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The jamming constant of uniform random graphs

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Abstract

By constructing jointly a random graph and an associated exploration process, we define the dynamics of a "parking process" on a class of uniform random graphs as a measure-valued Markov process, representing the empirical degree distribution of non-explored nodes. We then establish a functional law of large numbers for this process as the number of vertices grows to infinity, allowing us to assess the jamming constant of the considered random graphs, i.e. the size of the maximal independent set discovered by the exploration algorithm. This technique, which can be applied to any uniform random graph with a given – possibly unbounded – degree distribution, can be seen as a generalization in the space of measures, of the differential equation method introduced by Wormald.

MSC: primary 60J25; 05C80; 60B12

Keywords: Random graph; Configuration model; Parking process; Measure-valued Markov process; Hydrodynamic limit

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1. Introduction

Consider a finite graph G for which $\mathcal{V}$ is the set of nodes or sites. The parking process in continuous time on G may be described as follows. At time 0, all sites are vacant. They all have independent exponential clocks. When the clock of a given vacant site rings and all of its neighbors are vacant, this site turns occupied. Otherwise, nothing happens. Once occupied, a site remains so for ever. The process goes on until all sites are either occupied or have at least one of their neighbors occupied. The final state of the process is often referred to as the jamming limit of G, and the final proportion of occupied sites, its jamming constant.

Our motivation to study the parking process on random graphs is twofold. On the one hand, these dynamics are the simplest procedure to discover maximal independent sets and have been extensively studied for some specific graphs. The differential equation method developed by Wormald provides explicit results for regular graphs [29], exploiting their very specific structure (see also [13] for graphs with large girths) and can be directly extended to graphs with bounded degree [31]. In the Erdös–Rényi case, a similar differential method can be employed thanks to the great amount of independence and symmetry of the collection of edges, to get an explicit jamming constant (see Theorem 2.2 and the references in [21]). Hence, to look at "uniform" random graphs having a given asymptotic degree distribution, but much less structure and symmetry, is a natural continuation of this research avenue.

On the other hand, parking processes have received a great amount of attention in the case of spatial structures. It has been considered on discrete structures like $\mathbb{Z}^d$ [23,11] and on point processes [22,11,2,26]. In physics and biological sciences, where it is usually referred to as random sequential absorption, it models phenomena of deposition of colloidal particles or proteins on surfaces (see [10]). In communication sciences and in wireless networks in particular, it allows to represent the number of connections for CSMA-like algorithms in a given time-slot, for a given spatial configuration of terminals (see [17] for a classical reference on the definition of the protocol). The general idea of CSMA is to schedule transmissions in such a way that nodes that interfere each other would not transmit simultaneously, see for instance [2] for a stochastic geometry-based model in which CSMA is approximated by a Matérn-like process. Unfortunately, spatial models are in general very difficult to study theoretically and to the best of our knowledge, there is no efficient way to compute the jamming constant in most cases. Studying the jamming constant of uniformly chosen random graphs with a given asymptotic degree distribution (we make this notion more precise in the sequel) allows to make a first step in this direction, by studying a "first order" model which grasps only the bonds between points but no further correlations. The techniques and analysis presented here are at the core of the performance evaluation analysis of wireless systems, as developed in a companion applied paper [19].

In this paper we focus on the parking process for a class of random graphs having given deterministic asymptotic degree distributions, and derive a computable characterization of the jamming constant, as the number of vertices grows to infinity. To describe the evolution of the parking process in a Markovian fashion, without keeping track of a too large set of information, our strategy is the following: we start from the degree distribution of the graph, and then construct simultaneously the random graph and the associated parking process. In both cases, the underlying (multi-)graph is constructed similarly to the configuration model, see [30,27,14] and the references therein. A similar approach was considered in [5], to construct a random social network together with a SIR process which propagates on it.

Following these ideas, we first define these dynamics for a graph having a fixed number of nodes n and study the time-evolution of the empirical measure of the degrees of the vacant sites, which defines a measure-valued Markov process. Then, under the assumption that the initial empirical measure of degrees converges to a measure having mild moment assumptions, we take n to infinity and prove a functional law of large numbers on the evolution of the empirical measures of degrees. We show in particular that given our assumptions on the initial random degree distribution, the limit is unique, and defined as the solution of a non-linear infinite-dimensional system of differential equations.

Our results can be seen as a generalization in the space of point measures, of the differential equation method introduced by Wormald [29]. The techniques of Wormald as developed in [29], or [31] cannot be directly generalized to the case of generic unbounded degree distribution as they rely on approximating the drift of the exploration processes by uniformly bounded, Lipschitz and finite dimensional functions [29] or on a bound on the maximum degree [31] which in both cases are too strong assumptions for the present context. More specifically, our strategy relies on generator approximations for measure-valued Markov processes on the one hand and finding an adequate norm on the space of solutions for proving the uniqueness of the infinite-dimensional system on the other hand. The approximation of the generator of the infinite-dimensional Markov process relies on quantifying the probability to obtain self-loops and multi-edges. Note that this difficulty is inherent to configuration model constructions, which have the disadvantage of constructing a multi-graph rather than a simple one, though elegant arguments have shown that with a probability independent of the size of the graph, a simple graph is obtained [14] (see also the monograph [27]). To the best of our knowledge, we provide the first result for the case of unbounded degree distributions, embracing several particular cases investigated in the literature. Let us also mention that while the present manuscript was under review (and a version of it was available on the ArXiv), the problem of finding the jamming constant was also considered in [20] where a related, though not identical, construction was presented. This alternative construction does not allow to compute the scaling limit of the empirical measure of degrees but gives a more explicit expression for the jamming constant.

In the case of the Poisson distribution, we are able to explicitly compute the measure-valued flow of unexplored nodes, which turns out to be an inhomogeneous Poisson measure. We then retrieve the jamming constant of the Erdös–Rényi (ER) graph. (Note however that our construction does not lead to a proper ER graph, only to a random multi-graph having the same degree distribution.) We also retrieve a constant calculated by Rényi [12,25] for a spatial model on $\mathbb{Z}$, showing that both models share the same jamming limit.

The rest of the paper is organized as follows. In Section 2, we describe the simultaneous construction of the parking process and the random graph. In Section 2.2 we calculate the generator of the induced measure-valued Markov process, and the corresponding semi-martingale decomposition is introduced. In Section 3, we state our main result and its consequences. In particular we show how the latter leads to closed-forms, or at least to computable characterizations of the jamming constant in various cases. Section 4 is devoted to the proofs of our main results.

Notation. Let us introduce the main notation used throughout the paper.

- We denote by $\mathbb{R}$ the set of real numbers, and $\mathbb{R}+$ (respectively, $\mathbb{R}^*$) the subset of non-negative (resp., non-null) real numbers. Let also $\mathbb{N}$ be the set of non-negative integers and $\mathbb{N}^*$, the subset of positive integers. For any $x, y \in \mathbb{R}$, let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and $x^+ = x \vee 0$. Let also for any $a, b \in \mathbb{N}$, $[\![a, b]\!] = \{a, a+1, \dots, b\}$.
- Let $\mathcal{B}_b$ be the set of Borel bounded functions $\phi : \mathbb{R} \to \mathbb{R}$. For all $\phi \in \mathcal{B}_b$, denote

$$\|\phi\| = \sup_{x\in\mathbb{R}} |\phi(x)|.$$

Denote for any Borel set A, $\mathbb{1}_A$ the indicator function of A. Denote by $\mathbf{1}$, the real function constantly equal to 1 and for any $k \in \mathbb{N}$, χ^k the function $x \mapsto x^k$. For all $\phi \in \mathcal{B}_b$, we also denote by $\nabla\phi$ the *discrete gradient* of ϕ, *i.e.*

$$\nabla\phi(i) = \phi(i) - \phi(i-1), \quad \forall i \in \mathbb{N}^*.$$

- Let $\mathcal{M}_F(\mathbb{N})$ be the set of finite measures on $\mathbb{N}$, embedded with the topology of weak convergence. We write $\mu(i) := \mu(\{i\})$ for any $\mu \in \mathcal{M}_F(\mathbb{N})$ and any $i \in \mathbb{N}$. The null measure is denoted $\mathbf{0}$. For all $\mu \in \mathcal{M}_F(\mathbb{N})$ and all $\phi : \mathbb{R} \to \mathbb{R}$, $\langle \mu, \phi \rangle$ denotes the integral of ϕ with respect to μ:

$$\langle \mu, \phi \rangle = \int \phi(x)\mu(dx) = \sum_{i \in \mathbb{N}} \phi(i)\mu(i).$$

In this way, for any such μ and any $A \subset \mathbb{N}$, $\left\langle \mu, \mathbb{1}_A \right\rangle = \mu(A)$ is the measure of A, $\langle \mu, \mathbf{1} \rangle = \mu(\mathbb{N})$ is the total mass of μ, and for any $k \in \mathbb{N}^*$, $\left\langle \mu, \chi^k \right\rangle$ is the kth moment of μ. We denote $\mathcal{M}_p(\mathbb{N})$, the subset of *counting* measures on $\mathbb{N}$, i.e. of elements μ of $\mathcal{M}_F(\mathbb{N})$ such that $\mu(i) \in \mathbb{N}$ for all $i \in \mathbb{N}$. For any $\mu \in \mathcal{M}_p(\mathbb{N})$, we will be led to order and index the atoms of μ as follows:
  - we denote for any $\ell \in \{1, \ldots, \mu(0)\}$, $v_\ell(\mu)$ the ℓth atom of degree 0 ranked in arbitrary order;
  - by induction, for any $i \in \mathbb{N}$ and any $\ell \in [\![1, \mu(i+1)]\!]$, $v_{\sum_{j=1}^{i} \mu(j)+\ell}(\mu)$, the ℓth atom of degree $i+1$, in arbitrary order, in a way that

$$\mu = \sum_{j=1}^{\langle \mu, \mathbf{1} \rangle} \delta_{v_j(\mu)}. \tag{1}$$

- For $T > 0$ and a Polish space (E, d_E), we denote by $\mathcal{D}([0,T], E)$ the Skorokhod space of rcll (right-continuous left-limited) functions from $\mathbb{R}$ to E (*e.g.* [3,15]). It is equipped with the Skorokhod topology, induced by the metric

$$d_T(f, g) := \inf_{\alpha \in \Delta([0,T])} \left\{ \sup_{\substack{(s,t) \in [0,T]^2 \\ s<t}} \left| \log \frac{\alpha(s) - \alpha(t)}{s - t} \right| \vee \sup_{t \leq T} d_E\big(f(t), g(\alpha(t))\big) \right\}, \tag{2}$$

where the infimum is taken over the set $\Delta([0,T])$ of continuous increasing functions $\alpha : [0,T] \to [0,T]$ such that $\alpha(0) = 0$ and $\alpha(T) = T$. We denote by $\mathcal{C}([0,T], E)$, the subspace of continuous functions from $[0,T]$ to E.
- Unless explicitly mentioned, throughout all the random variables (r.v.'s, for short) are defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. On the latter, let us write "$\Rightarrow$" for weak convergence of r.v.'s, and "$\xrightarrow{(\mathcal{P})}$" for convergence in probability. Finally, let us denote $\big(\langle\!\langle M \rangle\!\rangle_t\big)_{t \geq 0}$ the quadratic variation of the $\mathcal{D}([0,T], E)$-valued martingale $(M_t)_{t \geq 0}$.

2. Construction of the graph and Markov representation

2.1. Construction

In this section we present our construction of a random graph of prescribed degree distribution, simultaneously with the parking process on the latter graph. The basic objects of our construction are:

(i) a probability measure ν on $\mathbb{N}$ having support $[\![0, n-1]\!]$ (where $n \geq 1$), which will be referred to as the *degree distribution*;

(ii) a n-independent sample $\mathbf{d} := (d(1), \ldots, d(n))$ of the distribution ν, termed *degree vector*.[1] The *empirical degree distribution* is the following random point measure,

$$\mu_0 = \sum_{i=1}^{n} \delta_{d(i)};$$

(iii) the set of *nodes* $\mathcal{V}$, whose elements are denoted $u_0(1), \ldots, u_0(n)$ (the use of this notation will become clear in a few lines). We set a one-to-one relation between the nodes and the atoms of μ_0 as follows: to the node $u_0(i)$ is associated the element $d(\gamma(i))$, where γ is a permutation of $[\![1, n]\!]$ arranging d in increasing order. We then say that $d(\gamma(i))$ is the *degree* of $u_0(i)$ and we write $d_{u_0(i)} := d(\gamma(i))$.

At time 0, all the nodes of $\mathcal{V}$ are disconnected: the *associated graph* of our construction at time 0, denoted G_0, thus consists in the set of nodes $\mathcal{V}$, without any edge. At this point, the nodes are all said *unexplored* (we say that they are "U-nodes"). We consider that each node has as many **unmatched half-edges** as its degree.

Let us define for all $t \geq 0$, $\mathcal{U}_t$, $\mathcal{A}_t$ and $\mathcal{B}_t$ the sets of *unexplored*, *active* and *blocked* nodes, respectively. At $t = 0$, we thus fix $\mathcal{U}_0 = \mathcal{V}$ (hence the notation above), and set $\mathcal{A}_0 = \mathcal{B}_0 = \emptyset$. Let us also define for all $t \geq 0$,

$$\mathcal{H}_t = \Big\{\text{unmatched half-edges at time } t\Big\}.$$

Let for any t and any $j \in \mathcal{U}_t$, $d_j(\mathcal{U}_t)$ denote the number of unmatched half-edges of j at time t. Define also the following element of $\mathcal{M}_F(\mathbb{N})$,

$$\mu_t = \sum_{j \in \mathcal{U}_t} \delta_{d_j(\mathcal{U}_t)},$$

termed *empirical degree distribution at t*. Notice that the cardinality U_t of $\mathcal{U}_t$, and the cardinality H_t of $\mathcal{H}_t$ can respectively be retrieved from μ_t by

$$U_t = \langle \mu_t, \mathbf{1} \rangle \quad \text{and} \quad H_t = \langle \mu_t, \chi \rangle .$$

All these time-dependent quantities will be updated, by induction on the event times, as will be described hereafter.

Fix $\lambda > 0$, and let ξ_0 be a random variable of exponential distribution of rate λn. As long as this exponential clock does not ring, the system remains unchanged: we set $\mu_t = \mu_0$ and likewise, $\mathcal{U}_t$, $\mathcal{A}_t$, $\mathcal{B}_t$ and $\mathcal{H}_t$ equal their initial value, for any $t \in [0, \xi_0)$. The dynamics of the system is then determined by induction, as follows: assume that a clock rings at time t. Then, several state changes occur instantaneously at t, but following a given sequence. To represent these changes, all characteristics of the system just before the event occurs are indexed by t^- and then, to distinguish between the different steps of the state actualization, by t^{-+}, t^{-++} and finally t. In the construction below, all uniform draws are performed independently of each other, and of everything else.

[1] We assume independence of the degrees for simplicity, however it should be noted that the results hereafter hold in larger generality: for assumption (21) to hold true, we only need the convergence of the Cesàro means of the vector $\mathbf{d}$ as n goes large.

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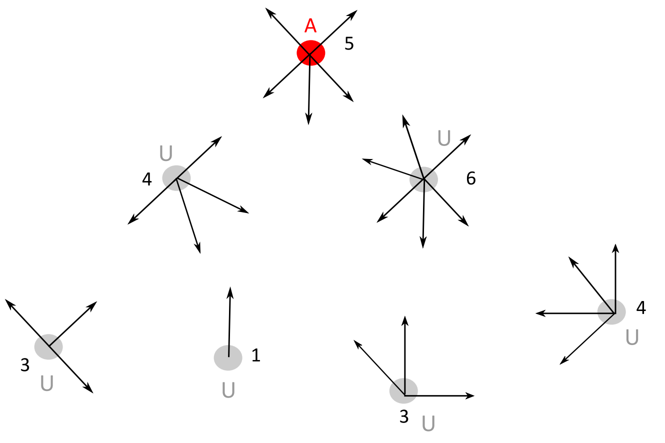

Fig. 1. Step 1—the new active node is selected.

Step 1. A node a becomes active: we draw uniformly an index i in $[\![1, U_{t^-}]\!]$. The corresponding node $a := u_{t^-}(i)$ of $\mathcal{U}_{t^-}$ becomes active and we set

$$\mathcal{U}_{t^{-+}} = \mathcal{U}_{t^-} \setminus \{a\}, \qquad \mathcal{A}_{t^{-+}} = \mathcal{A}_{t^-} \cup \{a\} \quad \text{and} \quad \mathcal{B}_{t^{-+}} = \mathcal{B}_{t^-}.$$

Let $K(\mu_{t^-}) = d_a$, the number of unmatched half-edges of a at t^-. As a is no longer a U-node, we update the measure μ_{t^-} as follows,

$$\mu_{t^{-+}} = \mu_{t^-} - \delta_{K(\mu_{t^-})}. \tag{3}$$

Example 2.1. Let the measure at t^- be

$$\mu_{t^-} = \delta_1 + 2\delta_3 + 2\delta_4 + \delta_5 + \delta_6,$$

so that the associated graph has $n = \langle \mu_{t^-}, \mathbf{1} \rangle = 7$ nodes and $\langle \mu_{t^-}, \chi \rangle = 26$ half-edges (see Fig. 1). A clock rings at t. The new A-node a has degree $K(\mu_{t^-}) = 5$ and the measure is updated to

$$\mu_{t^{-+}} = \delta_1 + 2\delta_3 + 2\delta_4 + \delta_6.$$

Step 2. The neighbors of a become of class B: the neighbors of the new A-node are blocked, we say that they become B-nodes. The identity of these new B-nodes is determined by matching the $K(\mu_{t^-})$ elements of $\mathcal{H}_{t^-}$ emanating from a, with half-edges of $\mathcal{H}_{t^-}$, as follows:
- a first half-edge of $\mathcal{H}_{t^-}$ emanating from a is matched with another one, drawn uniformly among the $\langle \mu_{t^-}, \chi \rangle - 1$ possible ones;
- on and on, as long as all half-edges of $\mathcal{H}_{t^-}$ emanating from a have not been matched, we take one of those, and match it with another half-edge of $\mathcal{H}_{t^-}$ which has not yet been matched, drawn uniformly in the latter set.

Notice that at each step, we may match couples of half-edges emanating from a together – hence creating self-loops around a. At the end of this procedure, we let $\tilde{K}(\mu_{t^-})$ be the number of edges linking a to other nodes. Clearly, $\tilde{K}(\mu_{t^-})$ cannot exceed the number of half-edges emanating from a, nor the number of half-edges of $\mathcal{H}_{t^-}$ emanating from nodes different to a, in other words

$$\tilde{K}(\mu_{t^-}) \le K(\mu_{t^-}) \wedge \Big(\langle \mu_{t^-}, \chi\rangle - K(\mu_{t^-})\Big). \tag{4}$$

We have thus fixed the identity of the q new B-nodes (where $q \le \tilde{K}(\mu_{t^-})$), which are the emanating nodes $u_{t^-}(i_1), u_{t^-}(i_2), \dots, u_{t^-}(i_q)$ different from a, of the $\tilde{K}(\mu_{t^-})$ half-edges matched with the $\tilde{K}(\mu_{t^-})$ half-edges of ego a. We then set

$$\begin{cases} \mathcal{U}_{t^{-++}} = \mathcal{U}_{t^{-+}} \setminus \left\{u_{t^-}(i_1), u_{t^-}(i_2), \dots, u_{t^-}(i_q)\right\}; \\ \mathcal{A}_{t^{-++}} = \mathcal{A}_{t^{-+}}; \\ \mathcal{B}_{t^{-++}} = \mathcal{B}_{t^{-+}} \cup \left\{u_{t^-}(i_1), u_{t^-}(i_2), \dots, u_{t^-}(i_q)\right\}. \end{cases}$$

For all $j \in [\![1, q]\!]$, let $N_j(\mu_{t^-})$ be the number of edges shared by $u_{t^-}(i_j)$ with a. Let us define the two following point measures,

$$Y(\mu_{t^-}) = \sum_{j=1}^{q} N_j(\mu_{t^-})\, \delta_{d_{u_{t^-}(i_j)}}; \tag{5}$$

$$\tilde{Y}(\mu_{t^-}) = \sum_{j=1}^{q} \delta_{d_{u_{t^-}(i_j)}}. \tag{6}$$

In other words, for any i, $\tilde{Y}(\mu_{t^-})(i)$ (resp., $Y(\mu_{t^-})(i)$) is the number of neighbors of a (resp., of edges shared by a with its neighbors) having i unmatched half-edges at t^-. Thus, $\langle Y(\mu_{t^-}), \mathbf{1}\rangle = \tilde{K}(\mu_{t^-})$ is the number of half-edges emanating from a and $\left\langle \tilde{Y}(\mu_{t^-}), \mathbf{1}\right\rangle = q$ is the number of neighbors of a. As the new B-nodes are no longer unexplored, their degree must be erased from the measure $\mu_{t^{-+}}$, which is updated as follows,

$$\mu_{t^{-++}} = \mu_{t^{-+}} - \tilde{Y}(\mu_{t^-}). \tag{7}$$

There remain $\langle \mu_{t^-}, \chi\rangle - K(\mu_{t^-}) - \tilde{K}(\mu_{t^-})$ unmatched half-edges at this point.

Example 2.2 (Example 2.1 *Continued*)**.** The uniform selection of the neighbors of the new A-node a results in a loop around it, so we have $\tilde{K}(\mu_{t^-}) = 3$, $q = 2$,

$$Y(\mu_{t^-}) = \delta_4 + 2\delta_6 \quad \text{and} \quad \tilde{Y}(\mu_{t^-}) = \delta_4 + \delta_6,$$

as there exists a double-edge between a and its neighbor of degree 6 (see Fig. 2). We then have

$$\mu_{t^{-++}} = \delta_1 + 2\delta_3 + \delta_4.$$

Observe that $\tilde{K}(\mu_{t^-}) = 3$ and the number of unmatched half-edges at this point is $\langle \mu_{t^-}, \chi\rangle - K(\mu_{t^-}) - \tilde{K}(\mu_{t^-}) = 26 - 5 - 3 = 18$. Between parenthesis is indicated the number of still unmatched half-edges of the B-nodes.

Step 3. Updating of the number of unmatched half-edges: the available half-edges at this point, *i.e.* the elements of $\mathcal{H}_{t^{-++}}$, either emanate from B-nodes and do not point to a, or emanate from nodes of $\mathcal{U}_{t^{-++}}$. Let us denote

$$\mathcal{H}^{\mathrm{B}}_{t^{-++}} := \left\{\text{half-edges of } \mathcal{H}_{t^{-++}} \text{ emanating from nodes in } \mathcal{B}_{t^{-++}}\right\} \subset \mathcal{H}_{t^{-++}},$$

and observe that

$$\left|\mathcal{H}^{\mathrm{B}}_{t^{-++}}\right| = \left\langle \tilde{Y}\left(\mu_{t^-}\right), \chi \right\rangle - \tilde{K}\left(\mu_{t^-}\right) = \left\langle \tilde{Y}\left(\mu_{t^-}\right), \chi \right\rangle - \left\langle Y\left(\mu_{t^-}\right), \mathbf{1} \right\rangle. \tag{8}$$

We now fully attach the new B-nodes to the associated graph, *i.e.* we match all the half-edges of $\mathcal{H}^{\mathrm{B}}_{t^{-++}}$, either with other elements of $\mathcal{H}^{\mathrm{B}}_{t^{-++}}$, or with elements of $\mathcal{H}_{t^{-++}}$ emanating from nodes in $\mathcal{U}_{t^{-++}}$. This is done according to the following procedure:

– draw an integer uniformly at random in $[\![1, q]\!]$, say ℓ. We match the remaining $d_{u_{t^-}(i_\ell)} - N_\ell\left(\mu_{t^-}\right)$ open half-edges of $u_{t^-}(i_\ell)$ exactly as those of a: we take these open half-edges one by one; each time, we draw uniformly at random a match for the latter in all available half-edges (emanating from a node of $\mathcal{U}_{t^{-++}}$, another node of $\mathcal{B}_{t^{-++}}$ or from $u_{t^-}(i_\ell)$ itself), until all half-edges of $u_{t^-}(i_\ell)$ are matched;
– then, draw at random another integer m in $[\![1, q]\!] \setminus \{\ell\}$, and match all available half-edges of $u_{t^-}(i_m)$ in the same manner, and so on… until all half-edges of $\mathcal{H}^{\mathrm{B}}_{t^{-++}}$ have been matched, to form edges of the associated graph.

At the end of this operation, we have possibly created edges between the new B-nodes and the remaining U-nodes. Let us denote $X\left(\mu_{t^-}\right)$ the number of such edges, in other words

$$\begin{aligned} &X\left(\mu_{t^-}\right) \\ &\quad = \mathrm{Card}\left\{\text{half-edges of } \mathcal{H}^{\mathrm{B}}_{t^{-++}} \text{ matched with half-edges of } \mathcal{H}_{t^{-++}} \setminus \mathcal{H}^{\mathrm{B}}_{t^{-++}}\right\}. \end{aligned} \tag{9}$$

Observe that by its very definition, $X(\mu_{t^-})$ is less than the number of half-edges emanating from the new B-nodes, and thus

$$X\left(\mu_{t^-}\right) \leq \left\langle \tilde{Y}\left(\mu_{t^-}\right), \chi \right\rangle \leq \left\langle Y\left(\mu_{t^-}\right), \chi \right\rangle. \tag{10}$$

To update $\mu_{t^{-++}}$, we have to subtract these $X\left(\mu_{t^-}\right)$ half-edges from the number of available half-edges of the remaining U-nodes. To formalize this operation, it is convenient to index the remaining U-nodes in the following way: for all $i \in [\![1, n-1]\!]$ and all $\ell \in [\![1, \mu_{t^{-++}}(i)]\!]$, we let $u_{t^{-++}}(i, \ell)$ be the ℓth node of $\mathcal{U}_{t^{-++}}$ having i unmatched half-edges at t^{-++} (if any), ranked in an arbitrary order. Then, we define

$$W\left(\mu_{t^-}\right)(i, \ell) = \mathrm{Card}\left\{\text{edges shared by } u_{t^{-++}}(i, \ell) \text{ with nodes of } \mathcal{B}_{t^{-++}}\right\}, \tag{11}$$

and observe that

$$\sum_{i=1}^{n-1} \sum_{\ell=1}^{\mu_{t^{-++}}(i)} W\left(\mu_{t^-}\right)(i, \ell) = X\left(\mu_{t^-}\right).$$

Let us define the counting measure

$$\mathbf{W}\left(\mu_{t^-}\right) = \sum_{i=1}^{n-1} \sum_{\ell=1}^{\mu_{t^{-++}}(i)} \left(\delta_i - \delta_{i - W\left(\mu_{t^-}\right)(i, \ell)}\right). \tag{12}$$

To represent the change at this step, the quantity $W\left(\mu_{t^-}\right)(i, \ell)$ must be subtracted from the number of open half-edges of each unexplored node $u_{t^{-++}}(i, \ell)$, hence we finally obtain

$$\mu_t = \mu_{t^{-++}} - \mathbf{W}\left(\mu_{t^-}\right). \tag{13}$$

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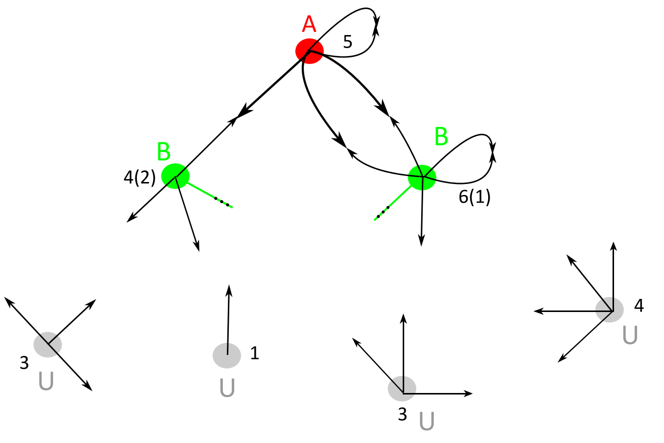

Fig. 2. Step 2—the blocked nodes are attached to the active node.

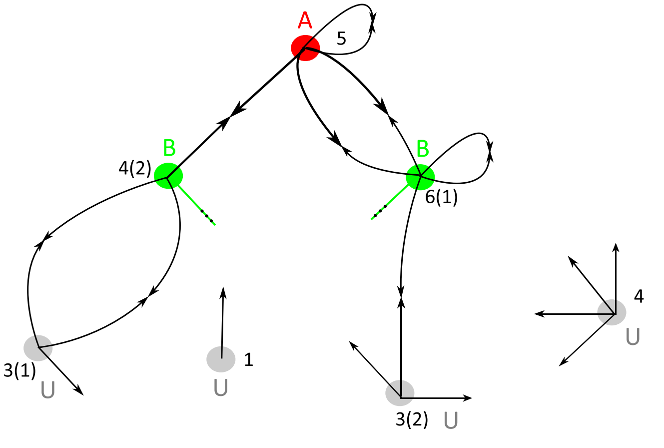

Fig. 3. Step 3—the new blocked nodes are connected between each other, and with remaining unexplored nodes.

Example 2.3 (Example 2.1 *Concluded*)**.** In Fig. 3 we see that $\left|\mathcal{H}^{\mathrm{B}}_{t-++}\right| = \left\langle \tilde{Y}\left(\mu_{t^+}\right), \chi\right\rangle - \tilde{K}\left(\mu_{t^-}\right) = 10 - 3 = 7$ and $X\left(\mu_{t^-}\right) = 3$, since there are 4 half-edges emanating from B-nodes that are matched together. The three remaining half-edges are matched with the remaining U-nodes as follows: if among the U-nodes of degree 3, the one on the left has label 1 and that on the right has label 2, then

$$W\left(\mu_{t^-}\right)(3, 1) = 2 \quad \text{and} \quad W\left(\mu_{t^-}\right)(3, 2) = 1.$$

The updated measure is then

$$\mu_t = 2\delta_1 + \delta_2 + \delta_4.$$

There remain $\langle \mu_t, \mathbf{1}\rangle = 4$ unexplored nodes and $\langle \mu_t, \chi\rangle = 8$ unmatched half-edges.

As a conclusion, gathering (3), (7) and (13), the resulting measure after Steps 1–3 is

$$\mu_t = \mu_{t^-} - \vartheta\left(\mu_{t^-}\right), \tag{14}$$

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where we denote for any counting measure μ,

$$\vartheta(\mu) = \delta_{K(\mu)} + \tilde{Y}(\mu) + \mathbf{W}(\mu). \tag{15}$$

In the counting measure μ_t, $1+q$ atoms have been erased with respect to μ_{t-}, and whose first moment (*i.e.* the number of open half-edges) has the same parity as that of μ_{t-}. The associated graph G_t equals G_{t-}, plus all edges that have been drawn between a and its neighbors, between its neighbors with one another, and between its neighbors and the remaining U-nodes. Moreover, only the remaining U-nodes still have open half-edges at t and the measure μ_t provides the repartition of the latter half-edges among $\mathcal{U}_t$. At this point, we re-index all elements of $\mathcal{U}_t$ in the order of increasing number of open half-edges, as was done above:

$$\mathcal{U}_t = \{u_t(1), \dots, u_t(U_t)\}.$$

We now draw a new exponential r.v. ξ_t of parameter λU_t, independently of everything else. As above, the system remains constant until time $t+\xi_t$, at which we re-iterate Step 1–3, and so on.

Remark 2.4. In (14) and (15), the shorthand notation ϑ denotes a random map from $\mathcal{M}_p(\mathbb{N})$ to itself. For a fixed $\mu \in \mathcal{M}_p(\mathbb{N})$, the distribution of the random measure $\vartheta(\mu)$ is fully determined by the value of μ and the uniform draws performed at the successive steps. In particular, as these uniform draws are independent of everything else, at any event time t the random measure $\vartheta(\mu_t)$ is drawn independently of all the random variables generating μ_t.

The procedure ends at the stopping time

$$T_0 = \inf\Big\{t \geq 0;\ U_t = 0\Big\},$$

which clearly is almost surely (a.s., for short) finite. At that instant, if $\sum_{i=1}^n d(i)$ was odd there remains a single unmatched half-edge, which we remove. At that time T_0, we thus end up with $\mu_{T_0} = \mathbf{0}, \mathcal{U}_{T_0} = \mathcal{H}_{T_0} = \emptyset$ and $\left|\mathcal{A}_{T_0} \cup \mathcal{B}_{T_0}\right| = n$. The final associated graph G_{T_0} is a multi-graph of degree vector $\mathbf{d}$ (up to the deletion of a single half-edge in the case mentioned above). The set $\mathcal{A}_{T_0}$ is the *jamming limit* of the latter graph (and of the parking process) and $\left|\mathcal{A}_{T_0}\right|/n$, its *jamming constant*.

2.2. Generator and semi-martingale decomposition

Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of $(\mu_t)_{t\geq 0}$. For all t, all $h \geq 0$ and all bounded measurable functions $F : \mathcal{M}_F(\mathbb{N}) \to \mathbb{R}$, we have from (14) that

$$\begin{aligned}\mathbb{E}\left[F(\mu_{t+h}) \mid \mathcal{F}_t\right] &= (1-\lambda h\,\langle \mu_t, \mathbf{1}\rangle)F(\mu_t) + \lambda h\,\langle \mu_t, \mathbf{1}\rangle\,\mathbb{E}\left[F\left(\mu_t - \vartheta(\mu_t)\right) \mid \mu_t\right] + o(h)\\ &=: T_h F(\mu_t).\end{aligned} \tag{16}$$

Therefore, according to the definition of [4, p.18], $(\mu_t)_{t\geq 0}$ is a weak homogeneous $\mathcal{M}_F(\mathbb{N})$-valued Markov process having transition operator $(T_h,\ h \geq 0)$. Moreover, in the present context it is routine to check, as in Proposition 1 of [6], that the assumptions of Lemma 3.5.1 and Corollary 3.5.2 of [4] are met. Therefore, $(\mu_t)_{t\geq 0}$ is a Feller–Dynkin process of $\mathcal{D}([0,\infty), \mathcal{M}_F(\mathbb{N}))$. Its infinitesimal generator $\mathcal{Q}$ is given for all $F : \mathcal{M}_F(\mathbb{N}) \to \mathbb{R}$ such that the following limit exists (we say that F belongs to the *domain* of $\mathcal{Q}$), by

$$\mathcal{Q}F(\mu) = \lim_{h\to 0}\frac{1}{h}\Big(T_h F(\mu) - F(\mu)\Big) = \lim_{h\to 0}\frac{1}{h}\Big(\mathbb{E}\left[F(\mu_h) \mid \mu_0 = \mu\right] - F(\mu)\Big),$$
$$\mu \in \mathcal{M}_F(\mathbb{N}). \tag{17}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bounded function, and denote for all $\mu \in \mathcal{M}_F(\mathbb{N})$, $\Pi_\phi(\mu) = \langle \mu, \phi\rangle$. We can easily check that Π_ϕ belongs to the domain of $\mathcal{Q}$, and explicitly write $\mathcal{Q}\Pi_\phi$: denoting for any counting measure μ, by $\mathbb{P}_\mu$, the probability measure conditional on $\{\mu_{t^-} = \mu\}$ at a jump time t, and $\mathbb{E}_\mu$ the corresponding expectation, from (16) we obtain that

$$\mathcal{Q}\Pi_\phi(\mu) = -\lambda \left\langle \mu, \mathbf{1}\right\rangle \mathbb{E}_\mu\left[\left\langle \vartheta(\mu), \phi\right\rangle\right], \tag{18}$$

for ϑ defined by (15).

For any bounded $\phi : \mathbb{R} \to \mathbb{R}$, applying Dynkin's Lemma (see [8,9]) to the Markov process $(\mu_t)_{t\ge 0}$ and the map Π_ϕ, we obtain from (18) a decomposition of the process $t \to \Pi_\phi(\mu_t)$ as a semi-martingale. Specifically, the following is a $\mathcal{F}_t$- local martingale:

$$\begin{aligned} t \mapsto M(\phi)_t &= \left\langle \mu_t, \phi\right\rangle - \left\langle \mu_0, \phi\right\rangle - \int_0^t \mathcal{Q}\Pi_\phi\left(\mu_s\right) ds. \\ &= \left\langle \mu_t, \phi\right\rangle - \left\langle \mu_0, \phi\right\rangle + \lambda \int_0^t \left\langle \mu_s, \mathbf{1}\right\rangle \mathbb{E}_{\mu_s}\left[\left\langle \vartheta(\mu_s), \phi\right\rangle\right] ds. \end{aligned} \tag{19}$$

We can then easily retrieve the quadratic variation $\langle\langle M(\phi)\rangle\rangle$ of the latter local martingale using the semi-martingale decomposition of $F := \Pi_\phi^2$ (obtained as in (19)), and Ito's integration by parts formula (see *e.g.* Theorem 8.15 in [7], and a specific example in the context of measure-valued processes in Proposition 2 of [6]). We obtain for all t,

$$\begin{aligned} \langle\langle M(\phi)\rangle\rangle_t &= \int_0^t \left(\mathcal{Q}\left(\Pi_\phi\right)^2(\mu_s) - 2\Pi_\phi(\mu_s)\mathcal{Q}\Pi_\phi\left(\mu_s\right)\right) ds \\ &= \lambda \int_0^t \left\langle \mu_s, \mathbf{1}\right\rangle \Big(\mathbb{E}_{\mu_s}\left[\left\langle \vartheta(\mu_s), \phi\right\rangle^2\right] - 2\left\langle \mu_s, \phi\right\rangle \mathbb{E}_{\mu_s}\left[\left\langle \vartheta(\mu_s), \phi\right\rangle\right] \\ &\quad + 2\left\langle \mu_s, \phi\right\rangle \mathbb{E}_{\mu_s}\left[\left\langle \vartheta(\mu_s), \phi\right\rangle\right]\Big) ds. \\ &= \lambda \int_0^t \left\langle \mu_s, \mathbf{1}\right\rangle \mathbb{E}_{\mu_s}\left[\left\langle \vartheta(\mu_s), \phi\right\rangle^2\right] ds. \end{aligned} \tag{20}$$

It follows in particular from (20), (10) and the fact that $\langle \mu_s, \mathbf{1}\rangle$ and $\langle \mu_s, \chi\rangle$ are non-increasing functions of s, that for all $t \ge 0$,

$$\begin{aligned} \mathbb{E}\left[\langle\langle M(\phi)\rangle\rangle_t\right] &\le \lambda\|\phi\|^2 \int_0^t \left\langle \mu_s, \mathbf{1}\right\rangle \mathbb{E}_{\mu_s}\left[\left(1 + \left\langle Y(\mu_s), \mathbf{1}\right\rangle + 2\left\langle Y(\mu_s), \chi\right\rangle\right)^2\right] ds \\ &\le \lambda\|\phi\|^2 \int_0^t \left\langle \mu_s, \mathbf{1}\right\rangle \mathbb{E}_{\mu_s}\left[\left(1 + \left\langle \mu_s, \mathbf{1}\right\rangle + 2\left\langle \mu_s, \chi\right\rangle\right)^2\right] ds \\ &\le 3\lambda t\|\phi\|^2 \left\langle \mu_0, \mathbf{1}\right\rangle \left(1 + \left\langle \mu_0, \mathbf{1}\right\rangle^2 + 4\left\langle \mu_0, \chi\right\rangle^2\right) \\ &\le 3\lambda t\|\phi\|^2 n(1 + n^2 + 4n^4) < \infty, \end{aligned}$$

hence $M(\phi)$ is a square integrable $\mathcal{F}_t$-martingale.

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3. Main results and consequences

3.1. Hydrodynamic limit

We are interested in the behavior of the measure-valued process $(\mu_t)_{t\geq 0}$ as the size of the graph grows to infinity. We consider a sequence of models, where the size of the nth graph equals n, and add a superscript ^{n} to all parameters and processes relative to the nth system. Then, we scale the nth process of empirical degree distributions as follows:

$$\bar{\mu}_t^n = \frac{1}{n}\mu_t^n, \quad t \geq 0$$

and we denote the normalized versions of the derived processes accordingly, *i.e.* for all $t \geq 0$,

$$\bar{U}_t^n = \frac{U_t^n}{n}; \qquad \bar{H}_t^n = \frac{H_t^n}{n}; \qquad \bar{M}^n(\phi)_t = \frac{1}{n}M^n(\phi)_t, \quad \phi \in \mathcal{B}_b.$$

Our main result is the following.

Theorem 3.1. *Let $\kappa > 3.5$. Assume that for all $\phi \in \mathcal{B}_b \cup \{\chi, \chi^\kappa\}$,*

$$\left\langle \bar{\mu}_0^n, \phi \right\rangle \xrightarrow[n\to\infty]{(\mathcal{P})} \langle \zeta, \phi \rangle, \tag{21}$$

where ζ is a deterministic element of $\mathcal{M}_F(\mathbb{N})$ such that

$$0 < \langle \zeta, \chi \rangle \wedge \langle \zeta, \mathbf{1} \rangle \quad \textit{and} \quad \left\langle \zeta, \chi^\kappa \right\rangle \vee \langle \zeta, \mathbf{1} \rangle < \infty. \tag{22}$$

Then, for all $T > 0$ and all $\phi \in \mathcal{B}_b \cup \{\chi^k : k \in (0, \kappa - 1]\}$ we have

$$\sup_{t\in[0,T]} \left| \left\langle \bar{\mu}_t^n, \phi \right\rangle - \langle \bar{\mu}_t, \phi \rangle \right| \xrightarrow[n\to\infty]{(\mathcal{P})} 0,$$

where $\bar{\mu}$ is the unique element of $\mathcal{C}(\mathbb{R}+, \mathcal{M}_F(\mathbb{N}))$ satisfying the following infinite dimensional differential system: for all $t \geq 0$ and all bounded ϕ,

$$\begin{aligned} &\langle \bar{\mu}_0, \phi \rangle = \langle \zeta, \phi \rangle; \\ &\frac{d}{dt}\langle \bar{\mu}_t, \phi \rangle = \langle \Psi(\bar{\mu}_t), \phi \rangle \\ &\qquad := \begin{cases} -\lambda \left[\langle \bar{\mu}_t, (\mathbf{1} + \chi)\phi \rangle + \langle \bar{\mu}_t, \chi \nabla \phi \rangle \left(\dfrac{\left\langle \bar{\mu}_t, \chi^2 \right\rangle}{\langle \bar{\mu}_t, \chi \rangle} - 1 \right) \right] & \textit{if } \langle \bar{\mu}_t, \chi \rangle > 0; \\ -\lambda \bar{\mu}_t(0)\phi(0) & \textit{if } \langle \bar{\mu}_t, \chi \rangle = 0. \end{cases} \end{aligned} \tag{23}$$

Remark 3.2. A consequence of the proof of this theorem is the existence of a solution to (23). Its uniqueness is proved separately.

Remark 3.3. By our very assumptions, $\left\langle \mu_0^n, \mathbf{1} \right\rangle = n$ and thus $\left\langle \bar{\mu}_0^n, \mathbf{1} \right\rangle = 1$ for all n. It thus follows from (21) and (22) that $\bar{\mu}_0$ is a probability measure. In particular we have

$$\langle \bar{\mu}_0, \mathbf{1} \rangle \vee \left\langle \bar{\mu}_0, \chi^\kappa \right\rangle < \infty,$$

a fact that will be used at several points of the proofs.

By specializing (23) to the functions $\phi_i(j) = \delta_{ij}$, for all $i \in \mathbb{N}$, we obtain the following system of ODE's for $\bar{\mu}_{.}(i)$, $i \in \mathbb{N}$,

$$\bar{\mu}_0(i) = \zeta(i);$$

$$\frac{d}{dt}\bar{\mu}_t(i) = \begin{cases} -\lambda \left[\Big(\bar{\mu}_t(i) + i\bar{\mu}_t(i)\Big) + \Big(i\bar{\mu}_t(i) - (i+1)\bar{\mu}_t(i+1)\Big) \right. \\ \left. \qquad \times \left(\dfrac{\sum_{i\in\mathbb{N}} i^2 \bar{\mu}_t(i)}{\sum_{i\in\mathbb{N}} i\bar{\mu}_t(i)} - 1 \right) \right] & \text{if } \sum_{i\in\mathbb{N}} i\bar{\mu}_t(i) > 0; \\ -\lambda\bar{\mu}_t(0)\mathbb{1}_{i=0} & \text{if } \sum_{i\in\mathbb{N}} i\bar{\mu}_t(i) = 0. \end{cases} \tag{24}$$

Note that the differential equation for the ith coordinate depends explicitly on the $(i+1)$th coordinate and implicitly on all the other coordinates through the first and the second moment of the measure.

3.2. Main characteristics, and the jamming constant

It follows from Theorem 3.1 that the sequence $\{\bar{U}^n\}$ tends uniformly over compact time sets to the deterministic functions $\bar{u}$ given for all t by $\bar{u}_t = \langle \bar{\mu}_t, \mathbf{1} \rangle$ and which, from (23), satisfies

$$\dot{\bar{u}}_t = -\lambda \left(\bar{u}_t + \bar{h}_t\right), \quad t \geq 0. \tag{25}$$

On another hand, applying Theorem 3.1 to the function $\phi = \chi$ yields the convergence of $\bar{H}^n$ to the deterministic function $\bar{h}$, defined by $\bar{h}_t = \langle \bar{\mu}_t, \chi \rangle$ for all $t \in [0, T]$. Again from (23), we can easily check that

$$\dot{\bar{h}}_t = -2\lambda \left\langle \bar{\mu}_t, \chi^2 \right\rangle, \quad t \geq 0.$$

Jamming constant. Denote for all $n \in \mathbb{N}^*$ and all $t \geq 0$,

$$J_t^n := \left|\mathcal{A}_t^n\right| \tag{26}$$

the number of active nodes at t. The *jamming constant* $\bar{J}^n$ of the associated graph, is the proportion of active nodes at the ending time T_0^n of the exploration process. In other words, it is given by

$$\bar{J}^n = \frac{J^n_{T_0^n}}{n}. \tag{27}$$

The following result can be deduced from Theorem 3.1. Its proof is provided in Section 4.5.

Corollary 3.4 (*Jamming Constant of Random Graphs*)**.** *Under the assumption of Theorem 3.1, we have that*

$$\bar{J}^n \underset{n\to\infty}{\longrightarrow} c_\zeta \quad \text{in } L^1,$$

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where

$$c_\zeta = \lambda \int_0^\infty \bar{u}_t \, dt = \lambda \int_0^\infty \langle \bar{\mu}_t, \mathbf{1} \rangle \, dt \tag{28}$$

and $(\bar{\mu}_t)_{t\geq 0}$ *is the only solution to* (23).

Remark 3.5. As expected, it readily follows from (28) and (25) that $c_\zeta = \int_0^\infty \bar{u}_{t/\lambda} \, dt$ does not depend on λ—we hence fix $\lambda = 1$ in Section 3.4 without loss of generality.

3.3. Connexion with the configuration model

As far as the structure of the associated random graph is concerned, our construction mimics the so-called *configuration model* (or uniform model). More precisely, denote $\mathrm{CM}(n, \mathbf{d}^n)$ the random (multi-)graph obtained by the uniform mapping of half-edges in a graph of n nodes having degree vector $\mathbf{d}^n$, as described in [27,30]. We have the following.

Proposition 3.6. *The associated random multi-graph constructed jointly with the exploration process in Section 2 equals* $\mathrm{CM}(n, \mathbf{d}^n)$ *in distribution.*

Proof. The result follows from the so-called *independence property* of the configuration model (see *e.g.* [30]): choosing whatever rule for matching the half-edges as long as a given half-edge is matched uniformly among all the unmatched half-edges at each step, provides a realization of $\mathrm{CM}(n, \mathbf{d}^n)$. This is exactly what is done here: at each step, first all half-edges of the new active node a are matched with other half-edges that are chosen uniformly among available ones (including other half-edges of a), and then all the remaining open half-edges of the neighbors b_1, and then $b_2, b_3, \ldots$ and finally b_q (q being the number of neighbors of a) are matched according to the same rule. Therefore, at the end of the algorithm the associated multi-graph we have drawn is nothing but a realization of $\mathrm{CM}(n, \mathbf{d}^n)$. □

We can now link our construction with the more usual one, consisting in fixing the graph beforehand, and then building an independent set on the latter. The *parking process* on a uniform graph, mentioned in the introduction, can be formalized as follows: we first fix $\tilde{G}$, a realization of $\mathrm{CM}(n, \mathbf{d}^n)$. We then construct sequentially the independent set on $\tilde{G}$ according to the following procedure,

(i) At time 0, set all nodes of $\tilde{G}$ as unexplored, and initiate an exponential clock of intensity λn;
(ii) Each time an exponential clock rings, select a new active node a uniformly at random among all unexplored nodes;
(iii) The neighbors of a in $\tilde{G}$ all become of class B;
(iv) We set another clock of intensity λU, where U is the cardinality of the set of unexplored nodes at that instant, and go to step (ii).

The algorithm is terminated as soon as the set of unexplored nodes of $\tilde{G}$ is empty, and the jamming constant of the graph is obtained as the proportion of active nodes at that time.

Let for all t, $\tilde{\mathcal{U}}_t^n$ the set of unexplored nodes at t and for any $j \in \tilde{\mathcal{U}}_t^n$, $d_j\left(\tilde{\mathcal{U}}_t^n\right)$ the number of neighbors of j in $\tilde{\mathcal{U}}_t^n$ at t. Let also the following random point measure,

$$\tilde{\mu}_t^n = \sum_{j \in \tilde{\mathcal{U}}_t^n} \delta_{d_j\left(\tilde{\mathcal{U}}_t^n\right)}. \tag{29}$$

Observe that only the order of exploration of the nodes differs in the present construction with respect to ours, this order being itself drawn according to a uniform choice on the set of unexplored nodes at each time in both cases. Therefore, as a simple consequence of the invariance in distribution of any permutation of the coordinates of $\mathbf{d}^n$, and of Proposition 3.6.

Corollary 3.7. *The sequence of measure-valued processes* $\{\tilde{\mu}^n\}$ *defined by* (29) *coincides in distribution with* $\{\mu^n\}$*, and so do the jamming constants of the two models.*

It is significant that, although the two processes μ^n and $\tilde{\mu}^n$ have the same distribution, the first one is Markov but the second one is not, since the knowledge of the multi-graph $\tilde{G}$ is needed.

3.4. Jamming constants of particular graphs

Characterizing the jamming constant of parking problems has a long history in mathematics, see e.g. [22]. One of the most studied problem in the field is the so-called random sequential absorption on discrete structures. We show hereafter how our result can be adapted to regular graphs, *i.e.* graphs with fixed degree. We then focus on the case of the Poisson distribution, and relate our approach to the Erdös–Rényi graph.

Regular graph of degree 2. When for any n, the root degree distributions ρ^n (and therefore, ζ) are deterministic and equal to δ_2, we can solve exactly the three-dimensional limiting differential system:

$$\begin{cases} \dfrac{d}{dt}\bar{\mu}_t(2) = -\bar{\mu}_t(2)(1+2L(\bar{\mu}_t)); \\ \dfrac{d}{dt}\bar{\mu}_t(1) = -\bar{\mu}_t(1)(1+L(\bar{\mu}_t)) + 2(L(\bar{\mu}_t)-1)\bar{\mu}_t(2); \\ \dfrac{d}{dt}\bar{\mu}_t(0) = -\bar{\mu}_t(0) + (L(\bar{\mu}_t)-1)\bar{\mu}_t(1), \end{cases} \tag{30}$$

with

$$L(\bar{\mu}_t) = \frac{\langle \bar{\mu}_t, \chi^2 \rangle}{\langle \bar{\mu}_t, \chi \rangle} = \frac{4\bar{\mu}_t(2)+\bar{\mu}_t(1)}{2\bar{\mu}_t(2)+\bar{\mu}_t(1)}, \quad t \geq 0.$$

After tedious but simple calculus, one obtains

$$\begin{cases} \bar{\mu}_t(2) = e^{-3t-2+2e^{-t}}; \\ \bar{\mu}_t(1) = 2(e^t-1)\bar{\mu}_t(2); \\ \bar{\mu}_t(0) = (e^t-1)^2\bar{\mu}_t(2). \end{cases}$$

Therefore the jamming constant for δ_2 equals

$$c_{\delta_2} = \int_0^\infty \sum_{i=0}^{2} \bar{\mu}_t(i)dt = \int_0^\infty e^{-t-2+2e^{-t}}dt = \frac{1-e^{-2}}{2},$$

which coincides with the famous Rényi parking constant of $\mathbb{Z}$ (see [12, Section 5.3.1]). Informally, this can be explained by the fact that in the resulting configuration graph with distribution δ_2, only cycles of different sizes can appear while the number of small cycles (say smaller than a given constant) present in the resulting random graph is going to be very small compared to n with overwhelming probability. We do not go further into these calculations, which are beyond the scope of this paper.

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Regular graphs for $d \geq 3$. More generally, jamming constants of regular graphs for $d \geq 3$ have been shown to be asymptotically equivalent to $\frac{1-(d-1)^{-\frac{2}{d-2}}}{2}$, for a random graph $\mathrm{CM}(n, \mathbf{d}^n)$ of degree distribution $\rho^n = \delta_d$ (i.e. $\mathbf{d}^n = (d, d, \ldots, d)$), see [29].

Let us quickly reformulate in our terminology the approach in [29], and compare it with ours. The algorithm of *random pairing* as termed in [29] is similar to the one that is presented here, the main difference in the construction of $\mathrm{CM}(n, \mathbf{d}^n)$ being that at any time, all unexplored nodes have exactly d available half-edges. This is obtained as follows: each time a new active node is selected (uniformly at random among all unexplored nodes), its d half-edges are matched uniformly at random with d other ones, exactly as we do so. At this point, according to the method of deferred decisions (see [18]), and unlike our construction, the connectivity of the neighbors of the new active node with the rest of the graph is not completed yet. At the following instant, a new active node is selected among all unexplored nodes, and the same procedure is reiterated on and on, until there is no more unexplored node. It is important to observe that, at the end of this algorithm, only the edges between active and blocked nodes have been build in the associated graph, and that the blocked nodes may have unmatched half-edges (precisely as many as d minus the number of edges they share with their active neighbor). Then, the associated graph is completed by creating edges between blocked nodes in arbitrary order, following uniform choices of half-edges (which does not change the jamming constant of the graph). Under such a dynamics, it appears clearly that the process $\left((J_t^n, U_t^n)\right)$ is Markov (recall the notation (26)). In particular, using the fact that the possible neighbors of the new active node must have d available half-edges, it is easy to observe that whenever a node becomes active in $[t, t+h)$ we have that

$$\mathbb{E}\left[U_{t+h}^n - U_t^n \mid (J_t^n, U_t^n)\right] = -1 - d\frac{U_t^n}{n - 2J_t^n}, \tag{31}$$

which leads to a simple one-dimensional asymptotic ODE for $t \mapsto U_t^n$, that is solved explicitly.

However, for a general degree distribution the relation (31) no longer holds, and the process $\left((J_t^n, U_t^n)\right)$ is no longer Markov. In fact, one can observe that the measure-valued process $\left(\mu_t^n\right)$ itself is not Markov, when constructing the graph as is done in [29], since one needs to know which nodes are active and which are blocked in the associated graph, to complete the connectivity of the new active node at each instant—which is why we need to complete the neighboring between blocked nodes at each step.

For the more general case of unbounded degree distribution, the assumptions needed in [29] and subsequent articles are not satisfied in the present context. In particular, it is assumed that the drift of its ith component can be uniformly approximated by a (globally) Lipschitz function of the states j with $j \leq i$. It is clear that in our case this condition does not hold: the drift of the ith component depends on all components through moments of the distribution and directly on component $i+1$ (see (24)). A second hypothesis is that the drift of the ith component be uniformly bounded, which is also not verified here.

The price to pay for working in such generality, is that, due to a more intricate dynamics we do not obtain a closed-form formula for the function $t \mapsto \bar{u}_t$ in the particular case of regular graphs. However, though an exact computation of $(\bar{u}_t)$ becomes more and more involved when the degree d grows, we can easily retrieve the asymptotic value of [29] by solving numerically the system corresponding to (30), as is shown for $d = 3$ and 4 in Table 1.

The Poisson distribution. In the case where the asymptotic initial empirical degree distribution is Poisson, we obtain also a closed-form expression for the function $t \mapsto \bar{\mu}_t$.

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Proposition 3.8. *If ζ is a Poisson distribution with parameter p (we denote $\zeta = \mathcal{P}(p)$), then*

$$\bar{\mu}_t(i) = v_t \frac{(pv_t)^i}{i!} \exp(-pv_t), \quad t \geq 0,\ i \in \mathbb{N},$$

where v is the solution of the differential equation

$$\dot{v} = -v(1 + pv).$$

Moreover the jamming constant reads

$$c_{\mathcal{P}(p)} = \frac{1}{p} \log(1 + p).$$

Proof. Let us define the following Poisson measures for all t,

$$\kappa_t(i) = \frac{(pv_t)^i}{i!} \exp(-pv_t), \quad i \in \mathbb{N}.$$

For all $t \geq 0$, using the definition of v_t and $\bar{\mu}_t$ we obtain

$$\begin{aligned}
\frac{d}{dt} \langle \bar{\mu}_t, \phi \rangle &= \dot{v}_t \exp(-pv_t) \left[\sum_{i \geq 0} (i+1) \frac{p^i}{i!} v_t^i \phi(i) - p \sum_{i \geq 0} \frac{p^i}{i!} v_t^{i+1} \phi(i) \right], \\
&= -\lambda v_t (1 + pv_t) \left[\langle \kappa_t, (1+\chi)\phi \rangle - \sum_{i \geq 0} (i+1) \kappa_t(i+1) \phi(i) \right], \\
&= -(1 + pv_t) \left[\langle \bar{\mu}_t, (1+\chi)\phi \rangle - \sum_{j \geq 1} j \bar{\mu}_t(j) \phi(j-1) \right], \\
&= -\left[\langle \bar{\mu}_t, (1+\chi)\phi \rangle + pv_t \langle \bar{\mu}_t, \chi \nabla \phi \rangle + pv_t \langle \bar{\mu}_t, \phi \rangle - \sum_{j \geq 1} j \bar{\mu}_t(j) \phi(j-1) \right], \\
&= -\left[\langle \bar{\mu}_t, (1+\chi)\phi \rangle + \left(\frac{\langle \bar{\mu}_t, \chi^2 \rangle}{\langle \bar{\mu}_t, \chi \rangle} - 1 \right) \langle \bar{\mu}_t, \chi \nabla \phi \rangle \right],
\end{aligned}$$

where we use in the last identity that

$$pv_t = \frac{\langle \bar{\mu}_t, \chi^2 \rangle}{\langle \bar{\mu}_t, \chi \rangle} - 1$$

and

$$\begin{aligned}
pv_t \langle \bar{\mu}_t, \phi \rangle &= \sum_{i \geq 0} (i+1) \frac{p^{i+1}}{(i+1)!} v_t^{i+1} \exp(-pv_t) \phi(i) = \sum_{j \geq 1} j \frac{p^j}{j!} v_t^j \exp(-pv_t) \phi(j-1) \\
&= \sum_{j \geq 1} j \bar{\mu}_t(j) \phi(j-1).
\end{aligned}$$

The jamming constant then simply follows by integration of $t \mapsto v_t$. □

The asymptotic empirical degree measure of unexplored nodes thus turns out to be an inhomogeneous Poisson measure. A typical example where Proposition 3.8 applies is that of a binomial degree distribution $\rho^n = \mathrm{Bin}\,(n-1, p^n)$ for all n (as is the case for Erdös–Rényi graphs), where $np^n \underset{n \to \infty}{\longrightarrow} p$.

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Table 1
Jamming constants for different degree distributions and their counterparts on deterministic graphs.
Source: Simulation values for deterministic graphs are taken from [28].

Degree distribution	JC of Random graphs	JC of specific deterministic graphs
Geometric $e^{-1/2}$	0.7599203270	
Poisson (1)	$\log(2) = 0.6931472$	
δ_2	$\frac{1-e^{-2}}{2} = 0.4323323583$	$\frac{1-e^{-2}}{2}$ ($\mathbb{Z}$)
δ_3	3/8	0.37913944 (Honeycomb)
δ_4	1/3	0.3641323 ($\mathbb{Z}^2$)

It is then interesting to observe that we retrieve the jamming constant of the Erdös–Rényi graph of parameter p, which was given in [21]. Notice however that we do *not* construct a proper Erdös–Rényi graph and in fact, no uniform construction based on a prescribed degree distribution can do so, since the independence assumption for the existence of the various edges cannot be fulfilled.

Table 1 gathers in the middle column, our results for several jamming constants (exact for $\zeta = \mathcal{P}(1)$ and δ_2, and numerically computed for geometric, δ_3 and δ_4).

4. Proof of Theorem 3.1

From (22), there exist two real numbers $\alpha > 0$ and $M > 1$ such that

$$\left\langle \zeta, \chi^{\kappa} \right\rangle \vee \langle \zeta, \mathbf{1} \rangle < M; \tag{32}$$

$$\langle \zeta, \chi \rangle \wedge \langle \zeta, \mathbf{1} \rangle > 2\alpha, \tag{33}$$

where ζ is defined by (21). Let us define the following subsets of $\mathcal{M}_F(\mathbb{N})$:

$$\mathcal{M}_{\alpha,M} = \left\{ \mu \in \mathcal{M}_F(\mathbb{N});\ \langle \mu, \mathbf{1} \rangle \vee \left\langle \mu, \chi^{\kappa} \right\rangle < M \text{ and } \langle \mu, \chi \rangle \wedge \langle \mu, \mathbf{1} \rangle > \alpha \right\} \tag{34}$$

and for all $n \in \mathbb{N}^*$,

$$n\mathcal{M}_{\alpha,M} = \left\{ \mu \in \mathcal{M}_F(\mathbb{N});\ \frac{1}{n}\mu \in \mathcal{M}_{\alpha,M} \right\}.$$

The main steps of the proofs are the following. After showing that the generator can be approximated for large n by the (non-linear) operator Ψ (at least on the subset $n\mathcal{M}_{\alpha,M}$), we show the tightness property of the family of measure-valued Markov processes under consideration. We then show the uniqueness of the possible limit – which is solution of the deterministic system of Eqs. (23) – and finally the convergence in probability towards this limit.

4.1. Generator approximations

Recall the definition of the mapping $\Psi : \mathcal{M}_F(\mathbb{N}) \to \mathcal{M}_F(\mathbb{N})$ in (23). We show in this section that the finite variation part $\frac{1}{n}\mathscr{Q}^n \Pi_\phi(\mu^n)$ of $\{\bar{\mu}^n\}$ can be approximated by $\frac{1}{n}\Psi(\mu^n)$ as long as $\bar{\mu}^n$ takes values in $\mathcal{M}_{\alpha,M}$. More precisely, we have the following result.

Proposition 4.1. *For any $\alpha, M > 0$, there exists a constant $C(\alpha, M)$ such that for all $\phi \in \mathcal{B}_b(\mathbb{N})$, all $n \in \mathbb{N}^*$ and all counting measures $\mu \in n\mathcal{M}_{\alpha,M}$, we have*

$$\left| \mathscr{Q}\Pi_\phi(\mu) - \langle \Psi(\mu), \phi \rangle \right| \le \sqrt{n} \|\phi\| C(\alpha, M). \tag{35}$$

Remark 4.2. Applying (35) to any measure $\bar{\mu} \in \mathcal{M}_{\alpha,M}$ and any $n \in \mathbb{N}^*$ such that $n\bar{\mu}$ is a counting measure, and using the fact that $\Psi(\bar{\mu}) = \Psi(n\bar{\mu})/n$, yields to

$$\left| \frac{1}{n} \mathcal{Q}^n \Pi_\phi(n\bar{\mu}) - \langle \Psi(\bar{\mu}), \phi \rangle \right| \leq \frac{1}{\sqrt{n}} \|\phi\| C(\alpha, M). \tag{36}$$

The rest of this section is devoted to the proof of Proposition 4.1. Our strategy is as follows: we first propose an alternative and simpler dynamics for updating the measure μ, whose key feature is that the various uniform random choices are made with replacement, instead of without, as in the original construction. We then show that the corresponding alternative update of μ amounts to the action of the operator Ψ (see Lemma 4.3). We finally construct a coupling between the two constructions, leading to the bound (35), as long as μ belongs to $n\mathcal{M}_{\alpha,M}$. The coupling relies on the property that conditionally to the event that all selected elements are different, the law of the sampling with replacement coincides with that of sampling without replacement. Moreover, we prove that the probability of this event tends to 1 as n goes large (see Lemma 4.4).

Alternative construction. Fix a counting measure $\mu \in \mathcal{M}_p(\mathbb{N})$. To each $i \in \mathbb{N}$ are associated $\mu(i)$ buckets of i items. (*Buckets* and *items* correspond to *nodes* and *half-edges* in the original construction.) So there are $b = \langle \mu, \mathbf{1} \rangle$ buckets and a total of $c = \langle \mu, \chi \rangle$ items. We label the buckets from 1 to b according to the labeling procedure proposed in Section 1. Denote for any $j = 1, \ldots, b$, $d(j)$ the cardinality of the bucket j. We also label the items from 1 to c as follows: items 1 to $d(1)$ are the elements of bucket 1, labeled arbitrarily, items $d(1) + 1, \ldots, d(1) + d(2)$ are the elements of bucket 2, and so on.... For any $i = 1, \ldots, c$, let $b(i)$ be the index of the bucket of item i. Note that we are identifying buckets with nodes and items with half-edges.

We perform the following random experiment, which mimics the original dynamics when the sampling is performed with replacement. The various uniform draws in the construction below are independent of one another.

(i) First, we draw uniformly at random an element $j \in [\![1, b]\!]$. Denote

$$\widehat{A} = j \quad \text{and} \quad \widehat{K} = d\left(\widehat{A}\right).$$

(ii) We draw uniformly at random, and with replacement, $\widehat{K}$ items among c. Let $i_1, \ldots, i_{\widehat{K}}$ be the indexes of these items. We then denote for any $\ell \in [\![1, \widehat{K}]\!]$,

$$\widehat{B}_\ell = b\left(i_\ell\right) \quad \text{and} \quad \widehat{Y}_\ell = d\left(\widehat{B}_\ell\right),$$

in other words $\widehat{Y}_\ell$ is the size of the bucket to which item i_ℓ belongs.

(iii) Let

$$\widehat{X} = \sum_{\ell=1}^{\widehat{K}} \left(\widehat{Y}_\ell - 1\right).$$

We draw, uniformly at random and with replacement, $\widehat{X}$ items among c, denoted $i_{\ell,m}$, $\ell \in [\![1, \widehat{K}]\!]$, $m \in [\![1, \widehat{Y}_\ell - 1]\!]$. We denote for all such ℓ, m,

$$\widehat{U}_{\ell,m} = b\left(i_{\ell,m}\right) \quad \text{and} \quad \widehat{W}_{\ell,m} = d\left(\widehat{U}_{\ell,m}\right).$$

Finally, denote the two following point measures,

$$\widehat{Y} = \sum_{\ell=1}^{\widehat{K}} \delta_{\widehat{Y}_\ell}; \qquad \widehat{\mathbf{W}} = \sum_{\ell=1}^{\widehat{K}} \sum_{m=1}^{\widehat{Y}_\ell - 1} \left(\delta_{\widehat{W}_{\ell,m}} - \delta_{\widehat{W}_{\ell,m}-1}\right); \quad \text{and} \quad \widehat{\vartheta}(\mu) = \delta_{\widehat{K}} + \widehat{Y} + \widehat{\mathbf{W}}. \tag{37}$$

The random measure $\widehat{\vartheta}(\mu)$ corresponds to the update of the measure μ in this alternative construction. As in Section 2.1 (see Remark 2.4), the uniform draws specifying $\widehat{\vartheta}(\mu)$ are assumed independent of the source of randomness generating μ, if any.

Lemma 4.3. *For any counting measure μ such that $\langle \mu, \chi \rangle > 0$ and any $\phi \in \mathcal{B}_b$,*

$$\lambda \langle \mu, \mathbf{1} \rangle \mathbb{E}_\mu \left[\langle \widehat{\vartheta}(\mu), \phi \rangle \right] = \langle \Psi(\mu), \phi \rangle, \tag{38}$$

for $\Psi(.)$ defined by (23).

Proof. Clearly, from the very construction above, $\widehat{K}$ follows a discrete uniform distribution on $[\![1, \langle \mu, \mathbf{1} \rangle]\!]$, in other words

$$\mathbb{P}_\mu \left(\widehat{K} = k \right) = \frac{\mu(k)}{\langle \mu, \mathbf{1} \rangle}, \quad k \in \mathbb{N}. \tag{39}$$

Also, all r.v.'s $\widehat{Y}_\ell$ and all $\widehat{W}_{\ell,m}$ above follow the size-biased distribution associated to μ: conditionally on $\widehat{K} = k$ and $\widehat{Y}_\ell = y_\ell$, $\ell = 1, \dots, k$,

$$\mathbb{P}_\mu \left(\widehat{Y}_\ell = y \right) = \frac{y\mu(y)}{\langle \mu, \chi \rangle}, \quad \ell = 1, \dots, k; \tag{40}$$

$$\mathbb{P}_\mu \left(\widehat{W}_{\ell,m} = w \right) = \frac{w\mu(w)}{\langle \mu, \chi \rangle}, \quad \ell = 1, \dots, k,\ m = 1, \dots, y_\ell - 1.$$

We therefore obtain that

$$\begin{aligned} \lambda \langle \mu, \mathbf{1} \rangle \mathbb{E}_\mu \left[\langle \widehat{\vartheta}(\mu), \phi \rangle \right] &= -\lambda \langle \mu, \mathbf{1} \rangle \sum_{k=0}^{+\infty} \frac{\mu(k)}{\langle \mu, \mathbf{1} \rangle} \\ &\times \left\{ \phi(k) + k \sum_{y=1}^{+\infty} \frac{y\mu(y)}{\langle \mu, \chi \rangle} \left(\phi(y) + (y-1) \sum_{w=1}^{+\infty} \frac{w\mu(w)}{\langle \mu, \chi \rangle} \Big(\phi(w) - \phi(w-1) \Big) \right) \right\} \\ &= -\lambda \left(\langle \mu, \phi \rangle + \langle \mu, \chi \rangle \left(\frac{\langle \mu, \chi\phi \rangle}{\langle \mu, \chi \rangle} + \frac{\langle \mu, \chi^2 - \chi \rangle}{\langle \mu, \chi \rangle} \frac{\langle \mu, \chi \nabla\phi \rangle}{\langle \mu, \chi \rangle} \right) \right) = \langle \Psi(\mu), \phi \rangle. \quad \blacksquare \end{aligned}$$

Coupling. Fix a counting measure $\mu \in \mathcal{M}_p(\mathbb{N})$. Consider the construction of Section 2, starting just before a change time t with the counting measure $\mu_{t^-} = \mu$, and recall the definitions of $K(\mu)$ and $\tilde{K}(\mu)$, of $W(\mu)(i, \ell)$, $i, \ell \in \mathbb{N}$ in (11), of $X(\mu)$ in (9), and of the point measures $Y(\mu)$, $\tilde{Y}(\mu)$ and $\mathbf{W}(\mu)$, respectively in (5), (6) and (12).

Now, let us define the event that of no self-loops and/or multiple edges are drawn in the construction of Section 2., i.e.

$$\mathcal{D}(\mu) = \left\{ \tilde{K}(\mu) = K(\mu), \tilde{Y}(\mu) = Y(\mu),\ W(\mu)(i, \ell) \in \{0, 1\} \text{ for all } i, \ell \right\}.$$

And the corresponding one in the alternative construction, that is the selected buckets and items are all distinct, i.e.

$$\widehat{\mathcal{D}}(\mu) = \left\{ \widehat{A}, \widehat{B}_1, \dots, \widehat{B}_{\widehat{K}}, \widehat{U}_{1,1}, \dots, \widehat{U}_{1,\widehat{Y}_1 - 1}, \widehat{U}_{2,1}, \dots \ \dots, \widehat{U}_{\widehat{K}, \widehat{Y}_{\widehat{K}} - 1} \text{ are all distinct} \right\}.$$

Clearly, identifying buckets with nodes and items with half-edges in the new construction defined above, we see that the r.v.'s $\widehat{K}(\mu)$, $\widehat{Y}(\mu)$ and $\widehat{\mathbf{W}}(\mu)$ play similar respective roles as $K(\mu)$, $\tilde{Y}(\mu)$ and $\mathbf{W}(\mu)$, substituting uniform random selections without replacement by selections with replacement. Also, observe that samples in $\widehat{\mathcal{D}}(\mu)$ prevent the selection of the same bucket, or the same item, twice. Therefore, as uniform sampling without replacement amounts, in law, to sampling with replacement conditionally on not selecting twice the same element, $\left(K(\mu), \tilde{Y}(\mu), \mathbf{W}(\mu)\right)$ has the same distribution conditionally on $\mathcal{D}(\mu)$ as $\left(\widehat{K}(\mu), \widehat{Y}(\mu), \widehat{\mathbf{W}}(\mu)\right)$ conditionally on $\widehat{\mathcal{D}}(\mu)$. In turn, recalling the definitions (15) and (37), we obtain that for any $v \in \mathcal{M}_p(\mathbb{N})$,

$$\mathbb{P}_\mu\left(\widehat{\vartheta}(\mu) = v \mid \widehat{\mathcal{D}}(\mu)\right) = \mathbb{P}_\mu\left(\vartheta(\mu) = v \mid \mathcal{D}(\mu)\right). \tag{41}$$

The proof of Proposition 4.1 relies on the coupling defined above and the key remark that obtaining self-loops and multiple edges is easier if the half-edges are drawn with, than without replacement, that is $\mathbb{P}_\mu\left(\widehat{\mathcal{D}}(\mu)\right) \leq \mathbb{P}_\mu\left(\mathcal{D}(\mu)\right)$, along with Lemma 4.4.

Before proving Lemma 4.4 and Proposition 4.1, recall the original construction in Section 2, and let us make precise the distributions of the random variables $K(\mu)$ and $Y(\mu)$ defined therein. In step 1, observe that the probability that a node of degree k is drawn, equals the proportion of atoms at level k among all atoms of the measure μ. In other words, for any $k \in \mathbf{N}$,

$$\mathbb{P}_\mu(K(\mu) = k) = \frac{\mu(k)}{\langle \mu, \mathbf{1}\rangle}. \tag{42}$$

Second, conditionally on $\{K(\mu) = k\}$ and $\{\tilde{K}(\mu) = \tilde{k}\}$, $Y(\mu)$ follows a multivariate hypergeometrical distribution on $\mathcal{M}_F(\mathbb{N})$, of parameters $\left(\tilde{k}, \langle\mu,\chi\rangle - k, \langle\mu,\mathbf{1}\rangle - 1, P\right)$ (see Appendix), where P is given by

$$P(i) = i\left(\mu(i) - \delta_k(i)\right), \quad i \in [\![0, \langle\mu,\mathbf{1}\rangle - 1]\!].$$

In other words, we have for all $y \in \mathcal{M}_F([\![0, \langle\mu,\mathbf{1}\rangle - 1]\!])$ such that $\langle y, \mathbf{1}\rangle = \tilde{k}$ and $y(i) \leq P(i)$ for all i,

$$\mathbb{P}_\mu\left(Y(\mu) = y \mid K(\mu) = k, \tilde{K}(\mu) = \tilde{k}\right) = \frac{\prod_{i \in [\![0,\langle\mu,\mathbf{1}\rangle-1]\!]} \binom{i(\mu(i)-\delta_k(i))}{y(i)}}{\binom{\langle\mu,\chi\rangle - k}{\tilde{k}}}. \tag{43}$$

Lemma 4.4. *Fix $\alpha > 0$, $M > 0$ and $n > 1$. Then, for any counting measure $\mu \in n\mathcal{M}_{\alpha,M}$, there exists a constant $\widehat{C}_1(\alpha, M)$ such that*

$$\mathbb{P}_\mu\left(\widehat{D}(\mu)^c\right) \leq \frac{\widehat{C}_1(\alpha, M)}{n}.$$

Proof. First, observe that for all $k > 0$,

$$\begin{aligned}
&\mathbb{P}_\mu\left(\widehat{B}_i = \widehat{B}_j \text{ for some } i \neq j \mid \widehat{K} = k\right) \leq \binom{k}{2} \mathbb{P}_\mu\left(\widehat{B}_2 = \widehat{B}_1\right) \\
&= \binom{k}{2} \sum_{y=1}^{+\infty} \mathbb{P}_\mu\left(\widehat{B}_2 = \widehat{B}_1 \mid \widehat{Y}_1 = y\right) \mathbb{P}_\mu\left(\widehat{Y}_1 = y\right) = \binom{k}{2} \sum_{y=1}^{+\infty} \frac{y}{\langle\mu,\chi\rangle} \frac{y\mu(y)}{\langle\mu,\chi\rangle} \\
&= \binom{k}{2} \frac{\langle\mu,\chi^2\rangle}{\langle\mu,\chi\rangle^2}.
\end{aligned} \tag{44}$$

Now, for all k we have that

$$\mathbb{P}_\mu\left(\widehat{B}_i = \widehat{A} \text{ for some } i \mid \widehat{K} = k\right) \le k\mathbb{P}_\mu\left(\widehat{B}_1 = \widehat{A} | \widehat{K} = k\right) = k\frac{k}{\langle \mu, \chi\rangle} = \frac{k^2}{\langle \mu, \chi\rangle}. \tag{45}$$

Further, as the buckets $\widehat{B}_1, \ldots, \widehat{B}_{\widehat{K}}$ gather altogether $\widehat{X} + \widehat{K}$ items, we have for all k and x,

$$\begin{aligned}&\mathbb{P}_\mu\left(\widehat{U}_{\ell,m} \in \left\{\widehat{A}, \widehat{B}_1, \ldots, \widehat{B}_{\widehat{K}}\right\} \text{ for some } \ell, m \mid \widehat{K} = k, \widehat{X} = x\right)\\ &\quad\le x\mathbb{P}_\mu\left(\widehat{U}_{1,1} \in \left\{\widehat{A}, \widehat{B}_1, \ldots, \widehat{B}_{\widehat{K}}\right\} \mid \widehat{K} = k, \widehat{X} = x\right) = x\frac{k+x+k}{\langle \mu, \chi\rangle} = \frac{x^2+2xk}{\langle \mu, \chi\rangle}.\end{aligned} \tag{46}$$

Finally, for all k and x, similarly to (44) we obtain that

$$\begin{aligned}&\mathbb{P}_\mu\left(\widehat{U}_{\ell,m} = \widehat{U}_{\ell',m'} \text{ for some } (\ell, m) \ne (\ell', m') \mid \widehat{K} = k, \widehat{X} = x\right)\\ &\quad\le \binom{x}{2}\mathbb{P}_\mu\left(\widehat{U}_{1,1} = \widehat{U}_{1,2}\right) = \binom{x}{2}\sum_{w=1}^{+\infty}\mathbb{P}_\mu\left(\widehat{U}_{11} = \widehat{U}_{12} \mid \widehat{W}_{11} = w\right)\mathbb{P}_\mu\left(\widehat{W}_{11} = w\right)\\ &\quad= \binom{x}{2}\frac{\langle \mu, \chi^2\rangle}{\langle \mu, \chi\rangle^2}.\end{aligned} \tag{47}$$

Now, observe that

$$\begin{aligned}\mathbb{E}_\mu\left[\widehat{X}^2 | \widehat{K} = k\right] &= \mathrm{Var}_\mu\left(\sum_{\ell=1}^{\widehat{K}}(\widehat{Y}_\ell - 1)\right) + \left(\mathbb{E}_\mu\left[\sum_{\ell=1}^{\widehat{K}}(\widehat{Y}_\ell - 1)\right]\right)^2\\ &= k\mathrm{Var}_\mu\left(\widehat{Y}_1 - 1\right) + k^2\left(\mathbb{E}_\mu\left[\widehat{Y}_1 - 1\right]\right)^2\\ &= k\mathbb{E}_\mu\left[(\widehat{Y}_1 - 1)^2\right] + (k^2 - k)\left(\mathbb{E}_\mu\left[\widehat{Y}_1 - 1\right]\right)^2.\end{aligned} \tag{48}$$

But it follows from (39) and (40) that

$$\begin{aligned}&\mathbb{E}_\mu\left[\widehat{K}\right] = \frac{\langle \mu, \chi\rangle}{\langle \mu, \mathbf{1}\rangle}, \qquad \mathbb{E}_\mu\left[\widehat{K}^2\right] = \frac{\langle \mu, \chi^2\rangle}{\langle \mu, \mathbf{1}\rangle}, \qquad \mathbb{E}_\mu\left[\widehat{Y}_1\right] = \frac{\langle \mu, \chi^2\rangle}{\langle \mu, \chi\rangle} \quad \text{and}\\ &\mathbb{E}_\mu\left[\left(\widehat{Y}_1\right)^2\right] = \frac{\langle \mu, \chi^3\rangle}{\langle \mu, \chi\rangle}\end{aligned} \tag{49}$$

which, together with (48), entails that

$$\begin{aligned}\mathbb{E}_\mu\left[\widehat{X}^2\right] &\le \mathbb{E}_\mu\left[\widehat{K}\right]\mathbb{E}_\mu\left[(\widehat{Y}_1)^2\right] + \mathbb{E}_\mu\left[\widehat{K}^2\right]\left(\mathbb{E}_\mu\left[\widehat{Y}_1\right]\right)^2\\ &= \frac{\langle \mu, \chi\rangle}{\langle \mu, \mathbf{1}\rangle}\frac{\langle \mu, \chi^3\rangle}{\langle \mu, \chi\rangle} + \frac{\langle \mu, \chi^2\rangle}{\langle \mu, \mathbf{1}\rangle}\frac{\langle \mu, \chi^2\rangle^2}{\langle \mu, \chi\rangle^2} = \frac{\langle \mu, \chi^3\rangle}{\langle \mu, \mathbf{1}\rangle} + \frac{\langle \mu, \chi^2\rangle}{\langle \mu, \mathbf{1}\rangle}\frac{\langle \mu, \chi^2\rangle^2}{\langle \mu, \chi\rangle^2}.\end{aligned} \tag{50}$$

Finally, gathering (44)–(47) and then using (49) and (50) and the fact that $\mu \in n\mathcal{M}_{\alpha,M}$, we conclude that

$$\begin{aligned}
\mathbb{P}_\mu\left(\left(\widehat{D}(\mu)\right)^c\right) &\le \left(\mathbb{E}_\mu\left[\widehat{K}^2\right]+\mathbb{E}_\mu\left[\widehat{X}^2\right]\right)\frac{\langle\mu,\chi^2\rangle}{\langle\mu,\chi\rangle^2}+\mathbb{E}_\mu\left[(\widehat{X}+\widehat{K})^2\right]\frac{1}{\langle\mu,\chi\rangle}\\
&\le\left(\frac{\langle\mu,\chi^2\rangle}{\langle\mu,\chi\rangle^2}+\frac{2}{\langle\mu,\chi\rangle}\right)\left(\mathbb{E}_\mu\left[\widehat{K}^2\right]+\mathbb{E}_\mu\left[\widehat{X}^2\right]\right)\le\frac{1}{n}\left(\frac{M}{\alpha^2}+\frac{2}{\alpha}\right)\left(\frac{2M}{\alpha}+\frac{M^3}{\alpha^3}\right)\\
&=\frac{\widehat{C}_1(\alpha,M)}{n}.
\end{aligned}$$

Hence the result. ■

We are now in position to prove Proposition 4.1.

Proof of Proposition 4.1. For notational simplicity, we skip the dependence in μ of ϑ, $\widehat{\vartheta}$, $\mathcal{D}$ and $\widehat{\mathcal{D}}$. Recall that $\mathbb{P}_\mu\left(\widehat{\mathcal{D}}\right)\le\mathbb{P}_\mu\left(\mathcal{D}\right)$. Then, we can define the following probability,

$$p_\mu := \frac{\mathbb{P}_\mu\left(\mathcal{D}\right)-\mathbb{P}_\mu\left(\widehat{\mathcal{D}}\right)}{1-\mathbb{P}_\mu\left(\widehat{\mathcal{D}}\right)}\in(0,1).$$

Now, define the following $\mathcal{M}_p(\mathbb{N})$-valued random variable,

$$\widehat{\vartheta}' = \widehat{\vartheta}\mathbb{1}_{\widehat{\mathcal{D}}}+\theta\mathbb{1}_{\widehat{\mathcal{D}}^c},$$

where θ is drawn from the distribution $\mathbb{P}_\mu\left(\vartheta = .\mid\mathcal{D}\right)$ with probability p_μ, and is drawn independently from the distribution $\mathbb{P}_\mu\left(\vartheta=.\mid\mathcal{D}^c\right)$ with probability $1-p_\mu$. So defined, $\widehat{\vartheta}'$ has the same distribution as ϑ, since we have for any v that

$$\begin{aligned}
\mathbb{P}_\mu\left(\widehat{\vartheta}'=v\right) &= P_\mu\left(\widehat{\vartheta}'=v\mid\widehat{\mathcal{D}}\right)\mathbb{P}_\mu\left(\widehat{\mathcal{D}}\right)+P_\mu\left(\widehat{\vartheta}'=v\mid\widehat{\mathcal{D}}^c\right)\left(1-\mathbb{P}_\mu\left(\widehat{\mathcal{D}}\right)\right)\\
&=\mathbb{P}_\mu\left(\widehat{\vartheta}=v\mid\widehat{\mathcal{D}}\right)\mathbb{P}_\mu\left(\widehat{\mathcal{D}}\right)+\Big(\mathbb{P}_\mu\left(\vartheta=v\mid\mathcal{D}\right)p_\mu+P_\mu\left(\vartheta=v\mid\mathcal{D}^c\right)(1-p_\mu)\Big)\\
&\quad\times\left(1-\mathbb{P}_\mu\left(\widehat{\mathcal{D}}\right)\right)\\
&=\mathbb{P}_\mu\left(\vartheta=v\mid\mathcal{D}\right)P_\mu\left(\widehat{\mathcal{D}}\right)+\mathbb{P}_\mu\left(\vartheta=v\mid\mathcal{D}\right)\left(\mathbb{P}_\mu\left(\mathcal{D}\right)-\mathbb{P}_\mu\left(\widehat{\mathcal{D}}\right)\right)\\
&\quad+\mathbb{P}_\mu\left(\vartheta=v\mid\mathcal{D}^c\right)\left(1-\mathbb{P}_\mu\left(\mathcal{D}\right)\right)\\
&=\mathbb{P}_\mu\left(\vartheta=v\right),
\end{aligned}$$

where we used (41) in the third equality. Therefore, as $\widehat{\mathcal{D}}\subset\{\widehat{\vartheta}=\widehat{\vartheta}'\}$, we obtain from (18) and Lemma 4.3 that

$$\begin{aligned}
\left|\mathscr{Q}\Pi_\phi(\mu)-\langle\Psi(\mu),\phi\rangle\right| &= \lambda\langle\mu,\mathbf{1}\rangle\left|\mathbb{E}_\mu\left[\langle\vartheta,\phi\rangle\right]-\mathbb{E}_\mu\left[\langle\widehat{\vartheta},\phi\rangle\right]\right|\\
&=\lambda\langle\mu,\mathbf{1}\rangle\left|\mathbb{E}_\mu\left[\langle\widehat{\vartheta}'-\widehat{\vartheta},\phi\rangle\mathbb{1}_{\widehat{\mathcal{D}}^c}\right]\right|\\
&\le\lambda\langle\mu,\mathbf{1}\rangle\left(\mathbb{E}_\mu\left[\langle\widehat{\vartheta}'-\widehat{\vartheta},\phi\rangle^2\right]\right)^{1/2}\left(\mathbb{P}_\mu\left(\widehat{\mathcal{D}}^c\right)\right)^{1/2}\\
&\le\lambda\langle\mu,\mathbf{1}\rangle\left(2\mathbb{E}_\mu\left[\langle\vartheta,\phi\rangle^2\right]+2\mathbb{E}_\mu\left[\langle\widehat{\vartheta},\phi\rangle^2\right]\right)^{1/2}\left(\mathbb{P}_\mu\left(\widehat{\mathcal{D}}^c\right)\right)^{1/2}.
\end{aligned}\tag{51}$$

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First, observe that using (49) and (50),

$$\mathbb{E}_\mu\left[\left\langle\widehat{\vartheta},\phi\right\rangle^2\right] \le \|\phi\|^2\mathbb{E}_\mu\left[\left\langle\widehat{\vartheta},\mathbf{1}\right\rangle^2\right] \le \|\phi\|^2\mathbb{E}_\mu\left[\left(1+\widehat{K}(\mu)+2\widehat{X}(\mu)\right)^2\right]$$
$$\le 3\|\phi\|^2\left(1+\mathbb{E}_\mu\left[\widehat{K}(\mu)^2\right]+4\mathbb{E}_\mu\left[\widehat{X}(\mu)^2\right]\right) \le 3\|\phi\|^2\left(1+\frac{5M}{\alpha}+\frac{4M^3}{\alpha^3}\right). \tag{52}$$

All the same, we have

$$\mathbb{E}_\mu\left[\langle\vartheta,\phi\rangle^2\right] \le 3\|\phi\|^2\left(1+\mathbb{E}_\mu\left[K(\mu)^2\right]+4\mathbb{E}_\mu\left[X(\mu)^2\right]\right). \tag{53}$$

Now, we clearly have from (42) that

$$\mathbb{E}_\mu\left[K(\mu)^2\right] = \sum_{k\in\mathbb{N}} k^2\frac{\mu(k)}{\langle\mu,\mathbf{1}\rangle} = \frac{\langle\mu,\chi^2\rangle}{\langle\mu,1\rangle} \le \frac{M}{\alpha}. \tag{54}$$

Also, from (10) we have

$$\mathbb{E}_\mu\left[X(\mu)^2\right] \le \mathbb{E}_\mu\left[\langle Y(\mu),\chi\rangle^2\right]$$
$$= \sum_{\substack{k\in\mathbb{N}^*:\\ k\le\langle\mu,\chi\rangle/4}} \mathbb{E}_\mu\left[\langle Y(\mu),\chi\rangle^2 \mid K(\mu)=k\right]\frac{\mu(k)}{\langle\mu,\mathbf{1}\rangle}$$
$$+ \sum_{\substack{k\in\mathbb{N}^*:\\ k>\langle\mu,\chi\rangle/4}} \mathbb{E}_\mu\left[\langle Y(\mu),\chi\rangle^2 \mid K(\mu)=k\right]\frac{\mu(k)}{\langle\mu,\mathbf{1}\rangle}. \tag{55}$$

But as $\langle\mu,\chi\rangle > n\alpha$ we have for all $k\in\mathbb{N}^*$ such that $k\le\langle\mu,\chi\rangle/4$,

$$\langle\mu,\chi\rangle-(k+1) \ge \langle\mu,\chi\rangle-2k \ge \frac{n\alpha}{2}.$$

Thus, in view of the distribution of $Y(.)$ in (43) and from (A.3), we obtain that

$$\sum_{\substack{k\in\mathbb{N}^*:\\ k\le\langle\mu,\chi\rangle/4}} \mathbb{E}_\mu\left[\langle Y(\mu),\chi\rangle^2 \mid K(\mu)=k\right]\frac{\mu(k)}{\langle\mu,\mathbf{1}\rangle}$$
$$\le \sum_{\substack{k\in\mathbb{N}^*:\\ k\le\langle\mu,\chi\rangle/4}} \frac{k(k-1)\langle\mu,\chi\rangle^2}{(\langle\mu,\chi\rangle-k)(\langle\mu,\chi\rangle-(k+1))}\frac{\mu(k)}{\langle\mu,\mathbf{1}\rangle} \le \frac{\langle\mu,\chi^2\rangle}{\langle\mu,\mathbf{1}\rangle}\frac{4M^2}{\alpha^2} \le \frac{4M^3}{\alpha^3}. \tag{56}$$

Also, as $\langle\mu,\chi\rangle-K(\mu)-\langle Y(\mu),\chi\rangle\ge 0$, we have for all k such that $k>\langle\mu,\chi\rangle/4$, conditionally on $\{K(\mu)=k\}$,

$$\langle Y(\mu),\chi\rangle \le \frac{3\langle\mu,\chi\rangle}{4} < 3k,$$

which entails that

$$\sum_{\substack{k\in\mathbb{N}^*:\\ k>\langle\mu,\chi\rangle/4}} \mathbb{E}_\mu\left[\langle Y(\mu),\chi\rangle^2 \mid K(\mu)=k\right]\frac{\mu(k)}{\langle\mu,\mathbf{1}\rangle}$$
$$\le \sum_{\substack{k\in\mathbb{N}^*:\\ k>\langle\mu,\chi\rangle/4}} \frac{9k^2\mu(k)}{\langle\mu,\mathbf{1}\rangle} \le 9\frac{\langle\mu,\chi^2\rangle}{\langle\mu,\mathbf{1}\rangle} \le \frac{9M}{\alpha}. \tag{57}$$

Gathering (54)–(57) in (53), and then (52) and (53) in (51), we obtain that for some constant $C_2(\alpha, M) > 0$,

$$\begin{aligned}\left|\mathscr{Q}\Pi_\phi(\mu) - \langle \Psi(\mu), \phi\rangle\right| &\le \lambda \langle \mu, \mathbf{1}\rangle \|\phi\| C_2(\alpha, M) \left(\mathbb{P}_\mu\left(\widehat{\mathcal{D}}^c\right)\right)^{1/2}\\ &\le \lambda \|\phi\| C_2(\alpha, M) n M \left(\mathbb{P}_\mu\left(\widehat{\mathcal{D}}^c\right)\right)^{1/2},\end{aligned}$$

and Lemma 4.4 allows to conclude. ■

4.2. Tightness

Recall the definition of $\alpha > 0$ in (33). For all $n \in \mathbb{N}^*$, define the stopping time

$$\tau_\alpha^n = \sup\left\{t \ge 0;\ \left\langle \mu_t^n, \mathbf{1}\right\rangle \wedge \left\langle \mu_t^n, \chi\right\rangle \ge n\alpha\right\},$$

and denote $\bar{\mu}^{n,\tau_\alpha^n}$ the version of the process $\bar{\mu}^n$ stopped at τ_α^n, in other words for all t,

$$\bar{\mu}_t^{n,\tau_\alpha^n} = \bar{\mu}^n_{t\wedge\tau_\alpha^n}.$$

Proposition 4.5. *For any $T > 0$, the sequence of measure-valued processes $\left\{\left(\bar{\mu}_t^{n,\tau_\alpha^n}\right)_{t\ge 0}\right\}_{\mathbb{N}^*}$ is tight in $D([0, T], \mathcal{M}_F(\mathbb{N}))$.*

Proof. First fix $t \ge 0$. It is clear that the family of random measure $\left\{\bar{\mu}_t^{n,\tau_\alpha^n}\right\}_{n\in\mathbb{N}^*}$ is tight. Indeed we have almost surely for any finite subset A of $\mathbb{N}$, $\mu_t^n(A) \le \mu_0^n(A)$ for all $\mathbb{N}$. Hence, the family of random variables $\left\{\bar{\mu}_t^{n,\tau_\alpha^n}(A)\right\}_{n\in\mathbb{N}^*}$ is tight for all such A, which implies in turn that the family of random measures $\left\{\bar{\mu}_t^{n,\tau_\alpha^n}\right\}_{n\in\mathbb{N}^*}$ is tight (see Lemma 14.15 in [16]).

Therefore, from Roelly's criterion [24], it suffices to show that $\left\{\left(\left\langle\bar{\mu}_t^{n,\tau_\alpha^n}, \phi\right\rangle_t\right)_{t\ge 0}\right\}_{n\in\mathbb{N}^*}$ is tight in $D([0, T], \mathbb{R})$ for all $\phi \in \mathcal{B}_b$. For this, we exploit the semi-martingale decomposition (19) as in Joffe and Métivier (Corollary 2.3.3 in [15]) and apply Rebolledo–Aldous's criterion [1] for the finite variation part $\frac{1}{n}\int_0^{\cdot\wedge\tau_\alpha^n} \mathscr{Q}\Pi_\phi\left(\mu_s^n\right) ds$ and the quadratic variation process $\langle\langle \bar{M}^n(\phi)\rangle\rangle_{\cdot\wedge\tau_\alpha^n}$. Specifically, we aim at showing that for all $\varepsilon > 0$ and $\eta > 0$, there exist $\delta > 0$ and n_0 such that

$$\sup_{n\ge n_0} \mathbb{P}\left[\left|\frac{1}{n}\int_{\sigma_n\wedge\tau_\alpha^n}^{\rho^n\wedge\tau_\alpha^n} \mathscr{Q}\Pi_\phi\left(\mu_s^n\right) ds\right| \ge \eta\right] \le \varepsilon; \tag{58}$$

$$\sup_{n\ge n_0} \mathbb{P}\left[\left|\langle\langle \bar{M}^n(\phi)\rangle\rangle_{\sigma^n\wedge\tau_\alpha^n} - \langle\langle \bar{M}^n(\phi)\rangle\rangle_{\rho_n\wedge\tau_\alpha^n}\right| \ge \eta\right] \le \varepsilon, \tag{59}$$

for any two sequences $\{\rho^n\}_{n\in\mathbb{N}^*}$ and $\{\sigma_n\}_{n\in\mathbb{N}^*}$ of stopping times such that $\rho^n < \sigma_n < \rho^n + \delta$ for all $n \in \mathbb{N}^*$.

Let us first address the finite variation part. We have for all $n \in \mathbb{N}^*$,

$$\begin{aligned}&\mathbb{P}\left[\left|\frac{1}{n}\int_{\rho^n\wedge\tau_\alpha^n}^{\sigma^n\wedge\tau_\alpha^n} \mathscr{Q}\Pi_\phi\left(\mu_s^n\right) ds\right| > \eta\right]\\ &\le \mathbb{P}\left[\left|\frac{1}{n}\int_{\rho^n\wedge\tau_\alpha^n}^{\sigma^n\wedge\tau_\alpha^n} \mathscr{Q}\Pi_\phi\left(\mu_s^n\right) ds\right| > \eta\,;\ \bar{\mu}_0^n \in \mathcal{M}_{\alpha,M}\right] + \mathbb{P}\left[\bar{\mu}_0^n \notin \mathcal{M}_{\alpha,M}\right].\end{aligned} \tag{60}$$

First, we have that

$$\begin{aligned}
&\mathbb{P}\left[\bar{\mu}_0^n \notin \mathcal{M}_{\alpha,M}\right] \\
&\le \mathbb{P}\left[\langle\bar{\mu}_0^n,\chi\rangle \le \alpha\right] + \mathbb{P}\left[\langle\bar{\mu}_0^n,\mathbf{1}\rangle \le \alpha\right] + \mathbb{P}\left[\langle\bar{\mu}_0^n,\chi^\kappa\rangle \ge M\right] + \mathbb{P}\left[\langle\bar{\mu}_0^n,\mathbf{1}\rangle \ge M\right] \\
&\le \mathbb{P}\left[\langle\bar{\mu}_0^n,\chi\rangle < \langle\zeta,\chi\rangle - \frac{\langle\zeta,\chi\rangle-\alpha}{2}\right] + \mathbb{P}\left[\langle\bar{\mu}_0^n,\mathbf{1}\rangle < \langle\zeta,\mathbf{1}\rangle - \frac{\langle\zeta,\mathbf{1}\rangle-\alpha}{2}\right] \\
&\quad + \mathbb{P}\left[\langle\bar{\mu}_0^n,\chi^\kappa\rangle > \langle\zeta,\chi^\kappa\rangle + \frac{M-\langle\zeta,\chi^\kappa\rangle}{2}\right] + \mathbb{P}\left[\langle\bar{\mu}_0^n,\mathbf{1}\rangle > \langle\zeta,\mathbf{1}\rangle + \frac{M-\langle\zeta,\mathbf{1}\rangle}{2}\right] \\
&\xrightarrow[n\to\infty]{} 0,
\end{aligned} \tag{61}$$

in view of (21), (32) and (33).

Second, from (18) and Markov's inequality we have that

$$\begin{aligned}
&\mathbb{P}\left[\left|\frac{1}{n}\int_{\rho^n\wedge\tau_\alpha^n}^{\sigma^n\wedge\tau_\alpha^n} \mathcal{Q}\Pi_\phi\left(\mu_s^n\right)ds\right| > \eta\,;\ \bar{\mu}_0^n \in \mathcal{M}_{\alpha,M}\right] \\
&\le \frac{1}{\eta}\mathbb{E}\left[\left(\frac{1}{n}\int_{\rho^n\wedge\tau_\alpha^n}^{\sigma^n\wedge\tau_\alpha^n} \left|\mathcal{Q}\Pi_\phi\left(\mu_s^n\right)\right| ds\right)\mathbb{1}_{\{\bar{\mu}_0^n\in\mathcal{M}_{\alpha,M}\}}\right] \\
&\le \frac{\lambda}{\eta}\mathbb{E}\left[\left(\int_{\rho^n\wedge\tau_\alpha^n}^{\sigma^n\wedge\tau_\alpha^n} \langle\bar{\mu}_s^n,\mathbf{1}\rangle\left|\mathbb{E}_{\mu_s^n}\left[\langle\vartheta\left(\mu_s^n\right),\phi\rangle\right]\right| ds\right)\mathbb{1}_{\{\bar{\mu}_0^n\in\mathcal{M}_{\alpha,M}\}}\right] \\
&\le \frac{\lambda\|\phi\|}{\eta}\mathbb{E}\Bigg[\left(\int_{\rho^n\wedge\tau_\alpha^n}^{\sigma^n\wedge\tau_\alpha^n} \sum_k \bar{\mu}_s^n(k)\left\{1+k+2\mathbb{E}_{\mu_s^n}\left[\langle Y(\mu_s^n),\chi\rangle \mid K(\mu_s^n)=k\right]\right\}ds\right) \\
&\quad \times \mathbb{1}_{\{\bar{\mu}_0^n\in\mathcal{M}_{\alpha,M}\}}\Bigg],
\end{aligned} \tag{62}$$

where we use the same bound as in (53). Observe that for all μ, $\langle Y(\mu),\chi\rangle \le \langle Y(\mu),\chi\rangle^2$. Thus for all $n \in \mathbb{N}^*$ and all $s \in \left[\rho^n\wedge\tau_\alpha^n, \sigma^n\wedge\tau_\alpha^n\right]$, if $\bar{\mu}_0^n \in \mathcal{M}_{\alpha,M}$ we have that

$$\begin{aligned}
&\sum_{k\in\mathbb{N}^*} \bar{\mu}_s^n(k)\mathbb{E}_{\mu_s^n}\left[\langle Y(\mu_s^n),\chi\rangle\right] \\
&\le \sum_{\substack{k\in\mathbb{N}^*:\\ k\le\langle\mu_s^n,\chi\rangle/4}} \bar{\mu}_s^n(k)\mathbb{E}_{\mu_s^n}\left[\langle Y(\mu_s^n),\chi\rangle^2 \mid K(\mu_s^n)=k\right] \\
&\quad + \sum_{\substack{k\in\mathbb{N}^*:\\ k>\langle\mu_s^n,\chi\rangle/4}} \bar{\mu}_s^n(k)\mathbb{E}_{\mu_s^n}\left[\langle Y(\mu_s^n),\chi\rangle^2 \mid K(\mu_s^n)=k\right] \\
&\le \sum_{\substack{k\in\mathbb{N}^*:\\ k\le\langle\mu^s,\chi\rangle/4}} \bar{\mu}_s^n(k)\frac{k(k-1)\langle\mu_s^n,\chi\rangle^2}{\left(\langle\mu_s^n,\chi\rangle-k\right)\left(\langle\mu_s^n,\chi\rangle-(k+1)\right)} + \sum_{\substack{k\in\mathbb{N}^*:\\ k>\langle\mu_s^n,\chi\rangle/4}} \bar{\mu}_s^n(k)9k^2 \\
&\le \frac{4\langle\mu_0^n,\chi\rangle^2}{n^2\alpha^2}\left\langle\bar{\mu}_0^n,\chi^2\right\rangle + 9\left\langle\bar{\mu}_0^n,\chi^2\right\rangle \le \frac{4M^2}{\alpha} + 9M =: C_3(\alpha,M),
\end{aligned} \tag{63}$$

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using the bounds of (56) and (57) and the fact that the functions $t \mapsto \langle \mu_t^n, \chi \rangle$ and $t \mapsto \langle \mu_t^n, \chi^2 \rangle$ are non-increasing for all n. Injecting this in (62) we obtain that

$$\begin{aligned}
&\mathbb{P}\left[\left| \frac{1}{n} \int_{\rho^n \wedge \tau_\alpha^n}^{\sigma^n \wedge \tau_\alpha^n} \mathcal{Q}\Pi_\phi\left(\mu_s^n\right) ds \right| > \eta \,;\; \bar{\mu}_0^n \in \mathcal{M}_{\alpha,M} \right] \\
&\le \frac{\lambda \|\phi\|}{\eta} \mathbb{E}\left[\left(\int_{\rho^n \wedge \tau_\alpha^n}^{\sigma^n \wedge \tau_\alpha^n} \sum_k \bar{\mu}_s^n(k) \left\{ 1 + k + 2C_3(\alpha, M) \right\} ds \right) \mathbb{1}_{\{\bar{\mu}_0^n \in \mathcal{M}_{\alpha,M}\}} \right] \\
&\le \frac{\lambda \|\phi\|}{\eta} \mathbb{E}\left[\left((1 + 2C_3(\alpha, M)) \left\langle \bar{\mu}_0^n, \mathbf{1} \right\rangle + \left\langle \bar{\mu}_0^n, \chi \right\rangle \right) \left(\sigma^n - \rho_n\right) \mathbb{1}_{\{\bar{\mu}_0^n \in \mathcal{M}_{\alpha,M}\}} \right] \\
&\le \frac{\lambda \|\phi\|}{\eta} \left(1 + 2C_3(\alpha, M) + M\right) \mathbb{E}\left[\sigma^n - \rho^n \right].
\end{aligned}$$

As a conclusion, plugging this together with (61) in (60), we can choose δ small enough so that (58) holds for a sufficiently large n_0. All the same, using (20) we obtain that for all $n \in \mathbb{N}^*$,

$$\begin{aligned}
&\mathbb{P}\left[\left| \langle\langle \bar{M}^n(\phi) \rangle\rangle_{\sigma^n} - \langle\langle \bar{M}^n(\phi) \rangle\rangle_{\rho^n} \right| \ge \eta \right] \\
&\le \frac{\lambda}{n\eta} \mathbb{E}\left[\left(\int_{\rho^n \wedge \tau_\alpha^n}^{\sigma^n \wedge \tau_\alpha^n} \left\langle \bar{\mu}_s^n, \mathbf{1} \right\rangle \left| \mathbb{E}_{\mu_s^n}\left[\left\langle \vartheta\left(\mu_s^n\right), \phi \right\rangle^2 \right] \right| ds \right) \mathbb{1}_{\{\bar{\mu}_0^n \in \mathcal{M}_{\alpha,M}\}} \right] \\
&\quad + \mathbb{P}\left[\bar{\mu}_0^n \notin \mathcal{M}_{\alpha,M} \right], \\
&\le \frac{3\lambda \|\phi\|^3}{n\eta} \mathbb{E}\left[\left(\int_{\rho^n \wedge \tau_\alpha^n}^{\sigma^n \wedge \tau_\alpha^n} \sum_k \bar{\mu}_s^n(k) \left\{ 1 + k^2 + 4\mathbb{E}_{\mu_s^n}\left[\left\langle Y(\mu_s^n), \chi \right\rangle^2 \mid K(\mu_s^n) = k \right] \right\} ds \right) \right. \\
&\quad \left. \times \mathbb{1}_{\{\bar{\mu}_0^n \in \mathcal{M}_{\alpha,M}\}} \right] + \mathbb{P}\left[\bar{\mu}_0^n \notin \mathcal{M}_{\alpha,M} \right]
\end{aligned} \tag{64}$$

and, similarly to (58), (61) together with the bound (63) allow to conclude that (59) also holds for a suitable choice of $\delta > 0$ and $n_0 \in \mathbb{N}^*$. □

4.3. Uniqueness

Let $(\bar{\mu}_t)_{t \ge 0}$ and $(\bar{\nu}_t)_{t \ge 0}$, be two solutions of (23) in $\mathcal{C}(\mathbb{R}+, \mathcal{M}_F(\mathbb{N}))$, with the same initial condition ζ, and let us denote $\gamma_t = \bar{\mu}_t - \bar{\nu}_t$ for all t. Denote also for all β such that $0 < \beta < \alpha$,

$$\begin{aligned}
t_\beta^\mu &= \sup\left\{ t \ge 0;\ \langle \bar{\mu}_t, \mathbf{1} \rangle \wedge \langle \bar{\mu}_t, \chi \rangle > \beta \right\}; \\
t_\beta^\nu &= \sup\left\{ t \ge 0;\ \langle \bar{\nu}_t, \mathbf{1} \rangle \wedge \langle \bar{\nu}_t, \chi \rangle > \beta \right\}.
\end{aligned}$$

Fix $\kappa > 3.5$. Observe that in view of (22) and using (23) for $\phi = \chi^\kappa$,

$$\begin{aligned}
\frac{d}{dt} \left\langle \bar{\mu}_t, \chi^\kappa \right\rangle &= -\lambda \left[\left\langle \bar{\mu}_t, (\mathbf{1} + \chi) \chi^\kappa \right\rangle + \left\langle \bar{\mu}_t, \chi^{\kappa+1} - \chi(\chi - 1)^\kappa \chi^\kappa \right\rangle \left(\frac{\langle \bar{\mu}_t, \chi^2 \rangle}{\langle \bar{\mu}_t, \chi \rangle} - 1 \right) \right] \\
&\le 0.
\end{aligned}$$

Hence, we deduce that

$$\left\langle \bar{\mu}_t, \chi^\kappa \right\rangle \vee \left\langle \bar{\nu}_t, \chi^\kappa \right\rangle \le \left\langle \bar{\mu}_0, \chi^\kappa \right\rangle \vee \left\langle \bar{\nu}_0, \chi^\kappa \right\rangle \le M, \qquad t \ge 0.$$

Thus, denoting again for all $\mu \in \mathcal{M}_F(\mathbb{N})$, $L(\mu) = \frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle}$ this leads to

$$L(\bar{\mu}_t) \vee L(\bar{\nu}_t) \leq \frac{M}{\beta}, \quad t \in \left[0, t_\beta^\mu \wedge t_\beta^\nu\right].$$

Let for all $t \geq 0$ and $\varepsilon = \kappa - 3.5$,

$$\Gamma_t = \sum_{i \geq 1} i^{5+\varepsilon} \gamma_t(i)^2.$$

Fix β such that $0 < \beta < \alpha$ and $t \in \left[0, t_\beta^\mu \wedge t_\beta^\nu\right]$. From (23), we have that

$$\begin{aligned}
\frac{d}{dt}\Gamma_t &= \sum_{i \geq 1} i^{5+\varepsilon} \frac{d}{dt} \gamma_t(i)^2 \\
&= 2 \sum_{i \geq 1} i^{5+\varepsilon} \gamma_t(i) \frac{d}{dt} \gamma_t(i) \\
&= 2\lambda \sum_{i \geq 1} \left\{ (i+1) \left(L(\bar{\mu}_t) - 1\right) \gamma_t(i) \gamma_t(i+1) - \left(1 + iL(\bar{\mu}_t)\right) \left(\gamma_t(i)\right)^2 \right\} i^{5+\varepsilon} \\
&\quad + 2\lambda \sum_{i \geq 1} \left\{ \left(L(\bar{\nu}_t) - L(\bar{\mu}_t)\right) \gamma_t(i) (i \bar{\nu}_t(i) - (i+1) \bar{\nu}_t(i+1)) \right\} i^{5+\varepsilon} \\
&= A_t + B_t.
\end{aligned} \tag{65}$$

We first address the term A_t. Using that

$$\begin{aligned}
(i+1)\gamma_t(i+1)\gamma_t(i) = &\frac{1}{2}(i+1)\gamma_t(i+1)^2 + \frac{1}{2}(i+1)\gamma_t(i)^2 \\
&- \frac{1}{2}\left(\sqrt{i+1}\gamma_t(i+1) - \sqrt{i+1}\gamma_t(i)\right)^2,
\end{aligned}$$

we get

$$\begin{aligned}
\sum_{i \geq 1} i^{5+\varepsilon}(i+1)\gamma_t(i+1)\gamma_t(i) &\leq \frac{1}{2} \sum_{i \geq 1} i^{5+\varepsilon}(i+1)\gamma_t(i+1)^2 + \frac{1}{2} \sum_{i \geq 1} i^{5+\varepsilon}(i+1)\gamma_t(i)^2 \\
&\leq \frac{1}{2} \sum_{i \geq 1} (i+1) i^{5+\varepsilon} \gamma_t(i+1)^2 + \frac{1}{2} \sum_{i \geq 1} i^{6+\varepsilon} \gamma_t(i)^2 + \frac{1}{2} \sum_{i \geq 1} i^{5+\varepsilon} \gamma_t(i)^2 \\
&\leq \sum_{i \geq 1} i^{6+\varepsilon} \gamma_t(i)^2 + \frac{1}{2} \sum_{i \geq 1} i^{5+\varepsilon} \gamma_t(i)^2.
\end{aligned}$$

Hence, as $L(\bar{\mu}_t) \geq 1$,

$$\begin{aligned}
A_t &= 2\lambda \sum_{i \geq 1} \left\{ \left(L(\bar{\mu}_t) - 1\right)(i+1)\gamma_t(i+1)\gamma_t(i) - \left(1 + iL(\bar{\mu}_t)\right)\left(\gamma_t(i)\right)^2 \right\} i^{5+\varepsilon} \\
&\leq 2\lambda(L(\bar{\mu}_t) - 1) \left(\sum_{i \geq 1} i^{6+\varepsilon} \gamma_t(i)^2 + \frac{1}{2} \sum_{i \geq 1} i^{5+\varepsilon} \gamma_t(i)^2 \right) - 2\lambda L(\bar{\mu}_t) \sum_{i \geq 1} i^{6+\varepsilon} \gamma_t(i)^2 \\
&\leq \lambda L(\bar{\mu}_t) \sum_{i \geq 1} i^{5+\varepsilon} \gamma_t(i)^2 \\
&\leq \frac{\lambda M}{\beta} \Gamma_t.
\end{aligned} \tag{66}$$

Let us now deal with B_t. First observe that

$$B_t \le 2\lambda |L(\bar{\mu}_t) - L(\bar{\nu}_t)| \sum_{i\ge 1} \bar{\nu}_t(i) i^{6+\varepsilon} |\gamma_t(i)| + \bar{\nu}_t(i+1)(i+1) i^{5+\varepsilon} |\gamma_t(i)|,$$

where the first term can be bounded as follows:

$$\begin{aligned}
&|L(\bar{\mu}_t) - L(\bar{\nu}_t)| \\
&= \left| \frac{\sum_i i^2 \bar{\mu}_t(i) \sum_i i \bar{\nu}_t(i) - \sum_i i \bar{\mu}_t(i) \sum_i i^2 \bar{\nu}_t(i)}{\sum_i i \bar{\mu}_t(i) \sum_i i \bar{\nu}_t(i)} \right| \\
&= \left| \frac{\sum_i i^2 \gamma_t(i) \sum_i i \bar{\nu}_t(i) + \sum_i i^2 \bar{\nu}_t(i) \sum_i i \bar{\nu}_t(i) - \sum_i i \gamma_t(i) \sum_i i^2 \bar{\nu}_t(i) - \sum_i i^2 \bar{\nu}_t(i) \sum_i i \bar{\nu}_t(i)}{\sum_i i \bar{\mu}_t(i) \sum_i i \bar{\nu}_t(i)} \right| \\
&\le \frac{\sum_i i^2 |\gamma_t(i)| \sum_i i \bar{\nu}_t(i) + \sum_i i |\gamma_t(i)| \sum_i i^2 \bar{\nu}_t(i)}{\sum_i i \bar{\mu}_t(i) \sum_i i \bar{\nu}_t(i)} \le \frac{2M}{\beta^2} \sum_i i^2 |\gamma_t(i)|.
\end{aligned}$$

Hence, using Cauchy Schwartz inequality,

$$\begin{aligned}
|L(\bar{\mu}_t) - L(\bar{\nu}_t)| &\le \frac{2M}{\beta^2} \sum_{i\ge 1} i^{\frac{5+\varepsilon}{2}} i^{-\frac{1+\varepsilon}{2}} |\gamma_t(i)|, \\
&\le \frac{2M}{\beta^2} \left(\sum_{i\ge 1} i^{5+\varepsilon} |\gamma_t(i)|^2 \right)^{1/2} \left(\sum_{i\ge 1} \frac{1}{i^{1+\varepsilon}} \right)^{1/2} \le \frac{2Mc}{\beta^2} \Gamma_t^{\frac{1}{2}}.
\end{aligned} \tag{67}$$

On the other hand, using again Cauchy Schwartz inequality and the moment condition previously underlined,

$$\sum_{i\ge 1} \bar{\nu}(i) i^{6+\varepsilon} |\gamma_t(i)| \le \left(\sum_{i\ge 1} i^{5+\varepsilon} \gamma_t(i)^2 \right)^{1/2} \left(\sum_{i\ge 1} i^{7+\varepsilon} \bar{\nu}_t(i)^2 \right)^{1/2} \le \Gamma_t^{\frac{1}{2}} M, \tag{68}$$

where for the last inequality we used that if $x_i \ge 0$ for all $i \ge 0$, then $\sum_i x_i^2 \le \left(\sum_i x_i\right)^2$. All the same, we have that

$$\begin{aligned}
\sum_{i\ge 1} \bar{\nu}_t(i+1)(i+1) i^{5+\varepsilon} |\gamma_t(i)| &\le \left(\sum_{i\ge 1} i^{5+\varepsilon} \gamma_t(i)^2 \right)^{1/2} \left(\sum_{i\ge 1} (i+1)^{7+\varepsilon} \bar{\nu}_t(i+1)^2 \right)^{1/2} \\
&\le \Gamma_t^{\frac{1}{2}} M.
\end{aligned} \tag{69}$$

Hence, using (67) and (68) we obtain that

$$B_t \le \frac{8\lambda M^2 c}{\beta^2} \Gamma_t. \tag{70}$$

Finally, using (66) and (70) in (65), we obtain that for some positive constant C, for all $t \le t_\beta^\mu \wedge t_\beta^\nu$,

$$\frac{d}{dt} \Gamma_t \le C \Gamma_t.$$

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Since $\Gamma(0) = 0$, Γ is a positive function and $t^{\mu}_{\beta} \wedge t^{\nu}_{\beta} > 0$, this shows using Gronwall's Lemma that $\Gamma_t = 0$ for all such t. Therefore, $t^{\mu}_{\beta} = t^{\nu}_{\beta} =: t_{\beta}$, and $\bar{\mu}_t$ and $\bar{\nu}_t$ coincide up to t_{β}. In other words there is at most one solution to (23) up to time t_{β}. Since this is true for all β, and since the only solution $\bar{\mu}$, if any, is such that $t \to \langle \bar{\mu}_t, \chi \rangle$ and $t \to \langle \bar{\mu}_t, \mathbf{1} \rangle$ are continuous, there is at most one solution up to the (positive, in view of (22)) instant

$$t_0 = \sup \left\{ t_{\beta};\ \beta > 0 \right\}.$$

Now observe that

$$\sup \left\{ t \geq 0;\ \langle \bar{\mu}_t, \chi \rangle > 0 \right\} \leq \sup \left\{ t \geq 0;\ \langle \bar{\mu}_t, \mathbf{1} \rangle > 0 \right\},$$

thus by the continuity of the paths of $t \mapsto \langle \bar{\mu}_t, \chi \rangle$ and $t \mapsto \langle \bar{\mu}_t, \mathbf{1} \rangle$, we obtain that

$$t_0 = \sup \left\{ t \geq 0;\ \langle \bar{\mu}_t, \chi \rangle > 0 \right\}. \tag{71}$$

The proof of uniqueness is thus completed by noticing that whenever $t_0 < \infty$, the only solution $\bar{\mu}$ to (23), if any, can be extended uniquely after t_0, as follows:

$$\bar{\mu}_t = \mu_{t_0}(0) e^{-\lambda(t - t_0)} \delta_0, \quad t \geq t_0. \tag{72}$$

4.4. Convergence

Recall that $\zeta \in \mathcal{M}_F(\mathbb{N})$, $M > 1$ and $\alpha > 0$ are respectively defined by (21), (32) and (33). Fix $T > 0$.

Convergence before reaching a given positive threshold.
We first prove the following result.

Proposition 4.6. *There exists a time $t_{2\alpha} > 0$, such that for all $\phi \in \mathcal{B}_b$,*

$$\sup_{t \in [0, t_{2\alpha}]} \left| \left\langle \bar{\mu}^n_t, \phi \right\rangle - \left\langle \bar{\mu}_t, \phi \right\rangle \right| \xrightarrow[n \to \infty]{(\mathcal{P})} 0,$$

where $(\bar{\mu}_t)_{t \geq 0}$ is the unique solution of (23) on $\mathcal{C}([0, t_{2\alpha}], \mathcal{M}_F(\mathbb{N}))$.

Proof. From Proposition 4.5 and Prohorov' Lemma (see *e.g.* [9, p.104]), the sequence of stopped processes $\left\{ \bar{\mu}^{n, \tau^n_{\alpha}} \right\}$ is relatively compact for the topology of weak convergence, and we let $\bar{\mu}^*$ be a sub-sequential limit. First observe that from (21), $\bar{\mu}^*_0 = \bar{\mu}_0 = \zeta$, and in particular from (33), that

$$\left\langle \bar{\mu}^*_0, \mathbf{1} \right\rangle \wedge \left\langle \bar{\mu}^*_0, \chi \right\rangle > 2\alpha \quad \text{a.s.}$$

Let us then define

$$\tau_{2\alpha} = \sup \left\{ t \geq 0;\ \left\langle \bar{\mu}^*_t, \mathbf{1} \right\rangle \wedge \left\langle \bar{\mu}^*_t, \chi \right\rangle > 2\alpha \right\},$$

which is set to $+\infty$ if the latter set is unbounded. Notice that the process $\bar{\mu}^*$ and hence the instant $\tau_{2\alpha}$, are *a priori* random. Fix $\phi \in \mathcal{B}_b$ throughout the proof. Let $n \in \mathbb{N}^*$. We have for all

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$t \in [0, T]$,

$$\int_0^{t\wedge\tau_{2\alpha}\wedge\tau_\alpha^n} \left\langle \Psi\left(\bar\mu_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds = \left(\int_0^{t\wedge\tau_{2\alpha}} \left\langle \Psi\left(\bar\mu_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds\right) \mathbb{1}_{\{\tau_\alpha^n > \tau_{2\alpha}\wedge T\}} + \left(\int_0^{t\wedge\tau_\alpha^n} \left\langle \Psi\left(\bar\mu_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds\right) \mathbb{1}_{\{\tau_\alpha^n \le \tau_{2\alpha}\wedge T\}}. \tag{73}$$

It is immediate to adapt the results concerning the subspace of $\mathcal{M}_F(\mathbb{N})$ defined by equation (3.7) in [5], to the subspace $\mathcal{M}_{\alpha,M}$ defined by (34). In particular the topological properties of Appendix A in [5] apply to $\mathcal{M}_{\alpha,M}$. In view of Lemma A.5 of [5], the following map is continuous for the Skorokhod topology:

$$\begin{cases} \mathcal{D}\left([0,T], \mathcal{M}_{\alpha,M}\right) & \longrightarrow \mathcal{D}([0,T],\mathbb{R}) \\ \mu & \longmapsto \langle \mu_., \chi\rangle, \end{cases}$$

and so does

$$\begin{cases} \mathcal{D}([0,T],\mathbb{R}) & \longrightarrow \mathbb{R} \\ x_. & \longmapsto \inf_{t\in[0,T]} x_t. \end{cases}$$

Therefore, from the Continuous Mapping Theorem [3], along the latter subsequence the following convergences in distribution hold:

$$\inf_{t\in[0,T]} \left\langle \bar\mu^{n,\tau_\alpha^n}_{t\wedge\tau_{2\alpha}}, \mathbf{1}\right\rangle \Rightarrow \inf_{t\in[0,T]} \left\langle \bar\mu^{*}_{t\wedge\tau_{2\alpha}}, \mathbf{1}\right\rangle;$$
$$\inf_{t\in[0,T]} \left\langle \bar\mu^{n,\tau_\alpha^n}_{t\wedge\tau_{2\alpha}}, \chi\right\rangle \Rightarrow \inf_{t\in[0,T]} \left\langle \bar\mu^{*}_{t\wedge\tau_{2\alpha}}, \chi\right\rangle.$$

Hence from Fatou's Lemma, by the very definition of $\tau_{2\alpha}$,

$$\begin{aligned} 1 &= \mathbb{P}\left[\inf_{t\in[0,T]} \left\langle \bar\mu^{*}_{t\wedge\tau_{2\alpha}}, \mathbf{1}\right\rangle \wedge \left\langle \bar\mu^{*}_{t\wedge\tau_{2\alpha}}, \chi\right\rangle > \alpha\right] \\ &\le \lim_{n\to\infty} \mathbb{P}\left[\inf_{t\in[0,T]} \left\langle \bar\mu^{n,\tau_\alpha^n}_{t\wedge\tau_{2\alpha}}, \mathbf{1}\right\rangle \wedge \left\langle \bar\mu^{n,\tau_\alpha^n}_{t\wedge\tau_{2\alpha}}, \chi\right\rangle > \alpha\right] \le \lim_{n\to\infty} \mathbb{P}\left[T\wedge\tau_{2\alpha} < \tau_\alpha^n\right]. \end{aligned} \tag{74}$$

Therefore,

$$\left(\int_0^{t\wedge\tau_\alpha^n} \left\langle \Psi\left(\bar\mu_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds\right) \mathbb{1}_{\{\tau_\alpha^n \le \tau_{2\alpha}\wedge T\}} \xrightarrow[n\to\infty]{(\mathcal{P})} 0. \tag{75}$$

Now, it follows again from Lemma A.5 of [5] that the following mappings are continuous for the Skorokhod topology:

$$\begin{cases} \mathcal{D}\left([0,T], \mathcal{M}_{\alpha,M}\right) & \longrightarrow \mathcal{D}([0,T],\mathbb{R}) \\ \mu & \longmapsto \left\langle \mu_., \chi^2\right\rangle; \\ & \qquad \langle \mu_., (\mathbf{1}+\chi)\phi\rangle; \\ & \qquad \langle \mu_., \chi\nabla\phi\rangle, \end{cases}$$

and it is a classical result that the following map is also continuous:

$$\begin{cases} \mathcal{D}\left([0,T], \mathbb{R}\times\mathbb{R}^*\right) & \longrightarrow \mathcal{D}([0,T],\mathbb{R}) \\ (x_., y_.) & \longmapsto \dfrac{x_.}{y_.}. \end{cases}$$

So from the Continuous Mapping Theorem, the map

$$\begin{cases} \mathcal{D}\left([0,T],\mathcal{M}_{\alpha,M}\right) & \longrightarrow \mathcal{D}\left([0,T],\mathbb{R}\right)\\ \quad\mu & \longmapsto \left\langle \Psi(\mu_.),\phi\right\rangle \end{cases}$$

is itself continuous, and it follows from the continuity of the map

$$\begin{cases} \mathcal{D}\left([0,T],\mathbb{R}\right) & \longrightarrow \mathcal{C}\left([0,T],\mathbb{R}\right)\\ \quad x_. & \longmapsto \displaystyle\int_0^{.} x_s\,ds, \end{cases}$$

together with (73)–(75), that along the same sub-sequence

$$\int_0^{.\wedge\tau_{2\alpha}\wedge\tau_\alpha^n} \left\langle \Psi\left(\bar{\mu}_s^{n,\tau_\alpha^n}\right),\phi\right\rangle ds \Longrightarrow \int_0^{.\wedge\tau_{2\alpha}} \left\langle \Psi\left(\bar{\mu}_s^{*}\right),\phi\right\rangle ds \quad \text{in } \mathcal{C}\left([0,T],\mathbb{R}\right). \tag{76}$$

On the other hand, whenever $\bar{\mu}_0^n \in \mathcal{M}_{\alpha,M}$ we clearly have that $\bar{\mu}_{t\wedge\tau_{2\alpha}}^{n,\tau_\alpha^n} \in \mathcal{M}_{\alpha,M}$ for all t. Therefore, as a consequence of Proposition 4.1 together with (19), we have a.s. for all $n \in \mathbb{N}^*$ and $t \geq 0$,

$$\begin{aligned}
\left\langle \bar{\mu}_{t\wedge\tau_{2\alpha}}^{n,\tau_\alpha^n},\phi\right\rangle &= \left\langle \bar{\mu}_0^n,\phi\right\rangle + \left(\frac{1}{n}\int_0^{t\wedge\tau_\alpha^n\wedge\tau_{2\alpha}} \mathcal{Q}^n \Pi_\phi\left(\mu_s^n\right)\,ds\right) + \bar{M}^n(\phi)_{t\wedge\tau_\alpha^n\wedge\tau_{2\alpha}},\\
&= \left\langle \bar{\mu}_0^n,\phi\right\rangle + \left(\int_0^{t\wedge\tau_\alpha^n\wedge\tau_{2\alpha}} \left\langle \Psi\left(\bar{\mu}_s^{n,\tau_\alpha^n}\right),\phi\right\rangle ds + o_t^{n,\alpha}\right)\mathbb{1}_{\{\bar{\mu}_0^n\in\mathcal{M}_{\alpha,\mathcal{M}}\}}\\
&\quad + \left(\frac{1}{n}\int_0^{t\wedge\tau_\alpha^n\wedge\tau_{2\alpha}} \mathcal{Q}^n \Pi_\phi\left(\mu_s^n\right)\,ds\right)\mathbb{1}_{\{\bar{\mu}_0^n\notin\mathcal{M}_{\alpha,M}\}} + \bar{M}^n(\phi)_{t\wedge\tau_\alpha^n\wedge\tau_{2\alpha}},
\end{aligned} \tag{77}$$

where $o^{n,\alpha}$ is a process converging to 0 in probability and uniformly over compact sets. Now, applying Doob's inequality to the stopped martingale $\bar{M}^n(\phi)_{.\wedge\tau_\alpha^n\wedge\tau_{2\alpha}}$ and using (20) as in (64) yields that

$$\sup_{t\in[0,T]} \left|\bar{M}^n(\phi)_{t\wedge\tau_\alpha^n}\right| \xrightarrow[n\to\infty]{(\mathcal{P})} 0. \tag{78}$$

Moreover, for all n, t and all ε we have that

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,T]}\left(\frac{1}{n}\int_0^{t\wedge\tau_\alpha^n\wedge\tau_{2\alpha}} \mathcal{Q}^n \Pi_\phi\left(\mu_s^n\right)\,ds\right)\mathbb{1}_{\{\bar{\mu}_0^n\notin\mathcal{M}_{\alpha,M}\}} > \varepsilon\right]\\
&\quad= \mathbb{P}\left[\sup_{t\in[0,T]}\frac{1}{n}\int_0^{t\wedge\tau_\alpha^n\wedge\tau_{2\alpha}} \mathcal{Q}^n \Pi_\phi\left(\mu_s^n\right)\,ds > \varepsilon,\ \bar{\mu}_0^n \notin \mathcal{M}_{\alpha,M}\right]\\
&\quad\leq \mathbb{P}\left[\bar{\mu}_0^n \notin \mathcal{M}_{\alpha,M}\right] \xrightarrow[n\to\infty]{} 0,
\end{aligned} \tag{79}$$

from (61). Plugging (21) together with (76), (78) and (79) into (77), and using Skorokhod Representation Theorem implies that on some probability space, almost surely

$$\left\langle \bar{\mu}_{t\wedge\tau_{2\alpha}}^*,\phi\right\rangle = \left\langle \zeta,\phi\right\rangle + \int_0^{t\wedge\tau_{2\alpha}} \left\langle \Psi\left(\bar{\mu}_s^*\right),\phi\right\rangle ds, \qquad t\geq 0.$$

In other words, $\bar{\mu}^*$ is a process of $\mathcal{C}\left([0,T],\mathcal{M}_F(\mathbb{N})\right)$ having initial deterministic value ζ, and solving (23) on $[0, T\wedge\tau_{2\alpha}]$. As the solution of the latter, if any, is unique, we conclude (i) that

$\tau_{2\alpha}$ is deterministic, hence (ii) that there exists a solution $\bar{\mu}$ to (23) on $[0, T \wedge \tau_{2\alpha}]$, with which $\bar{\mu}^*$ coincides almost surely and (iii) that $\tau_{2\alpha} = t_{2\alpha}$, where

$$t_{2\alpha} = \sup\Big\{t \geq 0;\ \langle \bar{\mu}_t, \mathbf{1}\rangle \wedge \langle \bar{\mu}_t, \chi\rangle > 2\alpha\Big\}, \tag{80}$$

which is strictly positive from (33) and the continuity of the path of $\bar{\mu}$. In particular, as the tightness of $\{\bar{\mu}^{n,\tau_\alpha^n}\}$ clearly implies that of $\left\{\int_0^{\cdot\wedge\tau_{2\alpha}\wedge\tau_\alpha^n}\left\langle \Psi\left(\bar{\mu}_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds\right\}$, we deduce from (76) that

$$\int_0^{\cdot\wedge t_{2\alpha}\wedge\tau_\alpha^n}\left\langle \Psi\left(\bar{\mu}_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds \Longrightarrow \int_0^{\cdot\wedge t_{2\alpha}} \langle \Psi(\bar{\mu}_s), \phi\rangle\, ds \quad \text{in } \mathcal{C}([0,T],\mathbb{R}).$$

Using once again the Representation Theorem we obtain that on some probability space,

$$\int_0^{\cdot\wedge t_{2\alpha}\wedge\tau_\alpha^n}\left\langle \Psi\left(\bar{\mu}_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds \underset{n\to\infty}{\longrightarrow} \int_0^{\cdot\wedge t_{2\alpha}} \langle \Psi(\bar{\mu}_s), \phi\rangle\, ds \quad \text{a.s. in } \mathcal{C}([0,T],\mathbb{R})$$

which, as the Skorokhod topology and the uniform topology coincide on $\mathcal{C}([0,T],\mathbb{R})$ (see [3, p.112]), implies that

$$\sup_{t\in[0,T]}\left|\int_0^{t\wedge t_{2\alpha}\wedge\tau_\alpha^n}\left\langle \Psi\left(\bar{\mu}_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds - \int_0^{t\wedge t_{2\alpha}} \langle \Psi(\bar{\mu}_s), \phi\rangle\, ds\right| \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.}$$

and therefore,

$$\sup_{t\in[0,T]}\left|\int_0^{t\wedge t_{2\alpha}\wedge\tau_\alpha^n}\left\langle \Psi\left(\bar{\mu}_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds - \int_0^{t\wedge t_{2\alpha}} \langle \Psi(\bar{\mu}_s), \phi\rangle\, ds\right| \xrightarrow[n\to\infty]{(\mathcal{P})} 0.$$

As the latter holds true for all $T > 0$, we obtain that

$$\sup_{t\in[0,t_{2\alpha}]}\left|\int_0^{t\wedge\tau_\alpha^n}\left\langle \Psi\left(\bar{\mu}_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds - \int_0^{t} \langle \Psi(\bar{\mu}_s), \phi\rangle\, ds\right| \xrightarrow[n\to\infty]{(\mathcal{P})} 0. \tag{81}$$

Consequently, from (77) we obtain that for all $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,t_{2\alpha}]}\left|\left\langle\bar{\mu}_t^n, \phi\right\rangle - \langle\bar{\mu}_t, \phi\rangle\right| > \varepsilon\right]\\
&\quad\leq \mathbb{P}\left[\left\{\sup_{t\in[0,t_{2\alpha}]}\left|\int_0^{t\wedge\tau_\alpha^n}\left\langle \Psi\left(\bar{\mu}_s^{n,\tau_\alpha^n}\right), \phi\right\rangle ds - \int_0^{t} \langle \Psi(\bar{\mu}_s), \phi\rangle\, ds\right| > \frac{\varepsilon}{3}\right\}\bigcap\left\{\tau_\alpha^n > t_{2\alpha}\right\}\right]\\
&\qquad+\mathbb{P}\left[\sup_{t\in[0,t_{2\alpha}]}\left|\bar{M}^n(\phi)_{t\wedge\tau_\alpha^n} + o_t^{n,\alpha}\right| > \frac{\varepsilon}{3}\right] + \mathbb{P}\left[\left|\left\langle\bar{\mu}_0^{n,\tau_\alpha^n}, \phi\right\rangle - \langle\zeta, \phi\rangle\right| > \frac{\varepsilon}{3}\right]\\
&\qquad+\mathbb{P}\left[\tau_\alpha^n \leq t_{2\alpha}\right].
\end{aligned}$$

The first term on the r.h.s. vanishes for large n thanks to (81), the second one from Doob's inequality, the third one in view of (21) and the last one from (74). This concludes the proof. □

Existence of the solution on $\mathbb{R}+$. A consequence of Proposition 4.6 is the existence of a solution $(\bar{\mu}_t)_{t\geq 0}$ of (23) until $t_{2\alpha}$ defined by (80). Clearly, the latter result can be generalized to any $0 < \beta < \alpha$. Therefore, by the continuity of its path the solution $(\bar{\mu}_t)_{t\geq 0}$ can be extended at least until t_0, the hitting time of 0 defined by (71). The existence of the solution $(\bar{\mu}_t)_{t\geq 0}$ after t_0 then follows from the explicit form (72).

Asymptotics of $\bar{\mu}^n_.(0)$. We now focus on the mass concentrated at 0 in the hydrodynamic limit. Let us first give the following result.

Lemma 4.7. *For all $t \geq t_{2\alpha}$,*

$$\lambda \int_{t_{2\alpha}}^{t\wedge t_0} \bar{\mu}_s(1) \left(\frac{\langle \bar{\mu}_s, \chi^2 \rangle}{\langle \bar{\mu}_s, \chi \rangle} - 1 \right) ds \leq \langle \bar{\mu}_{t_{2\alpha}}, \chi \rangle,$$

where $(\bar{\mu}_t)_{t\geq 0}$ is the only solution to (23) on $\mathbb{R}+$ and $t_{2\alpha}$ is defined by (80).

Proof. Let $t \geq t_{2\alpha}$. Plainly, applying (23) to $\phi := \chi$ leads to

$$\langle \bar{\mu}_t, \chi \rangle = \langle \bar{\mu}_{t_{2\alpha}}, \chi \rangle - \lambda \left\{ \int_{t_{2\alpha}}^{t} \langle \bar{\mu}_s, \chi \rangle \, ds + \int_{t_{2\alpha}}^{t\wedge t_0} \langle \bar{\mu}_s, \chi^2 \rangle \, ds \right.$$
$$\left. + \int_{t_{2\alpha}}^{t\wedge t_0} \langle \bar{\mu}_s, \chi \nabla \chi \rangle \left(\frac{\langle \bar{\mu}_s, \chi^2 \rangle}{\langle \bar{\mu}_s, \chi \rangle} - 1 \right) ds \right\}.$$

Therefore, as $\langle \bar{\mu}_t, \chi \rangle \geq 0$ we obtain that

$$\langle \bar{\mu}_{t_{2\alpha}}, \chi \rangle \geq \lambda \int_{t_{2\alpha}}^{t\wedge t_0} \langle \bar{\mu}_s, \chi \nabla \chi \rangle \left(\frac{\langle \bar{\mu}_s, \chi^2 \rangle}{\langle \bar{\mu}_s, \chi \rangle} - 1 \right) ds.$$

The proof is completed by noticing that for all s,

$$\langle \bar{\mu}_s, \chi \nabla \chi \rangle = \sum_{i\in\mathbb{N}} \bar{\mu}_s(i) i \left(i - (i-1)\right) = \sum_{i\in\mathbb{N}} \bar{\mu}_s(i) i \geq \bar{\mu}_s(1). \quad \square$$

Now, let for all $n \in \mathbb{N}^*$ and all $t \geq t_{2\alpha}$,

$$X_t^{n,\alpha} = \text{Card} \left\{ \text{U-vertices of degree 0 at } t_{2\alpha} \text{ that have become A before time } t \right\};$$
$$Y_t^{n,\alpha} = \text{Card} \left\{ \text{U-vertices of degree } \geq 1 \text{ at } t_{2\alpha}, \right.$$
$$\left. \text{having become of degree 0 at time } t \text{ and being still U at } t \right\}.$$

Denote also $\bar{X}_t^{n,\alpha} = \frac{1}{n} X_t^{n,\alpha}$ and $\bar{Y}_t^{n,\alpha} = \frac{1}{n} Y_t^{n,\alpha}$ for all n and t. Then, clearly

$$\bar{\mu}_t^n(0) = \bar{\mu}_{t_{2\alpha}}^n(0) - \bar{X}_t^{n,\alpha} + \bar{Y}_t^{n,\alpha}, \quad t \geq t_{2\alpha}. \tag{82}$$

As the U-vertices of degree 0 are independent of the rest of the graph and eventually all become A-vertices, it is clear that for any n, the process $X^{n,\alpha}$ is Markov on $\mathbb{N}$. It has rcll paths, its generator clearly reads

$$\tilde{\mathcal{Q}} F(x) = \lambda \left(\mu_{t_{2\alpha}}^n(0) - x \right)$$

for all functions $F : \mathbb{R} \to \mathbb{R}$ and all $x \in \mathbb{N}$, so it is routine to check that for all $n \in \mathbb{N}^*$, for some square integrable martingale $\bar{M}^{n,\alpha}$, for all $t \geq t_{2\alpha}$,

$$\bar{X}_t^{n,\alpha} = \lambda \int_{t_{2\alpha}}^{t} \left(\bar{\mu}_{t_{2\alpha}}^n(0) - \bar{X}_s^{n,\alpha} \right) ds + \bar{M}_t^{n,\alpha}. \tag{83}$$

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Therefore, with (82), we obtain that

$$\begin{aligned}\bar{\mu}^n_t(0) &= \bar{\mu}^n_{t_{2\alpha}}(0) - \lambda \int_{t_{2\alpha}}^t \left(\bar{\mu}^n_{t_{2\alpha}}(0) - \bar{X}^{n,\alpha}_s\right) ds - \bar{M}^{n,\alpha}_t + \bar{Y}^{n,\alpha}_t \\ &= \bar{\mu}^n_{t_{2\alpha}}(0) - \lambda \int_{t_{2\alpha}}^t \left(\bar{\mu}^n_s(0) - \bar{Y}^{n,\alpha}_s\right) ds - \bar{M}^{n,\alpha}_t + \bar{Y}^{n,\alpha}_t. \end{aligned} \tag{84}$$

It also readily follows once again by Doob's inequality that the martingale term vanishes uniformly in L^2, and in particular, that for all $t \geq t_{2\alpha}$,

$$\sup_{s\in[t_{2\alpha},t]} \left|\bar{M}^{n,\alpha}_s\right| \xrightarrow[n\to\infty]{(\mathcal{P})} 0. \tag{85}$$

On the other hand, applying (23) to $\phi \equiv \mathbb{1}_0$, and observing that for all $i \in \mathbb{N}$,

$$\chi(i)\nabla\mathbb{1}_0(i) = -\mathbb{1}_1(i),$$

we get that

$$\bar{\mu}_t(0) = \bar{\mu}_{t_{2\alpha}}(0) - \lambda \left\{ \int_{t_{2\alpha}}^t \bar{\mu}_s(0)\, ds - \int_{t_{2\alpha}}^{t\wedge t_0} \bar{\mu}_s(1) \left(\frac{\langle \bar{\mu}_s, \chi^2\rangle}{\langle \bar{\mu}_s, \chi\rangle} - 1 \right) du \right\}, \quad t \geq t_{2\alpha}.$$

Combining this with (84) leads for all n to

$$\begin{aligned}\left|\bar{\mu}^n_t(0) - \bar{\mu}_t(0)\right| \leq{}& \left|\bar{\mu}^n_{t_{2\alpha}}(0) - \bar{\mu}_{t_{2\alpha}}(0)\right| + \lambda \int_{t_{2\alpha}}^t \left|\bar{\mu}^n_s(0) - \bar{\mu}_s(0)\right| ds + \left|\bar{M}^{n,\alpha}_t\right| \\ &+ \bar{Y}^{n,\alpha}_t + \lambda \int_{t_{2\alpha}}^t \bar{Y}^{n,\alpha}_s\, ds + \lambda \int_{t_{2\alpha}}^t \bar{\mu}_s(1) \left(\frac{\langle \bar{\mu}_s, \chi^2\rangle}{\langle \bar{\mu}_s, \chi\rangle} - 1 \right) ds. \end{aligned} \tag{86}$$

But by its very definition, we have that for all $s \geq t_{2\alpha}$ and all n,

$$\bar{Y}^{n,\alpha}_s \leq \bar{\mu}^n_{t_{2\alpha}}(1) \leq \left\langle \bar{\mu}^n_{t_{2\alpha}}, \chi \right\rangle.$$

Plugging this together with Lemma 4.7 in (86) yields to

$$\begin{aligned}\left|\bar{\mu}^n_t(0) - \bar{\mu}_t(0)\right| \leq{}& \left|\bar{\mu}^n_{t_{2\alpha}}(0) - \bar{\mu}_{t_{2\alpha}}(0)\right| + \lambda \int_{t_{2\alpha}}^t \left|\bar{\mu}^n_s(0) - \bar{\mu}_s(0)\right| ds + \left|\bar{M}^{n,\alpha}_t\right| \\ &+ (1 + \lambda(t - t_{2\alpha})) \left\langle \bar{\mu}^n_{t_{2\alpha}}, \chi \right\rangle + \left\langle \bar{\mu}_{t_{2\alpha}}, \chi \right\rangle. \end{aligned}$$

Therefore, from Gronwall's Lemma we conclude that for all $n \in \mathbb{N}^*$ and all $t \geq t_{2\alpha}$,

$$\begin{aligned}&\left|\bar{\mu}^n_t(0) - \bar{\mu}_t(0)\right| \\ &\leq \left(\left|\bar{\mu}^n_{t_{2\alpha}}(0) - \bar{\mu}_{t_{2\alpha}}(0)\right| + \left|\bar{M}^{n,\alpha}_t\right| + (1 + \lambda(t - t_{2\alpha})) \left\langle \bar{\mu}^n_{t_{2\alpha}}, \chi \right\rangle + 2\alpha \right) e^{\lambda(t - t_{2\alpha})}. \end{aligned} \tag{87}$$

Proof of Theorem 3.1. We are now in position to prove Theorem 3.1. Fix $T > 0$. First consider a function $\phi \in \mathcal{B}_b$ such that $\phi(0) \neq 0$. Let $\varepsilon > 0$. We can chose α small enough in (33), and small enough positive numbers δ, η and ξ so that

$$\Big(\delta+\eta+\big(1+\lambda(T-t_{2\alpha})^+\big)(\alpha+\xi)+2\alpha\Big)e^{\lambda(T-t_{2\alpha})^+} < \frac{\varepsilon}{2|\phi(0)|}; \tag{88}$$

$$4\alpha+\xi < \frac{\varepsilon}{2\|\phi\|}. \tag{89}$$

First, if $T \le t_{2\alpha}$, Proposition 4.6 trivially implies that

$$\mathbb{P}\left[\sup_{t\in[0,T]}\left|\langle\bar{\mu}^n_t,\phi\rangle-\langle\bar{\mu}_t,\phi\rangle\right|>\varepsilon\right]\underset{n\to\infty}{\longrightarrow}0.$$

If $T > t_{2\alpha}$, define the following events for all $n \in \mathbb{N}^*$,

$$\Omega^n_{\alpha,\delta}=\left\{\sup_{t\in(0,t_{2\alpha}]}\left|\bar{\mu}^n_t(0)-\bar{\mu}_t(0)\right|\le\delta\right\};$$

$$\Omega^n_{\alpha,\eta}=\left\{\sup_{t\in(t_{2\alpha},T]}\left|\bar{M}^{n,\alpha}_t\right|\le\eta\right\};$$

$$\Omega^n_{\alpha,\xi}=\left\{\sup_{t\in(0,t_{2\alpha}]}\left|\langle\bar{\mu}^n_t,\chi\rangle-\langle\bar{\mu}_t,\chi\rangle\right|\le\xi\right\},$$

where $\bar{M}^{n,\alpha}$ is the martingale defined by (83). We have for all n,

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in(t_{2\alpha},T]}\left|\langle\bar{\mu}^n_t,\phi\rangle-\langle\bar{\mu}_t,\phi\rangle\right|>\varepsilon\right]\\
&\le\mathbb{P}\left[\sup_{t\in(t_{2\alpha},T]}|\phi(0)|\left|\bar{\mu}^n_t(0)-\bar{\mu}_t(0)\right|>\frac{\varepsilon}{2}\right]\\
&\quad+\mathbb{P}\left[\sup_{t\in(t_{2\alpha},T]}\|\phi\|\left|\langle\bar{\mu}^n_t,\mathbb{1}_{\mathbb{N}^*}\rangle-\langle\bar{\mu}_t,\mathbb{1}_{\mathbb{N}^*}\rangle\right|>\frac{\varepsilon}{2}\right]\\
&\le\mathbb{P}\left[\left\{\sup_{t\in(t_{2\alpha},T]}\left|\bar{\mu}^n_t(0)-\bar{\mu}_t(0)\right|>\frac{\varepsilon}{2|\phi(0)|}\right\}\cap\Omega^n_{\alpha,\delta}\cap\Omega^n_{\alpha,\eta}\cap\Omega^n_{\alpha,\xi}\right]\\
&\quad+\mathbb{P}\left[\left(\Omega^n_{\alpha,\delta}\right)^c\right]+\mathbb{P}\left[\left(\Omega^n_{\alpha,\eta}\right)^c\right]+2\mathbb{P}\left[\left(\Omega^n_{\alpha,\xi}\right)^c\right]\\
&\quad+\mathbb{P}\left[\left\{\sup_{t\in(t_{2\alpha},T]}\left|\langle\bar{\mu}^n_t,\mathbb{1}_{\mathbb{N}^*}\rangle-\langle\bar{\mu}_t,\mathbb{1}_{\mathbb{N}^*}\rangle\right|>\frac{\varepsilon}{2\|\phi\|}\right\}\cap\Omega^n_{\alpha,\xi}\right].
\end{aligned}\tag{90}$$

Clearly, from (87) and (88), for all $n \in \mathbb{N}^*$,

$$\mathbb{P}\left[\left\{\sup_{t\in(t_{2\alpha},T]}\left|\bar{\mu}^n_t(0)-\bar{\mu}_t(0)\right|>\frac{\varepsilon}{2|\phi(0)|}\right\}\cap\Omega^n_{\alpha,\delta}\cap\Omega^n_{\alpha,\eta}\cap\Omega^n_{\alpha,\xi}\right]=0. \tag{91}$$

On another hand, applying (85) and Proposition 4.6 respectively to $\phi = \mathbb{1}_0$ and $\phi = \chi$ yields that

$$\mathbb{P}\left[\left(\Omega^n_{\alpha,\delta}\right)^c\right]+\mathbb{P}\left[\left(\Omega^n_{\alpha,\eta}\right)^c\right]+2\mathbb{P}\left[\left(\Omega^n_{\alpha,\xi}\right)^c\right]\underset{n\to\infty}{\longrightarrow}0. \tag{92}$$

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Finally, notice that for all $t \geq t_{2\alpha}$ and for all n,

$$\begin{aligned}\left|\left\langle \bar{\mu}^n_t, \mathbb{1}_{\mathbb{N}^*}\right\rangle - \left\langle \bar{\mu}_t, \mathbb{1}_{\mathbb{N}^*}\right\rangle\right| &\leq \left\langle \bar{\mu}^n_{t_{2\alpha}}, \mathbb{1}_{\mathbb{N}^*}\right\rangle + \left\langle \bar{\mu}_{t_{2\alpha}}, \mathbb{1}_{\mathbb{N}^*}\right\rangle \leq \left\langle \bar{\mu}^n_{t_{2\alpha}}, \chi\right\rangle + \left\langle \bar{\mu}_{t_{2\alpha}}, \chi\right\rangle \\ &\leq \left\langle \bar{\mu}^n_{t_{2\alpha}}, \chi\right\rangle + 2\alpha,\end{aligned}$$

and therefore with (89),

$$\mathbb{P}\left[\left\{\sup_{t\in(t_{2\alpha},T]} \left|\left\langle \bar{\mu}^n_t, \mathbb{1}_{\mathbb{N}^*}\right\rangle - \left\langle \bar{\mu}_t, \mathbb{1}_{\mathbb{N}^*}\right\rangle\right| > \frac{\varepsilon}{2\|\phi\|}\right\} \cap \Omega^n_{\alpha,\xi}\right] = 0.$$

This together with (91) and (92) in (90), concludes the proof for all $\phi \in \mathcal{B}_b$ such that $\phi(0) \neq 0$.

We now treat the case of a bounded function ϕ such that $\phi(0) = 0$. For this, we fix $\varepsilon > 0$ and let $\alpha > 0$ be small enough in (33), and $\xi > 0$ small enough so that

$$4\alpha + \xi < \varepsilon. \tag{93}$$

Here again, if $T \leq t_{2\alpha}$ then Proposition 4.6 implies that

$$\mathbb{P}\left[\sup_{t\in[0,T]} \left|\left\langle \bar{\mu}^n_t, \phi\right\rangle - \left\langle \bar{\mu}_t, \phi\right\rangle\right| > \varepsilon\right] \underset{n\to\infty}{\longrightarrow} 0.$$

If $T > t_{2\alpha}$, just write that for all n,

$$\begin{aligned}&\mathbb{P}\left[\sup_{t\in(t_{2\alpha},T]} \left|\left\langle \bar{\mu}^n_t, \phi\right\rangle - \left\langle \bar{\mu}_t, \phi\right\rangle\right| > \varepsilon\right] \\ &\quad \leq \mathbb{P}\left[\left\{\sup_{t\in(t_{2\alpha},T]} \left|\left\langle \bar{\mu}^n_t, \phi\right\rangle - \left\langle \bar{\mu}_t, \phi\right\rangle\right| > \varepsilon\right\} \cap \Omega^n_{\alpha,\xi}\right] + \mathbb{P}\left[\left(\Omega^n_{\alpha,\xi}\right)^c\right].\end{aligned} \tag{94}$$

But then, we have for all $t \geq t_{2\alpha}$ that

$$\left|\left\langle \bar{\mu}^n_t, \phi\right\rangle - \left\langle \bar{\mu}_t, \phi\right\rangle\right| \leq \|\phi\| \left(\left\langle \bar{\mu}^n_{t_{2\alpha}}, \chi\right\rangle + 2\alpha\right), \tag{95}$$

and hence

$$\mathbb{P}\left[\left\{\sup_{t\in(t_{2\alpha},T]} \left|\left\langle \bar{\mu}^n_t, \phi\right\rangle - \left\langle \bar{\mu}_t, \phi\right\rangle\right| > \varepsilon\right\} \cap \Omega^n_{\alpha,\xi}\right] = 0$$

which, together with (92) in (94), concludes the proof for such a function ϕ.

Only the case of test functions of the form $\phi = \chi^k$, $k \in (0, \kappa - 1]$, remains to be treated. Fix such a k. Fix also $\varepsilon > 0$, $\alpha > 0$ small enough in (33), and $\xi > 0$ satisfying (93). Let us finally set L_ε such that

$$L_\varepsilon \geq 8M/\varepsilon. \tag{96}$$

First take $T \leq t_{2\alpha}$. We have

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,T]}\left|\left\langle\bar{\mu}_t^n,\chi^k\right\rangle-\left\langle\bar{\mu}_t,\chi^k\right\rangle\right|>\varepsilon\right]\\
&\quad\leq \mathbb{P}\left[\sup_{t\in[0,T]}\left|\left\langle\bar{\mu}_t^n-\bar{\mu}_t,\chi^k\right\rangle\right|\mathbb{1}_{\{t_{2\alpha}\leq\tau_\alpha^n\}}>\frac{\varepsilon}{2}\right]\\
&\qquad+\mathbb{P}\left[\sup_{t\in[0,T]}\left|\left\langle\bar{\mu}_t^n-\bar{\mu}_t,\chi^k\right\rangle\right|\mathbb{1}_{\{t_{2\alpha}>\tau_\alpha^n\}}>\frac{\varepsilon}{2}\right]\\
&\quad\leq \mathbb{P}\left[\sup_{t\in[0,T]}\left|\left\langle\bar{\mu}_{t\wedge\tau_\alpha^n}^n-\bar{\mu}_{t\wedge t_{2\alpha}},\chi^k\right\rangle\right|>\frac{\varepsilon}{2}\right]+\mathbb{P}\left[t_{2\alpha}>\tau_\alpha^n\right]\\
&\quad\leq \mathbb{P}\left[\sup_{t\in[0,T]}\left|\left\langle\bar{\mu}_t^n-\bar{\mu}_t,\chi^k\mathbb{1}_{[0,L_\varepsilon)}\right\rangle\right|>\frac{\varepsilon}{4}\right]\\
&\qquad+\mathbb{P}\left[\sup_{t\in[0,T]}\left|\left\langle\bar{\mu}_{t\wedge\tau_\alpha^n}^n-\bar{\mu}_{t\wedge t_{2\alpha}},\chi^k\mathbb{1}_{[L_\varepsilon,+\infty)}\right\rangle\right|>\frac{\varepsilon}{4}\right]+\mathbb{P}\left[t_{2\alpha}>\tau_\alpha^n\right].
\end{aligned}\tag{97}$$

Applying Proposition 4.6 to the bounded function $\chi^k\mathbb{1}_{[0,L_\varepsilon)}$, the first term on the right-hand side of (97) vanishes as n grows to infinity. The third term also goes to zero from (74). Regarding the second one, let us define for any measure μ on $\mathbb{N}$ such that $\langle\mu,\chi^{k+1}\rangle<M$, a random variable $I(\mu)$ having the following size-biased distribution on $\mathbb{N}$:

$$\mathbb{P}[I(\mu)=i]=\frac{i^k\mu(i)}{\langle\mu,\chi^k\rangle},\quad i\in\mathbb{N}.$$

Then, applying Markov Inequality to $I(\mu)$ we obtain that

$$\left\langle\mu,\chi^k\mathbb{1}_{[L_\varepsilon,+\infty)}\right\rangle\leq\frac{\sum_i i^{k+1}\mu(i)}{L_\varepsilon}\leq\frac{M}{L_\varepsilon}.$$

For any $n\geq 1$ and $t\leq T$, both $\bar{\mu}_{t\wedge\tau_\alpha^n}^n$ and $\bar{\mu}_{t\wedge t_{2\alpha}}$ belong to $\mathcal{M}_{\alpha,M}$, in particular we can apply the argument above to both measures. We obtain that

$$\sup_{t\in[0,T]}\left|\left\langle\bar{\mu}_{t\wedge\tau_\alpha^n}^n-\bar{\mu}_{t\wedge t_{2\alpha}},\chi^k\mathbb{1}_{[L_\varepsilon,+\infty)}\right\rangle\right|\leq\frac{2M}{L_\varepsilon},$$

so the second term in the r.h.s. of (97) vanishes in view of (96). We thus proved our claim for $\phi=\chi^k$ and $T\leq t_{2\alpha}$.

Finally, just as in (95), we have for all $t\geq t_{2\alpha}$ that

$$\left|\left\langle\bar{\mu}_t^n,\chi^k\right\rangle-\left\langle\bar{\mu}_t,\chi^k\right\rangle\right|\leq\left\langle\bar{\mu}_{t_{2\alpha}}^n,\chi^k\right\rangle+\left\langle\bar{\mu}_{t_{2\alpha}},\chi^k\right\rangle\leq\left\langle\bar{\mu}_{t_{2\alpha}}^n,\chi^k\right\rangle+2\alpha.$$

So the result for $\phi=\chi^k$ and $T>t_{2\alpha}$ follows exactly as for a bounded function vanishing at the origin. □

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4.5. Proof of Corollary 3.4

We conclude with the proof of Corollary 3.4. Recall the definitions (26) and (27). Let for all $t \geq 0$,

$$c_\zeta^t = \lambda \int_0^t \langle \bar{\mu}_s, \mathbf{1} \rangle \, ds.$$

Let $\varepsilon > 0$. Using simple manipulations of the limiting differential system, we have that for all $t \geq 0$, $\langle \bar{\mu}_t, \mathbf{1} \rangle \leq \exp(-\lambda t)$. Similarly, applying (19) to $\phi \equiv \mathbf{1}$ and taking expectations yields that for any $t \geq 0$ and $n \in \mathbb{N}^*$,

$$\frac{d}{dt}\mathbb{E}\left[\langle \mu_t^n, \mathbf{1} \rangle\right] \leq -\lambda \mathbb{E}\left[\langle \mu_t^n, \mathbf{1} \rangle\right].$$

Consequently, there exists $S > 0$ such that for all $n \in \mathbb{N}^*$,

$$\max\left\{\int_S^\infty \langle \bar{\mu}_t, \mathbf{1} \rangle \, dt; \int_S^\infty \mathbb{E}\left[\langle \bar{\mu}_t^n, \mathbf{1} \rangle\right] dt\right\} \leq \frac{\varepsilon}{4\lambda}. \tag{98}$$

Observe now that for all n, (μ^n, J^n) is a Markov jump process on $\mathcal{M}_F(\mathbb{N}) \times \mathbb{N}$, whose infinitesimal generator can be readily deduced from (18). Applying Dynkin's lemma to the test function

$$F : \begin{cases} \mathcal{M}_F(\mathbb{N}) \times \mathbb{R} & \to \mathbb{R} \\ (\mu, x) & \mapsto x \end{cases}$$

clearly entails that for all $n \in \mathbb{N}^*$ and $t \geq 0$,

$$\frac{J_t^n}{n} = \lambda \int_0^t \langle \bar{\mu}_s^n, \mathbf{1} \rangle \, ds + \frac{N_t^n}{n},$$

where N^n is a $\mathcal{F}_t^n$-martingale such that, uniformly over compact time sets

$$\frac{N^n}{n} \underset{n\to\infty}{\longrightarrow} \mathbf{0} \quad \text{in } L^2,$$

as can be proven using Doob's inequality. We obtain that for n large enough,

$$\begin{aligned}
\mathbb{E}\left[\left|\bar{J}^n - c_\zeta\right|\right] &\leq \mathbb{E}\left[\left|\frac{J_S^n}{n} - c_\zeta^S\right|\right] + \lambda \int_S^\infty \langle \bar{\mu}_t, \mathbf{1} \rangle \, dt + \lambda \mathbb{E}\left[\int_S^\infty \langle \bar{\mu}_t^n, \mathbf{1} \rangle \, dt\right] \\
&\leq \mathbb{E}\left[\left|\lambda \int_0^S \langle \bar{\mu}_t^n, \mathbf{1} \rangle \, dt - c_\zeta^S\right|\right] + \left(\mathbb{E}\left[\left(\frac{N_S^n}{n}\right)^2\right]\right)^{1/2} + \varepsilon/2 \\
&\leq \varepsilon.
\end{aligned}$$

In the latter, the second inequality follows from (98), the third one from the martingale convergence and from Theorem 3.1 applied to $\phi \equiv \mathbf{1}$, and by observing that the family of r.v.'s $\left\{\int_0^S \langle \bar{\mu}_t^n, \mathbf{1} \rangle \, dt;\ n \in \mathbb{N}^*\right\}$ is bounded by S, and hence uniformly integrable. □

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Appendix. Combinatorial results

Let us fix the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We first introduce the definition, and several basic properties which are used in Section 4.1, for the so called *hypergeometrical distribution*:

Definition A.1. Let n, N and p be three integers such that $p \geq 1$ and $N \geq 1$. Let $P := (P(1), \ldots, P(p)) \in \mathbb{N}^p$ such that $\sum_{i=1}^{p} P(i) = N$. We say that the measure-valued random variable $Y \in \mathcal{M}_F(\mathbb{N})$ follows a *multivariate hypergeometrical* distribution of parameters (n, N, p, P) if for all $y \in \mathcal{M}_F(\mathbb{N})$ of support in $[\![1, p]\!]$ such that $y(i) \leq P(i)$ for all i and $\sum_{i=1}^{p} y(i) = n$,

$$\mathbb{P}[Y = y] = \frac{\prod_{i=1}^{p} \binom{P(i)}{y(i)}}{\binom{N}{n}}.$$

The following main characteristics are well-known and easily calculated:

$$\mathbb{E}[Y(i)] = \frac{nP(i)}{N}, \quad i \in [\![1, p]\!] \tag{A.1}$$

$$\mathbf{Cov}(Y(i), Y(j)) = \frac{nP(i)P(j)}{N^2}\frac{N-n}{N-1}, \quad i, j \in [\![1, p]\!],$$

$$\mathbb{E}\left[Y(i)^3\right] = n(n-1)(n-2)\frac{P(i)^3}{N^3} + 3n(n-1)\frac{P(i)^2}{N^2} + \frac{nP(i)}{N}, \quad i \in [\![1, p]\!]. \tag{A.2}$$

In particular, we readily deduce from the latter that

$$\begin{aligned}
\mathbb{E}\left[\langle Y, \chi\rangle^2\right] &= \sum_i \sum_j ij\mathbb{E}[y(i)y(j)] \\
&= \sum_i \sum_j ij\left\{\mathbf{Cov}(y(i), y(j)) - \mathbb{E}[y(i)]\mathbb{E}[y(j)]\right\} \\
&= \sum_i \sum_j ij\left\{\frac{nP(i)P(j)}{N^2}\frac{N-n}{N-1} - \frac{n^2P(i)P(j)}{N^2}\right\} \\
&= \sum_i \sum_j ij\frac{n(n-1)P(i)P(j)}{N(N-1)}.
\end{aligned} \tag{A.3}$$

On another hand, as there are at most p integers i such that $y(i) > 0$, a simple computation gives

$$\mathbb{E}\left[\langle Y, \chi\rangle^3\right] = \sum_{(i,j,\ell)\in[1,n-1]\cap\mathbb{N}} ij\ell\mathbb{E}[y(i)y(j)y(\ell)] \leq 3n^2 \sum_i i^3\mathbb{E}\left[Y(i)^3\right].$$

So with (A.2), we obtain

$$\begin{aligned}
\mathbb{E}\left[\langle Y, \chi\rangle^3\right] \leq{}& \frac{3n^5}{N^3}\sum_{i\in[1,n-1]\cap\mathbb{N}} i^3P(i)^3 + \frac{3n^4}{N^2}\sum_{i\in[1,n-1]} i^3P(i)^2 \\
&+ \frac{3n^3}{N}\sum_{i\in[1,n-1]} i^3P(i).
\end{aligned} \tag{A.4}$$

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