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A lift of spatially inhomogeneous Markov process to extensions of the field of p-adic numbers

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Abstract

A Markov process on a local field which can be projected to a Markov process on a smaller local field is regarded as a lift of the one on the smaller field. The first part of this article is concerned with a Markov process on a local field which is obtained as the one projected from a larger field by means of the algebraic trace. Since the explicit expression of the transition probability plays important roles in a study of Markov processes on local fields, the second part is devoted to finding an explicit expression for the Markov process.

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1. Introduction

Various aspects of stochastic processes with rotationally invariant probability laws on non-Archimedean metric spaces were found. Some of them are not always similar to ones on the Euclidean space. In fact, the infinitesimal generators associated with stochastic processes in a typical class are written as Vladimirov operators and can be given as derivatives with order higher than 2 on the field $\boldsymbol{Q_p}$ of p-adic numbers. In [8], Kochubei found solutions of parabolic

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partial differential equations on $\boldsymbol{Q_p}$ where the infinitesimal generator of a stochastic process was involved.

Stochastic processes on finite extension of $\boldsymbol{Q_p}$ have been addressed by focusing mainly on rotationally invariant probability laws. For instance, based on the method initiated by Albeverio and Karwowski, a family of additive processes on local fields was introduced by Yasuda in [10]. On the other hand, Haran showed some results related to operators by focusing on important function spaces in [3,4].

A probabilistic counterpart of Sobolev spaces was introduced by Fukushima and the author in [2], and it has been investigated by many researchers. The first author proposed some function spaces over local fields including Dirichlet spaces and showed potential theoretic coverage of the fields including non-linear capacity. The study relied on the explicit description of rotationally invariant transition probabilities found by Yasuda in [10]. In [5], after some preliminary observations on non-linear capacities on finite extensions over $\boldsymbol{Q_p}$, Yasuda and the author found some properties of non-linear capacities on an infinite extension over $\boldsymbol{Q_p}$ introduced in [9,11]. Yasuda pointed out some difference in two particular infinite extensions of $\boldsymbol{Q_p}$ and some facts on non-trivial probability measures with no rotational invariance in [11].

As for probability laws with no rotational invariance on a local field, Karwowski and Mendes constructed a family of Markov processes on $\boldsymbol{Q_p}$ by introducing spatially inhomogeneous factors given as a function on $\boldsymbol{Q_p}$ in [7]. On finite extensions of $\boldsymbol{Q_p}$, Zhao and the first author made an attempt to construct Markov processes with spatial inhomogeneity based on an explicit expression of a counterpart of a Poisson process on the state space in [6].

Accordingly, one natural question raised here could be whether any rotationally non-invariant Markov process $\{X(t)\}$ can be found on a finite extension K_2 of $\boldsymbol{Q_p}$ whose natural projection $\{T(X(t))\}$ to another extension K_1 contained in K_2 is again a Markov process, where the projection T is given by $\frac{1}{[K_2:K_1]}\mathrm{Tr}_{K_2/K_1}(x)$.

In this article, we will give a construction of a Markov process $\{X(t)\}$ on a finite extension K_2 of K_1 which admits a Markov process $\{T(X(t))\}$ on K_1. Then, the Markov process $\{X(t)\}$ will be considered as a lift of $\{T(X(t))\}$. In deriving an explicit description of the transition probability of $\{X(t)\}$, we will see that the existing methods as in [1,7] are not directly applicable. For the construction of a Markov process $\{X(t)\}$, we will obtain an explicit description of the transition probability of the Markov process, by applying a modified method.

2. Kolmogorov's equations

For two finite separable extensions K_1 and K_2 of $\boldsymbol{Q_p}$ satisfying $K_1 \subset K_2$, we see that the p-adic valuation on $\boldsymbol{Q_p}$ is extended to the norm on K_2 which coincides with the original valuation for any element in $\boldsymbol{Q_p}$ in what follows. We denote the norm of $x \in K_2$ by $\|x\|$. The maximal ideal $P_i = \{x \in K_i | \, \|x\| < 1\}$ of the ring $R_i = \{x \in K_i | \, \|x\| \le 1\}$ has an element π_i with maximal norm and with the property $\pi_i R_i = P_i$ for $i = 1, 2$.

Since the residue field R_i/P_i is a finite extension of $\boldsymbol{F_p} = \boldsymbol{Z}/\boldsymbol{Z_p}$, one can choose a family $\{s_j^{(i)}\}_{j=1}^{f_{K_i}} \subset R$ so that their natural images in the residue field R_i/P_i are the bases over the finite field $\boldsymbol{F_p}$. In what follows, $p^{f_{K_i}}$ will be denoted by q_{K_i} and the extension degree of K_i over $\boldsymbol{Q_p}$ by m_{K_i} for $i = 1$ and 2. Then, the normalized Haar measure μ on K_2 is characterized by $\mu(B(x, q_{K_2}^{\ell/m_{K_2}})) = q_{K_2}^{\ell}$ for any integer ℓ and $x \in K_2$, where $B(x, q_{K_2}^{\ell/m_{K_2}})$ stands for the ball $\{x \in K_2 | \, \|x - a\| \le q_{K_2}^{\ell/m_{K_2}}\}$ in K_2.

Lemma 1. *For any ball $B(a, q_{K_1}^{M/m_{K_1}})$ in K_2, $T(B(a, q_{K_1}^{M/m_{K_1}}))$ is a ball in K_1 whose radius is given as $q_{K_1}^{(M+L)/m_{K_1}}$ with some fixed non-negative integer L.*

Proof. Every non-empty ball in K_2 contains at least one element K_1. Therefore, we may assume that a is an element of K_1. Accordingly, $T(B(a, q_{K_1}^{M/m_{K_1}}))$ contains $B(a, q_{K_1}^{M/m_{K_1}}) \cap K_1$. Since the real valued function $\Phi(x) = \|T(x-a)\|$ defined on K_2 is continuous, there exists a point x_0 in $B(a, q_{K_1}^{M/m_{K_1}})$ satisfying $\|T(x_0 - a)\| = \max\{\|T(x-a)\| \mid x \in B(a, q_{K_1}^{M/m_{K_1}})\}$. We can find some non-negative integer L such that $\|T(x_0 - a)\| = q_{K_1}^{(M+L)/m_{K_1}}$. On the other hand, any element $x \in K_1$ with $\|x - a\| \le \|T(x_0 - a)\|$ enjoys $\|\frac{x-a}{T(x_0-a)}(x_0 - a)\| \le \|x_0 - a\| \le q_{K_1}^{M/m_{K_1}}$ and $\frac{x-a}{T(x_0-a)}(x_0 - a) + a$ is mapped to x by T. As a result, it turns out that $T(B(a, q_{K_1}^{M/m_{K_1}}))$ is the ball centered at a with radius $q_{K_1}^{(M+L)/m_{K_1}}$. $\square$

Let us introduce a non-increasing sequence $\{u(k)\}_{k=-\infty}^{\infty}$ satisfying $\lim_{k\to\infty} u(k) = 0$ and a non-negative locally integrable function ρ defined on K_2. Here and in what follows, we will fix the radius $q_{K_1}^{M/m_{K_1}}$ of balls in our focus and choose a family $\{B_j\}_{j=1}^{\infty}$ of disjoint balls with the radius $q_{K_1}^{M/m_{K_1}}$ satisfying $K_2 = \cup_j B_j$. Then, we define $E_{M+m}(B_i) = \cup_{\mathrm{diam}(T(B_j)\cup T(B_i)) \le q_{K_1}^{(M+L+m)/m_{K_1}}} B_j$ for each non-negative integer m. We denote the integral $\int_{E_{M+m}(B_i)} \rho(\mathrm{d}x)$ by $\rho_{M+m}(B_i)$ and impose the following condition on the function ρ:

$$\sum_{k=0}^{\infty}(u(M+k) - u(M+k+1))\rho_{M+k}(B_f) < \infty.$$

In what follows, $-\sum_{k=m}^{\infty}(u(M+k) - u(M+k+1))\rho_{M+k}(B_f)$ will be denoted by $\mathcal{W}_{M,m}(B_f)$ for each ball B_f, namely, we introduce the notation

$$\mathcal{W}_{M,m}(B_f) = -\sum_{k=m}^{\infty}(u(M+k) - u(M+k+1))\rho_{M+k}(B_f) \tag{1}$$

for each ball B_f in the family $\{B_j\}_{j=1}^{\infty}$.

For topological Borel set E in K_2, we denote $\int_E \rho(\mathrm{d}x)$ by ρ_E. For any pair B_f, B_j of two balls in the family $\{B_j\}_{j=1}^{\infty}$, we define $\tilde{u}(B_f, B_j)$ by $\tilde{u}(B_f, B_j) = u(M + m(B_f, B_j))\rho_{B_f}$ and $\tilde{a}(B_f)$ by $\tilde{a}(B_f) = \sum_{j\ne f} \tilde{u}(B_j, B_f)$. Here and in what follows, if a pair of balls B_f and B_j with radius $q_{K_1}^{M/m_{K_1}}$ satisfies $\mathrm{diam}(T(B_f) \cup T(B_j)) = q_{K_1}^{(M+L+k)/m_{K_1}}$, this integer k will be denoted by $m(B_j, B_f)$. We introduce the notation $P_{E,F}(t) = P(X(t) \in E | X(0) \in F)$ for topological Borel sets $E, F \subset K_2$ and start with Kolmogorov's forward equation for a Markov process $\{X(t)\}$ on K_2. Here, we note that Kolmogorov's forward equation on the state space K_2 provides us with the one on the smaller field K_1:

Proposition 1. *If Kolmogorov's forward equation*

$$\dot{P}_{B_f,B_i}(t) = -\tilde{a}(B_f)P_{B_f,B_i}(t) + \sum_{j\ne f}\tilde{u}(B_f, B_j)P_{B_j,B_i}(t) \tag{2}$$

holds for a Markov process on the field K_2, then

$$\begin{aligned}
&\dot{P}_{E_M(B_f),E_M(B_i)}(t)\\
&\quad= -\sum_{E_M(B_{j(n)})\neq E_M(B_f)} u(M+m(B_{j(n)},B_f))\rho_{E_M(B_{j(n)})}P_{E_M(B_f),E_M(B_i)}(t)\\
&\qquad+\sum_{E_M(B_{j(n)})\neq E_M(B_f)} u(M+m(B_f,B_{j(n)}))\rho_M(B_f)P_{E_M(B_{j(n)}),E_M(B_i)}(t)
\end{aligned}$$

holds on K_2, where $\{B_{j(n)}\}_{n=1}^{\infty}$ stands for a subfamily of balls $\{B_j\}_{j=1}^{\infty}$ in K_2 satisfying $\cup_n E_M(B_{j(n)}) = K_2$ and $E_M(B_{j(n)}) \cap E_M(B_{j(m)}) = \emptyset$ for any distinct positive integers n and m.

Proof. From (2), we derive that

$$\begin{aligned}
\dot{P}_{E_M(B_f),B_i}(t) &= -\sum_{\ell\in\Lambda}\tilde{a}(B_{f_\ell})P_{B_{f_\ell},B_i}(t)+\sum_{\ell\in\Lambda}\sum_{j\neq f_\ell}\tilde{u}(B_{f_\ell},B_j)P_{B_j,B_i}(t)\\
&= -\sum_{\ell\in\Lambda}\left\{\sum_{j\neq f_\ell}u(M+m(B_j,B_{f_\ell}))\rho_{B_j}P_{B_{f_\ell},B_i}(t)\right\}\\
&\quad+\sum_{\ell\in\Lambda}\left\{\sum_{j\neq f_\ell}u(M+m(B_{f_\ell},B_j))\rho_{B_{f_\ell}}P_{B_j,B_i}(t)\right\}\\
&= -\sum_{\ell\in\Lambda}\left\{\sum_{j\notin\{f_\ell|\ell\in\Lambda\}}u(M+m(B_j,B_{f_\ell}))\rho_{B_j}P_{B_{f_\ell},B_i}(t)\right\}\\
&\quad+\sum_{\ell\in\Lambda}\left\{\sum_{j\notin\{f_\ell|\ell\in\Lambda\}}u(M+m(B_{f_\ell},B_j))\rho_{B_{f_\ell}}P_{B_j,B_i}(t)\right\},
\end{aligned}$$

where $\{B_{f_\ell}\}_{\ell\in\Lambda}$ stands for the subfamily of $\{B_i\}$ satisfying $T(B_{f_\ell}) = T(B_f)$.
Since $m(B_j, B_{f_\ell}) = m(B_j, B_f)$ for any $\ell \in \Lambda$, we have

$$\begin{aligned}
\dot{P}_{E_M(B_f),B_i}(t) &= -\sum_{j\notin\{f_\ell|\ell\in\Lambda\}}u(M+m(B_j,B_f))\rho_{B_j}P_{E_M(B_f),B_i}(t)\\
&\quad+\sum_{j\notin\{f_\ell|\ell\in\Lambda\}}u(M+m(B_f,B_j))\rho_M(B_f)P_{B_j,B_i}(t).
\end{aligned}$$

Multiplying both sides of the identity

$$\begin{aligned}
\dot{P}_{E_M(B_f),B_{i'}}(t) &= -\sum_{j\notin\{f_\ell|\ell\in\Lambda\}}u(M+m(B_j,B_f))\rho_{B_j}P_{E_M(B_f),B_{i'}}(t)\\
&\quad+\sum_{j\notin\{f_\ell|\ell\in\Lambda\}}u(M+m(B_f,B_j))\rho_M(B_f)P_{B_j,B_{i'}}(t)
\end{aligned}$$

by $P(X(0) \in B_{i'})$ with $T(B_{i'}) = T(B_i)$ and by taking the sum with respect to the family $\{B_{i'}\}$ of balls satisfying this condition, we have

$$\begin{aligned}
\dot{P}_{E_M(B_f),E_M(B_i)}(t) &= -\sum_{j\notin\{f_\ell|\ell\in\Lambda\}}u(M+m(B_j,B_f))\rho_{B_j}P_{E_M(B_f),E_M(B_i)}(t)\\
&\quad+\sum_{j\notin\{f_\ell|\ell\in\Lambda\}}u(M+m(B_f,B_j))\rho_M(B_f)P_{B_j,E_M(B_i)}(t).
\end{aligned}$$

By taking the sum first with respect to B_j mapped to an identical ball in K_1 by T, secondly choosing subsequence $\{B_{j(n)}\}$ of balls such that $K_1 = \cup_n T(B_{j(n)})$ and $T(B_{j(n)}) \cap T(B_{j(m)}) = \emptyset$ with $n \neq m$, we obtain

$$\begin{aligned}
&\dot{P}_{E_M(B_f),E_M(B_i)}(t)\\
&\quad = - \sum_{E_M(B_{j(n)}) \neq E_M(B_f)} u(M + m(B_{j(n)}, B_f))\rho_M(B_{j(n)}) P_{E_M(B_f),E_M(B_i)}(t)\\
&\qquad + \sum_{E_M(B_{j(n)}) \neq E_M(B_f)} u(M + m(B_f, B_{j(n)}))\rho_M(B_f) P_{E_M(B_{j(n)}),E_M(B_i)}(t). \quad \square
\end{aligned}$$

In the choice of $\{B_{j(n)}\}$, we may assume that $k \leq \ell$ implies $\mathrm{dist}(T(B_f), T(B_{j(k)})) \leq \mathrm{dist}(T(B_f), T(B_{j(\ell)}))$. As for this reordered family of balls, we obtain the following lemma:

Lemma 2. *For any ball B_f, there exists an increasing sequence $\{n_m\}$ of positive integers such that $E_M(B_f) = E_M(B_{j(n_0)})$ and $E_{M+m+1}(B_f) \setminus E_{M+m}(B_f) = \cup_{k=n_m+1}^{n_{m+1}} E_{M+m}(B_{j(k)})$ are satisfied for each non-negative integer m, where M stands for the integer satisfying* $\mathrm{diam}(B_f) = q_{K_1}^{M/m_{K_1}}$.

Proof. We note that the family $\{T(B_j)\}$ of balls satisfying $\mathrm{diam}(T(B_j) \cup T(B_i)) \leq q_{K_1}^{(M+L+m+1)/m_{K_1}}$ consists of finitely many elements for any B_i. Therefore, we can divide the family of balls into finitely many subfamilies so that any pair $T(B_j)$ and $T(B_{j'})$ of balls in the same subfamily enjoys $\mathrm{diam}(T(B_j) \cup T(B_{j'})) \leq q_{K_1}^{(M+L+m)/m_{K_1}}$. Therefore, we can take a subfamily $\{B_{j(k)}\}_{k=1}^{\infty}$ of balls $\{B_j\}_{j=1}^{\infty}$ satisfying $E_{M+m+1}(B_f) \setminus E_{M+m}(B_f) = \cup_{k=n_m+1}^{n_{m+1}} E_{M+m}(B_{j(k)})$ for any non-negative integer m. $\square$

Since we have started with Kolmogorov's forward equation which is similar to the one in [7], we can define $P_{K_2}(t) = \lim_{m\to\infty} P_{E_{M+m}(B_i),B_i}(t)$ independently of B_i and employ the conventional notation $\rho_{K_2}^{-1} = 0$ when $\rho_{K_2} = \infty$.

Proposition 2. (i) *If $E_M(B_i) \subset E_{M+m}(B_f)$ i.e. $m \geq m(B_i, B_f)$, then*

$$P_{E_{M+m}(B_f),E_M(B_i)}(t) = \rho_{M+m}(B_f)\left\{\rho_{K_2}^{-1} P_{K_2}(t) + \sum_{k=0}^{\infty}\left(\frac{1}{\rho_{M+m+k}(B_f)} - \frac{1}{\rho_{M+m+k+1}(B_f)}\right) e^{t\mathcal{W}_{M,m+k+1}(B_f)}\right\},$$

with $\mathcal{W}_{M,m}(B_f)$ defined by (1).

(ii) *If $E_M(B_i) \not\subset E_{M+m}(B_f)$ i.e. $m < m(B_i, B_f)$, then*

$$\begin{aligned}
P_{E_{M+m}(B_f),E_M(B_i)}(t) = \rho_{M+m}(B_f)\Bigg\{&\rho_{K_2}^{-1} P_{K_2}(t)\\
&+ \sum_{k=0}^{\infty}\left(\frac{1}{\rho_{M+m+k}(B_f)} - \frac{1}{\rho_{M+m+k+1}(B_f)}\right) e^{t\mathcal{W}_{M,m+k+1}(B_i)}\\
&- \frac{1}{\rho_{M+m(B_j,B_f)}(B_i)} e^{t\mathcal{W}_{M,m(B_j,B_f)}(B_i)}\Bigg\}.
\end{aligned}$$

Proof. The right-hand side of the equality

$$\dot{P}_{E_M(B_f),B_i}(t) = -\sum_{j\notin\{f_\ell|\ell\in\Lambda\}} u(M+m(B_j,B_f))\rho_{B_j}P_{E_M(B_f),B_i}(t) + \sum_{j\notin\{f_\ell|\ell\in\Lambda\}} u(M+m(B_f,B_j))\rho_M(B_f)P_{B_j,B_i}(t)$$

given in the proof of Proposition 1 admits the following expressions after taking the sum of the terms associated with balls in $\{B_j \mid m(B_j,B_f)=k\}$ for each positive integer k:

$$\begin{aligned}
&-\sum_{k=1}^{\infty} u(M+k)P_{E_M(B_f),B_i}(t)(\rho_{M+k}(B_f)-\rho_{M+k-1}(B_f)) \\
&\quad+\sum_{k=1}^{\infty} u(M+k)\rho_M(B_f)(P_{E_{M+k}(B_f),B_i}(t)-P_{E_{M+k-1}(B_f),B_i}(t)) \\
&= -P_{E_M(B_f),B_i}(t)\sum_{k=1}^{\infty} u(M+k)(\rho_{M+k}(B_f)-\rho_{M+k-1}(B_f)) \\
&\quad+\rho_M(B_f)\sum_{k=1}^{\infty} u(M+k)(P_{E_{M+k}(B_f),B_i}(t)-P_{E_{M+k-1}(B_f),B_i}(t)) \\
&= -P_{E_M(B_f),B_i}(t)\sum_{k=1}^{\infty}(u(M+k)-u(M+k+1))\rho_{M+k}(B_f) \\
&\quad+P_{E_M(B_f),B_i}(t)u(M+1)\rho_M(B_f) \\
&\quad+\rho_M(B_f)\sum_{k=1}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t) \\
&\quad-\rho_M(B_f)u(M+1)P_{E_M(B_f),B_i}(t) \\
&= -P_{E_M(B_f),B_i}(t)\sum_{k=1}^{\infty}(u(M+k)-u(M+k+1))\rho_{M+k}(B_f) \\
&\quad+\rho_M(B_f)\sum_{k=1}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t).
\end{aligned}$$

The right-hand side can be written with $\mathcal{W}_{M,1}(B_f)$ as in (1),

$$\mathcal{W}_{M,1}(B_f)P_{E_M(B_f),B_i}(t)+\rho_M(B_f)\sum_{k=1}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t).$$

By combining this with $E_{M+1}(B_f) = E_M(B_{j(n_0)}) \cup E_M(B_{j(n_0+1)}) \cup \cdots \cup E_M(B_{j(n_1)})$ obtained by Lemma 2, we see that

$$\begin{aligned}
\dot{P}_{E_{M+1}(B_f),B_i}(t) &= \sum_{\ell=n_0}^{n_1} \dot{P}_{E_M(B_{j(\ell)}),B_i}(t) \\
&= \sum_{\ell=n_0}^{n_1}\Bigg\{\mathcal{W}_{M,1}(B_{j(\ell)})P_{E_M(B_{j(\ell)}),B_i}(t)+\rho_M(B_{j(\ell)}) \\
&\quad\times\sum_{k=1}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_{j(\ell)}),B_i}(t)\Bigg\}
\end{aligned}$$

$$= -\sum_{\ell=n_0}^{n_1}\sum_{k=1}^{\infty}\{(u(M+k)-u(M+k+1))\rho_{M+k}(B_{j(\ell)})P_{E_M(B_{j(\ell)}),B_i}(t)\}$$
$$+\sum_{\ell=n_0}^{n_1}\sum_{k=1}^{\infty}\{(u(M+k)-u(M+k+1))P_{E_{M+k}(B_{j(\ell)}),B_i}(t)\rho_M(B_{j(\ell)})\}.$$

Since $E_{M+k}(B_{j(\ell)}) = E_{M+k}(B_f)$ for any $k \geq 1$, the right-hand side is equal to

$$-\sum_{k=1}^{\infty}\{(u(M+k)-u(M+k+1))\rho_{M+k}(B_f)P_{E_{M+1}(B_f),B_i}(t)\}$$
$$+\sum_{k=1}^{\infty}\{(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t)\rho_{M+1}(B_f)\}$$
$$= -\sum_{k=2}^{\infty}\{(u(M+k)-u(M+k+1))\rho_{M+k}(B_f)P_{E_{M+1}(B_f),B_i}(t)\}$$
$$+\sum_{k=2}^{\infty}\{(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t)\rho_{M+1}(B_f)\}$$
$$= \mathcal{W}_{M,2}(B_f)P_{E_{M+1}(B_f),B_i}(t)$$
$$+\rho_{M+1}(B_f)\sum_{k=2}^{\infty}(u(M+k)-u(M+k+1))P_{E_M+k(B_f),B_i}(t).$$

By iterating this procedure, we have

$$\dot{P}_{E_{M+m}(B_f),B_i}(t) = \mathcal{W}_{M,m+1}(B_f)P_{E_{M+m}(B_f),B_i}(t)+\rho_{M+m}(B_f)$$
$$\times\sum_{k=m+1}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t).$$

Since the initial condition is given as $P_{B_i,B_i}(0) = 1$, we can derive from these identities that

$$P_{E_{M+m}(B_f),B_i}(t) = P_{E_{M+m}(B_f),B_i}(0)\mathrm{e}^{t\mathcal{W}_{M,m+1}(B_f)},$$

in case $\rho_{M+m}(B_f) = 0$. If $\rho_{M+m}(B_f) \neq 0$, then we can derive from

$$\dot{P}_{E_{M+\ell}(B_f),B_i}(t) = \mathcal{W}_{M,\ell+1}(B_f)P_{E_{M+\ell}(B_f),B_i}(t)+\rho_{M+\ell}(B_f)$$
$$\times\sum_{k=\ell+1}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t),\quad (\ell = m, m+1, \ldots)$$

that

$$\frac{\rho_{M+m+1}(B_f)}{\rho_{M+m}(B_f)}\dot{P}_{E_{M+m}(B_f),B_i}(t)-\dot{P}_{E_{M+m+1}(B_f),B_i}(t)$$
$$= \frac{\rho_{M+m+1}(B_f)}{\rho_{M+m}(B_f)}\mathcal{W}_{M,m+1}(B_f)P_{E_{M+m}(B_f),B_i}(t)-\mathcal{W}_{M,m+2}(B_f)P_{E_{M+m+1}(B_f),B_i}(t)$$
$$+\rho_{M+m+1}(B_f)(u(M+m+1)-u(M+m+2))P_{E_{M+m+1}(B_f),B_i}(t).$$

This shows that

$$\frac{\rho_{M+m+1}(B_f)}{\rho_{M+m}(B_f)}\dot{P}_{E_{M+m}(B_f),B_i}(t)-\dot{P}_{E_{M+m+1}(B_f),B_i}(t)$$
$$=\mathcal{W}_{M,m+1}(B_f)\left(\frac{\rho_{M+m+1}(B_f)}{\rho_{M+m}(B_f)}P_{E_{M+m}(B_f),B_i}(t)-P_{E_{M+m+1}(B_f),B_i}(t)\right).$$

If $\rho_{M+m}(B_f)\neq 0$ and $E_{M+m}(B_f)\supset E_M(B_i)$ i.e. $m\geq m(B_i,B_f)$, this identity implies

$$\frac{\rho_{M+m+1}(B_i)}{\rho_{M+m}(B_i)}P_{E_{M+m}(B_f),B_i}(t)-P_{E_{M+m+1}(B_f),B_i}(t)=\left(\frac{\rho_{M+m+1}(B_i)}{\rho_{M+m}(B_i)}-1\right)\mathrm{e}^{t\mathcal{W}_{M,m+1}(B_f)}$$

and so

$$P_{E_{M+m}(B_f),B_i}(t)-\frac{\rho_{M+m}(B_i)}{\rho_{M+m+1}(B_i)}P_{E_{M+m+1}(B_f),B_i}(t)$$
$$=\rho_{M+m}(B_i)\left(\frac{1}{\rho_{M+m}(B_i)}-\frac{1}{\rho_{M+m+1}(B_i)}\right)\mathrm{e}^{t\mathcal{W}_{M,m+1}(B_f)}.$$

By replacing m with $m+1$, we see that

$$P_{E_{M+m+1}(B_f),B_i}(t)-\frac{\rho_{M+m+1}(B_i)}{\rho_{M+m+2}(B_i)}P_{E_{M+m+2}(B_f),B_i}(t)$$
$$=\rho_{M+m+1}(B_i)\left(\frac{1}{\rho_{M+m+1}(B_i)}-\frac{1}{\rho_{M+m+2}(B_i)}\right)\mathrm{e}^{t\mathcal{W}_{M,m+2}(B_i)}.$$

By combining these two identities, we get

$$P_{E_{M+m}(B_f),B_i}(t)-\frac{\rho_{M+m}(B_i)}{\rho_{M+m+2}(B_i)}P_{E_{M+m+2}(B_f),B_i}(t)$$
$$=\rho_{M+m}(B_i)\sum_{k=0}^{1}\left(\frac{1}{\rho_{M+m+k}(B_i)}-\frac{1}{\rho_{M+m+k+1}(B_i)}\right)\mathrm{e}^{t\mathcal{W}_{M,m+k+1}(B_i)}.$$

Taking a similar control over m and by combining the identities obtained by the procedures, we have

$$P_{E_{M+m}(B_f),B_i}(t)-\frac{\rho_{M+m}(B_i)}{\rho_{M+m+m'}(B_i)}P_{E_{M+m+m'}(B_f),B_i}(t)$$
$$=\rho_{M+m}(B_i)\sum_{k=0}^{m'-1}\left(\frac{1}{\rho_{M+m+k}(B_i)}-\frac{1}{\rho_{M*m+k+1}(B_i)}\right)\mathrm{e}^{t\mathcal{W}_{M,m+k+1}(B_i)}.$$

By passing to the limit as $m\to\infty$, we have

$$P_{E_{M+m}(B_f),B_i}(t)$$
$$=\rho_{M+m}(B_i)\left\{\rho_{K_2}^{-1}P_{K_2}(t)+\sum_{k=0}^{\infty}\left(\frac{1}{\rho_{M+m+k}(B_i)}-\frac{1}{\rho_{M+m+k+1}(B_i)}\right)\mathrm{e}^{t\mathcal{W}_{M,m+k+1}(B_i)}\right\}.$$

Since $m=m(B_f,B_I)-1$ implies $E_{M+m}(B_f)\not\supset E_M(B_i)$, in this case, the following identity is valid:

$$\frac{\rho_{M+m+1}(B_f)}{\rho_{M+m}(B_f)}\dot{P}_{E_{M+m}(B_f),B_i}(t)-\dot{P}_{E_{M+m+1}(B_f),B_i}(t)$$
$$=\mathcal{W}_{M,m+1}(B_f)\left(\frac{\rho_{M+m+1}(B_f)}{\rho_{M+m}(B_f)}P_{E_{M+m}(B_f),B_i}(t)-P_{E_{M+m+1}(B_f),B_i}(t)\right).$$

Because $\rho_{M+m}(B_f) = \rho_{M+m}(B_i), \mathcal{W}_{M,m}(B_f) = \mathcal{W}_{M,m}(B_i)$ and $P_{E_{M+m}(B_f),B_i}(t) = P_{E_{M+m}(B_i),B_i}(t)$ for any $m \geq m(B_f, B_i)$, we see that

$$\frac{\rho_{M+m+1}(B_i)}{\rho_{M+m}(B_f)} \dot{P}_{E_{M+m}(B_f),B_i}(t) - \dot{P}_{E_{M+m+1}(B_i),B_i}(t)$$
$$= \mathcal{W}_{M,m+1}(B_i) \left(\frac{\rho_{M+m+1}(B_i)}{\rho_{M+m}(B_f)} P_{E_{M+m}(B_f),B_i}(t) - P_{E_{M+m+1}(B_i),B_i}(t) \right).$$

The initial condition is given as $P_{E_{M+m+1}(B_f),B_i}(0) = 1$ and $P_{E_{M+m}(B_f),B_i}(0) = 0$. Therefore, we can deduce from

$$\frac{\rho_{M+m+1}(B_i)}{\rho_{M+m}(B_f)} P_{E_{M+m}(B_f),B_i}(t) - P_{E_{M+m+1}(B_i),B_i}(t) = -\mathrm{e}^{t\mathcal{W}_{M,m+1}(B_i)}$$

and

$$P_{E_{M+m(B_f,B_i)}(B_f),B_i}(t) = \rho_{M+m(B_f,B_i)}(B_i) \Bigg\{ \rho_{K_2}^{-1} P_{K_2}(t) + \sum_{k=0}^{\infty} \left(\frac{1}{\rho_{M+m(B_f,B_i)+k}(B_i)} - \frac{1}{\rho_{M+m(B_f,B_i)+k+1}(B_i)} \right) \mathrm{e}^{t\mathcal{W}_{M,m(B_f,B_i)+k+1}(B_i)} \Bigg\}$$

that

$$P_{E_{M+m(B_f,B_i)-1}(B_f),B_i}(t) = \rho_{M+m(B_f,B_i)-1}(B_i) \Bigg\{ \rho_{K_2}^{-1} P_{K_2}(t) + \sum_{k=0}^{\infty} \left(\frac{1}{\rho_{M+m(B_f,B_i)+k}(B_i)} - \frac{1}{\rho_{M+m(B_f,B_i)+k+1}(B_i)} \right) \mathrm{e}^{t\mathcal{W}_{M,m(B_f,B_i)+k+1}(B_i)} - \frac{1}{\rho_{M+m(B_f,B_i)}(B_i)} \mathrm{e}^{t\mathcal{W}_{M,m(B_f,B_i)}(B_i)} \Bigg\}.$$

For any non-negative integer m with $m \leq m(B_f, B_i) - 2$, we see that

$$\frac{\rho_{M+m+1}(B_i)}{\rho_{M+m}(B_f)} \dot{P}_{E_{M+m}(B_f),B_i}(t) - \dot{P}_{E_{M+m+1}(B_f),B_i}(t)$$
$$= \left(\frac{\rho_{M+m+1}(B_i)}{\rho_{M+m}(B_f)} P_{E_{M+m}(B_f),B_i}(0) - P_{E_{M+m+1}(B_f),B_i}(0) \right) \mathrm{e}^{t\mathcal{W}_{M,m+1}(B_i)}$$

and that the initial condition is given by $P_{E_{M+m+1}(B_f),B_i}(0) = P_{E_{M+m}(B_f),B_i}(0) = 0$.

Therefore, $P_{E_{M+m}(B_f),B_i}(t) = \frac{\rho_{M+m}(B_f)}{\rho_{M+m+1}(B_i)} P_{E_{M+m+1}(B_f),B_i}(t)$.

As a result, we obtain the following expression for the transition probability:

$$P_{E_{M+m}(B_f),B_i}(t) = \rho_{M+m}(B_f) \Bigg\{ \rho_{K_2}^{-1} P_{K_2}(t) + \sum_{k=0}^{\infty} \left(\frac{1}{\rho_{M+m(B_f,B_i)+k}(B_i)} - \frac{1}{\rho_{M+m(B_f,B_i)+k+1}(B_i)} \right) \times \mathrm{e}^{t\mathcal{W}_{M,m(B_f,B_i)+k+1}(B_i)} - \frac{1}{\rho_{M+m(B_f,B_i)}(B_i)} \mathrm{e}^{t\mathcal{W}_{M,m(B_f,B_i)}(B_i)} \Bigg\}. \quad \square$$

3. On transition probabilities with no rotational invariance

In this section, we will obtain the existence of a Markov process $\{X(t)\}$ on K_2 which is mapped to a Markov process on K_1 by T. In deriving an explicit description of the transition probabilities of $\{X(t)\}$, we will need to add an extra procedure to the existing methods in [1,7] as shown in the proof of the following theorem.

Theorem 1. *If $\sum_{k=0}^{\infty}(u(M+k)-u(M+k+1))\rho_{M+k}(B_i)<\infty$ for any i, then there exists a Markov process $\{X(t)\}$ on K_2 such that $\{T(X(t))\}$ is a Markov process on K_1.*

Proof. From Kolmogorov's forward equation in Section 2 we get

$$\dot{P}_{B_f,B_i}(t)=-\tilde{a}(B_f)P_{B_f,B_i}(t)+\sum_{j\neq f}\tilde{u}(B_f,B_j)P_{B_j,B_i}(t),$$

where $\tilde{u}(B_f,B_j)=u(M+m(B_j,B_f))\rho(B_f)$ and $\tilde{a}(B_f)=\sum_{j\neq f}\tilde{u}(B_j,B_f)$. We can derive

$$\begin{aligned}
\dot{P}_{B_f,B_i}(t) &= -\Bigg\{u(M)(\rho_M(B_f)-\rho(B_f)) \\
&\quad+\sum_{k=1}^{\infty}u(M+k)(\rho_{M+k}(B_f)-\rho_{M+k-1}(B_f))\Bigg\}P_{B_f,B_i}(t) \\
&\quad+\rho_{B_f}\Bigg\{u(M)(P_{E_M(B_f),B_i}-P_{B_f,B_i}(t)) \\
&\quad+\sum_{k=1}^{\infty}u(M+k)(P_{E_{M+k}(B_f),B_i}(t)-P_{E_{M+k-1}(B_f),B_i}(t))\Bigg\} \\
&= -\Bigg\{u(M)\rho_M(B_f)+\sum_{k=1}^{\infty}u(M+k)(\rho_{M+k}(B_f)-\rho_{M+k-1}(B_f))\Bigg\}P_{B_f,B_i}(t) \\
&\quad+\rho_{B_f}\Bigg\{u(M)P_{E_M(B_f),B_i}(t)+\sum_{k=1}^{\infty}u(M+k)(P_{E_{M+k}(B_f),B_i}(t) \\
&\quad-P_{E_{M+k-1}(B_f),B_i}(t))\Bigg\} \\
&= -\Bigg\{u(M)\rho_M(B_f)-u(M+1)\rho_M(B_f) \\
&\quad+\sum_{k=1}^{\infty}(u(M+k)-u(M+k+1))\rho_{M+k}(B_f)\Bigg\} \\
&\quad\times P_{B_f,B_i}(t)+\rho_{B_f}\Bigg\{u(M)P_{E_M(B_f),B_i}(t)-u(M+1)P_{E_M(B_f),B_i}(t) \\
&\quad+\sum_{k=1}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t)\Bigg\}
\end{aligned}$$

$$= -\sum_{k=0}^{\infty}(u(M+k)-u(M+k+1))\rho_{M+k}(B_f)P_{B_f,B_i}(t)$$
$$+\rho_{B_f}\sum_{k=0}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t).$$

Due to a basic formula in the theory of ordinary differential equation, by recalling the definition of $\mathcal{W}_{M,m}(B_f)$ given by (1), we can explicitly find $P_{B_f,B_i}(t)$ satisfying

$$\dot{P}_{B_f,B_i}(t) = \mathcal{W}_{M,0}(B_f)P_{B_f,B_i}(t) + \rho_{B_f}\sum_{k=0}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t)$$

as

$$P_{B_f,B_i}(t) = \mathrm{e}^{\int_0^t \mathcal{W}_{M,0}(B_f)\mathrm{d}s}\left\{P_{B_f}(0) + \rho_{B_f}\int_0^t \mathrm{e}^{-\int_0^t \mathcal{W}_{M,0}(B_f)\mathrm{d}s}\right.$$
$$\left.\times\sum_{k=0}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t)\mathrm{d}t\right\}.$$

Therefore, we have

$$P_{B_f,B_i}(t) = \mathrm{e}^{t\mathcal{W}_{M,0}(B_f)}\left\{P_{B_f,B_i}(0) + \rho_{B_f}\int_0^t \mathrm{e}^{-t\mathcal{W}_{M,0}(B_f)}\right.$$
$$\left.\times\sum_{k=0}^{\infty}(u(M+k)-u(M+k+1))P_{E_{M+k}(B_f),B_i}(t)\mathrm{d}t\right\}.$$

By combining this expression with the explicit expression of $P_{E_{M+k}(B_f),B_i}(t)$ in Proposition 2, an explicit expression of transition probability of $\{X(t)\}$ has been obtained. The transition probabilities of $\{T(X_t)\}$ are completely ruled by Kolmogorov's forward equation described in Proposition 1. Accordingly, $\{T(X_t)\}$ is a Markov process on K_1. □

4. The lift of Markov process

Definition. A Markov process $\{X(t)\}$ on K_2 is said to be a lift of Markov process $\{Y(t)\}$ on K_1, if $P_x(Y(t) = T(X(t))$ for all $t \geq 0) = 1$ is satisfied for all x in K_1.

Theorem 1 gives an explicit expression of the transition probabilities of a lift of Markov process which is determined by Kolmogorov's forward equation on the smaller field K_1 described in Proposition 1. Finally, we present an example of $E_M(B_f)$ which consists of infinite balls.

Example. A quadratic extension $\boldsymbol{Q_p}(\sqrt{\epsilon})$ of $\boldsymbol{Q_p}$ is obtained by choosing p, η or $p\eta$ as ϵ when $p = 2$ and $-1, \pm 2, \pm 3,$ or ± 6 as ϵ when $p \neq 2$. For $b = \sum_{j=-m}^{\infty}(\alpha_{-M+j}+\beta_{-M+j}\sqrt{\epsilon})p^{-M+j} \in \boldsymbol{Q_p}(\sqrt{\epsilon})$ with digits $\alpha_{-M+j}, \beta_{-M+j} = 0, 1, \ldots$ or $p-1$, the minimal polynomial of the element b over $\boldsymbol{Q_p}$ is given as $f(x) = (x-\sum_{j=-m}^{\infty}(\alpha_{-M+j}+\beta_{-M+j}\sqrt{\epsilon})p^{-M+j})(x-\sum_{j=-m}^{\infty}(\alpha_{-M+j}-\beta_{-M+j}\sqrt{\epsilon})p^{-M+j})$. Therefore, $T(b) = \mathrm{Tr}_{\boldsymbol{Q_p}(\sqrt{\epsilon})/\boldsymbol{Q_p}}(b)/2 = \sum_{j=-m}^{\infty}\alpha_{-M+j}p^{-M+j}$ and this shows that $E_M(B_f) = \cup_{m=0}^{\infty}\cup_{\beta_{-M-1}=1}^{p-1}\cdots\cup_{\beta_{-M-m}=1}^{p-1}B(\sum_{j=0}^{\infty}(\alpha_{-M+j}+\beta_{-M+j}\sqrt{\epsilon})p^{-M+j} + \sum_{j=1}^{m}\beta_{-M-j}\sqrt{\epsilon}p^{-M-j}, p^M)$ for $B_f = B(b, p^M)$. By taking $\boldsymbol{Q_p}$ as K_1 and $\boldsymbol{Q_p}(\sqrt{\epsilon})$ as K_2, we see an example of $E_M(B_f)$ which consists of infinitely many balls in K_2.

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