(\frac{1}{\sqrt{n}} (Z_{t+1} - Z_1) I(|Z_{t+1} - Z_1| \leq \sqrt{n}\delta) \right) \left(\frac{\varepsilon_t}{a_n} I(|\varepsilon_t| \geq Ma_n) \right) \right| > \zeta \right\}
\end{aligned}$$

and

$$I_{53} = P \left\{ \left| \sum_{i=1}^n \left(\frac{1}{\sqrt{n}} (Z_{t+1} - Z_1) I(|Z_{t+1} - Z_1| \leq \sqrt{n}\delta) \right) \left(\frac{\varepsilon_t}{a_n} I(|\varepsilon_t| < Ma_n) \right) \right| > \zeta \right\}.$$

Since $EZ_t^2 < \infty$, it follows that

$$I_{51} \leq P(\max_{1 \leq t \leq n} |Z_{t+1} - Z_1| > \sqrt{n}\delta) \rightarrow 0.$$

Next, we use Karamata's theorem to show that $I_{52} \rightarrow 0$. For this quantity, we split α into two cases: $\alpha > 1$ and $0 < \alpha \leq 1$. For $\alpha > 1$, by Karamata's theorem, we have

$$\begin{aligned}
I_{52} &\leq P \left\{ \sum_{i=1}^n \delta |\varepsilon_t| I(|\varepsilon_t| \geq Ma_n) > \zeta a_n \right\} \\
&\leq \frac{\delta}{\zeta a_n} \sum_{i=1}^n E[|\varepsilon_t| I(|\varepsilon_t| \geq Ma_n)] \\
&\leq C\delta \left(\frac{\alpha}{\alpha - 1} \right).
\end{aligned} \tag{A.22}$$

For $0 < \alpha \leq 1$ and $\nu < \alpha$, we have

$$I_{52} \leq \frac{\delta^\nu}{(\zeta a_n)^\nu} \sum_{i=1}^n E[|\varepsilon_t|^\nu I(|\varepsilon_t|^\nu \geq (Ma_n)^\nu)]. \tag{A.23}$$

Since ε_t^ν has index α/ν , similar to the argument of (A.22), the right-hand side of (A.23) is no bigger than $C\delta^\nu \alpha/(\alpha - \nu)$. Thus, by taking $\delta \rightarrow 0$ small enough in (A.22) and (A.23), we have $I_{52} \rightarrow 0$. Finally, using Karamata's theorem again, we have

$$\begin{aligned}
I_{53} &\leq \frac{1}{\zeta} E \left[\left(\frac{1}{n} \sum_{i=1}^n (Z_{t+1} - Z_1)^2 \right)^{1/2} \left(\frac{1}{a_n^2} \sum_{i=1}^n \varepsilon_t^2 I(|\varepsilon_t| < Ma_n) \right)^{1/2} \right] \\
&\leq \frac{1}{\zeta} \left[E \left(\frac{1}{n} \sum_{i=1}^n (Z_{t+1} - Z_1)^2 \right) \right]^{1/2} \left[E \left(\frac{1}{a_n^2} \sum_{i=1}^n \varepsilon_t^2 I(|\varepsilon_t| < Ma_n) \right) \right]^{1/2} \\
&\rightarrow 0,
\end{aligned}$$

by first letting $n \rightarrow \infty$ and then $M \rightarrow 0$. Thus $I_5 \xrightarrow{p} 0$. Note that since $1 - \gamma_n = \gamma/n$, similar to I_5 , I_6 can be shown converging to zero in probability. Combining these with $I_4 \xrightarrow{p} 0$ and (A.20) yields $\int_0^1 T_n d(W_n - W_n^*) \xrightarrow{p} 0$. We can therefore work with $\int T_n dW_n^*$ instead of $\int T_n dW_n$. Since $\sup_{1 \leq i \leq n} |\varphi_\tau(\varepsilon_i - \beta_0(\tau))| \leq 2$, it follows that W_n^* is a martingale with bounded jumps. By $T_n(s) = \lambda \sum_{j=1}^{[ns]} \gamma_n^{[ns]-j} \eta_j / a_n + o_p(1)$, we have

$$(T_n(s), W_n^*(s)) \xrightarrow{f.d.d} (S(s), W(\tau, s)).$$

Since $T_n(s) \Rightarrow^{M_1} S(s)$ (see Theorem 2 of [1] and $W_n^*(s) \Rightarrow^{J_1} \sigma W(\tau, s)$ on $D[0, 1]$ by Lemma A.1 and (A.21). It follows from Theorem 3 of [15] that

$$\int_0^1 T_n(s-) dW_n^*(\tau, s) \xrightarrow{d} \sigma \int_0^1 S(s-) dW(\tau, s).$$

This gives (A.9). The joint convergence follows from the joint weak convergence of $\{S_n(\cdot)\}$ and $\{W_n(\tau, \cdot)\}$. The proof of Lemma A.4 is complete. \square

Lemma A.4. Let $L_n(s) = a_n^{-1/\beta} \sum_{i=1}^{[ns]} \zeta(\eta_i, \beta_0(\tau))$, $S_n(s) = \sum_{i=1}^{[ns]} \eta_i / a_n$ and $W_n'(\tau, s) = \frac{1}{n^{\alpha\beta}} \sum_{t=1}^n \varphi_\tau(\varepsilon_t - \beta_0(\tau))$. When $1 + \sqrt{1 - 1/\alpha} < \beta < \alpha/2$, then under condition H_1 and H_2 , we have

$$(S_n(s), L_n(s)) \Rightarrow^{J_1} (Z_\alpha(s), L_{\alpha\beta}(s)) \quad \text{in } D[0, 1] \times D[0, 1] \quad (\text{A.24})$$

and

$$\begin{aligned} & \left(\int_0^1 T_n(s) ds, \int_0^1 T_n^2(s) ds, \int_0^1 T_n(s-) dW_n'(s) \right) \\ & \xrightarrow{d} \left(\int_0^1 S(s) ds, \int_0^1 S^2(s) ds, \int_0^1 S(s) dL_{\alpha\beta}(s) \right). \end{aligned} \quad (\text{A.25})$$

Proof. For the proof of (A.24), let $\kappa_i = (a_n^{-1} \eta_i, a_n^{-1/\beta} \zeta(\eta_i, \beta_0(\tau)))$, $1 \leq i \leq n$. Then $\{\kappa_i\}$ is a sequence of i.i.d random vectors. Since η_i belongs to the domain of attraction of a stable law with index α , it follows that $\zeta(\eta_i, \beta_0(\tau))$ belongs to the domains of attraction of a stable law with index $\alpha\beta$ (see [32]). By a similar argument of Theorem 1 in [25], it can be shown that κ_i belongs to a generalized domain of an operator stable law on R^2 . That is, there exists a stable vector process $\kappa(t) = (\kappa_1(t), \kappa_2(t))$ with $\kappa_1(t) \stackrel{d}{=} Z_\alpha(t)$, and $\kappa_2(t) \stackrel{d}{=} L_{\alpha\beta}(t)$ such that

$$Z_n(t) \Rightarrow^{J_1} \kappa(t),$$

where $Z_n(t) = \sum_{i=1}^{[nt]} \kappa_i$. This gives (A.24).

For (A.24), put $T_n'(s) = \lambda \sum_{i=1}^{[ns]} \gamma_n^{[ns]-j} \eta_i / a_n$. By a similar argument of [26]), we have $T_n'(s)$ is a semi-martingale satisfying the so-called UT conditions defined in Kurtz and Protter [21] (see also [16]). Further, by (A.24) and the continuous mapping theorem (see Lemma 2 of [8]), $T_n'(s) \Rightarrow^{J_1} S(s)$ on $D[0, 1]$. Thus, by Theorem 2.7 of [21] (see also [14]), it follows that

$$(T_n'(s), L_n(1)T_n'(1) - \int_0^1 L_n(s-) dT_n'(s)) \Rightarrow^{J_1} (S(s), \int_0^1 S(s) dL_{\alpha\beta}(s)).$$

Therefore, for the proof of (A.24), it is enough to show

$$\left(\int_0^1 T_n(s) ds, \int_0^1 T_n^2(s) ds, \int_0^1 T_n(s-) dW_n'(s) \right)$$

$$= \left(\int_0^1 T'_n(s) ds, \int_0^1 T_n'^2(s) ds, L_n(1)T'_n(1) - \int_0^1 L_n(s-) dT'_n(s) \right) + o_p(1). \quad (\text{A.26})$$

Similar to the argument of Lemma A.3 for the case of $\alpha > 1$, we have

$$\left(\int_0^1 T_n(s) ds, \int_0^1 T_n^2(s) ds \right) = \left(\int_0^1 T'_n(s) ds, \int_0^1 T_n'^2(s) ds \right) + o_p(1).$$

In the following, we show that when $1 + \sqrt{1 - 1/\alpha} < \beta < 2/\alpha$,

$$\int_0^1 T_n(s-) dW'_n(s) = L_n(1)T'_n(1) - \int_0^1 L_n(s-) dT'_n(s) + o_p(1).$$

By (A.15), we have

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{n^{1/(\alpha\beta)}} \sum_{t=1}^n [Y_{t-1}/a_n - T'_n((t-1)/n)] \varphi_\tau(\varepsilon_t - \beta_0(\tau)) \right| \\ & \leq \frac{2}{n^{1/(\alpha\beta)}} \sum_{t=1}^n \mathbb{E} \left| \frac{1}{a_n} \sum_{j=1}^t \gamma_n^{t-j} (\varepsilon_j - \lambda \eta_j) \right| \\ & \leq \frac{2}{n^{1/(\alpha\beta)}} \sum_{t=1}^n \left[\mathbb{E} \left| \frac{1}{a_n} \left(\sum_{j=1}^t \sum_{i=t-j+1}^\infty c_i \eta_j - \sum_{j=-\infty}^0 \sum_{i=1-j}^{t-j} c_i \eta_j \right) \right| \right. \\ & \quad \left. + \frac{\gamma}{n} \sum_{j=1}^t \mathbb{E} \left| \frac{1}{a_n} \left(\sum_{k=1}^j \sum_{l=j-k+1}^\infty c_l \eta_k - \sum_{k=-\infty}^0 \sum_{l=1-k}^{j-k} c_l \eta_k \right) \right| \right] \\ & \leq C n^{-1/(\alpha\beta)+1} a_n^{-1} \max_{1 \leq t \leq n} \mathbb{E} \left| \left(\sum_{j=1}^t \sum_{i=t-j+1}^\infty c_i \eta_j - \sum_{j=-\infty}^0 \sum_{i=1-j}^{t-j} c_i \eta_j \right) \right| \\ & \leq C n^{-1/(\alpha\beta)+1} a_n^{-1} n^{1-\beta+1/\alpha} = C n^{2-\beta-1/(\alpha\beta)} = o(1). \end{aligned} \quad (\text{A.27})$$

Therefore, by (A.27), it follows that

$$\begin{aligned} \sum_{t=1}^n Y_{t-1} \varphi_\tau(\varepsilon_t - \beta_0(\tau)) &= \frac{\lambda}{a_n n^{1/(\alpha\beta)}} \sum_{t=1}^n \left(\sum_{j=1}^{t-1} \gamma_n^{t-1-j} \eta_j \right) \varphi_\tau(\varepsilon_t - \beta_0(\tau)) + o_p(1) \\ &= \frac{\lambda}{a_n n^{1/(\alpha\beta)}} \left(\sum_{t=1}^{n-1} \gamma_n^{n-1-t} \eta_t \right) \left(\sum_{t=1}^n \varphi_\tau(\varepsilon_t - \beta_0(\tau)) \right) \\ &\quad - \frac{\lambda}{a_n n^{1/(\alpha\beta)}} \sum_{t=2}^{n-1} \left(\sum_{j=1}^{t-1} \varphi_\tau(\varepsilon_j - \beta_0(\tau)) \right) \eta_t + o_p(1). \end{aligned} \quad (\text{A.28})$$

By Lemma A.2, we have $\sum_{t=1}^n \varphi_\tau(\varepsilon_t - \beta_0(\tau)) = L_n(1) + o_p(1)$ and when $1 + \sqrt{1 - 1/\alpha} < \beta < 2/\alpha$,

$$\begin{aligned} & \mathbb{E} \left| \frac{\lambda}{a_n n^{1/(\alpha\beta)}} \sum_{t=1}^{n-1} \sum_{j=1}^{t-1} \left(\zeta(\eta_j, \beta_0(\tau)) - \zeta'(\eta_j, \beta_0(\tau)) \right) \eta_t \right| \\ & \leq \frac{\lambda}{a_n n^{1/(\alpha\beta)}} \sum_{t=1}^{n-1} \mathbb{E} \left| \sum_{j=1}^t \zeta(\eta_j, \beta_0(\tau)) - \zeta'(\eta_j, \beta_0(\tau)) \right| = o(1). \end{aligned}$$

Further,

$$\frac{1}{n^{1/(\alpha\beta)}} \sum_{i=1}^{[nt]} [\varphi_{\tau}(\varepsilon_j - \beta_0(\tau)) - \zeta'(\eta_j, \beta_0(\tau))] \Rightarrow^J 0.$$

Invoking Theorem 2.7 of [21] yields

$$\frac{\lambda}{a_n n^{1/(\alpha\beta)}} \sum_{i=1}^{n-1} \sum_{j=1}^{t-1} [\varphi_{\tau}(\varepsilon_j - \beta_0(\tau)) - \zeta'(\eta_j, \beta_0(\tau))] \eta_i \xrightarrow{d} 0.$$

Thus, by (A.28), we have (A.26) and the proof of Lemma 5.5 is completed. \square

References

- [1] F. Avram, M.S. Taqqu, Weak convergence of sums of moving averages in the α -stable domain of attraction, *Ann. Probab.* 20 (1992) 483–503.
- [2] E. Bayraktar, U. Horst, K.R. Sircar, A limiting theorem for financial markets with inert investors, Working Paper, Princeton University, 2003.
- [3] L. Bel, G. Oppenheim, L. Robbiao, M.C. Viano, Linear distribution processes, *J. Appl. Math. Stoch. Anal.* 11 (1998) 43–58.
- [4] P. Billingsley, *Convergence of Probability Measures*, 2nd ed., Wiley, New York, 1999.
- [5] N.H. Chan, Time series with roots on or near the unit circle, in: T.G. Andersen, R.A. Davis, J. Kreiss, T. Mikosch (Eds.), *Springer Handbook of Financial Time Series*, Springer, New York, 2009, pp. 695–707.
- [6] N.H. Chan, R.M. Zhang, M -estimation in nonparametric regression under strong dependence with infinite variance, *Ann. Inst. Statist. Math.* 60 (2008) 391–411.
- [7] N.H. Chan, R.M. Zhang, Inference for nearly nonstationary processes under strong dependence with infinite variance, *Statistica Sinica* 19 (2009) 925–947.
- [8] N.H. Chan, L. Peng, Y.C. Qi, Quantile inference for near-integrated autoregressive time series with infinite variance, *Statistica Sinica* 16 (2006) 15–28.
- [9] D.R. Cox, Long-range dependence: A review, in: H.A. David, H.T. David (Eds.), *Statistics: An Appraisal. Proceedings 50th Anniversary Conference*, The Iowa State University Press, 1984, pp. 55–74.
- [10] R.A. Davis, K. Knight, J. Liu, M -estimation for autoregressions with infinite variance, *Stochastic Process. Appl.* 40 (1992) 145–180.
- [11] P. Doukhan, G. Oppenheim, M. Taqqu, *Theory and Applications of Long-range Dependence*, Birkhäuser, Boston, 2003.
- [12] E. Fama, The behavior of stock market price, *J. Business* 38 (1965) 34–105.
- [13] B. Finkenstädt, H. Rootzén, *Extreme Values in Finance, Telecommunications, and the Environment*, Chapman and Hall/CRC, New York, 2004.
- [14] J. Jacod, A.N. Shiryaev, *Limit Theorems for Stochastic Processes*, 2nd ed., Springer-Verlag, Berlin, 2003.
- [15] A. Jakubowski, Convergence in various topologies for stochastic integrals driven by semimartingales, *Ann. Probab.* 24 (1996) 2141–2153.
- [16] A. Jakubowski, J. Mémin, G. Pagès, Convergence en loi des suites d'intégrales stochastiques sur l'espace \mathbb{D}^1 de Skorohod, *Probab. Theory Related Fields* 81 (1989) 111–137.
- [17] K. Knight, Limit theory for autoregressive parameter estimates in an infinite variance random walk, *Canad. J. Statist.* 17 (1989) 261–278.
- [18] K. Knight, Limit theory for M -estimates in an integrated infinite variance processes, *Econometric Theory* 7 (1991) 200–212.
- [19] R. Koenker, *Quantile Regression*, Cambridge University Press, Cambridge, 2005.
- [20] R. Koenker, G. Bassett, Regression quantile, *Econometrica* 46 (1978) 33–49.
- [21] T.G. Kurtz, P. Protter, Weak limit theorems for stochastic integrals and stochastic differential equations, *Ann. Probab.* 19 (1991) 1035–1070.
- [22] T. Lux, M. Marchesi, Volatility clustering in financial markets: A microsimulation of interesting agents, *Int. J. Theor. Appl. Finance* 3 (2000) 675–702.
- [23] B.B. Mandelbrot, The variation of certain speculative price, *J. Business* 36 (1963) 394–419.
- [24] B.B. Mandelbrot, The variation of some other speculative price, *J. Business* 40 (1967) 393–413.

- [25] V. Mandrekar, M.M. Meerschaert, Sample moments and symmetric statistics, in: H. Kunita, H.H. Kuo (Eds.), *Stochastic Analysis on Infinite Dimensional Spaces*, Pitman, London, 1994.
- [26] V. Paulauskas, S.T. Rachev, Cointegrated processes with infinite variance innovations, *Ann. Appl. Probab.* 8 (1998) 775–792.
- [27] P.C.B. Phillips, Time series regression with a unit root and infinite-variance errors, *Econometric Theory* 6 (1990) 44–62.
- [28] S.T. Rachev, Mittnik, *Stable Paretian Models in Finance*, Wiley, New York, 2000.
- [29] S. Resnick, P. Greenwood, A bivariate stable characterization and domains of attraction, *J. Multivariate Anal.* 9 (1979) 206–221.
- [30] G. Samorodnitsky, M.S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models With Infinite Variance*, Chapman and Hall, New York, 1994.
- [31] D. Surgailis, On L^2 and non- L^2 multiple stochastic integration, in: *Stochastic Differential Systems*, in: *Lecture Notes in Control and Information Science*, vol. 36, 1981, pp. 212–226.
- [32] D. Surgailis, Stable limits of empirical processes of moving averages with infinite variance, *Stochastic. Process. Appl.* 100 (2002) 255–274.
- [33] D. Surgailis, Stable limits of sums of bounded functions of long-memory moving averages with finite variance, *Bernoulli* 10 (2004) 327–355.
- [34] M. Woodroffe, A central limit theorem for functional of a Markov chain with applications to shifts, *Stochastic Process. Appl.* 41 (1992) 33–44.
- [35] W.B. Wu, Additive functionals of infinite-variance moving averages, *Statistica Sinica* 13 (2003) 1259–1267.