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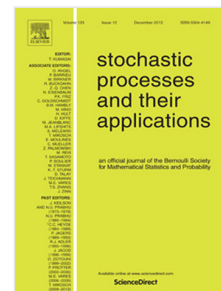
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QUASILINEAR PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS: EXISTENCE, UNIQUENESS

MARTINA HOFMANOVÁ AND TUSHENG ZHANG

ABSTRACT. We present a direct approach to existence and uniqueness of strong (in the probabilistic sense) and weak (in the PDE sense) solutions to quasilinear stochastic partial differential equations, which are neither monotone nor locally monotone. The proof of uniqueness is very elementary, based on a new method of applying Itô's formula for the L^1 -norm. The proof of existence relies on a recent regularity result and is direct in the sense that it does not rely on the stochastic compactness method.

Mathematics Subject Classification(2010). 60H15, 60F10, 35R60.

Keywords. Quasilinear stochastic partial differential equations, Strong solutions, Energy inequality.

1. INTRODUCTION

We consider a quasilinear parabolic stochastic partial differential equation of the form

$$(1.1) \quad \begin{aligned} du + \operatorname{div}(B(u)) dt &= \operatorname{div}(A(u)\nabla u) dt + \sigma(u) dW(t), \quad x \in \mathbb{T}^d, t \in [0, T], \\ u(0) &= u_0, \end{aligned}$$

where W is a cylindrical Wiener process in $H = L^2(\mathbb{T}^d)$ and σ a mapping with values in the space of γ -radonifying operators from H to certain Sobolev spaces. The coefficients $B : \mathbb{R} \rightarrow \mathbb{R}^d$ and $A : \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ are nonlinear functions satisfying suitable hypotheses, in particular, the diffusion matrix A is uniformly elliptic. The precise description of the problem setting will be given in the next section.

Equations of this type model the phenomenon of convection-diffusion of ideal fluids and therefore arise in a wide variety of important applications, including for instance two or three phase flows in porous media or sedimentation-consolidation processes (for a thorough exposition of this area given from a practical point of view we refer the reader to [GMT96] and the references therein). The addition of a stochastic noise to this physical model is fully natural as it represents external perturbations or a lack of knowledge of certain physical parameters.

Our aim is to establish existence of a unique solution to (1.1) that is strong in the probabilistic sense and weak in the PDEs sense. That is, we consider solutions that satisfy (1.1) with a given driving Wiener process and underlying stochastic basis in the sense of distributions. Recall that from the probabilistic point of view, two concepts of solution are typically considered in the theory of stochastic evolution equations, namely, pathwise (or strong) solutions and martingale (or weak) solutions. In the former notion the underlying probability space as well as the driving process is fixed in advance while in the latter case these stochastic elements become part of the solution of the problem.

The existence and uniqueness of a weak pathwise solution, i.e. strong in the probabilistic sense and weak in the PDE sense, was obtained as a by-product in [DHV16]. In this work, the authors were concerned with more general equations, namely degenerate parabolic SPDEs, and made use of the so-called kinetic approach and the notion of kinetic solution. The reason is that in such general cases, the degeneracy of the diffusion matrix introduces further difficulties and the classical notion of PDE weak solution does not provide enough information in order to prove uniqueness and furthermore in some cases the equation might not even be well defined in the sense of distributions. The proof of existence in [DHV16] relies on a Yamada-Watanabe-type argument (see e.g. [GK96], [PR07]). Pathwise uniqueness was established in the context of kinetic solutions which in particular implies uniqueness for PDE weak solutions in case they exist. Existence of a martingale solution was established in [DHV16] via stochastic compactness method and these two results were combined and existence of a weak martingale solution deduced.

In the present paper we put forward a direct (and therefore much simpler) approach towards existence and uniqueness of (1.1). Concerning uniqueness, we provide a fairly elementary proof based on a new way of applying Itô's formula to the L^1 -norm. This can be regarded as an alternative method for uniqueness in the L^1 setting that does not rely on the concept of kinetic solution and is therefore much simpler than that in [DHV16]. Existence of a pathwise solution to (1.1) is proved first under a stronger assumption upon the coefficient σ . The proof is based on a suitable approximation procedure: existence of unique approximating solutions is established using the theory of locally monotone operators [LR10] and then we show that these approximations satisfy several uniform bounds. In particular we prove that their gradients are uniformly bounded in space-time and have arbitrarily high moments. This part is based on the recent regularity result of [DdMH15]. Consequently, we are able to show strong convergence of the approximations and the limit is identified with a solution to (1.1). In the final section, we explain how the additional assumption on σ can be avoided using the method from [DHV16]. We believe that the methods presented in this paper are also useful for tackling other type of fully nonlinear SPDEs.

To conclude, let us mention several further references where similar problems were studied. In [DS04], analytical methods were used to prove existence and uniqueness of quasilinear parabolic SPDEs with second order operator having a dominating linear part and with gradient dependence in the noise. In the case of monotone coefficients, the literature is quite extensive, see for instance [PR07], [BDPR08], [BDPR09] and the references therein. For locally monotone coefficients, see [LR10]. For equations of gradient type see [Ges12].

2. MATHEMATICAL FRAMEWORK

2.1. Notations. In this paper, we adopt the following conventions. We work on a finite-time interval $[0, T]$, $T > 0$, and consider periodic boundary conditions, that is, $x \in \mathbb{T}^d$ where $\mathbb{T}^d = [0, 1]^d$ denotes the d -dimensional torus. C_b^1 denotes the space of continuously differentiable functions, not necessarily bounded but having bounded first order derivative. For $r \in [1, \infty]$, L^r are the Lebesgue spaces and the corresponding norm is denoted by $\|\cdot\|_{L^r}$. In order to measure higher regularity of functions (in the space variable) we make use of the Bessel potential spaces $H^{a,r}(\mathbb{T}^d)$, $a \in \mathbb{R}$ and $r \in (1, \infty)$. Throughout the paper we will mostly work with the L^2 -scale and so we will write H^a for $H^{a,2}(\mathbb{T}^d)$ and H for $L^2(\mathbb{T}^d)$. Recall that for all $a \geq 0$, the space H^a

is the usual Sobolev space of order a with the norm

$$\|u\|_{H^a}^2 = \sum_{|\alpha| \leq a} \int_{\mathbb{T}^d} |D^\alpha u|^2 dx$$

and that H^{-a} is the topological dual of H^a .

2.2. Hypotheses. Let us now introduce the precise setting of (1.1). We assume that the flux function

$$B = (B_1, \dots, B_d) : \mathbb{R} \longrightarrow \mathbb{R}^d$$

is of class C_b^1 . The diffusion matrix $A = (A_{ij})_{i,j=1}^d : \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ is of class C_b^1 , uniformly positive definite and bounded, i.e. $\delta I \leq A \leq CI$.

Regarding the stochastic term, let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The driving process W is a cylindrical Wiener process: it admits the following decomposition

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) \bar{e}_k,$$

where $(\bar{e}_k)_{k \geq 1}$ is some orthonormal system of the Hilbert space H , $(\beta_k)_{k \geq 1}$ is a sequence of independent real-valued Brownian motions relative to (\mathcal{F}_t) . For each $u \in H$ we consider a mapping $\sigma(u) : H \rightarrow H$ defined by $\sigma(u) \bar{e}_k = \sigma_k(u(\cdot)) \bar{e}_k$, where $\sigma_k(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ are real-valued functions. In particular, we suppose that σ satisfies the usual Lipschitz condition

$$(2.2) \quad \sum_{k=1}^{\infty} |\sigma_k(y_1) - \sigma_k(y_2)|^2 \leq C |y_1 - y_2|^2.$$

This assumption implies in particular that σ maps H to $L_2(H, H)$ where $L_2(H, H)$ denotes the collection of Hilbert-Schmidt operators from H to H . Thus, given a predictable process u that belongs to $L^2(\Omega, L^2(0, T; H))$, the stochastic integral $t \mapsto \int_0^t \sigma(u) dW$ is a well defined process taking values in H (see [?, Chapter 4] for a thorough exposition). The equation (1.1) can be rewritten as

$$(2.3) \quad \begin{aligned} du + \operatorname{div}(B(u)) dt &= \operatorname{div}(A(u) \nabla u) dt + \sum_{k=1}^{\infty} \sigma_k(u) d\beta_k(t), \quad x \in \mathbb{T}^d, t \in [0, T], \\ u(0) &= u_0. \end{aligned}$$

2.3. Additional assumption on σ . Stronger assumptions on σ will be taken in Section 4. More precisely, it will be needed to ensure the existence of the stochastic integral in (1.1) as an $H^{a,r}$ -valued process. We recall that the Bessel potential spaces $H^{a,r}$ with $a \geq 0$ and $r \in [2, \infty)$ belong to the class of 2-smooth Banach spaces and hence they are well suited for the stochastic Itô integration (see [Brz97], [BP99] for the precise construction of the stochastic integral). So, let us denote by $\gamma(H, X)$ the space of the γ -radonifying operators from H to a 2-smooth Banach space X . We recall that $\Psi \in \gamma(H, X)$ if the series

$$\sum_{k \geq 0} \gamma_k \Psi(e_k)$$

converges in $L^2(\tilde{\Omega}, X)$, for any sequence $(\gamma_k)_{k \geq 0}$ of independent Gaussian real-valued random variables on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and any orthonormal basis $(e_k)_{k \geq 0}$ of H . Then, this

space is endowed with the norm

$$\|\Psi\|_{\gamma(K,X)} := \left(\tilde{\mathbb{E}} \left\| \sum_{k=1}^{\infty} \gamma_k \Psi(e_k) \right\|_X^2 \right)^{\frac{1}{2}}, \quad \Psi \in \gamma(H, X),$$

(which does not depend on $(\gamma_k)_{k \geq 1}$, nor on $(e_k)_{k \geq 1}$) and is a Banach space.

With this notation in hand, we state our last assumption upon the coefficient σ . It coincides with the hypothesis $(H_{a,r})$ from [DdMH15]: we assume that for all $a < 2$ and $r \in [2, \infty)$

$$(2.4) \quad \|\sigma(u)\|_{\gamma(H, H^{a,r})} \leq \begin{cases} C(1 + \|u\|_{H^{a,r}}), & a \in [0, 1], \\ C(1 + \|u\|_{H^{a,r}} + \|u\|_{H^{1,a,r}}^a), & a > 1. \end{cases}$$

Detailed discussion of this condition was provided in [DdMH15] so let us just make a few comments here. First of all we observe that there is an overlap between the Lipschitz assumption (2.2) and the assumption (2.4). For instance, it follows immediately that (2.2) implies (2.4) for $a = 0$ and all $r \in [2, \infty)$. Nevertheless, for the purposes of our proof it proved useful to keep the two assumptions separate and thus we believe that it would not cause any confusion for the reader. Indeed, assumption (2.2) is used several times throughout the paper whereas the use of (2.4) is somewhat hidden in Theorem 4.4 which is an application of the regularity result from [DdMH15].

2.4. Solution. We end this section with a definition.

Definition 2.1. An (\mathcal{F}_t) -adapted, H -valued continuous process $(u(t), t \geq 0)$ is said to be a solution to equation (1.1) if

- (i) $u \in L^2(\Omega \times [0, T], H^1)$ for any $T > 0$,
- (ii) for any $\phi \in C^\infty(\mathbb{T}^d)$, $t > 0$, the following holds almost surely

$$\begin{aligned} \langle u(t), \phi \rangle - \langle u_0, \phi \rangle &= \int_0^t \langle B(u(s)), \nabla \phi \rangle ds \\ &= - \int_0^t \langle A(u(s)) \nabla u(s), \nabla \phi \rangle ds + \int_0^t \langle \sigma(u(s)) dW(s), \phi \rangle. \end{aligned}$$

Remark that the solution is a weak solution in the sense of PDEs.

3. UNIQUENESS

In this section we present our uniqueness result.

Theorem 3.1. Under the assumptions of Subsection 2.2, there is at most one solution to (1.1).

Proof. The aim is to apply the Itô formula to the L^1 -norm. However, this cannot be done directly since this norm is not smooth enough and therefore we introduce a suitable approximation that allows to overcome this issue.

Let $1 > a_1 > a_2 > \dots > a_n \dots > 0$ be a fixed sequence of decreasing positive numbers such that

$$\int_{a_1}^1 \frac{1}{u} du = 1, \dots, \int_{a_n}^{a_{n-1}} \frac{1}{u} du = n, \dots$$

Let $\psi_n(u)$ be a continuous function such that $\text{supp } \psi_n \subset (a_n, a_{n-1})$ and

$$0 \leq \psi_n(u) \leq 2 \frac{1}{n} \times \frac{1}{u}, \quad \int_{a_n}^{a_{n-1}} \psi_n(u) du = 1.$$

Define

$$\phi_n(x) = \int_0^{|x|} \int_0^y \psi_n(u) \, du \, dy.$$

We have

$$(3.5) \quad |\phi'_n(x)| \leq 1, \quad 0 \leq \phi''_n(x) \leq 2 \frac{1}{n} \times \frac{1}{|x|}.$$

Introduce a functional $\Phi_n : H \rightarrow \mathbb{R}$ by

$$\Phi_n(u) = \int_{\mathbb{T}^d} \phi_n(u(z)) \, dz, \quad u \in H.$$

Then we have

$$\Phi'_n(u)(h) = \int_{\mathbb{T}^d} \phi'_n(u(z)) h(z) \, dz,$$

and

$$(3.6) \quad \Phi''_n(u)(h, g) = \int_{\mathbb{T}^d} \phi''_n(u(z)) h(z) g(z) \, dz.$$

Suppose that u_1, u_2 are two solutions to equation (2.3). We may apply the generalized Itô formula [DHV16, Proposition A.1] to deduce

$$\begin{aligned} \Phi_n(u_1(t) - u_2(t)) &= \int_{\mathbb{T}^d} \phi_n(u_1(t, z) - u_2(t, z)) \, dz \\ &= \int_0^t \Phi'_n(u_1(s) - u_2(s)) (-\operatorname{div}(B(u_1(s))) + \operatorname{div}(B(u_2(s)))) \, ds \\ &\quad + \int_0^t \Phi'_n(u_1(s) - u_2(s)) (\operatorname{div}(A(u_1(s)) \nabla u_1(s)) - \operatorname{div}(A(u_2(s)) \nabla u_2(s))) \, ds \\ (3.7) \quad &\quad + \int_0^t \Phi'_n(u_1(s) - u_2(s)) (\sigma(u_1(s)) - \sigma(u_2(s))) \, dW(s) \\ &\quad + \frac{1}{2} \int_0^t \operatorname{tr}[(\sigma(u_1(s)) - \sigma(u_2(s)))^* \circ \Phi''_n(u_1(s) - u_2(s)) \circ (\sigma(u_1(s)) - \sigma(u_2(s)))] \, ds \\ &= I_n^1(t) + I_n^2(t) + I_n^3(t) + I_n^4(t). \end{aligned}$$

We will bound each of the terms on the right. Now

$$\begin{aligned} (3.8) \quad I_n^1(t) &= \int_0^t \int_{\mathbb{T}^d} \phi''_n(u_1(s, z) - u_2(s, z)) \nabla(u_1(s, z) - u_2(s, z)) \cdot B(u_1(s, z)) - B(u_2(s, z)) \, dz \, ds \\ &\leq \frac{C}{n} \int_0^t \int_{\mathbb{T}^d} |\nabla(u_1(s, z) - u_2(s, z))| \, dz \, ds \\ &\leq \frac{C}{n} \int_0^t \int_{\mathbb{T}^d} |\nabla u_1(s, z)| \, dz \, ds + \frac{C}{n} \int_0^t \int_{\mathbb{T}^d} |\nabla u_2(s, z)| \, dz \, ds \end{aligned}$$

where the Lipschitz continuity of B and (3.5) have been used. For I_n^2 , we have

$$\begin{aligned}
 I_n^2(t) &= - \int_0^t \int_{\mathbb{T}^d} \phi_n''(u_1(s, z) - u_2(s, z)) \nabla(u_1(s, z) - u_2(s, z)) \\
 &\quad \cdot A(u_1(s, z)) \nabla(u_1(s, z) - u_2(s, z)) \, dz \, ds \\
 &\quad - \int_0^t \int_{\mathbb{T}^d} \phi_n''(u_1(s, z) - u_2(s, z)) \nabla(u_1(s, z) - u_2(s, z)) \\
 &\quad \cdot (A(u_1(s, z)) - A(u_2(s, z))) \nabla u_2(s, z) \, dz \, ds \\
 (3.9) \quad &\leq -\delta \int_0^t \int_{\mathbb{T}^d} \phi_n''(u_1(s, z) - u_2(s, z)) |\nabla(u_1(s, z) - u_2(s, z))|^2 \, dz \, ds \\
 &\quad + \frac{C}{n} \int_0^t \int_{\mathbb{T}^d} |\nabla(u_1(s, z) - u_2(s, z))| |\nabla u_2(s, z)| \, dz \, ds \\
 &\leq -\delta \int_0^t \int_{\mathbb{T}^d} \phi_n''(u_1(s, z) - u_2(s, z)) |\nabla(u_1(s, z) - u_2(s, z))|^2 \, dz \, ds \\
 &\quad + \frac{C}{n} \int_0^t \int_{\mathbb{T}^d} |\nabla u_2(s, z)|^2 \, dz \, ds + \frac{C}{n} \int_0^t \int_{\mathbb{T}^d} |\nabla u_1(s, z)|^2 \, dz \, ds,
 \end{aligned}$$

where the Lipschitz continuity of A and (3.5) have been used. The fourth term in (3.6) can be estimated as follows.

$$\begin{aligned}
 I_n^4(t) &= \sum_{k=1}^{\infty} \int_0^t \langle \Phi_n''(u_1(s) - u_2(s)) \circ (\sigma(u_1(s)) - \sigma(u_2(s))) \bar{e}_k, (\sigma(u_1(s)) - \sigma(u_2(s))) \bar{e}_k \rangle \, ds \\
 (3.10) \quad &= \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}^d} \phi_n''(u_1(s, z) - u_2(s, z)) |(\sigma(u_1(s)) - \sigma(u_2(s))) \bar{e}_k(z)|^2 \, dz \, ds \\
 &= \int_0^t \int_{\mathbb{T}^d} \phi_n''(u_1(s, z) - u_2(s, z)) \sum_{k=1}^{\infty} |(\sigma_k(u_1(s)) - \sigma_k(u_2(s)))|^2 \, dz \, ds \\
 &\leq \frac{C}{n} \int_0^t \int_{\mathbb{T}^d} |u_1(s, z) - u_2(s, z)| \, dz \, ds.
 \end{aligned}$$

Substituting (3.8), (3.9) and (3.10) into (3.7) we get

$$\begin{aligned}
 \mathbb{E} \Phi_n(u_1(t) - u_2(t)) &= \mathbb{E} \int_{\mathbb{T}^d} \phi_n(u_1(t, z) - u_2(t, z)) \, dz \\
 &\leq \frac{C}{n} \left(\mathbb{E} \int_0^t \int_{\mathbb{T}^d} |\nabla u_1(s, z)|^2 \, dz \, ds + \mathbb{E} \int_0^t \int_{\mathbb{T}^d} |\nabla u_2(s, z)|^2 \, dz \, ds \right) \\
 &\quad + \frac{C}{n} \mathbb{E} \int_0^t \int_{\mathbb{T}^d} |u_1(s, z) - u_2(s, z)| \, dz \, ds
 \end{aligned}$$

Hence by (4.1),

$$\mathbb{E} \int_{\mathbb{T}^d} \phi_n(u_1(t, z) - u_2(t, z)) \, dz \leq \frac{C}{n}.$$

Letting $n \rightarrow \infty$ we obtain

$$\mathbb{E} \int_{\mathbb{T}^d} |u_1(t, z) - u_2(t, z)| \, dz = 0$$

This completes the proof. \square

4. EXISTENCE

In this section, we establish existence under the additional assumption from Subsection 2.3. The result then reads as follows.

Theorem 4.1. *Let $u_0 \in L^m(\Omega, \mathcal{F}_0; C^{1+l}(\mathbb{T}^d))$ for some $l > 0$ and all $m \in [2, \infty)$. Under the assumptions of Subsection 2.2 and 2.3, there exists a solution to the quasi-linear SPDE (1.1) that satisfies the following energy inequality*

$$(4.1) \quad \mathbb{E} \sup_{0 \leq t \leq T} \|u(t)\|_H^2 + \int_0^T \mathbb{E} \|u(t)\|_{H^1}^2 dt < \infty.$$

To begin with the proof, write

$$F(u) := -\operatorname{div}(B(u)) + \operatorname{div}(A(u)\nabla u).$$

We have the following estimate:

$$(4.2) \quad \|F(u)\|_{H^{-1}} \leq C(1 + \|u\|_{H^1}), \quad u \in H^1.$$

Indeed, for $v \in H^1$, it holds that

$$\begin{aligned} |\langle F(u), v \rangle| &= |\langle B(u(x)), \nabla v(x) \rangle - \langle A(u(x))\nabla u(x), \nabla v(x) \rangle| \\ &\leq C(1 + \|u\|_H) \|v\|_{H^1} + C\|u\|_{H^1} \|v\|_{H^1}. \end{aligned}$$

This implies (4.2).

Let $P_\varepsilon, \varepsilon > 0$ denote the semigroup on H generated by the Laplacian on \mathbb{T}^d . Recall that

$$P_\varepsilon f(x) = \int_{\mathbb{T}^d} P_\varepsilon(x, z) f(z) dz,$$

here $P_\varepsilon(x, z)$ stands for the heat kernel, $x, z \in \mathbb{T}^d$. For $\eta > 0$, denote by $C^\eta(\mathbb{T}^d)$ the space of functions that are η -Hölder continuous. We will use the following properties of the semigroup in the sequel.

$$(4.3) \quad \|P_\varepsilon f\|_{L^\infty} \leq C_\varepsilon \|f\|_H, \quad f \in H.$$

$$(4.4) \quad \left| \int_{\mathbb{T}^d} (P_{\varepsilon_1}(x, z) - P_{\varepsilon_2}(x, z)) h(z) dz \right| \leq C \|h\|_{C^\eta(\mathbb{T}^d)} (\varepsilon_1 - \varepsilon_2)^{\alpha_\eta}, \quad h \in C^\eta(\mathbb{T}^d),$$

for some $\alpha_\eta > 0$. We refer the reader to [Gri09] for these two properties. (4.4) can also be seen through the relation $\int_{\mathbb{T}^d} P_\varepsilon(x, z) h(z) dz = \mathbb{E}[h(B_\varepsilon^x)]$, where B_ε^x is the Brownian motion on the torus \mathbb{T}^d . For $\varepsilon > 0$, $u \in H$, set

$$A_\varepsilon(u)(x) = P_\varepsilon(A(u))(x), \quad x \in \mathbb{T}^d,$$

here

$$P_\varepsilon(A(u))(x) = (P_\varepsilon(A_{ij}(u))(x))_{i,j=1}^d.$$

Consider the following stochastic partial differential equation:

$$(4.5) \quad \begin{aligned} du^\varepsilon(t) + \operatorname{div}(B(u^\varepsilon(t))) dt &= \operatorname{div}(A_\varepsilon(u^\varepsilon(t))\nabla u^\varepsilon(t)) dt + \sigma(u^\varepsilon(t)) dW(t) \\ u^\varepsilon(0) &= u_0. \end{aligned}$$

Theorem 4.2. *Let $u_0 \in L^2(\Omega, H)$. Under the assumptions of Subsection 2.2, there exists a unique solution to the quasi-linear SPDE (4.5) that satisfies the following energy inequality:*

$$(4.6) \quad \sup_\varepsilon \left\{ \mathbb{E} \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_H^2 + \int_0^T \mathbb{E} \|u^\varepsilon(t)\|_{H^1}^2 dt \right\} < \infty.$$

Proof. First we claim that there exists a constant C such that

$$(4.7) \quad \delta|\xi|^2 \leq A_\varepsilon(u)(x)\xi \cdot \xi \leq C|\xi|^2 \quad \text{for all } \varepsilon > 0, u \in H^1, x \in \mathbb{T}^d, \xi \in \mathbb{R}^d.$$

By (H.2), one can find a constant C such that

$$(4.8) \quad \delta|\xi|^2 \leq A(y)\xi \cdot \xi \leq C|\xi|^2 \quad \text{for all } y \in \mathbb{R}, \xi \in \mathbb{R}^d.$$

Now,

$$(4.9) \quad A_\varepsilon(u)(x)\xi \cdot \xi = \int_{\mathbb{T}^d} P_\varepsilon(x, z) A(u(z))\xi \cdot \xi \, dz.$$

Since $\int_{\mathbb{T}^d} P_\varepsilon(x, z) \, dz = 1$, (4.7) follows from (4.8) and (4.9). Set

$$F_\varepsilon(u) := -\operatorname{div}(B(u)) + \operatorname{div}(A_\varepsilon(u)\nabla u), \quad u \in H^1.$$

For $u \in H^1$, by (4.7) we have

$$(4.10) \quad \begin{aligned} \langle F_\varepsilon(u), u \rangle &= \langle B(u), \nabla u \rangle - \langle A_\varepsilon(u)\nabla u, \nabla u \rangle \\ &\leq (C + C\|u\|_H\|u\|_{H^1}) - \delta\|u\|_{H^1}^2 \\ &\leq C + C\|u\|_H^2 - \delta_1\|u\|_{H^1}^2 \end{aligned}$$

for some constant $\delta_1 > 0$. Moreover,

$$(4.11) \quad \begin{aligned} &\langle F_\varepsilon(u) - F_\varepsilon(v), u - v \rangle \\ &= \langle B(u) - B(v), \nabla(u - v) \rangle - \langle A_\varepsilon(u)\nabla u - A_\varepsilon(v)\nabla v, \nabla(u - v) \rangle \\ &= \langle B(u) - B(v), \nabla(u - v) \rangle - \langle A_\varepsilon(u)\nabla(u - v), \nabla(u - v) \rangle \\ &\quad - \langle (A_\varepsilon(u) - A_\varepsilon(v))\nabla v, \nabla(u - v) \rangle \\ &\leq C + C\|u - v\|_H^2 - \delta_1\|u - v\|_{H^1}^2 - \langle (A_\varepsilon(u) - A_\varepsilon(v))\nabla v, \nabla(u - v) \rangle. \end{aligned}$$

By (4.3) and the Lipschitz continuity of A we have

$$(4.12) \quad \begin{aligned} &-\langle (A_\varepsilon(u) - A_\varepsilon(v))\nabla v, \nabla(u - v) \rangle \\ &\leq \|A_\varepsilon(u) - A_\varepsilon(v)\|_{L^\infty(\mathbb{T}^d)} \|v\|_{H^1} \|u - v\|_{H^1} \\ &\leq \|P_\varepsilon[A(u) - A(v)]\|_{L^\infty(\mathbb{T}^d)} \|v\|_{H^1} \|u - v\|_{H^1} \\ &\leq C_\varepsilon \|A(u) - A(v)\|_H \|v\|_{H^1} \|u - v\|_{H^1} \\ &\leq C\|u - v\|_H^2 \|v\|_{H^1}^2 + \delta_2\|u - v\|_{H^1}^2, \end{aligned}$$

for some constant $\delta_2 < \delta_1$.

Putting (4.11), (4.12) together we arrive at

$$(4.13) \quad \begin{aligned} &\langle F_\varepsilon(u) - F_\varepsilon(v), u - v \rangle \\ &\leq C + C\|u - v\|_H^2 + C\|u - v\|_H^2 \|v\|_{H^1}^2 - \delta_3\|u - v\|_{H^1}^2, \end{aligned}$$

for some constant $\delta_3 > 0$. (4.13) shows that F_ε satisfies the local monotonicity conditions imposed in [LR10]. Applying Theorem 1.1 in [LR10], we obtain the existence and uniqueness

of the solution u^ε . Next we prove the uniform bound in (4.6). By Itô's formula,

$$\begin{aligned} \|u^\varepsilon(t)\|_H^2 &= \|u_0\|_H^2 - 2 \int_0^t \langle \operatorname{div}(B(u^\varepsilon(s))), u^\varepsilon(s) \rangle ds \\ &\quad + 2 \int_0^t \langle \operatorname{div}(A_\varepsilon(u^\varepsilon(s)) \nabla u^\varepsilon(s)), u^\varepsilon(s) \rangle ds \\ &\quad + 2 \int_0^t \langle u^\varepsilon(s), \sigma(u^\varepsilon(s)) dW(s) \rangle + \sum_{k=1}^{\infty} \int_0^t \|\sigma_k(u^\varepsilon(s))\|_H^2 ds. \end{aligned}$$

By (4.10) it follows that

$$\begin{aligned} (4.14) \quad &\|u^\varepsilon(t)\|_H^2 + \delta_1 \int_0^t \|u^\varepsilon(s)\|_{H^1}^2 ds \\ &\leq \|u_0\|_H^2 + \int_0^t (C + C\|u^\varepsilon(s)\|_H^2) ds \\ &\quad + 2 \int_0^t \langle u^\varepsilon(s), \sigma(u^\varepsilon(s)) dW(s) \rangle + \sum_{k=1}^{\infty} \int_0^t \|\sigma_k(u^\varepsilon(s))\|_H^2 ds. \end{aligned}$$

Because the constants involved in the above equation are independent of ε , the uniform bound (4.6) follows from (4.14), Burkholder's inequality and Gronwall's inequality. The proof is complete. \square

Proposition 4.3. *Let $u_0 \in L^p(\Omega, \mathcal{F}_0; L^p(\mathbb{T}^d))$ for some $p \in [2, \infty)$. Under the assumptions of Subsection 2.2, the solution to (4.5) satisfies the following estimate*

$$(4.15) \quad \sup_{\varepsilon} \mathbb{E} \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{L^p}^p < \infty.$$

Proof. This result is obtained by a suitable version of the Itô formula using similar arguments as in the proof of (4.6). For further details we refer the reader to [DHV16, Proposition 5.1]. \square

As the next step we establish higher regularity of solutions to (4.5) which holds true uniformly in ε .

Theorem 4.4. *Let $u_0 \in L^p(\Omega, \mathcal{F}_0; C^{1+l}(\mathbb{T}^d))$ for some $l > 0$ and all $p \in [2, \infty)$. Under the assumptions of Subsection 2.2 and 2.3, it holds true that*

$$(4.16) \quad \sup_{\varepsilon} \mathbb{E} \sup_{0 \leq t \leq T} \|\nabla u^\varepsilon(t)\|_{L^\infty(\mathbb{T}^d)}^p < \infty.$$

Proof. The proof is based on Theorem 2.6 and Theorem 2.7 in [DdMH15]. Since the setting in [DdMH15] is slightly different, let us explain why the same ideas apply here.

First of all, it is easy to observe that since the arguments for Dirichlet boundary conditions are more involved, considering periodic boundary conditions simplifies the proofs and does not cause any additional difficulties. The main difference between our equation (4.5) and the model problem from [DdMH15] is that our second order operator A_ε is by definition nonlocal. Let us thus repeat the main ideas from [DdMH15] and justify each step.

We consider the following auxiliary problem

$$(4.17) \quad \begin{aligned} dz^\varepsilon &= \Delta z^\varepsilon dt + \sigma(u^\varepsilon) dW, \\ z^\varepsilon(0) &= 0, \end{aligned}$$

and define $y^\varepsilon = u^\varepsilon - z^\varepsilon$. Then y^ε solves

$$(4.18) \quad \begin{aligned} \partial_t y^\varepsilon &= \operatorname{div}(A_\varepsilon(u^\varepsilon) \nabla y^\varepsilon) - \operatorname{div}(B(u^\varepsilon)) + \operatorname{div}((A_\varepsilon(u^\varepsilon) - I) \nabla z^\varepsilon), \\ y(0) &= u_0, \end{aligned}$$

which is a (pathwise) deterministic linear parabolic PDE.

Step 1: Our aim now is to show that there exists $\eta \in (0, 1)$ such that for all $p \in [2, \infty)$ there holds

$$(4.19) \quad \sup_{\varepsilon} \mathbb{E} \|u^\varepsilon\|_{C^\eta([0, T] \times \mathbb{T}^d)}^p < \infty.$$

To this end, we follow the proof of [DdMH15, Theorem 2.6].

Regularity of z^ε : First, the solution to (4.17) is given by the stochastic convolution

$$z^\varepsilon(t) = \int_0^t S(t-s) \sigma(u^\varepsilon(s)) dW_s,$$

where $S = (S(t))_{t \geq 0}$ denotes the semigroup generated by the Laplacian with periodic boundary conditions. The regularity estimates for the stochastic convolution apply (see [DdMH15, Proposition 3.1]) and together with an interpolation argument yield

$$(4.20) \quad \mathbb{E} \|z^\varepsilon\|_{L^r(0, T; H^{c, r})}^r \leq C (\mathbb{E} \|u^\varepsilon\|_{L^2(0, T; H^{1, 2})}^2, \mathbb{E} \|u^\varepsilon\|_{L^p(0, T; L^p)}^p).$$

Here $H^{c, r}$ denotes the Bessel potential space which coincides with the domain of $(-\Delta)^{c/2}$ if the periodic Laplacian is considered as an operator on L^r ; the exponent r can be arbitrary in $[2, \infty)$ and c can be chosen bigger than 1. The right hand side of (4.20) can be further estimated independently of ε due to Theorem 4.2 and Proposition 4.3. As a consequence of the boundedness of A , we thus obtain

$$(A_\varepsilon(u^\varepsilon) - I) \nabla z^\varepsilon \in L^r(\Omega; L^r(0, T; L^r))$$

uniformly in ε .

Regularity of y^ε : Next, we are able to establish the Hölder regularity of y^ε using [DdMH15, Theorem 3.2]. This auxiliary result depends on the coefficient $A_\varepsilon(u^\varepsilon)$ in the second order term in (4.18) only through the ellipticity and boundedness constants from (4.7) and is therefore uniform in ε . Hence it follows for some $\sigma \in (0, 1)$ that

$$\mathbb{E} \|y^\varepsilon\|_{C^{\sigma/2, \sigma}}^p \leq C (\mathbb{E} \|u_0\|_{C^\sigma}^p + \mathbb{E} \|B(u^\varepsilon) + (A_\varepsilon(u^\varepsilon) - I) \nabla z^\varepsilon\|_{L^r}^p),$$

which is again bounded uniformly in ε .

Hölder regularity of z^ε : In order to complete the proof of (4.19) we again apply the stochastic convolution estimates from [DdMH15, Proposition 3.1] to obtain

$$\mathbb{E} \|z^\varepsilon\|_{C^\gamma([0, T]; H^{\delta, r})}^r \leq C (1 + \mathbb{E} \|u^\varepsilon\|_{L^r(0, T; L^r)}^r),$$

where $r \in (2, \infty)$, $\delta \in (0, 1 - 2/r)$ and $\gamma \in [0, 1/2 - 1/r - \delta/2)$. With a suitable choice of these parameters, Sobolev's embedding implies the existence of $\lambda \in (0, 1)$ such that

$$\mathbb{E} \|z^\varepsilon\|_{C^\lambda([0, T]; C^\lambda)}^r \leq C$$

uniformly in ε . Thus (4.19) follows.

Step 2: The next step follows the proof of [DdMH15, Theorem 2.7, case $k = 1$] and has a similar structure as in *Step 1*. First, we increase the space regularity of z^ε to get sufficient regularity of the term

$$(A_\varepsilon(u^\varepsilon) - I) \nabla z^\varepsilon.$$

This is then used to establish the required regularity of y^ε .

To be more precise, according to [DdMH15, Proposition 3.1] and Sobolev's embedding we obtain for some $\gamma > 0$, $\sigma > 1$ and all p sufficiently large that

$$\mathbb{E}\|z^\varepsilon\|_{C^\gamma([0,T];C^\sigma)}^p \leq C(1 + \mathbb{E}\|u^\varepsilon\|_{L^p(0,T;H^{\kappa,p})}^p).$$

The right hand side can be estimated uniformly in ε due to (4.19) provided $\kappa < \eta$. Concerning the estimates of y^ε , we intend to apply [DdMH15, Theorem 3.3]. Thus, we need to verify that

$$(4.21) \quad A_\varepsilon(u^\varepsilon), B(u^\varepsilon), (A_\varepsilon(u^\varepsilon) - I)\nabla z^\varepsilon \in L^p(\Omega; C^{\alpha/2,\alpha}([0,T] \times \mathbb{T}^d))$$

for some $\alpha \in (0, 1)$, where $C^{\alpha/2,\alpha}([0,T] \times \mathbb{T}^d)$ denotes the space of functions that are α -Hölder continuous with respect to the parabolic distance

$$d((t, x), (s, y)) = \max\{|t - s|^{1/2}, |x - y|\}.$$

To this end, we observe that if $f \in C^{\alpha/2,\alpha}([0,T] \times \mathbb{T}^d)$ then $P_\varepsilon f \in C^{\alpha/2,\alpha}([0,T] \times \mathbb{T}^d)$ uniformly in ε . Indeed, since the convolution kernel $P_\varepsilon(x, z)$ depends only on the difference $x - z$, we have

$$\begin{aligned} \frac{|P_\varepsilon f(t, x) - P_\varepsilon f(s, y)|}{d((t, x), (s, y))} &= \frac{|\int_{\mathbb{T}^d} P_\varepsilon(x - z)f(t, z)dz - \int_{\mathbb{T}^d} P_\varepsilon(y - z)f(s, z)dz|}{d((t, x), (s, y))} \\ &\leq \int_{\mathbb{T}^d} P_\varepsilon(z) \frac{|f(t, x - z) - f(s, y - z)|}{d((t, x), (s, y))} dz \\ &\leq \|f\|_{C^{\alpha/2,\alpha}([0,T] \times \mathbb{T}^d)}. \end{aligned}$$

Therefore due to (4.19) and the Lipschitz continuity of A we conclude that $A_\varepsilon(u^\varepsilon)$ as well as $(A_\varepsilon(u^\varepsilon) - I)\nabla z^\varepsilon$ possess the regularity required in (4.21) uniformly in ε . The corresponding statement for $B(u^\varepsilon)$ follows immediately since B is Lipschitz. Therefore, [DdMH15, Theorem 3.3] applies and yields for all $p \in [2, \infty)$ that

$$\mathbb{E}\|y^\varepsilon\|_{C^{(1+\alpha)/2, 1+\alpha}([0,T] \times \mathbb{T}^d)}^p \leq C$$

with a constant independent of ε . The proof of (4.16) is complete. \square

Finally, we have all in hand to proceed with the proof of the main result of this section.

Proof of Theorem 4.1. Let u^ε be the solution to equation (4.5). We will show that u^ε converges to a solution to equation (2.3). The estimate (4.6) implies that there exist a sequence $(u^{\varepsilon_n}, n \geq 1)$, and a process

$$u \in L^2(\Omega \times [0, T], H^1) \cap L^2(\Omega, L^\infty(0, T, H)),$$

for which the following holds:

- (i) $u^{\varepsilon_n} \rightarrow u$ weakly in $L^2(\Omega \times [0, T], H^1)$, hence weakly in $L^2(\Omega \times [0, T], H)$.
- (ii) $u^{\varepsilon_n} \rightarrow u$ in $L^2(\Omega, L^\infty(0, T, H))$ with respect to the weak star topology,

Next we show that u^ε actually converges to u in $L^1(\Omega, H)$ as $\varepsilon \rightarrow 0$. It is sufficient to prove that u^{ε_n} is a Cauchy sequence.

Let $0 < \varepsilon_1 < \varepsilon_2$. By Itô's formula,

$$\begin{aligned}
 & \|u^{\varepsilon_1}(t) - u^{\varepsilon_2}(t)\|_H^2 \\
 &= 2 \int_0^t \langle B(u^{\varepsilon_1}(s)) - B(u^{\varepsilon_2}(s)), \nabla(u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)) \rangle ds \\
 &\quad - 2 \int_0^t \langle A_{\varepsilon_1}(u^{\varepsilon_1}(s)) \nabla u^{\varepsilon_1}(s) - A_{\varepsilon_2}(u^{\varepsilon_2}(s)) \nabla u^{\varepsilon_2}(s), \nabla(u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)) \rangle ds \\
 &\quad + 2 \int_0^t \langle u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s), (\sigma(u^{\varepsilon_1}(s)) - \sigma(u^{\varepsilon_2}(s))) dW(s) \rangle \\
 &\quad + \sum_{k=1}^{\infty} \int_0^t \|\sigma_k(u^{\varepsilon_1}(s)) - \sigma_k(u^{\varepsilon_2}(s))\|_H^2 ds \\
 &:= I_1(t) + I_2(t) + I_3(t) + I_4(t).
 \end{aligned} \tag{4.22}$$

By the Lipschitz continuity of B , for any $\delta_1 > 0$ there exists a constant C_1 such that

$$I_1(t) \leq C_1 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_H^2 ds + \delta_1 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds. \tag{4.23}$$

By (4.7), we have

$$\begin{aligned}
 I_2(t) &= -2 \int_0^t \langle A_{\varepsilon_1}(u^{\varepsilon_1}(s)) \nabla(u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)), \nabla(u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)) \rangle ds \\
 &\quad - 2 \int_0^t \langle (A_{\varepsilon_1}(u^{\varepsilon_1}(s)) - A_{\varepsilon_2}(u^{\varepsilon_2}(s))) \nabla u^{\varepsilon_2}(s), \nabla(u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)) \rangle ds \\
 &\leq -2\delta \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 &\quad - 2 \int_0^t \langle (A_{\varepsilon_1}(u^{\varepsilon_1}(s)) - A_{\varepsilon_2}(u^{\varepsilon_1}(s))) \nabla u^{\varepsilon_2}(s), \nabla(u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)) \rangle ds \\
 &\quad - 2 \int_0^t \langle (A_{\varepsilon_2}(u^{\varepsilon_1}(s)) - A_{\varepsilon_2}(u^{\varepsilon_2}(s))) \nabla u^{\varepsilon_2}(s), \nabla(u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)) \rangle ds.
 \end{aligned} \tag{4.24}$$

Let δ_2 be a small constant to be fixed later. In view of (4.4) we have

$$\begin{aligned}
 (4.25) \quad & -2 \int_0^t \langle (A_{\varepsilon_1}(u^{\varepsilon_1}(s)) - A_{\varepsilon_2}(u^{\varepsilon_1}(s))) \nabla u^{\varepsilon_2}(s), \nabla(u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)) \rangle ds \\
 & \leq \delta_2 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 & \quad + C \int_0^t \int_{\mathbb{T}^d} |A_{\varepsilon_1}(u^{\varepsilon_1}(s))(x) - A_{\varepsilon_2}(u^{\varepsilon_1}(s))(x)|^2 |\nabla u^{\varepsilon_2}(s)|^2(x) dx ds \\
 & \leq \delta_2 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 & \quad + C \int_0^t \int_{\mathbb{T}^d} \left| \int_{\mathbb{T}^d} (P_{\varepsilon_1}(x, z) - P_{\varepsilon_2}(x, z)) A(u^{\varepsilon_1}(s))(z) dz \right|^2 |\nabla u^{\varepsilon_2}(s)|^2(x) dx ds \\
 & \leq \delta_2 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 & \quad + C \int_0^t \int_{\mathbb{T}^d} \|A(u^{\varepsilon_1}(s))\|_{C^\eta(\mathbb{T}^d)}^2 (\varepsilon_1 - \varepsilon_2)^{2\alpha_\eta} |\nabla u^{\varepsilon_2}(s)|^2(x) dx ds \\
 & \leq \delta_2 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds + C(\varepsilon_1 - \varepsilon_2)^{2\alpha_\eta} \int_0^t \|u^{\varepsilon_2}(s)\|_{H^1}^2 (1 + \|u^{\varepsilon_1}(s)\|_{C^\eta(\mathbb{T}^d)}^2) ds.
 \end{aligned}$$

Due to Lipschitz continuity of A , for any positive constant δ_3 we have

$$\begin{aligned}
 (4.26) \quad & -2 \int_0^t \langle (A_{\varepsilon_2}(u^{\varepsilon_1}(s)) - A_{\varepsilon_2}(u^{\varepsilon_2}(s))) \nabla u^{\varepsilon_2}(s), \nabla(u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)) \rangle ds \\
 & \leq \delta_3 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 & \quad + C \int_0^t \int_{\mathbb{T}^d} |A_{\varepsilon_2}(u^{\varepsilon_1}(s))(x) - A_{\varepsilon_2}(u^{\varepsilon_2}(s))(x)|^2 |\nabla u^{\varepsilon_2}(s)|^2(x) dx ds \\
 & \leq \delta_3 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 & \quad + C \int_0^t \|\nabla u^{\varepsilon_2}(s)\|_{L^\infty}^2 \int_{\mathbb{T}^d} |P_{\varepsilon_2}[A(u^{\varepsilon_1}(s)) - A(u^{\varepsilon_2}(s))](x)|^2 dx ds \\
 & \leq \delta_3 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 & \quad + C \int_0^t \|\nabla u^{\varepsilon_2}(s)\|_{L^\infty(\mathbb{T}^d)}^2 \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds.
 \end{aligned}$$

Choose δ_2, δ_3 sufficiently small and substitute (4.26), (4.25) into (4.24) to obtain

$$\begin{aligned}
 (4.27) \quad I_2(t) &\leq -\delta_4 \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 &\quad + C(\varepsilon_1 - \varepsilon_2)^{2\alpha_\eta} \int_0^t \|u^{\varepsilon_2}(s)\|_{H^1}^2 (1 + \|u^{\varepsilon_1}(s)\|_{C^\eta(\mathbb{T}^d)}^2) ds \\
 &\quad + C \int_0^t \|\nabla u^{\varepsilon_2}(s)\|_{L^\infty}^2 \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_H^2 ds,
 \end{aligned}$$

for some $\delta_4 > 0$. Choosing $\delta_1 < \delta_4$ it follows from (4.22), (4.23) and (4.27) that

$$\begin{aligned}
 (4.28) \quad \|u^{\varepsilon_1}(t) - u^{\varepsilon_2}(t)\|_H^2 &\leq -(\delta_4 - \delta_1) \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 &\quad + C \int_0^t \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_H^2 ds \\
 &\quad + C(\varepsilon_1 - \varepsilon_2)^{2\alpha_\eta} \int_0^t (1 + \|u^{\varepsilon_1}(s)\|_{C^\eta(\mathbb{T}^d)}^2) \|u^{\varepsilon_2}(s)\|_{H^1}^2 ds \\
 &\quad + C \int_0^t \|\nabla u^{\varepsilon_2}(s)\|_{L^\infty}^2 \|u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s)\|_H^2 ds \\
 &\quad + 2 \int_0^t \langle u^{\varepsilon_1}(s) - u^{\varepsilon_2}(s), (\sigma(u^{\varepsilon_1}(s)) - \sigma(u^{\varepsilon_2}(s))) dW(s) \rangle.
 \end{aligned}$$

For any $M > 0$, define

$$\tau_M^{1,2} = \inf\{t > 0; \|\nabla u^{\varepsilon_2}(t)\|_{L^\infty} \geq M \quad \text{or} \quad \|u^{\varepsilon_1}(t)\|_{C^\eta(\mathbb{T}^d)} \geq M\}$$

with the convention $\inf \emptyset = T$. Then $\tau_M^{1,2}$ is an (\mathcal{F}_t) -stopping time. Keeping the bound (4.6) in mind and replacing t by $t \wedge \tau_M^{1,2}$ in (4.28) we deduce that

$$\begin{aligned}
 (4.29) \quad &\mathbb{E}\|u^{\varepsilon_1}(t \wedge \tau_M^{1,2}) - u^{\varepsilon_2}(t \wedge \tau_M^{1,2})\|_H^2 \\
 &\leq C \int_0^t \mathbb{E}\|u^{\varepsilon_1}(s \wedge \tau_M^{1,2}) - u^{\varepsilon_2}(s \wedge \tau_M^{1,2})\|_H^2 ds \\
 &\quad + C_M \mathbb{E} \int_0^{t \wedge \tau_M^{1,2}} \|u^{\varepsilon_2}(s)\|_{H^1}^2 ds (\varepsilon_1 - \varepsilon_2)^{2\alpha_\eta} \\
 &\quad + CM^2 \int_0^t \mathbb{E}\|u^{\varepsilon_1}(s \wedge \tau_M^{1,2}) - u^{\varepsilon_2}(s \wedge \tau_M^{1,2})\|_H^2 ds.
 \end{aligned}$$

By the Gronwall's inequality we obtain from (4.29) that

$$(4.30) \quad \mathbb{E}\|u^{\varepsilon_1}(t \wedge \tau_M^{1,2}) - u^{\varepsilon_2}(t \wedge \tau_M^{1,2})\|_H^2 \leq C_M(\varepsilon_1 - \varepsilon_2)^{2\alpha_\eta} \exp(CT + CM^2T).$$

By (4.6),

$$\begin{aligned}
 (4.31) \quad &\mathbb{E}\|u^{\varepsilon_1}(t) - u^{\varepsilon_2}(t)\|_H \\
 &= \mathbb{E}[\|u^{\varepsilon_1}(t \wedge \tau_M^{1,2}) - u^{\varepsilon_2}(t \wedge \tau_M^{1,2})\|_H \mathbf{1}_{\tau_M^{1,2} \geq t}] + \mathbb{E}[\|u^{\varepsilon_1}(t) - u^{\varepsilon_2}(t)\|_H \mathbf{1}_{\tau_M^{1,2} < t}] \\
 &\leq \mathbb{E}\|u^{\varepsilon_1}(t \wedge \tau_M^{1,2}) - u^{\varepsilon_2}(t \wedge \tau_M^{1,2})\|_H + C \sup_{\varepsilon} (\mathbb{E}\|u^\varepsilon(t)\|_H^2)^{\frac{1}{2}} [\mathbb{P}(\tau_M^{1,2} < t)]^{\frac{1}{2}} \\
 &\leq \mathbb{E}\|u^{\varepsilon_1}(t \wedge \tau_M^{1,2}) - u^{\varepsilon_2}(t \wedge \tau_M^{1,2})\|_H + C[\mathbb{P}(\tau_M^{1,2} < t)]^{\frac{1}{2}}.
 \end{aligned}$$

Given any $\eta > 0$. In view of (4.16), (4.19) and (4.6) we can first choose M such that

$$\begin{aligned} & C[\mathbb{P}(\tau_M^{1,2} < t)]^{\frac{1}{2}} \\ & \leq C\{\mathbb{P}(\sup_{0 \leq s \leq T} \|\nabla u^{\varepsilon_2}(s)\|_{L^\infty(\mathbb{T}^d)} \geq M)^{\frac{1}{2}} + \mathbb{P}(\sup_{0 \leq s \leq T} \|u^{\varepsilon_1}(s)\|_{C^\eta(\mathbb{T}^d)} \geq M)^{\frac{1}{2}}\} \\ & \leq \frac{\eta}{2}, \quad \text{for all } \varepsilon_2, \varepsilon_1 > 0. \end{aligned}$$

Then use (4.30) to find ε_0 so that for $\varepsilon_1, \varepsilon_2 \leq \varepsilon_0$,

$$(4.32) \quad \mathbb{E}\|u^{\varepsilon_1}(t \wedge \tau_M^{1,2}) - u^{\varepsilon_2}(t \wedge \tau_M^{1,2})\|_H \leq \frac{\eta}{2}.$$

Because η is arbitrary, we conclude from (4.31), (4.32) and (4.32) that for all $t \in [0, T]$, $u^\varepsilon(t) \rightarrow u(t)$ in $L^1(\Omega, H)$ and according to (4.6) and Vitali's convergence theorem, we deduce that $u^\varepsilon \rightarrow u$ in $L^1(\Omega \times [0, T], H)$, which can be further improved using (4.15) and Vitali's convergence theorem to

$$(4.33) \quad u^\varepsilon \rightarrow u \quad \text{in } L^p(\Omega \times [0, T], H) \quad \forall p \in [1, \infty).$$

Next we show that the limit process u is a solution to equation (2.3). To this end, take a test function $\phi \in C^\infty(\mathbb{T}^d)$ and use equation (4.5) to get

$$\begin{aligned} (4.34) \quad & \langle u^\varepsilon(t), \phi \rangle - \langle u_0, \phi \rangle - \int_0^t \langle B(u^\varepsilon(s)), \nabla \phi \rangle ds \\ & = - \int_0^t \langle A_\varepsilon(u^\varepsilon(s)) \nabla u^\varepsilon(s), \nabla \phi \rangle ds + \int_0^t \langle \sigma(u^\varepsilon(s)) dW(s), \phi \rangle. \end{aligned}$$

Taking (4.33) into account and letting $\varepsilon \rightarrow 0$ in (4.34) we obtain

$$\begin{aligned} & \mathbb{E}|\langle u^\varepsilon(t) - u(t), \phi \rangle| \rightarrow 0, \\ & \mathbb{E} \left| \int_0^t \langle B(u^\varepsilon) - B(u), \nabla \phi \rangle ds \right| \leq \|\nabla \phi\|_{L^\infty} \mathbb{E} \int_0^t \|u^\varepsilon - u\|_H ds \rightarrow 0. \end{aligned}$$

For the stochastic integral, we have

$$\begin{aligned} \mathbb{E} \left| \int_0^t \langle (\sigma(u^\varepsilon) - \sigma(u)) dW, \phi \rangle \right| & \leq C \mathbb{E} \left(\int_0^t \sum_{k=1}^\infty \langle \sigma_k(u^\varepsilon) - \sigma_k(u), \phi \rangle^2 ds \right)^{\frac{1}{2}} \\ & \leq C \|\phi\|_H \mathbb{E} \left(\int_0^t \|u^\varepsilon - u\|_H^2 ds \right)^{\frac{1}{2}} \rightarrow 0. \end{aligned}$$

It remains to pass to the limit in the second order term. Write

$$\begin{aligned} (4.35) \quad & \int_0^t \langle A_\varepsilon(u^\varepsilon(s)) \nabla u^\varepsilon(s) - A(u(s)) \nabla u(s), \nabla \phi \rangle ds \\ & = \int_0^t \langle (A_\varepsilon(u^\varepsilon(s)) - A_\varepsilon(u(s))) \nabla u^\varepsilon(s), \nabla \phi \rangle ds \\ & \quad + \int_0^t \langle (A_\varepsilon(u(s)) - A(u(s))) \nabla u^\varepsilon(s), \nabla \phi \rangle ds \\ & \quad + \int_0^t \langle A(u(s)) (\nabla u^\varepsilon(s) - \nabla u(s)), \nabla \phi \rangle ds. \end{aligned}$$

By the contraction property of the semigroup P_ε , Lipschitz continuity of A , (4.6) and (4.33), we have

$$\begin{aligned} & \mathbb{E} \left| \int_0^t \langle (A_\varepsilon(u^\varepsilon(s)) - A_\varepsilon(u(s))) \nabla u^\varepsilon(s), \nabla \phi \rangle ds \right| \\ & \leq \|\nabla \phi\|_{L^\infty} \mathbb{E} \int_0^t \int_{\mathbb{T}^d} |P_\varepsilon(A(u^\varepsilon(s)) - A(u(s)))(x)| \|\nabla u^\varepsilon(s)(x)\| dx ds \\ & \leq C \left(\mathbb{E} \int_0^t \|A(u^\varepsilon(s)) - A(u(s))\|_H^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \mathbb{E} \|u^\varepsilon(s)\|_{H^1}^2 ds \right)^{\frac{1}{2}} \rightarrow 0. \end{aligned}$$

By the strong continuity of the semigroup P_ε and boundedness of $A(u)$, we have

$$\begin{aligned} & \mathbb{E} \left| \int_0^t \langle (A_\varepsilon(u(s)) - A(u(s))) \nabla u^\varepsilon(s), \nabla \phi \rangle ds \right| \\ & \leq \|\nabla \phi\|_{L^\infty(\mathbb{T}^d)} \mathbb{E} \int_0^t \int_{\mathbb{T}^d} |P_\varepsilon(A(u(s))(x) - A(u(s))(x))| \|\nabla u^\varepsilon(s)(x)\| dx ds \\ & \leq C \left(\mathbb{E} \int_0^t \|P_\varepsilon A(u(s)) - A(u(s))\|_H^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \mathbb{E} \|u^\varepsilon(s)\|_{H^1}^2 ds \right)^{\frac{1}{2}} \rightarrow 0. \end{aligned}$$

By the weak convergence of u^ε ,

$$(4.36) \quad \mathbb{E} \int_0^t \langle A(u(s))(\nabla u^\varepsilon(s) - \nabla u(s)), \nabla \phi \rangle ds \rightarrow 0.$$

Putting together (4.35)–(4.36) we arrive at

$$\begin{aligned} & \langle u(t), \phi \rangle - \langle u_0, \phi \rangle - \int_0^t \langle B(u(s)), \nabla \phi \rangle ds \\ & = - \int_0^t \langle A(u(s)) \nabla u(s), \nabla \phi \rangle ds + \int_0^t \langle \sigma(u(s)) dW(s), \phi \rangle \end{aligned}$$

proving the existence of a solution. \square

Observe that actually a stronger result holds true as a consequence of Theorem 4.4.

Corollary 4.5. *Let $u_0 \in L^p(\Omega, \mathcal{F}_0; C^{1+l}(\mathbb{T}^d))$ for some $l > 0$ and all $p \in [2, \infty)$. Under the assumptions of Subsection 2.2 and 2.3, the unique solution u to (1.1) satisfies*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\nabla u(t)\|_{L^\infty(\mathbb{T}^d)}^p < \infty.$$

5. EXISTENCE - WITHOUT THE ADDITIONAL HYPOTHESIS ON σ

As a short addendum to our existence result presented in Section 4, let us point out that one can actually avoid the assumption introduced in Subsection 2.3. Recall, that this was used in order to obtain sufficiently high uniform regularity of the approximate solutions, cf. Theorem 4.4, which then allowed to obtain directly strong convergence of these approximations in $L^1(\Omega \times [0, T]; H)$ leading to existence. However, in view of the uniqueness proof presented in Section 3, which did not rely on this additional assumption, it is natural to ask whether the strong convergence in $L^1(\Omega \times [0, T] \times \mathbb{T}^d)$ cannot also be obtained in weaker setting.

Although it does not seem to be possible using the method of Section 3, the method of [DHV16, Subsection 6.2] actually gives such a tool that also does not rely on the stochastic

compactness method: let σ satisfy the basic assumptions from Subsection 2.2, let σ^n be suitable approximations satisfying the stronger assumptions from Subsection 2.3 and let u^n be the corresponding solutions constructed in Theorem 4.1. In order to show that (u^n) is Cauchy in $L^1(\Omega \times [0, T] \times \mathbb{T}^d)$ we proceed similarly as in [DHV16, Theorem 6.2] and observe that we only need to modify the estimation of the term K . This can be done provided, for instance,

$$\sup_{y \in \mathbb{R}} \sum_{k \geq 1} |\sigma_k(y) - \sigma_k^n(y)|^2 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Having the L^1 -convergence in hand together with the uniform estimates from Theorem 4.2 and Proposition 4.3 is sufficient to pass to the limit in the equation and to conclude the proof of existence.

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