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The speed of convergence of the Threshold estimator of integrated variance

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Abstract

In this paper we consider a semimartingale model for the evolution of the price of a financial asset, driven by a Brownian motion (plus drift) and possibly infinite activity jumps. Given discrete observations, the Threshold estimator is able to separate the integrated variance IV from the sum of the squared jumps. This has importance in measuring and forecasting the asset risks. In this paper we provide the exact speed of convergence of $\hat{IV}_h$, a result which was known in the literature only in the case of jumps with finite variation. This has practical relevance since many models used have jumps of infinite variation (see e.g. Carr et al. (2002) [4]).

Keywords: Integrated variance; Threshold estimator; Convergence speed; Infinite activity stable Lévy jumps

1. Definitions and notation

We consider a semimartingale $(X_t)_{t\in[0,T]}$, defined on a (filtered) probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}, P)$ with paths in $D([0,T],\mathbb{R})$, the space of càdlàg functions, driven by a (standard) Brownian motion W and a pure jump Lévy process L:

$$X_t = x_0 + \int_0^t a_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s + L_t, \quad t \in]0,T], \tag{1}$$

where a, σ are any adapted càdlàg processes such that (1) admits a unique strong solution X on $[0,T]$ which is adapted and càdlàg [7]. L has Lévy measure ν and may be decomposed as

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$L_t = J_t + M_t$, where

$$J_t := \int_0^t \int_{|x|>1} x\mu(\mathrm{d}x\,\mathrm{d}s) = \sum_{\ell=1}^{N_t} \gamma_\ell, \qquad M_t := \int_0^t \int_{|x|\le 1} x[\mu(\mathrm{d}x\,\mathrm{d}s) - \nu(\mathrm{d}x)\mathrm{d}t]. \tag{2}$$

J is a compound Poisson process representing the "large" jumps of L (and X), i.e. with absolute value larger than 1, μ is a Poisson random measure on $[0, T] \times \mathbb{R}$ with intensity measure $\nu(\mathrm{d}x)\mathrm{d}t$, N is a Poisson process with intensity $\nu(\{x, |x| > 1\}) < \infty$, γ_ℓ are IID and independent of N and the martingale M is the compensated sum of small jumps of L. We will define as $\mu(\mathrm{d}x, \mathrm{d}t) - \nu(\mathrm{d}x)\mathrm{d}t =: \tilde{\mu}(\mathrm{d}x, \mathrm{d}t)$ the compensated Poisson random measure associated with μ. We allow for *infinite activity* (IA) jumps, where small jumps of L occur infinitely often, i.e. $\nu(\mathbb{R}) = \infty$. This work contributes to the existing literature [5,8] precisely in the case where the jumps have also infinite variation.

The *Blumenthal–Getoor (BG) index* of L, defined as

$$\alpha := \inf\left\{\delta \ge 0, \int_{|x|\le 1} |x|^\delta \nu(\mathrm{d}x) < +\infty\right\} \le 2,$$

measures the degree of *activity* of the small jumps.

We will work under the following assumption, which allows us to control the behavior of the small jumps (like in Lemma 2 in [2]).

Assumption A1. L is symmetric α-stable.

A1 means that (see [6]) ν has a density of the form $\frac{A}{|x|^{1+\alpha}}$, for some constants $A \in \mathbb{R}, \alpha \in]0, 2[$. α is the BG index of L. Note that L has finite variation (fV) if and only if $\alpha \in]0, 1[$.

Remark 1.1. **A1** implies that

$$\int_{|x|\le c\sqrt{r_h}} x^k \nu(\mathrm{d}x) \sim r_h^{\frac{k-\alpha}{2}}, \quad k = 2, 3, 4$$
$$\int_{2\sqrt{r_h}<|x|\le 1} x\nu(\mathrm{d}x) \sim \left[c + c r_h^{\frac{1-\alpha}{2}}\right] I_{\{\alpha\ne 1\}} + c\left[\ln \frac{1}{2\sqrt{r_h}}\right] I_{\{\alpha=1\}}$$
$$\int_{2\sqrt{r_h}<|x|\le 1} \nu(\mathrm{d}x) \sim r_h^{-\alpha/2},$$

where c indicates a generic constant and $f(h) \sim g(h)$ means that both $f(h) = O(g(h))$ and $g(h) = O(f(h))$ as $h \to 0$.

Notation. We denote by $IV = \int_0^T \sigma_u^2 \mathrm{d}u$ the *integrated variance* of X and write

$$X_{0t} = \int_0^t a_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s, \qquad X_{1t} = X_{0t} + J_t.$$

For a semimartingale Z we denote its increments by $\Delta_i Z = Z_{t_i} - Z_{t_{i-1}}$ and its (eventual) jump at time t by $\Delta Z_t = Z_t - Z_{t-}$. $f(\omega, h) \overset{P}{\sim} g(\omega, h)$ means that $f(\omega, h) = O_P(g(\omega, h))$ and $g(\omega, h) = O_P(f(\omega, h))$ as $h \to 0$.

We observe X_t on a time grid $t_i = ih$, for a given resolution $h = T/n$, $i = 1, \ldots, n$. Since X is a semimartingale, the *realized variance* $RV_h = \sum_{i=1}^n (\Delta_i X)^2$ converges in probability (see e.g. [13], Theorem 22, p. 266) to

$$[X]_T := \int_0^T \sigma_t^2 \mathrm{d}t + \int_0^T \int_{\mathbb{R}-\{0\}} x^2 \mu(\mathrm{d}x, \mathrm{d}s).$$

The *Threshold estimator* [10] of IV is based on the idea of summing only some of the squared increments of X, those whose absolute value is smaller than some *threshold* r_h:

$$\hat{IV}_h := \sum_{i=1}^{n} (\Delta_i X)^2 I_{\{(\Delta_i X)^2 \le r_h\}}. \tag{3}$$

The term $\int_0^T \int_{\mathbb{R}-\{0\}} x^2 \mu(\mathrm{d}x, \mathrm{d}s)$ due to jumps is asymptotically eliminated as $h \to 0$ by an appropriate choice of r_h, which is possible in the light of the following consequence of the Paul Lévy law for the modulus of continuity of the Brownian motion paths [14, p. 10]:

$$P\left(\lim_{h\to 0} \sup_{i\in\{1..n\}} \frac{|\Delta_i W|}{\sqrt{2h \ln \frac{1}{h}}} \le 1\right) = 1.$$

It is shown in [10, Corollary 2 and Theorem 4] that, in the framework described and in particular under **A1**, if r_h is a deterministic function of h such that

$$\lim_{h\to 0} r_h = 0 \quad \text{and} \quad \lim_{h\to 0} \frac{h \ln h}{r_h} = 0 \tag{4}$$

then $\hat{IV}_h \xrightarrow{P} IV$, as $h \to 0$.

Note that the functions $r_h = ch^\beta$ satisfy condition (4) for any $\beta \in]0, 1[$ and any constant c.

Assessment of the speed of convergence of $\hat{IV}_h$ is important from a practical point of view because in finite samples, i.e. for fixed finite n, a theoretically faster estimator, at least in principle, is expected to be closer to the true IV. This is the case for instance for the Threshold estimator versus the bipower variation of [3] (see a finite sample performance comparison in [10]). As the risk induced by W is modulated by the σ amplitude, a better estimate of $\hat{IV}$ gives more precise information on the risks assumed when buying asset X.

In particular, efficient estimators are in general desirable. In [5,8] it has been shown that $\hat{IV}_h$ is efficient when the jumps have fV, provided $r_h = ch^\beta$ with β sufficiently close to 1, the speed of $\hat{IV}_h - IV$ being $\sqrt{2h\, IQ}$, where $IQ := \int_0^T \sigma_s^4 \mathrm{d}s$ is the *integrated quarticity* of X. [8] proves it when σ and L are Ito semimartingales and L has constant jump index α, while [5] proves it for any càdlàg σ and α-stable L. However in [5] it is also shown that when J has infinite variation (iV) then $\hat{IV}_h$ is not efficient, the efficiency rate still being $\sqrt{h}$ [1]. In [8] it is shown that for $r_h = ch^\beta$, $c \in \mathbb{R}$, $\beta \in]0, 1[$, then for any $\delta > \alpha$ the speed of $\hat{IV}_h - IV$ is higher than $r_h^{1-\delta/2}$, while in [5] it is shown that it is lower than $\sqrt{h}$. It is just by virtue of the different speeds under fV jumps or iV jumps that it was possible in [5] to construct the two tests for $\alpha < 1$ versus $\alpha \ge 1$ and for $\sigma \equiv 0$ versus $\sigma \not\equiv 0$. Note that under **A1** we have $\int_{|x|\le 1} |x|^\alpha \nu(\mathrm{d}x) = +\infty$.

In this paper we show that $\hat{IV}_h - IV$ has speed equal to $r_h^{1-\alpha/2}$. The result is stated and proved in Section 2, which also illustrates some consequences and some issues in considering a bivariate framework, and concludes the paper.

2. Speed of convergence of $\hat{IV}_h$

Theorem 2.1. *Take* $r_h = ch^\beta$, $\beta \in]0, 1[$, $c \in \mathbb{R}$. *Under* **A1**, *as* $h \to 0$,

$$\hat{IV}_h - IV \overset{P}{\sim} \sqrt{h} Z_h + r_h^{1-\alpha/2}, \tag{5}$$

where $Z_h \overset{st}{\to} \mathcal{N}$, *and* $\mathcal{N}$ *denotes a standard normal random variable.*

Remark. The first term in the right hand side of (5) is due to the presence of a Brownian component within X, while the last term is led by the sum of the jumps of X smaller in absolute value than $\sqrt{r_h}$.

Proof of the Theorem. Since $X = X_1 + M$, we use the decomposition

$$\begin{aligned}\hat{IV}_h - IV &= \sum_{i=1}^n (\Delta_i X)^2 I_{\{(\Delta_i X)^2 \le r_h\}} - IV \\ &= \left[\sum_{i=1}^n (\Delta_i X_1)^2 I_{\{(\Delta_i X_1)^2 \le 4r_h\}} - IV\right] \\ &\quad + \sum_{i=1}^n (\Delta_i X_1)^2 \left(I_{\{(\Delta_i X)^2 \le r_h\}} - I_{\{(\Delta_i X_1)^2 \le 4r_h\}}\right) \\ &\quad + 2\sum_{i=1}^n \Delta_i X_1 \Delta_i M I_{\{(\Delta_i X)^2 \le r_h\}} + \sum_{i=1}^n (\Delta_i M)^2 I_{\{(\Delta_i X)^2 \le r_h\}} := \sum_{j=1}^4 I_j(h).\end{aligned} \tag{6}$$

Inspection of the proof of Theorem 2 in [10] shows that $I_1(h)/\sqrt{h}$ converges stably in law to a mixed normal random variable, implying stable convergence of $I_1(h)/\sqrt{2h\, IQ}$ to a standard Gaussian r.v.

We now show that $I_2(h) = o_P(\sqrt{h}) + o_P(r_h^{1-\alpha/2})$. In [5] (Proof of Theorem 2.5) it is shown that $I_2(h)/\sqrt{h}$ has the same limit in probability as

$$\frac{\sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u\right)^2 I_{\{(\Delta_i X)^2 > r_h, (\Delta_i X_1)^2 \le 4r_h\}}}{\sqrt{h}}.$$

Note that this last term equals

$$\frac{1}{\sqrt{h}} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u\right)^2 I_{\{(\Delta_i X)^2 > r_h\}} - \frac{1}{\sqrt{h}} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u\right)^2 I_{\{(\Delta_i X)^2 > r_h, (\Delta_i X_1)^2 > 4r_h\}}.$$

However the last term is negligible since if $(\Delta_i X_1)^2 > 4r_h$ then (by (18) in [5]: for any fixed $c > 0$ a.s. for sufficiently small h we have $\sup_{i=1..n} |\Delta_i X_0| < c\sqrt{r_h}$) $\Delta_i N \neq 0$ and thus, by assuming wlog σ bounded on $\Omega \times [0, T]$ (through localization, similarly to in [8], Lemma 4.6) and because W and N are independent [7], we have

$$\begin{aligned}&\frac{1}{\sqrt{h}} E\left[\sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u\right)^2 I_{\{(\Delta_i X)^2 > r_h, (\Delta_i X_1)^2 > 4r_h\}}\right] \\ &\quad \le \frac{1}{\sqrt{h}} E\left[\sum_{i=1}^{N_T} \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u\right)^2\right] \le cE[N_T]\sqrt{h} \to 0,\end{aligned}$$

as $h \to 0$. So now we deal with $\sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u\right)^2 I_{\{(\Delta_i X)^2 > r_h\}}$. If $|\Delta_i J + \Delta_i M| + |\Delta_i X_0| > |\Delta_i X| > \sqrt{r_h}$ then $|\Delta_i J + \Delta_i M| > \sqrt{r_h} - |\Delta_i X_0|$ and, using (7) in [5] and the boundedness of σ, then for sufficiently small h, $\sqrt{r_h} - |\Delta_i X_0| > c\sqrt{r_h}$, c a constant less than 1, and then either $\Delta_i J \neq 0$ or $\Delta_i M > c\sqrt{r_h}/2$. However

$$\frac{\sum_i \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u\right)^2 I_{\{|\Delta_i J| \neq 0\}}}{r_h^{1-\alpha/2}} \leq c \frac{N_T h \ln \frac{1}{h}}{r_h^{1-\alpha/2}} \to 0,$$

and, by (20) in [5] and because σ is bounded and W and M are independent, we have

$$\frac{E\left[\sum_i \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u\right)^2 I_{\{|\Delta_i M| > c\sqrt{r_h}/2\}}\right]}{r_h^{1-\alpha/2}} \leq h \frac{\sum_i P\{|\Delta_i M| > c\sqrt{r_h}/2\}}{r_h^{1-\alpha/2}} \sim \frac{h^{1-\alpha\beta/2}}{r_h^{1-\alpha/2}} \to 0.$$

Then our result on $I_2(h)$ behavior is reached.

In [5] (Proof of Theorem 2.5) it is shown that $I_3(h)/\sqrt{h} \stackrel{P}{\to} 0$.

We now show that $I_4(h)$ has the same asymptotic behavior as $r_h^{1-\alpha/2}$. Fix any $q > 1$ and define

$$\tilde{N}_s = \sum_{u \leq s} I_{\{|\Delta X_u| > \frac{\sqrt{r_h}}{q}\}},$$

$$\xi_{ni} := \left(\int_{t_{i-1}}^{t_i} \int_{|x| \leq \frac{\sqrt{r_h}}{q}} x \tilde{\mu}(\mathrm{d}x, \mathrm{d}t) - h \int_{\frac{\sqrt{r_h}}{q} < |x| \leq 1} x \nu(\mathrm{d}x)\right)^2.$$

We can write

$$\begin{aligned} I_4(h) &= \sum_{i=1}^n (\Delta_i M)^2 I_{\{(\Delta_i X)^2 \leq r_h\}} \\ &= \sum_{i=1}^n (\Delta_i M)^2 \left[I_{\{\Delta_i \tilde{N} = 0\}} - I_{\{\Delta_i \tilde{N} = 0, (\Delta_i X)^2 > r_h\}} + I_{\{\Delta_i \tilde{N} \geq 1, (\Delta_i X)^2 \leq r_h\}}\right]. \end{aligned}$$

On $\{\Delta_i \tilde{N} = 0\}$ the squared increment $(\Delta_i M)^2$ equals ξ_{ni}, so we can write the rhs term above as

$$\begin{aligned} &\sum_i \xi_{ni} - \sum_i \xi_{ni} I_{\{\Delta_i \tilde{N} \geq 1\}} - \sum_i \xi_{ni} I_{\{\Delta_i \tilde{N} = 0, (\Delta_i X)^2 > r_h\}} + \sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N} \geq 1, (\Delta_i X)^2 \leq r_h\}} \\ &\doteq \sum_{k=1}^4 I_{4,k}(h). \end{aligned}$$

We are now going to show that

$$I_{4,1}(h) = \sum_i \xi_{ni}$$

is the leading term of $I_4(h)$ and that it has the same asymptotic behavior as

$$n E[\xi_{n1}] \sim r_h^{1-\alpha/2}.$$

In fact Theorem 2.4 in [5] states the following CLT:

$$\frac{\sum_i \xi_{ni} - n E[\xi_{n1}]}{\sqrt{n \,\mathrm{Var}[\xi_{n1}]}} \stackrel{d}{\to} \mathcal{N}.$$

Since $nE[\xi_{n1}] \sim r_h^{1-\alpha/2} + h\left(1 - r_h^{\frac{1-\alpha}{2}}\right)^2 I_{\{\alpha\neq 1\}} + h\ln^2\frac{1}{\sqrt{r_h}} I_{\{\alpha=1\}} \sim r_h^{1-\alpha/2} \to 0$, and $\sqrt{nVar[\xi_{n1}]} \sim r_h^{1-\alpha/4} \to 0$ we reach that $I_{4,1}(h) = \sum_i \xi_{ni}$ tends to zero in probability at speed $r_h^{1-\alpha/2}$.

We now show that $I_{4,2}(h) = -\sum_i \xi_{ni} I_{\{\Delta_i \tilde{N} \geq 1\}}$ is negligible wrt $r_h^{1-\alpha/2}$. In fact, by the independence of ξ_{ni} on $\{\Delta_i \tilde{N} \geq 1\} = \left\{\mu\left(\{|x| > \sqrt{r_h}/q\} \times]t_{i-1}, t_i]\right) \geq 1\right\}$, we have

$$\frac{E\left[\left|\sum_i \xi_{ni} I_{\{\Delta_i \tilde{N} \geq 1\}}\right|\right]}{r_h^{1-\alpha/2}} \leq \frac{nE[|\xi_{ni}|]P\{\Delta_i \tilde{N} \geq 1\}}{r_h^{1-\alpha/2}} \leq c\frac{n\theta\sqrt{E[\xi_{ni}^2]}}{r_h^{1-\alpha/2}}$$

$$\sim \frac{n\theta\sqrt{hr_h^{2-\alpha/2}}}{r_h^{1-\alpha/2}} = (hr_h^{-\alpha/2})^{1/2} \to 0,$$

where $\theta = h^{1-\alpha\beta/2} = hr_h^{-\alpha/2}$.

Now we prove that also $I_{4,3}(h) = -\sum_i \xi_{ni} I_{\{\Delta_i \tilde{N}=0, (\Delta_i X)^2 > r_h\}}$ is negligible wrt $r_h^{1-\alpha/2}$.

First we take $\eta > 0$: $\gamma \doteq 1/q + \eta < 1$. Now on $\{\Delta_i \tilde{N} = 0, (\Delta_i X)^2 > r_h\}$ we necessarily have that $|\Delta_i M| > \sqrt{r_h}/q$, because otherwise, for sufficiently small h we would have $|\Delta_i X_0 + \Delta_i M| < \eta\sqrt{r_h} + \sqrt{r_h}/q = \sqrt{r_h}\gamma$ and

$$|\Delta_i J| = |\Delta_i X - \Delta_i M - \Delta_i X_0| \geq |\Delta_i X| - |\Delta_i M + \Delta_i X_0| > \sqrt{r_h}(1-\gamma) > 0,$$

implying that $|\Delta_i J| \geq 1$, which is impossible since J only moves by jumps bigger than 1, while $\Delta_i \tilde{N} = 0$ indicates that no jumps bigger than $\sqrt{r_h}/q < 1$ happened. Second, note that on the set where X has no jumps bigger than $\sqrt{r_h}/q$, the same is true for M and for L, and $P\{\Delta_i \tilde{N} = 0, (\Delta_i X)^2 > r_h\} \leq P\{\Delta_i \tilde{N} = 0, |\Delta_i M| > \sqrt{r_h}/q\} = P\{\tilde{N}_h = 0, |M_h| > \sqrt{r_h}/q\}$, by the Lévy property of M, and this equals $P\{\tilde{N}_h = 0, |L'_h| > \sqrt{r_h}/q\}$, where L' is the L process deprived of its jumps bigger in absolute value than $\sqrt{r_h}/q$. In fact $M_0 = L'_0 = 0$ and M, L' have same compensation and differ only by jumps, but on the given set they made no jumps bigger than $\sqrt{r_h}/q$, so they made the same jumps and $M_h = L'_h$. Moreover the last probability is dominated by $P\{|L'_h| > \sqrt{r_h}/q\} \sim \theta^{4/3}$, by [2], the end of the proof of Lemma 2 (with β there in place of α here, $\delta/2$ there in place of $\sqrt{r_h}/q$ here, Y there in place of L here, Y'' there in place of L' here[1]). We then reach that $P\{\Delta_i \tilde{N} = 0, (\Delta_i X)^2 > r_h\} \leq c\theta^{4/3}$, and thus

$$\frac{E[|I_{4,3}(h)|]}{r_h^{1-\alpha/2}} \leq \frac{E\left[\sum_i |\xi_{ni}| I_{\{\Delta_i \tilde{N}=0, (\Delta_i X)^2 > r_h\}}\right]}{r_h^{1-\alpha/2}}$$

$$\leq \frac{\sum_i \sqrt{E[\xi_{ni}^2]}\sqrt{P\{\Delta_i \tilde{N} = 0, (\Delta_i X)^2 > r_h\}}}{r_h^{1-\alpha/2}} \leq c\frac{n\sqrt{hr_h^{2-\alpha/2}}\theta^{2/3}}{r_h^{1-\alpha/2}} \to 0.$$

[1] Within the last part of the proof of Lemma 2 in [2] we noticed a minor misprint which, however, is corrected by simply replacing $D' = \{|Y'| > \delta/2\}$ with $\tilde{D}' = \{|Y''| > \delta/2\}$, and does not substantially affect the statement in the Lemma.

Finally we show that $I_{4,4} = \sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N} \geq 1, (\Delta_i X)^2 \leq r_h\}}$ is also negligible wrt $r_h^{1-\alpha/2}$. In fact we use the decomposition

$$\frac{\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N} \geq 1, (\Delta_i X)^2 \leq r_h\}}}{r_h^{1-\alpha/2}} = \frac{1}{r_h^{1-\alpha/2}} \sum_i (\Delta_i M)^2$$
$$\times \left[I_{\{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1, (\Delta_i X)^2 \leq r_h\}} + I_{\{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| > 1, (\Delta_i X)^2 \leq r_h\}} + I_{\{\Delta_i \tilde{N} \geq 2, (\Delta_i X)^2 \leq r_h\}} \right], \tag{7}$$

where $\bar{s}$ is the time instant of the unique jump of X bigger than $\sqrt{r_h}/q$ within $]t_{i-1}, t_i]$ when $\Delta_i \tilde{N} = 1$.

Let us deal with the first term above. As, for small h, $\sqrt{r_h}/q < 1$, on $\{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1\}$ within $]t_{i-1}, t_i]$ we only have jumps less than 1, so $\Delta_i J = 0$. Fix now any $p > 0$. If also $(\Delta_i X)^2 \leq r_h$ then for sufficiently small h we have $\sup_i |\Delta_i X_0| < p\sqrt{r_h}$ and $\sqrt{r_h} \geq |\Delta_i X| = |\Delta_i X_0 + \Delta_i M| > |\Delta_i M| - |\Delta_i X_0|$, so $|\Delta_i M| < \sqrt{r_h} + |\Delta_i X_0| \leq \sqrt{r_h}(1+p)$ uniformly in $i = 1..n$. Thus $\{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1, |\Delta_i X| \leq \sqrt{r_h}\} \subset \{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1, |\Delta_i M| \leq \sqrt{r_h}(1+p)\}$, and the probability of this last set equals $P\{\tilde{N}_h = 1, |\Delta M_{\bar{s}}| \leq 1, |M_h| \leq \sqrt{r_h}(1+p)\}$ by the Lévy property of M, and in turn this equals $P\{\tilde{N}_h = 1, |\Delta L_{\bar{s}}| \leq 1, |L_h| \leq \sqrt{r_h}(1+p)\}$, since $M_0 = L_0 = 0$ and M, L have same compensation and differ only by jumps, but on the given set they made only jumps smaller than 1 and so they made the same jumps. Moreover the last probability is dominated by

$$\begin{aligned} &P\{\tilde{N}_h = 1, |L_h| \leq \sqrt{r_h}(1+p), |\Delta L_{\bar{s}}| > \sqrt{r_h}(1+p)\} \\ &\quad + P\{\tilde{N}_h = 1, |L_h| \leq \sqrt{r_h}(1+p), |\Delta L_{\bar{s}}| \leq \sqrt{r_h}(1+p)\} \\ &\leq P\{\tilde{N}_h' = 1, |L_h| \leq \sqrt{r_h}(1+p)\} + P\{\tilde{N}_h'' = 1\}, \end{aligned} \tag{8}$$

where $\tilde{N}_h' \doteq \sum_{u \leq h} I_{\{|\Delta L_u| > \sqrt{r_h}(1+p)\}}$, and $\tilde{N}_h'' \doteq \sum_{u \leq h} I_{\{|\Delta L_u| \in]\sqrt{r_h}/q, \sqrt{r_h}(1+p)]\}}$. The first term of (8), by Lemma 2 in [2], is $O(\theta^{4/3})$. As for the second one we have

$$P\{\tilde{N}_h'' = 1\} = 2h \int_{\sqrt{r_h}/q}^{\sqrt{r_h}(1+p)} 1\nu(\mathrm{d}x) = \frac{2A}{\alpha}\theta[q^\alpha - (1+p)^{-\alpha}],$$

and so

$$\begin{aligned} &\frac{1}{r_h^{1-\alpha/2}} E\left[\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1, (\Delta_i X)^2 \leq r_h\}}\right] \\ &\leq \frac{r_h(1+p)^2}{r_h^{1-\alpha/2}} \left(nP\left\{\tilde{N}_h' = 1, |L_h| \leq \sqrt{r_h}(1+p)\right\} + nP\{\tilde{N}_h'' = 1\}\right) \\ &\leq r_h^{\alpha/2}\left(nc\theta^{4/3} + n\theta\frac{2A}{\alpha}[q^\alpha - (1+p)^{-\alpha}]\right) = o(1) + \frac{2A}{\alpha}[q^\alpha - (1+p)^{-\alpha}]. \end{aligned}$$

So we obtained that for any $q > 1$, $p > 0$, for sufficiently small h,

$$\frac{1}{r_h^{1-\alpha/2}} E\left[\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1, (\Delta_i X)^2 \leq r_h\}}\right] \leq \frac{2A}{\alpha}[q^\alpha - (1+p)^{-\alpha}].$$

Letting $q \to 1$ and $p \to 0$ we find that

$$\lim_{h \to 0} \frac{1}{r_h^{1-\alpha/2}} E\left[\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1, (\Delta_i X)^2 \leq r_h\}}\right] = 0.$$

Let us now deal with the second term within (7). If $\Delta_i \tilde{N} = 1$ and $|\Delta X_{\bar{s}}| > 1$ then $\Delta_i J \neq 0$. If also $|\Delta_i X| \leq \sqrt{r_h}$ then for sufficiently small h we have $|\Delta_i M| > \sqrt{r_h}$ uniformly on i. In fact $|\Delta_i J + \Delta_i M| - |\Delta_i X_0| < |\Delta_i X| \leq \sqrt{r_h}$, so $|\Delta_i J + \Delta_i M| < |\Delta_i X_0| + \sqrt{r_h}$, which, for any positive p, for sufficiently small h, is dominated by $\sqrt{r_h}(1+p)$. Moreover, since $|\Delta X_{\bar{s}}| = |\Delta_i J| > 1$ and within $]t_{i-1}, t_i]$ exactly one jump of J occurred, $|\Delta_i J| - |\Delta_i M| < |\Delta_i J + \Delta_i M| \leq \sqrt{r_h}(1+p)$ implies $|\Delta_i M| > |\Delta_i J| - \sqrt{r_h}(1+p) > 1 - \sqrt{r_h}(1+p) > \sqrt{r_h}$, for sufficiently small h, uniformly on i. As a consequence

$$P\left(\frac{1}{r_h^{1-\alpha/2}} \sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N}=1, |\Delta X_{\bar{s}}|>1, (\Delta_i X)^2 \leq r_h\}} \neq 0\right)$$
$$\leq n P(\Delta_i N \neq 0, |\Delta_i M| > \sqrt{r_h}) \to 0$$

by Lemma 6.1 (ii) in [5].

Finally, we consider the last term in (7). On $|\Delta_i X| \leq \sqrt{r_h}$ either we have $\Delta_i J = 0$, and consequently $|\Delta_i M| \leq \sqrt{r_h}(1+p)$, or we have $\Delta_i J \neq 0$, and then as before $|\Delta_i M| > \sqrt{r_h}$, as for sufficiently small h in $]t_{i-1}, t_i]$ at most one jump of J occurs. Therefore

$$\frac{1}{r_h^{1-\alpha/2}} \sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N} \geq 2, (\Delta_i X)^2 \leq r_h\}}$$
$$= \sum_i \frac{(\Delta_i M)^2}{r_h^{1-\alpha/2}} \left(I_{\{\Delta_i \tilde{N} \geq 2, (\Delta_i X)^2 \leq r_h, \Delta_i J = 0, |\Delta_i M| \leq (1+p)\sqrt{r_h}\}} + I_{\{\Delta_i \tilde{N} \geq 2, (\Delta_i X)^2 \leq r_h, \Delta_i J \neq 0, |\Delta_i M| > \sqrt{r_h}\}} \right).$$

The expectation of the first term is dominated by $\frac{r_h (1+p)^2}{r_h^{1-\alpha/2}} n\theta^2 \to 0$, with $P\{\Delta_i \tilde{N} \geq 2\} \leq c\theta^2$, while the probability that the second term differs from zero is dominated by $nP\{\Delta_i N \neq 0, |\Delta_i M| > \sqrt{r_h}\}$ which tends to zero similarly to before.

Therefore, $I_4 \stackrel{P}{\sim} r_h^{1-\alpha/2}$ is proved.

We can summarize as follows:

$$\hat{IV}_h - IV \stackrel{P}{\sim} \sqrt{h} Z_h + o_P(\sqrt{h}) + r_h^{1-\alpha/2} + o_P(r_h^{1-\alpha/2}),$$

where the first term in the right hand side comes from I_1 and is due to the presence of a Brownian component within X, while the third term is determined by I_4, which is led by $nE[\xi_{n1}]$, where in turn the main term is the sum of the jumps of X smaller in absolute value than $\sqrt{r_h}$, and our theorem is proved. □

Corollary 2.2. *Under* **A1** *we have*

$$\begin{cases} \text{if } \sigma \equiv 0 \text{ then} & \hat{IV}_h - IV \stackrel{P}{\sim} r_h^{1-\alpha/2} \\ \text{if } \sigma \not\equiv 0 \text{ and } \alpha < 1, \beta > \dfrac{1}{2-\alpha} \text{ then} & \dfrac{\hat{IV}_h - IV}{\sqrt{2h\, IQ}} \stackrel{st}{\to} \mathcal{N} \\ \text{if } \sigma \not\equiv 0 \text{ and } \alpha < 1, \beta \leq \dfrac{1}{2-\alpha} \text{ then} & \hat{IV}_h - IV \stackrel{P}{\sim} r_h^{1-\alpha/2} \\ \text{if } \sigma \not\equiv 0 \text{ and } \alpha \geq 1 \text{ then} & \hat{IV}_h - IV \stackrel{P}{\sim} r_h^{1-\alpha/2}. \end{cases} \tag{9}$$

Proof. If $\sigma \not\equiv 0$, note that as $h \to 0$

$$\frac{\sqrt{h}}{r_h^{1-\alpha/2}} = h^{\frac{1}{2} - \beta(1-\frac{\alpha}{2})} \to \begin{cases} 0 & \text{if } \alpha \geq 1 \\ +\infty & \text{if } \alpha < 1 \text{ and } \beta > \dfrac{1}{2-\alpha} \in \left]\dfrac{1}{2}, 1\right[\end{cases}$$

since the h exponent above is positive if and only if $\beta < \frac{1}{2-\alpha}$, which is always the case when $\alpha \geq 1$, since $\frac{1}{2-\alpha} \in]1, \infty[$, while the exponent is negative when $\alpha < 1$ and β is close to 1, since $\frac{1}{2-\alpha} \in]1/2, 1[$. Therefore if $\alpha \geq 1$ we have $\sqrt{h} Z_h + o_P(\sqrt{h}) = o_P(r_h^{1-\alpha/2})$ and

$$I\hat{V}_h - IV \overset{P}{\sim} r_h^{1-\alpha/2}.$$

If $\alpha < 1$ and β is close to 1 ($\beta > \frac{1}{2-\alpha}$) then $r_h^{1-\alpha/2} + o_P(r_h^{1-\alpha/2}) = o_P(\sqrt{h})$ and

$$I\hat{V}_h - IV \overset{P}{\sim} \sqrt{h} Z_h.$$

If $\alpha < 1$ and $\beta \leq \frac{1}{2-\alpha}$ then $\sqrt{h} = O(r_h^{1-\alpha/2})$ and

$$I\hat{V}_h - IV \overset{P}{\sim} r_h^{1-\alpha/2}.$$

We now consider the case of $\sigma \equiv 0$. Recall decomposition (6). We have $IV \equiv 0$ and that $I_1(h) = O_p(h)$. In fact

$$I_1(h) = \sum_i (\Delta_i X_1)^2 I_{\{(\Delta_i X_1)^2 \leq 4r_h\}} \tag{10}$$

and, assuming wlog that a is bounded on $\Omega \times [0, T]$, we have that for sufficiently small h, for all $i = 1..n$, $I_{\{(\Delta_i X_1)^2 \leq 4r_h\}} = I_{\{\Delta_i N = 0\}}$, since if $|\Delta_i J| - |\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u| < |\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u + \Delta_i J| = |\Delta_i X_1| \leq 2\sqrt{r_h}$ then $|\Delta_i J| \leq 2\sqrt{r_h} + |\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u| = O_P(\sqrt{r_h}) \to 0$ and then, for sufficiently small h, $\Delta_i J = 0$. If otherwise $|\Delta_i J| + |\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u| \geq |\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u + \Delta_i J| = |\Delta_i X_1| > 2\sqrt{r_h}$ then $|\Delta_i J| > 2\sqrt{r_h} - |\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u| > 0$ and $\Delta_i J \neq 0$. Therefore (10) equals

$$\sum_i (\Delta_i X_1)^2 I_{\{\Delta_i N = 0\}} = \sum_i \left(\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u \right)^2 (1 - I_{\{\Delta_i N \neq 0\}}) = O_P(h).$$

Now we show that $I_2(h) = o_P(h)$. In fact, like for I_2 in the proof of Theorem 2.5 in [5] on $\{(\Delta_i X_1)^2 > 4r_h, (\Delta_i X)^2 \leq r_h\}$ we have $\Delta_i N \neq 0$ and $|\Delta_i M| > \sqrt{r_h}$, so

$$P\left\{ \frac{1}{h} \sum_{i=1}^n (\Delta_i X_1)^2 I_{\{(\Delta_i X)^2 \leq r_h, (\Delta_i X_1)^2 > 4r_h\}} \neq 0 \right\} \leq n P\{\Delta_i N \neq 0, |\Delta_i M| > \sqrt{r_h}\} \to 0.$$

Moreover on $\{(\Delta_i X)^2 > r_h, (\Delta_i X_1)^2 \leq 4r_h\}$ we have $\Delta_i N = 0$, so $\Delta_i X = \int_{t_{i-1}}^{t_i} a_u \mathrm{d}u + \Delta_i M$, $\Delta_i X_1 = \int_{t_{i-1}}^{t_i} a_u \mathrm{d}u$ and $|\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u| + |\Delta_i M| > |\Delta_i X| > \sqrt{r_h}$ implying that, for sufficiently small h, $|\Delta_i M| > c\sqrt{r_h}$. Therefore

$$\frac{1}{h} \sum_{i=1}^n (\Delta_i X_1)^2 I_{\{(\Delta_i X)^2 > r_h, (\Delta_i X_1)^2 \leq 4r_h\}} \leq \frac{\sum_i (\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u)^2 I_{\{|\Delta_i M| > c\sqrt{r_h}\}}}{h} = O_P(\theta) \to_P 0.$$

We then have $I_4(h) \overset{P}{\sim} r_h^{1-\alpha/2}$, as in the proof of the previous theorem, as a fortiori, for sufficiently small h, $\sup_{i=1..n} |\Delta X_0| = \sup_{i=1..n} |\int_{t_{i-1}}^{t_i} a_s \mathrm{d}s| < \sqrt{r_h}$.

Finally we see that $I_3(h) = \sum_{i=1}^n \Delta_i X_1 \Delta_i M I_{\{(\Delta_i X)^2 \leq r_h, \Delta_i J \neq 0\}} + \sum_{i=1}^n \Delta_i X_1 \Delta_i M$ $I_{\{(\Delta_i X)^2 \leq r_h, \Delta_i J = 0\}} = o_P(r_h^{1-\alpha/2})$. In fact, like for I_3 in the proof of Theorem 2.5 in [5], we have that if $\Delta_i J \neq 0$ then $|\Delta_i X_1| > \sqrt{r_h}$ and if further $|\Delta_i X| \leq \sqrt{r_h}$ then $|\Delta_i M| > \sqrt{r_h}$ and then

$$P\left\{\frac{1}{r_h^{1-\alpha/2}}\sum_{i=1}^{n}\Delta_i X_1 \Delta_i M I_{\{(\Delta_i X)^2 \leq r_h, \Delta_i J \neq 0\}} \neq 0\right\} \leq nP\{\Delta_i J \neq 0, |\Delta_i M| > \sqrt{r_h}\} \to 0.$$

Moreover

$$\sum_{i=1}^{n}\Delta_i X_1 \Delta_i M I_{\{(\Delta_i X)^2 \leq r_h, \Delta_i J = 0\}} = \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u \; \Delta_i M I_{\{(\Delta_i X)^2 \leq r_h, \Delta_i J = 0\}}$$

$$\leq \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u \; \Delta_i M I_{\{(\Delta_i X)^2 \leq r_h\}} \leq \sqrt{\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u\right)^2}\sqrt{\sum_i (\Delta_i M)^2 I_{\{(\Delta_i X)^2 \leq r_h\}}}$$

$$= \sqrt{\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u\right)^2}\sqrt{I_4(h)}$$

and

$$\frac{1}{r_h^{1-\alpha/2}}\sqrt{\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i} a_u \mathrm{d}u\right)^2}\sqrt{I_4(h)} = O_P\left(\sqrt{\frac{h}{r_h^{1-\alpha/2}}}\right) \to 0.$$

Summarizing, when $\sigma \equiv 0$ we have

$$\hat{IV}_h - IV = \sum_{j=1}^{4} I_j(h) \overset{P}{\sim} h + o_P(h) + o_P(r_h^{1-\alpha/2}) + r_h^{1-\alpha/2} \sim r_h^{1-\alpha/2}$$

and the final behavior of the estimation error is determined by $I_4(h)$. □

Remarks. (i) When $\alpha < 1$ and $\beta > 1/(2-\alpha)$, result (9) is consistent with [5,8] where, under some different assumptions on X in the two cases, we find that in the presence of a Brownian part within X and for threshold exponent β sufficiently close to 1, $\hat{IV}_h - IV/\sqrt{2h\, IQ} \overset{st}{\to} \mathcal{N}$.

(ii) Result (9) is also consistent with [5,8] when $\alpha \geq 1$ and in the presence of a Brownian component within X. In fact in [5] we have that $\frac{\hat{IV}_h - IV}{\sqrt{h}} \overset{P}{\to} +\infty$ and in [8] we have that $\frac{\hat{IV}_h - IV}{r_h^{1-s/2}} \overset{P}{\to} 0$, for all exponents s such that $\int 1 \wedge |x|^s \nu(\mathrm{d}x) < \infty$, i.e. for all $s > \alpha$.

(iii) The new features here are giving the exact speed at which the estimation error $\hat{IV}_h - IV$ converges to zero when $\alpha \geq 1$, both in the presence and in the absence of a Brownian component, and when $\alpha < 1$ in the absence of it. Such a speed depends both on the jump activity index α of X and on the threshold exponent β.

(iv) In the bivariate case things are more complicated. Given two processes such that $\mathrm{d}X_t^{(q)} = a_t^{(q)}\mathrm{d}t + \sigma_t^{(q)}\mathrm{d}W_t^{(q)} + \mathrm{d}L_t^{(q)}$, $q = 1, 2$, for $t \in [0, T]$, where $W_t^{(2)} = \rho_t W_t^{(1)} + \sqrt{1-\rho_t^2}W_t^{(3)}$ with independent Brownian motions $W^{(1)}$ and $W^{(3)}$, the speed of convergence of the Threshold estimator $\hat{IC}_h = \sum_{j=1}^{n}\Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2 \leq r(h)\}}\Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2 \leq r(h)\}}$ to the integrated covariance $IC = \int_0^T \rho_t \sigma_t^{(1)}\sigma_t^{(2)}\mathrm{d}t$ turns out to have some features in common with the univariate case [11], but a complete framework has to separately account for some different cases. More precisely: in the presence of Brownian parts and $\alpha_1, \alpha_2 < 1$ we still have $\frac{\hat{IV}_h - IV}{\sqrt{2h\, IQ}} \overset{st}{\to} \mathcal{N}$ (see [12,9]).

Otherwise the speed still depends on the jumps of both $M^{(1)}$ and $M^{(2)}$ smaller in absolute value than the threshold, but now such a speed differs according to different relations among $\alpha_1, \alpha_2, \beta$ and to the magnitude of a further parameter γ measuring the degree of dependence among the jumps of the two components [11].

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