

On tails of fixed points of the smoothing transform in the boundary case[☆]

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Received 19 December 2008; received in revised form 8 September 2009; accepted 10 September 2009

Available online 16 September 2009

Abstract

Let $\{A_i\}$ be a sequence of random positive numbers, such that only N first of them are strictly positive, where N is a finite a.s. random number. In this paper we investigate nonnegative solutions of the distributional equation $Z =_d \sum_{i=1}^N A_i Z_i$, where Z, Z_1, Z_2, \dots are independent and identically distributed random variables, independent of N, A_1, A_2, \dots . We assume $\mathbb{E}\left[\sum_{i=1}^N A_i\right] = 1$ and $\mathbb{E}\left[\sum_{i=1}^N A_i \log A_i\right] = 0$ (the boundary case), then it is known that all nonzero solutions have infinite mean. We obtain new results concerning behavior of their tails.

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MSC: Primary 60J80; secondary 60G42

Keywords: Smoothing transform; Branching random walk; Distributional equations; Random difference equation

1. Introduction

Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of random positive numbers. We assume that only first N of them are nonzero, where N is some random number, finite almost surely. For any random variable Z , let $\{Z_n\}_{n \in \mathbb{N}}$ be a sequence of i.i.d. copies of Z independent both on N and $\{A_n\}_{n \in \mathbb{N}}$. We define

[☆] This research project has been partially supported by Marie Curie Transfer of Knowledge Fellowship *Harmonic Analysis, Nonlinear Analysis and Probability* (contract number MTKD-CT-2004-013389) and by MNiSW grant N201 012 31/1020.

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the random variable $Z^* = \sum_{i=1}^N A_i Z_i$ and the map $Z \rightarrow Z^*$ is called the smoothing transform. A random variable Z is said to be fixed point of the smoothing transform if Z^* has the same distribution as Z , i.e.

$$Z \stackrel{d}{=} \sum_{i=1}^N A_i Z_i. \quad (1.1)$$

There exists an extensive literature, where the problems of existence, uniqueness and asymptotic behavior of solutions of (1.1) were studied. The answer is formulated in terms of the function $v(\theta) = \log \mathbb{E}[\sum_{i=1}^N A_i^\theta]$. If $v(1) = 0$ and $v'(1) < 0$ and N is nonrandom, Durrett and Liggett [6] proved existence of solutions of (1.1). Their results were later extended by Liu [10] to the case where N is random. Let us mention that in this case all nonzero solutions of (1.1) have finite mean. Fixed points of the smoothing transform were characterized by Biggins and Kyprianou [2]. Also their asymptotic properties are well described. Durrett and Liggett [6] studied behavior of the Laplace transform of Z close to 0. The tail of Z was described by Guivarc'h [8] (for nonrandom N) and Liu [11,12] (for random N). They proved that if $v(\chi) = 0$ for some $\chi > 1$ and some further hypotheses are satisfied then the limit $\lim_{x \rightarrow \infty} x^\chi \mathbb{P}(Z > x)$ exists, is strictly positive and finite.

In this paper we study ‘the boundary case’, when $v(1) = 0$ and $v'(1) = 0$. Existence of fixed points of (1.1) was proved by Durrett and Liggett [6] and Liu [10]. Uniqueness was studied by Biggins and Kyprianou [3]. It is known that all the solutions have infinite mean. To our knowledge, up until now, only partial results concerning tails of fixed points in the boundary case have been obtained. Under some further hypotheses Durrett and Liggett [6], Liu [12], Biggins and Kyprianou [3] investigated behavior near zero of the Laplace transform of Z , a solution of Eq. (1.1), and proved that:

$$\lim_{\lambda \rightarrow 0^+} \frac{1 - \mathbb{E}[e^{-\lambda Z}]}{\lambda |\log \lambda|} = C_1$$

for some positive constant C_1 . Moreover, Liu [10] (Corollary 1.6) showed that

$$\int_0^x \mathbb{P}[Z > t] dt \sim C_2 \log x$$

as x goes to infinity, for some constant $C_2 \in (0, \infty)$. Formal derivation of the last formula suggest that

$$\mathbb{P}[Z > x] \sim \frac{C_2}{x}.$$

The purpose of the present paper is to establish the formula above. Our main result is the following

Theorem 1.2. Assume

$$\mathbb{E} \left[\sum_{i=1}^N A_i \right] = 1, \quad (1.3)$$

$$\mathbb{E} \left[\sum_{i=1}^N A_i \log A_i \right] = 0, \quad (1.4)$$

$$\mathbb{E} \left[\sum_{i=1}^N A_i^{1-\delta_1} \right] < \infty, \quad \text{for some } \delta_1 > 0, \quad (1.5)$$

$$\mathbb{E} \left[\left(\sum_{i=1}^N A_i \right)^{1+\delta_2} \right] < \infty, \quad \text{for some } \delta_2 > 0, \quad (1.6)$$

$$\mathbb{E}[N] > 1 \text{ (it could be infinite)}. \quad (1.7)$$

Let Z be a nonnegative and nonzero solution of (1.1) then if A_i are aperiodic (i.e. there is no positive number h such that $\log A_i$ is a.s. an integer multiple of h , for $1 \leq i \leq N$)

$$\lim_{x \rightarrow \infty} x \mathbb{P}[Z > x] = C_0,$$

for some finite and strictly positive constant C_0 .

Otherwise, if A_i are periodic, then there exist two positive constants C_1 and C_2 such that

$$C_1 = \liminf_{x \rightarrow \infty} x \mathbb{P}[Z > x] \leq \limsup_{x \rightarrow \infty} x \mathbb{P}[Z > x] = C_2.$$

To prove the theorem, we follow arguments of Guivarc'h [8] and Liu [12], reducing the problem to study behavior at infinity of an invariant measure of a random difference equation. In Section 2 we describe the random difference equation in the critical case and in Section 3 we prove Theorem 1.2.

2. Random difference equation in the critical case

Given a probability measure μ on $\mathbb{R}^+ \times \mathbb{R}$ we define the Markov chain on \mathbb{R}

$$X_0 = 0,$$

$$X_n = A_n X_{n-1} + B_n,$$

where the random pairs $\{(A_n, B_n)\}$ are independent and identically distributed according to the measure μ . If $\mathbb{E}[\log A_1] < 0$ and $\mathbb{E}[\log^+ |B|] < \infty$, then there exists a unique stationary measure ν of $\{X_n\}$. The tail of ν is well understood, namely it was proved by Kesten [9] (see also Goldie [7]) that $\lim_{t \rightarrow \infty} t^\alpha \nu(|x| > t) = C_+$ for some positive constant C_+ , where α is the unique positive number such that $\mathbb{E}[A_1^\alpha] = 1$. This result was used by Guivarc'h [8] and Liu [12] to study solutions of (1.1) with finite mean.

In this paper we will refer to the ‘critical case’, when $\mathbb{E}[\log A_1] = 0$. Then there is no finite stationary measure, but Babbilot, Bougerol, Elie [1] proved that if

- $\mathbb{P}[A_1 = 1] < 1$ and $\mathbb{P}[A_1 x + B_1 = x] < 1$ for all $x \in \mathbb{R}$,
- $\mathbb{E}[(|\log A_1| + \log^+ |B_1|)^{2+\varepsilon}] < \infty$, for some $\varepsilon > 0$

then there exists a unique (up to a constant factor) invariant Radon measure ν of $\{X_n\}$, i.e. the measure ν on \mathbb{R} satisfying

$$\mu * \nu(f) = \nu(f), \quad (2.1)$$

for any positive measurable function f , where

$$\mu * \nu(f) = \int_{\mathbb{R}^+ \times \mathbb{R}} \int_{\mathbb{R}} f(ax + b) \nu(dx) \mu(dadb).$$

Recently behavior of the measure ν at infinity has been described:

Lemma 2.2 ([4,5]). Assume that hypotheses above are satisfied and $\mathbb{E}[A_1^{-\delta} + A_1^\delta + |B_1|^\delta] < \infty$ for some $\delta > 0$. If A_1 is aperiodic (i.e. there is no positive number h such that $\log A_1$ is a.s. an integer multiple of h), then there exists a strictly positive and finite constant C_+ such that

$$\lim_{t \rightarrow \infty} \nu \{x : \alpha t < |x| \leq \beta t\} = C_+ \log \frac{\beta}{\alpha},$$

for any pair $0 < \alpha < \beta$.

Furthermore, if A_1 is periodic and the group generated by the support of A_1 is $\{e^{np} : n \in \mathbb{Z}\}$ for some $p > 0$, then

$$\lim_{t \rightarrow \infty} \nu \{x : t < |x| \leq e^{np} t\} = nC_+,$$

for every $n \geq 1$ and some positive constant C_+ .

3. Proof of Theorem 1.2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which random variables $\{A_i\}_{i \in \mathbb{N}}$, N , $\{Z_i\}_{i \in \mathbb{N}}$ are supported. We denote by \mathbb{E} the expected value with respect to \mathbb{P} . Let η be the law of Z , being a positive solution of (1.1). We define the measure ν on \mathbb{R}^+ putting $\nu(dx) = x\eta(dx)$, then $\nu(f) = \mathbb{E}[f(Z)Z]$ for any bounded and compactly supported function f . The measure ν is unbounded on \mathbb{R}^+ , however it is a Radon measure. Using ideas of Guivarc'h [8] and Liu [12] we will prove that ν satisfies (2.1) for an appropriately chosen probability measure μ on $\mathbb{R}^+ \times \mathbb{R}$. We cannot use directly their proofs. Guivarc'h assumed A_i to be independent of each other, N to be a constant and obtained the random recurrence equation just by a simple algebraic transformation of measures. Liu introduced the Peyriere measure, which cannot be defined here. Nevertheless, we follow the approach of Liu [12, p. 276].

Define $\tilde{\Omega} = \Omega \times \mathbb{N}$, and let $\tilde{\mathcal{F}}$ be the σ -field on $\tilde{\Omega}$ being the direct product of \mathcal{F} and \mathcal{B} , where \mathcal{B} is the Borel σ -field on \mathbb{N} . We denote by ω an element of Ω and by (ω, i) an element of $\tilde{\Omega}$. Let δ_i be the Dirac measure on \mathbb{N} , i.e. $\delta_i(k) = 0$ if $k \neq i$ and $\delta_i(i) = 1$. For every $U \in \tilde{\mathcal{F}}$ we define

$$\tilde{\mathbb{P}}(U) = \mathbb{E} \left[\sum_{i=1}^N A_i(\omega) \int_{\mathbb{N}} \mathbf{1}_U(\omega, j) \delta_i(dj) \right],$$

then, in view of (1.3), $\tilde{\mathbb{P}}$ is a probability measure on $\tilde{\Omega}$. We write $\mathbb{E}_{\tilde{\mathbb{P}}}$ for its expected value. Thus, we have defined a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Given $(\omega, i) \in \tilde{\Omega}$ we define $\tilde{Z}(\omega, i) = Z_i(\omega)$, $\tilde{A}(\omega, i) = A_i(\omega)$, $\tilde{B}(\omega, i) = \sum_{j \neq i} A_j(\omega) Z_j(\omega)$.

Lemma 3.1. Random variables \tilde{Z} and (\tilde{A}, \tilde{B}) are $\tilde{\mathbb{P}}$ independent. Moreover for every nonnegative functions h and g on $\mathbb{R}^+ \times \mathbb{R}$ and \mathbb{R} respectively:

$$\mathbb{E}_{\tilde{\mathbb{P}}}[g(\tilde{Z})] = \mathbb{E}[g(Z)], \quad (3.2)$$

$$\mathbb{E}_{\tilde{\mathbb{P}}}[h(\tilde{A}, \tilde{B})] = \mathbb{E} \left[\sum_{i=1}^N A_i h \left(A_i, \sum_{j \neq i} A_j Z_j \right) \right]. \quad (3.3)$$

In particular \tilde{Z} and Z have the same law η .

Proof. We write

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{P}}}[h(\tilde{A}, \tilde{B})g(\tilde{Z})] &= \mathbb{E}\left[\sum_{i=1}^N A_i(\omega) \int_{\mathbb{N}} (h(\tilde{A}(\omega, j), \tilde{B}(\omega, j))g(\tilde{Z}(\omega, j))) \delta_i(dj)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^N A_i h\left(A_i, \sum_{j \neq i} A_j Z_j\right) g(Z_i)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^N A_i h\left(A_i, \sum_{j \neq i} A_j Z_j\right)\right] \mathbb{E}[g(Z_i)].\end{aligned}$$

Putting $g = 1$ we obtain (3.3) and next taking $h = 1$ we have (3.2). Therefore

$$\mathbb{E}_{\tilde{\mathbb{P}}}[h(\tilde{A}, \tilde{B})g(\tilde{Z})] = \mathbb{E}_{\tilde{\mathbb{P}}}[h(\tilde{A}, \tilde{B})] \mathbb{E}_{\tilde{\mathbb{P}}}[g(\tilde{Z})],$$

that proves independence of \tilde{Z} and (\tilde{A}, \tilde{B}) . \square

Lemma 3.4. For every $\alpha < 1$:

$$\mathbb{E}[Z^\alpha] < \infty.$$

Proof. Denote by ϕ the Laplace transform of η , the law of Z . Then, for all $t > 0$

$$1 - \phi(t) \geq (1 - e^{-1})\eta(1/t, \infty)$$

(see part (a) of Lemma 10.1 in [10]). Next we write

$$\begin{aligned}\mathbb{E}[Z^\alpha] &= \int_{\mathbb{R}^+} x^\alpha \eta(dx) = \alpha \int_{\mathbb{R}^+} x^{\alpha-1} \eta(x, \infty) dx \\ &\leq \frac{\alpha}{1 - e^{-1}} \int_{\mathbb{R}^+} x^{\alpha-1} (1 - \phi(1/x)) dx = \frac{\alpha}{1 - e^{-1}} \int_{\mathbb{R}^+} x^{-\alpha-1} (1 - \phi(x)) dx.\end{aligned}$$

The integral above is finite since by Theorem 5 in [3] (its assumptions are satisfied because of (1.5) and (1.6)): $\lim_{x \rightarrow 0} \frac{1 - \phi(x)}{x |\log x|} \in (0, \infty)$. \square

Next we define a probability measure μ on $\mathbb{R}^+ \times \mathbb{R}$: $\mu(U) = \mathbb{E}_{\tilde{\mathbb{P}}}[\mathbf{1}_U(\tilde{A}, \tilde{B})]$, for every Borel set $U \subset \mathbb{R}^+ \times \mathbb{R}$.

Lemma 3.5. The measure μ satisfies hypotheses of Lemma 2.2. Moreover the measure ν is μ -invariant, i.e. (2.1) is fulfilled.

Proof. First notice that by (1.4) we have $\mathbb{E}_{\tilde{\mathbb{P}}}[\log \tilde{A}] = \mathbb{E}[\sum_{i=1}^N A_i \log A_i] = 0$. Next if \tilde{A} was equal to 1 almost surely, then we would have $\mathbb{E}[\sum_{i=1}^N A_i] = \mathbb{E}[N]$. But the left-hand side of this equation by (1.3) is equal to 1, whereas the right one, by (1.7) is strictly larger than 1.

Next we check moments conditions. Assumptions (1.5) and (1.6) imply $\mathbb{E}_{\tilde{\mathbb{P}}}[\tilde{A}^{\delta_2} + \tilde{A}^{-\delta_1}] < \infty$. To estimate moments of \tilde{B} we consider the σ -field generated by N and $\{A_i\}$: $\mathcal{F}_0 = \sigma(N, A_1, A_2, \dots)$. Take $\alpha = 1 - \delta_1$ and ε such that $\frac{\alpha}{\alpha - \varepsilon} = 1 + \delta_2$. We may assume $\frac{\varepsilon}{\alpha} < 1$. First, we are going to estimate for every i the conditional expectation of $|\sum_{j \neq i} A_j Z_j|^\varepsilon$ with respect to \mathcal{F}_0 . For this purpose, using the Jensen inequality for the concave function $x \mapsto x^{\frac{\varepsilon}{\alpha}}$ and the inequality $|a + b|^\alpha \leq |a|^\alpha + |b|^\alpha$, valid for $\alpha < 1$, we have

$$\mathbb{E}\left[\left|\sum_{j \neq i} A_j Z_j\right|^\varepsilon \middle| \mathcal{F}_0\right] \leq \left(\mathbb{E}\left[\left|\sum_{j \neq i} A_j Z_j\right|^\alpha \middle| \mathcal{F}_0\right]\right)^{\frac{\varepsilon}{\alpha}} \leq \left(\mathbb{E}\left[\sum_{j \neq i} A_j^\alpha Z_j^\alpha \middle| \mathcal{F}_0\right]\right)^{\frac{\varepsilon}{\alpha}}.$$

Therefore using independence of Z_j and \mathcal{F}_0 , and [Lemma 3.4](#) we obtain

$$\mathbb{E} \left[\left| \sum_{j \neq i} A_j Z_j \right|^\varepsilon \middle| \mathcal{F}_0 \right] \leq C \left(\sum_{j=1}^N A_j^\alpha \right)^{\frac{\varepsilon}{\alpha}}.$$

Next we use the Hölder inequality with parameters $p = \frac{\alpha}{\alpha - \varepsilon}$ and $q = \frac{\alpha}{\varepsilon}$ and we estimate

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{P}}} [|\tilde{B}|^\varepsilon] &= \mathbb{E} \left[\sum_{i=1}^N A_i \left| \sum_{j \neq i} A_j Z_j \right|^\varepsilon \right] = \mathbb{E} \left[\mathbb{E} \left[\sum_{i=1}^N A_i \left| \sum_{j \neq i} A_j Z_j \right|^\varepsilon \middle| \mathcal{F}_0 \right] \right] \\ &= \mathbb{E} \left[\sum_{i=1}^N A_i \mathbb{E} \left[\left| \sum_{j \neq i} A_j Z_j \right|^\varepsilon \middle| \mathcal{F}_0 \right] \right] \leq C \mathbb{E} \left[\sum_{i=1}^N A_i \cdot \left(\sum_{j=1}^N A_j^\alpha \right)^{\frac{\varepsilon}{\alpha}} \right] \\ &\leq C \left(\mathbb{E} \left[\left(\sum_{i=1}^N A_i \right)^{1+\delta_2} \right] \right)^{\frac{1}{p}} \cdot \left(\mathbb{E} \left[\sum_{j=1}^N A_j^{1-\delta_1} \right] \right)^{\frac{1}{q}} \end{aligned}$$

and in view of [\(1.5\)](#) and [\(1.6\)](#) the value above is finite.

Finally, to prove that the measure ν is μ -invariant, take arbitrary compactly supported function f on \mathbb{R} and let $h((a, b), x) = f(ax + b)x$ be a function on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. In [Lemma 3.1](#) we proved that (\tilde{A}, \tilde{B}) and \tilde{Z} are independent, and moreover that \tilde{Z} and Z have the same distribution. Hence

$$\begin{aligned} \mu * \nu(f) &= \int_{\mathbb{R}^+ \times \mathbb{R}} \int_{\mathbb{R}^+} f(ax + b) \nu(dx) \mu(da db) \\ &= \int_{\mathbb{R}^+ \times \mathbb{R}} \int_{\mathbb{R}^+} h((a, b), x) \eta(dx) \mu(da db) \\ &= \mathbb{E}_{\tilde{\mathbb{P}}} [h((\tilde{A}, \tilde{B}), \tilde{Z})] = \mathbb{E}_{\tilde{\mathbb{P}}} [f(\tilde{A}\tilde{Z} + \tilde{B})\tilde{Z}] \\ &= \mathbb{E} \left[\sum_{i=1}^N A_i f \left(A_i Z_i + \sum_{j \neq i} A_j Z_j \right) Z_i \right] \\ &= \mathbb{E} \left[f \left(\sum_{i=1}^N A_i Z_i \right) \sum_{i=1}^N A_i Z_i \right] = \mathbb{E}[f(Z)Z] = \nu(f). \quad \square \end{aligned}$$

Now we are ready to conclude.

Proof of Theorem 1.2. Assume that A_i are aperiodic. Fix $\beta > 1$. In view of [Lemma 3.5](#) hypotheses of [Lemma 2.2](#) are fulfilled, therefore for every ε there exists M such that

$$|\nu(t, \beta t) - C_0 \log \beta| < \varepsilon$$

for every $t > M$. Next we estimate the tail of Z . We have

$$\begin{aligned} t\mathbb{P}[Z > t] &= t \cdot \sum_{n=0}^{\infty} \mathbb{P}[t\beta^n < Z \leq t\beta^{n+1}] = t \cdot \sum_{n=0}^{\infty} \int_{t\beta^n}^{t\beta^{n+1}} \eta(dx) \\ &\leq \sum_{n=0}^{\infty} \frac{1}{\beta^n} \int_{t\beta^n}^{t\beta^{n+1}} x \eta(dx) = \sum_{n=0}^{\infty} \frac{1}{\beta^n} \nu(t\beta^n, t\beta^{n+1}] \end{aligned}$$

$$\leq \sum_{n=0}^{\infty} \frac{C_0 \log \beta + \varepsilon}{\beta^n} = \frac{\beta(C_0 \log \beta + \varepsilon)}{\beta - 1}.$$

Hence passing with ε to 0 and next with β to 1 we obtain

$$\limsup_{t \rightarrow \infty} t \mathbb{P}[Z > t] \leq C_0.$$

Analogously we justify

$$\liminf_{t \rightarrow \infty} t \mathbb{P}[Z > t] \geq C_0,$$

that proves the theorem in the aperiodic case. If A_i are periodic, then the same arguments give the result. \square

Acknowledgements

The author is grateful to anonymous referees for many comments and suggestions on the presentation of this paper.

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