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Finite dimensional Fokker–Planck equations for continuous time random walk limits

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Abstract

Continuous Time Random Walk (CTRW) is a model where particle's jumps in space are coupled with waiting times before each jump. A Continuous Time Random Walk Limit (CTRWL) is obtained by a limit procedure on a CTRW and can be used to model anomalous diffusion. The distribution $p\,(dx,t)$ of a CTRWL X_t satisfies a Fractional Fokker–Planck Equation (FFPE). Since CTRWLs are usually not Markovian, their one dimensional FFPE is not enough to completely determine them. In this paper we find the FFPEs of the distribution of X_t at multiple times, i.e. the distribution of the random vector $\left(X_{t_1},\ldots,X_{t_n}\right)$ for $t_1<\cdots<t_n$ for a large class of CTRWLs. This allows us to define CTRWLs by their finite dimensional FFPEs.

MSC: 60G50; 60F17; 82C31

Keywords: Continuous time random walk limits; Fractional Fokker–Planck equation; Anomalous diffusion

1. Introduction

Continuous Time Random Walk (CTRW) models the movement of a particle in space, where the k'th jump J_k of the particle in space succeeds the k'th waiting time W_k. We let $N_t=\sup\{k: T_k\le t\}$ where $T_k=\sum_{i=1}^{k}W_i$, if $T_1>t$ then we set N_t to be 0. N_t is just the number of jumps of the particle up to time t. Then

$$X'_t=\sum_{k=1}^{N_t}J_k,$$

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is the CTRW associated with the time–space jumps $\{(J_k, W_k)\}_{k\in\mathbb{N}}$. Let us now assume that $\{J_k\}$ and $\{W_k\}$ are independent i.i.d. sequences of random variables. In order to model the long time behaviour of the CTRW we write $\left\{\left(J_k^c, W_k^c\right)\right\}_{k\in\mathbb{N}}$ for $c > 0$. Here the purpose of c is to facilitate the convergence of the trajectories of $\left\{\left(J_k^c, W_k^c\right)\right\}_{k\in\mathbb{N}}$ weakly on a proper space. More precisely, we let $\mathcal{D}\left([0,\infty), \mathbb{R}^2\right)$ be the space of cádlág functions $f : [0,\infty) \to \mathbb{R}^2$ equipped with the Skorokhod J_1 topology. We assume that

$$\left(S_u^c, T_u^c\right) = \sum_{k=1}^{\lfloor cu\rfloor}\left(J_k^c, W_k^c\right) \Rightarrow \left(A_u, D_u\right) \quad c\to\infty,$$

where $\Rightarrow$ denotes weak convergence of measures with respect to the J_1 topology. We further assume that the processes A_t and D_t are independent Lévy processes and that D_t is a strictly increasing subordinator. Denote by X_t^c the CTRW associated with $\left\{\left(J_k^c, W_k^c\right)\right\}_{k\in\mathbb{N}}$. We then have ([15, Theorem 3.6] and [14, Lemma 2.4.5])

$$X_t^c \Rightarrow X_t = A_{E_t} \quad c\to\infty, \tag{1.1}$$

where $E_t = \inf\{s : D_s > t\}$ is the inverse of D_t and $\Rightarrow$ means weak convergence on $\mathcal{D}([0,\infty), \mathbb{R})$ equipped with the J_1 topology. It is well known that X_t is usually not Markovian, a fact that makes the task of finding basic properties of X_t nontrivial. One such task is finding the finite dimensional distributions (FDDs) of the process X_t, i.e. $P\left(X_{t_1}\in dx_1,\ldots,X_{t_n}\in dx_n\right)$. In the physics literature, there is much emphasis put on the FDDs and correlation functions of the Continuous Time Random Walk Limit (CTRWL). Correlation functions are a vital experimental tool for distinguishing CTRWL from other fractional diffusion (such as fractional Brownian motion) [2]. In [11], Meerschaert and Straka used a semi-Markov approach to find the FDDs for a large class of CTRWL. It turns out that the discrete regeneration times of X_t^c converge to a set of points where X_t is renewed. Once we know the next time of regeneration of X_t, we no longer need older observations in order to determine the future behaviour of X_t. More mathematically, denote by $R_t = D_{E_t} - t$ the time left before regeneration of X_t then (X_t, R_t) is a Markov process. One can then use the transition probabilities of (X_t, R_t) along with the Chapman–Kolmogorov Equations in order to find $P\left(X_{t_1}\in dx_1,\ldots,X_{t_n}\in dx_n\right)$ for $t_1 < \cdots < t_n$ and $n\in\mathbb{N}$. This method was used in [6] in order to find the FDD of the aged process $X_t^{t_0} = X_t - X_{t_0}$. It is well known [9, Section 4.5] that the one dimensional distribution $p(dx,t) = P(X_t\in dx)$ satisfies a Fractional Fokker–Planck Equation (FFPE). Once again, as X_t is non Markovian the FFPE satisfied by $p(dx,t)$ is not enough to fully describe X_t (as it does when X_t is Markovian). Hence, a dual problem to finding the FDDs is that of finding the finite dimensional FFPEs of the FDDs of X_t. In this paper we obtain the finite dimensional FFPEs for a large class of CTRWL. We use the expression of the FDDs found in [11] along with Fourier–Laplace transform to find the FFPEs of these FDD. This is done first by investigating the multivariable Fourier–Laplace transform on relevant distributions on certain subsets of $\mathbb{R}_+^n$, developing multivariable space–time pseudo-differential operators (PDOs) and applying these results to the expression found in [11]. Results on the finite dimensional FFPEs of CTRWL exist in the literature [3,4,7], however, the methods used there are somewhat limited (cf. Remark 4). For example, these methods can only be used to find the FFPEs of the distribution $h(dx_1,\ldots,dx_n;t_1,\ldots,t_n)$ of the inverse of a subordinator on $x_1 < x_2 < \cdots < x_n$, whereas the distribution's support is $x_1\le x_2\le\cdots\le x_n$. Moreover, these methods are ill-suited for coupled CTRWs. Our results generalize prior results to find the FFPE of the inverse subordinator on $x_1\le x_2\le\cdots\le x_n$ as well as for the

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coupled case. We also provide results on PDOs that appear naturally in the finite dimensional FFPEs.

In Section 2 we present relevant mathematical background for this paper and prepare the way for our main result. It is divided into 4 subsections; Section 2.1 introduces the notation to be used throughout the paper, Section 2.2 presents the Caputo and Riemann–Liouville fractional derivatives, Section 2.3 establishes results regarding PDOs on certain multivariable functions which facilitate the proof of Theorem 1 and Section 2.4 presents briefly the work in [11] upon which we establish our results.

Section 3 presents our main results; Theorem 1 gives the finite dimensional FFPEs of E_t, Theorem 2 states the finite dimensional FFPEs of the process $X_t = A_{E_t}$ where the outer process A_t and the subordinator D_t are independent. Finally, Theorem 3 gives the finite dimensional FFPEs of the coupled case. Section 4 compares our results with the well known finite dimensional case.

In Section 5 we show that if $\xi\left(-k, s\right)$ is the symbol of a PDO on a suitable Banach space then $\xi\left(-\sum_{i=1}^{n} k_i, \sum_{i=1}^{n} s_i\right)$ is also a symbol of a PDO on another Banach space. This complements the results in Section 3.

For the reader's convenience, we list here the abbreviations that appear in this paper:

- CTRW—Continuous Time Random Walk.
- CTRWL—Continuous Time Random Walk Limit.
- FFPE—Fractional Fokker–Planck Equation.
- FDD—finite dimensional distribution.
- PDO—pseudo-differential operators.
- LT—Laplace Transform.
- FT—Fourier Transform.
- FLT—Fourier–Laplace Transform.
- LLT—Laplace–Laplace Transform.
- RL—Riemann–Liouville.

2. Mathematical background

2.1. Notations

A well known method of solving partial differential equations of distributions $p(dx_1, \ldots, dx_n; t_1, \ldots, t_n)$ on $\mathbb{R}^n$ is taking the Fourier Transform (FT) of the distribution with respect to the spatial variables and then the Laplace Transform (LT) with respect to the time variables. This is referred to as the Fourier–Laplace Transform (FLT) of $p\left(dx_1, \ldots, dx_n; t_1, \ldots, t_n\right)$. More generally, for $m, n \in \mathbb{N}$ let $f\left(dx_1, \ldots, dx_m; t_1, \ldots, t_n\right)$ be a finite measure on $\mathbb{R}^m$ for every $\mathbf{t} = (t_1, \ldots, t_n)$ s.t. $0 < t_1 \le \cdots \le t_n$ and assume that $\int_{\mathbf{x} \in A} f\left(dx_1, \ldots, dx_m; t_1, \ldots, t_n\right)$ is measurable as a function of $\mathbf{t}$ for every measurable $A \subset \mathbb{R}^m$. We denote the FT of f by

$$\widetilde{f}\left(k_1, \ldots, k_m; t_1, \ldots, t_n\right) = \int_{x_1 \in \mathbb{R}} \cdots \int_{x_m \in \mathbb{R}} e^{-i \sum_{j=1}^{m} k_j x_j} f\left(dx_1, \ldots, dx_m; t_1, \ldots, t_n\right).$$

When f has density $f\left(x_1, \ldots, x_m; t_1, \ldots, t_n\right)$ we denote the LT of f by

$$\hat{f}\left(x_1, \ldots, x_m; s_1, \ldots, s_n\right) = \int_{t_1=0}^{\infty} \cdots \int_{t_n=0}^{\infty} e^{-\sum_{j=1}^{n} s_j t_j} f\left(x_1, \ldots, x_m; t_1, \ldots, t_n\right) dt_1 \cdots dt_n.$$

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The FLT of f is

$$\bar{f}(k_1,\ldots,k_m;s_1,\ldots,s_n) = \int_{t_1=0}^{\infty}\cdots\int_{t_n=0}^{\infty}\int_{x_1\in\mathbb{R}}\cdots\int_{x_n\in\mathbb{R}} e^{-i\sum_{j=1}^{m}k_jx_j-\sum_{j=1}^{n}s_jt_j} \times f(dx_1,\ldots,dx_m;t_1,\ldots,t_n)\,dt_1\cdots dt_n.$$

We also denote by $\tilde{f}$ the FT of f with respect to some of its spatial variables, therefore, $\widetilde{f}(dx_1,k_2;t_1,t_2)$ is the FT of f w.r.t. x_2. Similarly, $\hat{f}(dx_1,dx_2;s_1,t_2)$ is the LT of f w.r.t. t_1 and $\bar{f}(k_1,dx_2;s_1,t_2)$ is the FLT of f w.r.t. x_1 and t_1. When using the hat symbol is cumbersome we also use $\hat{f} = \mathscr{L}(f)$. We occasionally use bold font to represent the vector $\mathbf{x} = (x_1,\ldots,x_n)$ where the size of the vector is clear.

2.2. Caputo and Riemann–Liouville fractional derivatives

The Riemann–Liouville (RL) fractional derivative of index $0<\alpha<1$ is given by

$$\mathfrak{D}_t^{\alpha} f(t) = \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha} f(r)\,dr, \tag{2.1}$$

for a suitable function f defined on $\mathbb{R}_+$. When the variable with respect to which we take the derivative is obvious we drop the subscript and just write $\mathfrak{D}^{\alpha} f(t)$. It can be verified that the LT of (2.1) is

$$\widehat{\mathfrak{D}^{\alpha} f}(s) = s^{\alpha}\hat{f}(s).$$

Hence, the RL derivative is a PDO of symbol s^{α}. Caputo's derivative is obtained by moving the derivative in (2.1) under the integral to obtain

$$\mathbb{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}\frac{\partial}{\partial r} f(r)\,dr. \tag{2.2}$$

The LT of (2.2) is

$$\widehat{\mathbb{D}^{\alpha} f}(s) = s^{\alpha}\hat{f}(s) - s^{\alpha-1} f\left(0^+\right).$$

We denote the classic derivative by $\frac{\partial}{\partial t} = \mathbb{D}^1$, and note that $\mathbb{D}^1 = \mathfrak{D}^1$ iff $f\left(0^+\right) = 0$. For simplicity we drop the superscript and write $\frac{\partial}{\partial t} = \mathcal{D}$ (or $\frac{\partial}{\partial t} = \mathfrak{D}$ when that is the case).

2.3. Pseudo-differential operators of multivariable functions

Here we investigate the PDOs acting on measures $f(dx_1,\ldots,dx_n)$ on $\mathbb{R}_+^n$ with support in $A^n = \{\mathbf{x} : 0 \le x_1 \le x_2 \le \cdots \le x_n\}$ with LT $\hat{f}$. Let $\mathbf{k} = (k_1,\ldots,k_l)$ be a strictly increasing l-tuple where $1 \le k_i \le n$ for $1 \le i \le l \le n$ and s.t. $k_1 = 1$. We shall sometimes abuse notation and write $i \in \mathbf{k}$ where we mean that $i = k_j$ for some $1 \le j \le l$. We also write $\mathbf{k}^c$ for the increasing vector s.t. $i \in \mathbf{k}^c$ iff $2 \le i \le n$ and $i \notin \mathbf{k}$. If $\mathbf{x}$ is a vector of length n we write $\mathbf{x_k}$ for the vector of length l whose i'th element is x_{k_i}. Let $A_{\mathbf{k}}^n$ be the set of all $\mathbf{x} \in A^n$ s.t. $x_{i-1} < x_i$ iff $i \in \mathbf{k}$ and where $x_0 = 0$. For example, for $n = 3$ $A_{(1,2)}^3 = \{\mathbf{x} : 0 < x_1 < x_2 = x_3\}$. Since our interest in these distributions comes from the FDDs of the process E_t, i.e. $h(dx_1,\ldots,dx_n;t_1,\ldots,t_n) = P\left(E_{t_1}\in dx_1,\ldots,E_{t_n}\in dx_n\right)$ we also assume in this subsection that $f(dx_1,\ldots,dx_n)$ can be written as $f(dx_1,\ldots,dx_n) = f\left(x_{k_1},\ldots,x_{k_l}\right)\delta_{k_1^c-1}\left(dx_{k_1^c}\right)\times\cdots\times\delta_{k_{n-l}^c-1}\left(dx_{k_{n-l}^c}\right)dx_{k_1}\cdots dx_{k_l}$ where $f\left(x_{k_1},\ldots,x_{k_l}\right)$ is absolutely continuous (a.c.) in each of its variables, i.e. $x_{k_i}\to f(x_1,\ldots,x_n)$ is a.c. with respect to Lebesgue measure on $\mathbb{R}$ for each $1\le i\le l$. We occasionally refer to such f as a.c., not to be confused with the concept of a.c. measure on $\mathbb{R}^n$. We abbreviate by writing

$$f_{\mathbf{k}}(d\mathbf{x}) = f(\mathbf{x}_{\mathbf{k}})\,\delta_{\mathbf{k}^c-1}(d\mathbf{x}_{\mathbf{k}^c})\,d\mathbf{x}_{\mathbf{k}}, \tag{2.3}$$

so that $\mathbf{k}$ points out the indices for which f has absolutely continuous density. For example, $f_{(1,2,4)}(dx_1,dx_2,dx_3,dx_4)$ can be written as $f(x_1,x_2,x_4)\,\delta_{x_2}(dx_3)\,dx_1dx_2dx_4$. To motivate this assumption cf. (3.1) and note that by (2.8) $h(dx_1,\ldots,dx_n;t_1,\ldots,t_n)$ is of the form $f(\mathbf{x}_{\mathbf{k}})$ on $A^n_{\mathbf{k}}$. The set $A^n_{\mathbf{k}}$ is a manifold of dimension l, and represents the event where the process E_t has been stuck at the point x_i since the time t_{i-1} to t_i for $i\notin\mathbf{k}$. For example, $A^4_{(1,3)}$ represents the event $\left\{E_{t_1}=x_1\in(0,\infty),E_{t_2}=x_1,E_{t_3}=x_3\in(x_1,\infty),E_{t_4}=x_3\right\}$, and it helps to think of $\mathbf{k}$ as the indices of mobilized points of the particle. Let us define a derivative operator on $f_{\mathbf{k}}$ distributions. We define the derivative operator to be

$$\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}(d\mathbf{x}) = \sum_{i=1}^{l}\frac{\partial}{\partial x_{k_i}} f(\mathbf{x}_{\mathbf{k}})\,\delta_{\mathbf{k}^c-1}(d\mathbf{x}_{\mathbf{k}^c})\,d\mathbf{x}_{\mathbf{k}}.$$

For example, if $f_{(1,2)}(d\mathbf{x}) = f(x_1,x_2)\,\delta_{x_2}(dx_3)\,dx_1dx_2$ then

$$\begin{aligned}\mathbb{D}_{\mathbf{x}} f_{(1,2)}(d\mathbf{x}) &= \frac{\partial}{\partial x_1} f_{(1,2)}(x_1,x_2)\,\delta_{x_2}(dx_3)\,dx_1dx_2\\ &\quad+\frac{\partial}{\partial x_2} f_{(1,2)}(x_1,x_2)\,\delta_{x_2}(dx_3)\,dx_1dx_2.\end{aligned}$$

Note that $\mathbb{D}_{\mathbf{x}}$ is well defined as we assume that $f_{\mathbf{k}}$ has a.c. density in x_i for $i\in\mathbf{k}$. We also assume that $\lim_{x_{k_l}\to\infty} e^{-x_{k_l}} f\left(x_{k_1},\ldots,x_{k_l}\right)=0$ where f is as in (2.3). This is not a strong assumption as f has LT.

Lemma 1. *Let $f_{\mathbf{k}}$ be such that $l=n$. Then the LT of $\mathbb{D}_{\mathbf{x}} f(\mathbf{x})$ is*

$$\widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{s}) = \left(\sum_{i=1}^{n} s_i\right)\widehat{f_{\mathbf{k}}}(s_1,\ldots,s_n) - \lim_{x_1\to0^+}\hat{f}_{\mathbf{k}}(x_1,s_2,\ldots,s_n). \tag{2.4}$$

Proof. In the following, we use $\check{a}_i$ to indicate that a_i is absent from where it normally should be. Since here $f_{\mathbf{k}}(d\mathbf{x}) = f(\mathbf{x})\,d\mathbf{x}$, for $1\le i\le n$ we have

$$\begin{aligned}&\int_{x_1=0}^{\infty}\int_{x_2=0}^{\infty}\cdots\int_{x_n=0}^{\infty} e^{-\langle\mathbf{s},\mathbf{x}\rangle}\frac{\partial f(\mathbf{x})}{\partial x_i}d\mathbf{x}\\ &=\int_{x_1=0}^{\infty}\cdots\int_{x_i=0}^{\check{\infty}}\cdots\int_{x_n=0}^{\infty} e^{-s_1x_1\cdots-s_i\check{x}_i\cdots-s_nx_n}\\ &\quad\times\left[\int_{x_i=0}^{\infty} e^{-s_ix_i}\frac{\partial f(\mathbf{x})}{\partial x_i}dx_i\right]dx_1\cdots d\check{x}_i\cdots dx_n\\ &=\int_{x_1=0}^{\infty}\cdots\int_{x_i=0}^{\check{\infty}}\cdots\int_{x_n=0}^{\infty} e^{-s_1x_1\cdots-s_i\check{x}_i\cdots-s_nx_n}\left[e^{-s_ix_i} f(\mathbf{x})\Big|^{(x_1,\ldots,x_{i-1},x_{i+1},x_{i+1},\ldots,x_n)}_{(x_1,\ldots,x_{i-1},x_{i-1},x_{i+1},\ldots,x_n)}\right.\end{aligned}$$

$$
\begin{aligned}
&+ s_i \int_{x_i=0}^{\infty} e^{-s_i x_i} f(\mathbf{x})\, dx_i \Bigg] dx_1 \cdots d\check{x}_i \cdots dx_n \\
&= \int_{x_1=0}^{\infty} \cdots \int_{x_i=0}^{\check{\infty}} \cdots \int_{x_n=0}^{\infty} e^{-s_1 x_1 \cdots - s_i \check{x}_i \cdots - s_n x_n} \\
&\quad \times \Bigg[e^{-s_i x_{i+1}} f\Bigg(x_1 \ldots, x_{i-1}, \underbrace{x_{i+1}}_{\text{i'th coordinate}}, x_{i+1} \ldots, x_n \Bigg) \\
&\quad - e^{-s_i x_{i-1}} f\Bigg(x_1 \ldots, x_{i-1}, \underbrace{x_{i-1}}_{\text{i'th coordinate}}, x_{i+1} \ldots, x_n \Bigg) \\
&\quad + s_i \int_{x_i=0}^{\infty} e^{-s_i x_i} f(x_1, x_2, \ldots, x_n)\, dx_i \Bigg] \\
&\quad \times dx_1 \cdots d\check{x}_i \cdots dx_n \\
&= \int_{x_{i+1}=0}^{\infty} e^{-(s_i + s_{i+1}) x_{i+1}} \hat{f}\Bigg(s_1 \ldots, s_{i-1}, \underbrace{x_{i+1}}_{\text{i'th coordinate}}, x_{i+1}, s_{i+2} \ldots, s_n \Bigg) dx_{i+1} \\
&\quad - \int_{x_{i-1}=0}^{\infty} e^{-(s_i + s_{i-1}) x_{i-1}} \hat{f}\Bigg(s_1 \ldots, x_{i-1}, \underbrace{x_{i-1}}_{\text{i'th coordinate}}, s_{i+1}, s_{i+2} \ldots, s_n \Bigg) dx_{i-1} \\
&\quad + s_i \hat{f}(s_1, s_2, \ldots, s_n). \qquad (2.5)
\end{aligned}
$$

Note that since $\lim_{x_n \to \infty} e^{-x_n} f(x_1, \ldots, x_n) = 0$, summing over the variable i the first two terms in the last equality in (2.5) cancel out for every $i \neq 1$. For $i = 1$ only the second term in the brackets cancels out and the result follows. □

Lemma 2. *The LT of* $\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}(d\mathbf{x})$ *is*

$$
\widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{s}) = \left(\sum_{i=1}^{n} s_i \right) \widehat{f_{\mathbf{k}}}(\mathbf{s}) - \lim_{x_1 \to 0^+} \hat{f_{\mathbf{k}}}(x_1, s_2, \ldots, s_n).
$$

Proof. Taking the LT of $f_{\mathbf{k}}(dx_1, \ldots, dx_n)$ first w.r.t. the indices that are not in $\mathbf{k}$ we see that

$$
\widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{x}_{\mathbf{k}}, \mathbf{s}_{\mathbf{k}^c}) = \int_{\mathbb{R}_+^{n-l}} e^{-\sum_{i \in \mathbf{k}^c} s_i x_i} \mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}(d\mathbf{x}), \qquad (2.6)
$$

and can be written as

$$
\widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{x}_{\mathbf{k}}, \mathbf{s}_{\mathbf{k}^c}) = \mathbb{D}_{\mathbf{x}} f(\mathbf{x}_{\mathbf{k}})\, e^{-\sum_{i=1}^{l-1} \left(\sum_{j=k_i+1}^{k_{i+1}-1} s_j \right) x_{k_i} - \sum_{j=k_l+1}^{n} s_j x_{k_l}},
$$

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where f is an a.c. function. It follows that

$$\widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{s}) = \int_{\mathbb{R}_+^l} e^{-\sum_{i\in\mathbf{k}} s_i x_i} \widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{x}_{\mathbf{k}}, \mathbf{s}_{\mathbf{k}^c})\, d\mathbf{x}_{\mathbf{k}}$$

$$= \int_{\mathbb{R}_+^l} e^{-\sum_{i\in\mathbf{k}} s_i x_i} \mathbb{D}_{\mathbf{x}} f(\mathbf{x}_{\mathbf{k}})\, e^{-\sum_{i=1}^{l-1}\left(\sum_{j=k_i+1}^{k_{i+1}-1} s_j\right) x_{k_i} - \sum_{j=k_l+1}^{n} s_j x_{k_l}} d\mathbf{x}_{\mathbf{k}}$$

$$= \int_{\mathbb{R}_+^l} e^{-\sum_{i=1}^{l-1}\left(\sum_{j=k_i}^{k_{i+1}-1} s_j\right) x_{k_i} - \sum_{j=k_l}^{n} s_j x_{k_l}} \mathbb{D}_{\mathbf{x}} f(\mathbf{x}_{\mathbf{k}})\, d\mathbf{x}_{\mathbf{k}}.$$

By Lemma 1 for $n = l$ we see that

$$\widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{s}) = \left(\sum_{i=1}^{n} s_i\right) \int_{\mathbb{R}_+^l} e^{-\sum_{i=1}^{l-1}\left(\sum_{j=k_i}^{k_{i+1}-1} s_j\right) x_{k_i} - \sum_{j=k_l}^{n} s_j x_{k_l}} f(\mathbf{x}_{\mathbf{k}})\, d\mathbf{x}_{\mathbf{k}}$$

$$- \lim_{x_1\to 0^+} \int_{\mathbb{R}_+^{l-1}} e^{-\sum_{i=2}^{l-1}\left(\sum_{j=k_i}^{k_{i+1}-1} s_j\right) x_{k_i} - \sum_{j=k_l}^{n} s_j x_{k_l}} f(\mathbf{x}_{\mathbf{k}})\, d\mathbf{x}_{\mathbf{k}'}$$

where $\mathbf{k}'$ is just the vector of length $l-1$ s.t. $k'_i = k_{i+1}$ for $1 \le i \le l-1$. Since

$$\widehat{f_{\mathbf{k}}}(\mathbf{s}) = \int_{\mathbb{R}_+^n} e^{-\sum_{i=1}^{n} s_i x_i} f(\mathbf{x}_{\mathbf{k}})\, \delta_{\mathbf{k}^c-1}(d\mathbf{x}_{\mathbf{k}^c})\, d\mathbf{x}_{\mathbf{k}}$$

$$= \int_{\mathbb{R}_+^l} e^{-\sum_{i=1}^{l-1}\left(\sum_{j=k_i}^{k_{i+1}-1} s_i\right) x_{k_i} - \sum_{j=k_l+1}^{n} s_j x_{k_l}} f(\mathbf{x}_{\mathbf{k}})\, d\mathbf{x}_{\mathbf{k}},$$

the result follows. □

If $f(\mathbf{x})$ is a differentiable function then $\mathbb{D}_{\mathbf{x}}$ is just the directional derivative along the vector $v = (1, \dots, 1)$ of size n. Let Ψ_x be a PDO on $\mathbb{R}$ with symbol $\psi(k)$. Then $\psi(\sum_{i=1}^n k_i)$ is a symbol of the PDO $\Psi_{\mathbf{x}}$ to be defined later and where we use bold $\mathbf{x}$ subscript to emphasize the fact that $\Psi_{\mathbf{x}}$ is defined on functions on $\mathbb{R}^n$. One can think of $\Psi_{\mathbf{x}}$ as the directional version of Ψ_x with directional vector $v = (1, \dots, 1)$, this will be defined rigorously in Section 5.

Define the RL fractional derivative of index $0 < \alpha < 1$ of $f(\mathbf{x})$ to be

$$\mathfrak{D}_{\mathbf{x}}^{\alpha} f(\mathbf{x}) = \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\right) \int_0^{x_1} f(x_1 - r, x_2 - r, \dots, x_n - r) \frac{r^{-\alpha}}{\Gamma(1-\alpha)} dr. \tag{2.7}$$

Once again, Eq. (2.7) can be thought of as a fractional directional derivative.

As opposed to the one dimensional case where under certain conditions the derivative w.r.t. the time variable is defined on a function $p(x; t)$, in the finite dimensional case one cannot avoid the fact that $p(d\mathbf{x}; \mathbf{t})$ is $f_{\mathbf{k}}(d\mathbf{x})$ valued on $A_{\mathbf{k}}^n$. In order to describe the dynamics of $p(d\mathbf{x}; \mathbf{t})$ on $A_{\mathbf{k}}^n$ one should extend this notion to the functions $f_{\mathbf{k}}(\mathbf{t})$, where we now use the letter $\mathbf{t}$ in order to emphasize the context of this operator. Since on $A_{\mathbf{k}}^n$ the dynamics on $\mathbf{t}_{\mathbf{k}^c}$ are degenerate it is reasonable to apply $\mathbb{D}_{\mathbf{x}}^{\alpha}$ on $\mathbf{t}_{\mathbf{k}}$. More precisely, if $f_{\mathbf{k}}(d\mathbf{t}) = f(\mathbf{t}_{\mathbf{k}})\, \delta_{\mathbf{t}_{\mathbf{k}}-1}(d\mathbf{t}_{\mathbf{k}^c})\, d\mathbf{t}_{\mathbf{k}}$ (here $f(\mathbf{t}_{\mathbf{k}})$

need not be a.c.) then we define

$$\mathfrak{D}_{\mathbf{t}}^{\alpha} f_{\mathbf{k}}(d\mathbf{t}) := \mathbb{D}_{\mathbf{t}}\left[\int_0^{x_1} f\left(x_{k_1}-r,\ldots,x_{k_l}-r\right)\frac{r^{-\alpha}}{\Gamma(1-\alpha)}dr\,\delta_{\mathbf{t_k}-1}\left(d\mathbf{t}_{\mathbf{k}^c}\right)\right].$$

The analogue of Lemma 2 is the following.

Lemma 3. *The LT of* $\mathfrak{D}_{\mathbf{t}}^{\alpha} f_{\mathbf{k}}(d\mathbf{t})$ *is* $\left(\sum_{i=1}^n s_n\right)^{\alpha}\widehat{f_{\mathbf{k}}}(\mathbf{s})$.

Proof. As before, we start with $f_{\mathbf{k}}(d\mathbf{t})$ where $l = n$ so that $f_{\mathbf{k}}(d\mathbf{t}) = f(\mathbf{t})$. A simple computation shows that

$$\mathscr{L}\left(\int_0^{t_1} f(t_1-r,t_2-r,\ldots,t_n-r)\frac{r^{-\alpha}}{\Gamma(1-\alpha)}dr\right)(\mathbf{s}) = \left(\sum_{i=1}^n s_i\right)^{\alpha-1}\hat{f}(s_1,\ldots,s_n).$$

Next, note that

$$\mathscr{L}\left(\int_0^{t_1} f(t_1-r,t_2-r,\ldots,t_n-r)\frac{r^{-\alpha}}{\Gamma(1-\alpha)}dr\right)(t_1,s_2,\ldots,s_n)$$
$$= \int_{r=0}^{t_1} e^{-\left(\sum_{i=2}^n s_i\right)r}\hat{f}(t_1-r,s_2,\ldots,s_n)\frac{r^{-\alpha}}{\Gamma(1-\alpha)}dr$$

so that $\lim_{t_1\to 0^+}\mathscr{L}\left(\int_0^{t_1} f(t_1-r,t_2-r,\ldots,t_n-r)\frac{r^{-\alpha}}{\Gamma(1-\alpha)}dr\right)(t_1,s_2,\ldots,s_n) = 0$. It follows by Lemma 1 that

$$\widehat{\mathfrak{D}_{\mathbf{t}}^{\alpha} f_{\mathbf{k}}}(d\mathbf{t}) = \left(\sum_{i=1}^n s_n\right)^{\alpha}\hat{f}(s_1,\ldots,s_n).$$

The case where $l < n$ is similar to Lemma 2. □

Remark 1. There is nothing exceptional about the operator $\mathfrak{D}_{\mathbf{t}}^{\alpha}$, in fact it is better to think of it as an archetype of PDOs corresponding to Laplace symbols of Lévy measures on $\mathbb{R}_+$. Indeed, if $\phi(s) = \int_{\mathbb{R}_+}\left(e^{-sy}-1\right)K_2(y)\,dy$, then $\phi(s)$ is the symbol of the PDO $\Phi_t(f)(t) = \int_0^{\infty}(f(t-y)-f(t))K_2(y)\,dy$. A simple calculation then shows that $\phi\left(\sum_{i=1}^n s_i\right)$ is the symbol of $\Phi_{\mathbf{t}}(f)(\mathbf{t}) = \int_0^{\infty}(f(t_1-y,\ldots,t_n-y)-f(t))K_2(y)\,dy$. The extension to the functions $f_{\mathbf{k}}$ is obtained along similar lines to Lemma 3.

2.4. The semi-Markov approach

Since the process $X_t = A_{E_t}$ is not Markovian, knowing its one dimensional distribution in not enough to construct its FDDs. To circumvent this problem Meerschaert and Straka [11] constructed the Markov process (X_t, R_t), where $R_t = D_{E_t} - t$ is the time left before the next regeneration of the process X_t. Let $Q_t\left(x',r';dx,dr\right)$ be the transition probability of the process (X_t, R_t) and $0 < t_1 < t_2 < \cdots < t_n$ for some $n \in \mathbb{N}$. Then

$$P\left(X_{t_1}\in dx_1, X_{t_2}\in dx_2,\ldots,X_{t_n}\in dx_n\right)$$
$$= \int_{r_1=0}^{\infty}\int_{r_2=0}^{\infty}\cdots\int_{r_n=0}^{\infty} Q_{t_1}(0,0;dx_1,dr_1)$$
$$\times Q_{t_2-t_1}(x_1,r_1;dx_2,dr_2)\cdots\times Q_{t_n-t_{n-1}}(x_{n-1},r_{n-1};dx_n,dr_n)$$

$$= Q_{t_1}(0,0;dx_1,dr_1)\circ Q_{t_2-t_1}(x_1,r_1;dx_2,dr_2) \times\cdots Q_{t_n-t_{n-1}}(x_{n-1},r_{n-1};dx_n,dr_n)\circ. \tag{2.8}$$

Here, $Q_t(x',r';dx,dr)\circ f(x,r)=\int_{r=0}^{\infty}f(x,r)\,Q_t(x',r';dx,dr)$ and $Q_t(x',r';dx,dr)\circ=\int_{r=0}^{\infty}Q_t(x',r';dx,dr)$. In [11], the expression for Q_t is given for a large class of jump diffusions. Here, however, unless stated otherwise we consider processes of the form $X_t=A_{E_t}$, where A_t is a Lévy process and E_t is the inverse of a strictly increasing subordinator D_t that is independent of A_t. That is,

$$E_t=\inf\{s>0:D_s>t\}.$$

More precisely, the characteristic function of A_t and the Laplace transform of D_t are given respectively by

$$E\left(e^{ikA_t}\right)=\exp\left[t\left(ibk-\frac{1}{2}ak^2+\int_{\mathbb{R}}\left(e^{iky}-1-iky1_{\{|y|<1\}}\right)K_1(dy)\right)\right] \tag{2.9}$$
$$E\left(e^{-sD_t}\right)=\exp\left[t\left(\int_{\mathbb{R}_+}\left(e^{-sy}-1\right)K_2(dy)\right)\right].$$

Here, $a\geq 0$, $b\in\mathbb{R}$. K_1 is a Lévy measure while K_2 is a Lévy measure whose support is $[0,\infty)$ and satisfies $\int(y\wedge 1)\,K_2(dy)<\infty$, $K_2(\{0\})=0$ and $\int K_2(dy)=\infty$. By (2.9) it can be easily verified that the infinitesimal generator $\mathcal{A}$ of the process (A_t,D_t) is

$$\mathcal{A}(f)(x,t)=b\frac{\partial}{\partial x}f(x,t)+\frac{a}{2}\frac{\partial^2}{\partial x^2}f(x,t) + \int_{\mathbb{R}^2}\left(f(x+y,t+w)-f(x,t)-y\frac{\partial f(x,t)}{\partial x}1_{\{|(y,w)|<1\}}\right)K(dy,dw), \tag{2.10}$$

where K is again a Lévy measure. In [11], the case where the coefficients b and a as well as the measure K may be dependent on (x,t) is considered. However, when they do not (this is referred to as the homogeneous case), the transition probability Q_t is given by [11, Equation. 4.4]

$$Q_t(x',r';dx,dr)=1_{\{0<t<r'\}}\delta_0(dx-x')\,\delta_{r'-t}(dr)+1_{\{0\leq r'\leq t\}}Q_{t-r'}(x',0;dx,dr)$$
$$Q_t(x',0;dx,dr)=\int_{y\in\mathbb{R}}\int_{w\in[0,t]}U^{x'}(dy,dw)\,K(dx-y,dr+t-w), \tag{2.11}$$

where $U^{x'}(dy,dw)$ is the occupation measure of (A_t,D_t), i.e.

$$\int f(y,w)\,U^{x'}(dy,dw)=\mathbb{E}\left(\int_0^{\infty}f\left(A_u+x',D_u\right)du\right).$$

When the processes A_t and D_t are independent, it can be easily verified that

$$U^{x'}(dy,dw)=\int_0^{\infty}z\left(dy-x',u\right)g(dw,u)\,du, \tag{2.12}$$

where $z(dx,t)=P(A_t\in dx)$ and $g(dx,t)=P(D_t\in dx)$. Moreover, in the case of independence it was shown that [5, Corollary 2.3]

$$K(dy,dw)=K_1(dy)\,\delta_0(dw)+\delta_0(dy)\,K_2(dw).$$

Hence, Eqs. (2.11) translate into

$$\begin{aligned} Q_t\left(x',r';dx,dr\right) &= 1_{\{0<t<r'\}}\delta_0\left(dx-x'\right)\delta_{r'-t}\left(dr\right)\\ &+1_{\{0\le r'\le t\}}\int_{y\in\mathbb{R}}\int_{w\in[0,t-r']}\left(\int_0^\infty z\left(dy-x',u\right)g\left(dw,u\right)du\right)\\ &\times\left(\delta_0\left(dr+t-r'-w\right)K_1\left(dx-y\right)+\delta_0\left(dx-y\right)K_2\left(dr+t-r'-w\right)\right). \end{aligned}\tag{2.13}$$

However, since $\int K_2\left(dy\right)=\infty$, we see [13, Theorem. 27.4] that $g\left(dw,t\right)$ has no atoms. Therefore, (2.13) reduces to

$$\begin{aligned} Q_t\left(x',r';dx,dr\right) = 1_{\{0\le t<r'\}}\delta_0\left(dx-x'\right)\delta_{r'-t}\left(dr\right)&\\ +\ 1_{\{0\le r'\le t\}}\int_{w\in[0,t-r']}\left(\int_0^\infty z\left(dx-x',u\right)g\left(dw,u\right)du\right)&\\ \times\ K_2\left(dr+t-r'-w\right).& \end{aligned}\tag{2.14}$$

3. Fokker–Planck equations

Throughout this section, we let A_t be a Lévy process such that $E\left(e^{ikA_t}\right)=e^{t\psi(k)}$, its probability density is given by $z\left(dx,t\right)=P\left(A_t\in dx\right)$. E_t is the inverse of a subordinator D_t such that $E\left(e^{-sD_t}\right)=e^{t\phi(s)}$, its probability density is $h\left(dx,t\right)=P\left(E_t\in dx\right)$. We denote by Ψ and Φ the pseudo-differential operators of the symbols $\psi\left(-k\right)$ and $-\phi\left(s\right)$ respectively. We also denote the transition probability function of the Markov process $\left(X_t,R_t\right)$ by Q_t and that of $\left(E_t,R_t\right)$ by H_t. Next note that the occupation measure of $\left(t,E_t\right)$ is just $U^{x'}\left(dx,dw\right)=g\left(dw,x-x'\right)dx$ (cf. [11, Eq. 5.1]), and similarly to (2.14) we have

$$\begin{aligned} H_t\left(x',r';dx,dr\right) = 1_{\{0\le t<r'\}}\delta_0\left(dx-x'\right)\delta_{r'-t}\left(dr\right)&\\ +\ 1_{\{0\le r'\le t\}}\int_{w\in[0,t-r']}g\left(dw,x-x'\right)dx\times K_2\left(dr+t-r'-w\right).& \end{aligned}\tag{3.1}$$

The next theorem finds the FFPE of the FDD of E_t.

Theorem 1. *Let $h\left(dx_1,\ldots,dx_n;t_1,\ldots,t_n\right)$ be the FDD of E_t where $t_1<t_2<\cdots<t_n$, i.e.*

$$h\left(dx_1,\ldots,dx_n;t_1,\ldots,t_n\right)=P\left(E_{t_1}\in dx_1,\ldots,E_{t_n}\in dx_n\right).$$

Then

$$\Phi_{\mathbf{t}}h\left(d\mathbf{x};\mathbf{t}\right)=-\mathbb{D}_{\mathbf{x}}h\left(d\mathbf{x};\mathbf{t}\right).\tag{3.2}$$

Proof. Let us take LT with respect to the spatial variables and with respect to the time variables, this will be abbreviated by LLT. Before taking the LLT of $h\left(d\mathbf{x};\mathbf{t}\right)$ we note that since $H_t\left(x',r';dx,dr\right)$ is translation invariant with respect to the spatial variable we have

$$\begin{aligned} h\left(d\mathbf{x};\mathbf{t}\right) &= H_{t_1}\left(0,0;dx_1,dr_1\right)\circ H_{t_2-t_1}\left(0,r_1;dx_2-x_1,dr_2\right)\\ &\cdots H_{t_n-t_{n-1}}\left(0,r_{n-1};dx_n-x_{n-1},dr_n\right)\circ. \end{aligned}\tag{3.3}$$

Taking the LLT of (3.3), by the change of variables $x_i'=x_i-x_{i-1}$ for $i\ge 2$ we see that (to avoid confusion we now use λ instead of k)

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$$\widehat{h}\left(\lambda_1,\dots\lambda_n;s_1,\dots,s_n\right)=\int_{t_1=0}^{\infty}\int_{x_1=0}^{\infty}e^{-\left(\sum_{i=1}^{n}s_i\right)t_1-\left(\sum_{i=1}^{n}\lambda_i\right)x_1}H_{t_1}\left(0,0;dx_1,dr_1\right)\circ dt_1$$
$$\widehat{H}_{\sum_{i=2}^{n}s_i}\left(0,r_1;\sum_{i=2}^{n}\lambda_i,dr_2\right)$$
$$\times\ \circ\cdots\widehat{H}_{s_n+s_{n-1}}\left(0,r_{n-2};\lambda_n+\lambda_{n-1},dr_{n-1}\right)\circ\widehat{H}_{s_n}\left(0,r_{n-1};\lambda_n,dr_n\right)\circ. \tag{3.4}$$

Now, let us look at

$$\int_{t_1=0}^{\infty}\int_{x_1=0}^{\infty}e^{-\left(\sum_{i=1}^{n}s_i\right)t_1-\left(\sum_{i=1}^{n}\lambda_i\right)x_1}H_{t_1}\left(0,0;dx_1,dr_1\right)dt_1$$
$$=\int_{t_1=0}^{\infty}\int_{x_1=0}^{\infty}e^{-\left(\sum_{i=1}^{n}s_i\right)t_1-\left(\sum_{i=1}^{n}\lambda_i\right)x_1}\int_{w\in[0,t_1]}g\left(w,x_1\right)dx_1$$
$$\times K_2\left(dr_1+t_1-w\right)dw$$
$$=\int_{x_1=0}^{\infty}e^{-\left(\sum_{i=1}^{n}\lambda_i\right)x_1}\int_{w\in[0,\infty]}g\left(w,x_1\right)dx_1$$
$$\times\int_{t_1=w}^{\infty}e^{-\left(\sum_{i=1}^{n}s_i\right)t_1}K_2\left(dr_1+t_1-w\right)dw$$
$$=\int_{x_1=0}^{\infty}e^{-\left(\sum_{i=1}^{n}\lambda_i\right)x_1}\int_{w\in[0,\infty]}g\left(w,x_1\right)dx_1e^{-\left(\sum_{i=1}^{n}s_i\right)w}dw$$
$$\times\int_{t_1=0}^{\infty}e^{-\left(\sum_{i=1}^{n}s_i\right)t_1}K_2\left(dr_1+t_1\right)$$
$$=\frac{1}{\sum_{i=1}^{n}\lambda_i-\phi\left(\sum_{i=1}^{n}s_i\right)}\int_{t_1=0}^{\infty}e^{-\left(\sum_{i=1}^{n}s_i\right)t_1}K_2\left(dr_1+t_1\right). \tag{3.5}$$

Next note that,

$$\lim_{x_1\to0^+}\widehat{h}\left(x_1,\lambda_2,\dots\lambda_n;s_1,\dots,s_n\right)$$
$$=\lim_{x_1\to0^+}\int_{t_1=0}^{\infty}e^{-\left(\sum_{i=1}^{n}s_i\right)t_1-\left(\sum_{i=2}^{n}\lambda_i\right)x_1}\int_{w\in[0,t_1]}g\left(dw,x_1\right)\times\int_{r_1=0}^{\infty}K_2\left(dr_1+t_1-w\right)$$
$$\times\ \widehat{H}_{\sum_{i=2}^{n}s_i}\left(0,r_1;\sum_{i=2}^{n}\lambda_i,dr_2\right)$$
$$\times\ \circ\cdots\widehat{H}_{s_n+s_{n-1}}\left(0,r_{n-2};\lambda_n+\lambda_{n-1},dr_{n-1}\right)\circ\widehat{H}_{s_n}\left(0,r_{n-1};\lambda_n,dr_n\right)\circ.$$

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$$
\begin{aligned}
&= \int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right)t_1} \int_{r_1=0}^{\infty} K_2\left(dr_1+t_1\right) \\
&\quad \times \widehat{H}_{\sum_{i=2}^{n} s_i}\left(0, r_1; \sum_{i=2}^{n} \lambda_i, dr_2\right) \cdots \widehat{H}_{s_n+s_{n-1}}\left(0, r_{n-2}; \lambda_n+\lambda_{n-1}, dr_{n-1}\right) \\
&\quad \times \circ \widehat{H}_{s_n}\left(0, r_{n-1}; \lambda_n, dr_n\right) \circ . \qquad (3.6)
\end{aligned}
$$

Indeed, by the continuity of the measure K_2 and [13, Lemma 27.1] follows the continuity of the following function

$$
w \mapsto \int_{r_1=0}^{\infty} K_2\left(dr_1+t_1-w\right) \times \widehat{H}_{\sum_{i=2}^{n} s_i}\left(0, r_1; \sum_{i=2}^{n} \lambda_i, dr_2\right) \circ \cdots \widehat{H}_{s_n}\left(0, r_{n-1}; \lambda_n, dr_n\right) \circ,
$$

since $g\left(dw, x_1\right) dx_1$ converges weakly to $\delta_0\left(dw\right)$ as $x_1 \rightarrow 0^+$ (3.6) follows. Finally, plugging (3.5) in (3.4), using (3.6) and rearranging terms we arrive at

$$
\begin{aligned}
-\phi\left(\sum_{i=1}^{n} s_i\right) \widehat{h}\left(\lambda_1, \ldots \lambda_n; s_1, \ldots, s_n\right) &= -\left(\sum_{i=1}^{n} \lambda_i\right) \widehat{h}\left(\lambda_1, \ldots \lambda_n; s_1, \ldots, s_n\right) \\
&\quad + \widehat{h}\left(0^+, \lambda_2, \ldots \lambda_n; s_1, \ldots, s_n\right). \qquad (3.7)
\end{aligned}
$$

Taking the inverse LLT of (3.7) and using Lemmas 2 and 3 we obtain (3.2). □

Theorem 1 paves the way for the finite dimensional FFPEs of the process X_t. We denote the FDD of A_t by $z\left(dx_1, \ldots, dx_n; t_1, \ldots, t_n\right) = P\left(A_{t_1} \in dx_1, \ldots, A_{t_n} \in dx_n\right)$.

Theorem 2. *Let* $p\left(dx_1, \ldots, dx_n; t_1, \ldots, t_n\right) = P\left(X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n\right)$ *where* $t_1 < t_2 < \cdots < t_n$. *Then*

$$
\begin{aligned}
&\Phi_{\mathbf{t}} p\left(dx_1, \ldots, dx_n; t_1, \ldots, t_n\right) = \Psi_{\mathbf{x}} p\left(dx_1, \ldots, dx_n; t_1, \ldots, t_n\right) \\
&\quad + \int_{u_2=0}^{\infty} \int_{u_3=u_2}^{\infty} \cdots \int_{u_n=u_{n}-1}^{\infty} \delta_0\left(dx_1\right) z\left(dx_2, \ldots, dx_n; u_2, \ldots, u_n\right) \\
&\quad \times h\left(0^+, du_2, \ldots, du_n; t_1, \ldots, t_n\right) \qquad (3.8)
\end{aligned}
$$

Proof. By the independence of A_t and D_t

$$
\begin{aligned}
&p\left(dx_1, \ldots, dx_n; t_1, \ldots, t_n\right) \\
&\quad = \int_{u_1=0}^{\infty} \int_{u_2=u_1}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} z\left(dx_1, \ldots, dx_n; u_1, \ldots, u_n\right) h\left(du_1, \ldots, du_n; t_1, \ldots, t_n\right) \\
&\quad = \int_{u_1=0}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} z\left(dx_1, u_1\right) z\left(dx_2-x_1, dx_3-x_1, \ldots, dx_n\right. \\
&\quad \left.- x_1; u_2-u_1, u_3-u_1, \ldots, u_n-u_1\right) \\
&\quad \times H_{t_1}\left(0, 0; du_1, dr_1\right) \circ H_{t_2-t_1}\left(0, r_1; du_2-u_1, dr_2\right) \\
&\quad \times \circ \cdots H_{t_n-t_{n-1}}\left(0, r_{n-1}; du_n-u_{n-1}, dr_n\right) \circ . \qquad (3.9)
\end{aligned}
$$

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Taking the FLT of $p(dx_1, \ldots, dx_n; t_1, \ldots, t_n)$ and using the change of variables $u_i' = u_i - u_1$ for $2 \le i \le n$ we obtain

$$\begin{aligned}
&\overline{p}(k_1, \ldots, k_n; s_1, \ldots, s_n) \\
&\quad = \int_{u_1=0}^{\infty} \int_{t_1=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-\left(\sum_{i=1}^{n} s_i\right) t_1 - \left(i \sum_{i=1}^{n} k_i\right) x_1} z(dx_1, u_1) H_{t_1}(0, 0; du_1, dr_1) \circ dt_1 \\
&\quad\quad \times \int_{u_2=0}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \widetilde{z}(k_2, \ldots, k_n; u_2, \ldots, u_n) \hat{H}_{\sum_{i=2}^{n} s_i}(0, r_1; du_2, dr_2) \circ \cdots \\
&\quad\quad \times \hat{H}_{s_n}(0, r_{n-1}; du_n - u_{n-1}, dr_n) \circ
\end{aligned} \tag{3.10}$$

Let us look at

$$\begin{aligned}
&\int_{u_1=0}^{\infty} \int_{t_1=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-\left(\sum_{i=1}^{n} s_i\right) t_1 - \left(i \sum_{i=1}^{n} k_i\right) x_1} z(dx_1, u_1) H_{t_1}(0, 0; du_1, dr_1) dt_1 \\
&\quad = \int_{u_1=0}^{\infty} \int_{t_1=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-\left(\sum_{i=1}^{n} s_i\right) t_1 - \left(i \sum_{i=1}^{n} k_i\right) x_1} z(dx_1, u_1) \\
&\quad\quad \times \int_{w \in [0, t_1]} g(w, u_1) du_1 K_2(dr_1 + t_1 - w) dt_1 \\
&\quad = \int_{u_1=0}^{\infty} \int_{w=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-i\left(\sum_{i=1}^{n} k_i\right) x_1 - \left(\sum_{i=1}^{n} s_i\right) w} z(dx_1, u_1) g(w, u_1) du_1 \\
&\quad\quad \times \int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right) t_1} K_2(dr_1 + t_1) dt_1 \\
&\quad = \int_{u_1=0}^{\infty} \int_{w=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{u_1\left(\psi\left(-\sum_{i=1}^{n} k_i\right) + \phi\left(s \sum_{i=1}^{n} s_i\right)\right)} du_1 \\
&\quad\quad \times \int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right) t_1} K_2(dr_1 + t_1) dt_1
\end{aligned} \tag{3.11}$$

$$= \frac{1}{-\psi\left(-\sum_{i=1}^{n} k_i\right) - \phi\left(\sum_{i=1}^{n} s_i\right)} \int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right) t_1} K_2(dr_1 + t_1) dt_1. \tag{3.12}$$

Plugging (3.12) in (3.10) and using (3.6) we have

$$\begin{aligned}
\overline{p}(k_1, \ldots, k_n; s_1, \ldots, s_n) = &\frac{1}{-\psi\left(-\sum_{i=1}^{n} k_i\right) - \phi\left(\sum_{i=1}^{n} s_i\right)} \\
&\times \int_{u_2=0}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \widetilde{z}(k_2, \ldots, k_n; u_2, \ldots, u_n) \hat{h} \\
&\times \left(0^+, du_2, \ldots, du_n; s_1, \ldots, s_n\right).
\end{aligned}$$

Rearranging and taking the inverse FLT we arrive at (3.8). □

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Working along similar lines to the proof of Theorem 1 one can also obtain the finite dimensional FFPEs of the process $X_t = A_{E_t}$ where E_t is the inverse of a strictly increasing subordinator D_t and (A_t, D_t) is a Lévy process, i.e. the processes A_t and D_t are not necessarily independent. More precisely, suppose $E\left(e^{ikA_t - sD_t}\right) = e^{t\xi(k,s)}$ and that $\xi(k,s) = ibk - \frac{1}{2}ak^2 + \int_{\mathbb{R}}\left(e^{iky-sw} - 1 - iky1_{\{|(y,w)|<1\}}\right)K(dy,dw)$ and that Ξ is the operator whose symbol is $-\xi(-k,s)$.

Theorem 3. *Let (A_t, D_t) be a Lévy process s.t. $E\left(e^{ikA_t - sD_t}\right) = e^{t\xi(k,s)}$. Let E_t be the inverse of the strictly increasing subordinator D_t and let $p(dx_1,\dots,dx_n;t_1,\dots,t_n) = P\left(X_{t_1} \in dx_1,\dots,X_{t_n} \in dx_n\right)$. Then*

$$\Xi_{\mathbf{x},\mathbf{t}}p(dx_1,\dots,dx_n;t_1,\dots,t_n) = \int_{r_1=0}^{\infty} K(dx_1, dr_1+t_1) \times Q_{t_2-t_1}(x_1,r_1;dx_2,dr_2)\circ\cdots Q_{t_n-t_{n-1}}(x_{n-1},r_{n-1};dx_n,dr_n)\circ. \tag{3.13}$$

Proof. Using (2.11) we see that Q_t is again translation invariant with respect to the spatial variable. Note that here

$$U^{x'}(dy,dw) = \int_0^{\infty} v\left(dy-x',dw;u\right)du,$$

where $v(dy,dw;u) = P(A_u \in dy, D_u \in dw)$. Using the same ideas as in the proof of Theorem 1 we obtain

$$\begin{aligned}
\overline{p}(k_1,\dots,k_n;s_1,\dots,s_n) &= \int_{t_1=0}^{\infty}\int_{x_1\in\mathbb{R}} e^{-\left(\sum_{i=1}^n s_i\right)t_1 - \left(\sum_{i=1}^n k_i\right)x_1} \int_{u=0}^{\infty} v(dy,dw;u)\,du \\
&\quad\times \int_{r_1=0}^{\infty}\int_{y\in\mathbb{R}}\int_{w=0}^{t_1} K(dx_1-y, dr_1+t_1-w) \\
&\quad\times \overline{Q}_{\sum_{i=2}^n s_i}\left(0,dr_1;\sum_{i=2}^n k_i, dr_2\right)\circ\cdots\overline{Q}_{s_n}(0,r_{n-1};k_n,dr_n)\circ \\
&= \int_{y\in\mathbb{R}}\int_{w=0}^{\infty} e^{-\left(\sum_{i=1}^n s_i\right)w - i\left(\sum_{i=1}^n k_i\right)y}\int_{u=0}^{\infty} v(dy,dw;u)\,du \\
&\quad\times \int_{r_1=0}^{\infty}\int_{t_1=0}^{\infty}\int_{x_1\in\mathbb{R}} e^{-\left(\sum_{i=1}^n s_i\right)t_1 - \left(\sum_{i=1}^n k_i\right)x_1} \\
&\quad\times K(dx_1,dr_1+t_1)\,\overline{Q}_{\sum_{i=2}^n s_i}\left(0,dr_1;\sum_{i=2}^n k_i,dr_2\right)\circ\cdots \\
&\quad\times \overline{Q}_{s_n}(0,r_{n-1};k_n,dr_n)\circ \\
&= \frac{1}{-\xi\left(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i\right)}\int_{r_1=0}^{\infty}\int_{t_1=0}^{\infty}\int_{x_1\in\mathbb{R}} e^{-\left(\sum_{i=1}^n s_i\right)t_1 - \left(\sum_{i=1}^n k_i\right)x_1} K(dx_1,dr_1+t_1) \\
&\quad\times \overline{Q}_{\sum_{i=2}^n s_i}\left(0,dr_1;\sum_{i=2}^n k_i,dr_2\right)\circ\cdots\overline{Q}_{s_n}(0,r_{n-1};k_n,dr_n)\circ.
\end{aligned} \tag{3.14}$$

Rearrange and invert to obtain (3.13). □

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Remark 2. When the CTRW is coupled, a distinction between the limit of the CTRW $X'_t = \sum_{k=1}^{N_t} J_k$ and the Overshoot CTRW $X''_t = \sum_{k=1}^{N_t+1} J_k$ is needed. Indeed, in the case where the outer process A_t and the subordinator D_t are dependent it has been proven in [15] that the limits of the CTRW and the Overshoot CTRW are $A_{(E_t)-}$ and A_{E_t} respectively.

Remark 3. As was mentioned above, it is usually impossible to define CTRWL by their one-dimensional FFPE. However, since Eqs. (3.13) and (3.14) are equivalent and cádlág processes are characterized (up to their law) by their FDDs, we see that one can define the process A_{E_t} by specifying all its n dimensional FFPE.

Our next result gives a meaning to the measure $\int_{r_1=0}^{\infty} K(dx_1, dr_1 + t_1)$ in the context of CTRWL.

Proposition 1. *Let A_t and D_t as in Theorem 3. Then*

$$\frac{\partial}{\partial u} P(X_t \in dx, E_t \le u) \xrightarrow{w} K(dx, (t,\infty)) \qquad u \to 0. \tag{3.15}$$

Proof. Let $A'_t = (A_t, t)$ and note that $A'_{E_t} = (X_t, E_t)$. Using [11, Equation 4.4] (which is Eq. (2.11) for outer process in $\mathbb{R}^d$) for every $x_1 \in \mathbb{R}$ we have

$$\begin{aligned} P(X_t \in (-\infty, x_1], E_t \le q) = \int_{y_1\in\mathbb{R}} \int_{y_2\in\mathbb{R}} \int_{w=0}^{t} \int_{u=0}^{\infty} \int_{r=0}^{\infty} v'(dy_1, dy_2, dw; u) \\ \times\, du K'((-\infty, x_1 - y_1], dx_2 - y_2, r + t - w). \end{aligned} \tag{3.16}$$

It is not hard to see that here

$$\begin{aligned} v'(dy_1, dy_2, dw; u) &= v(dy_1, dw; u)\,\delta_u(dy_2) \\ K'(dx_1, dx_2, dw) &= K(dx_1, dw)\,\delta_0(dx_2). \end{aligned} \tag{3.17}$$

Plugging (3.17) in (3.16) we have

$$\begin{aligned} &\int_{y_1\in\mathbb{R}} \int_{y_2\in\mathbb{R}} \int_{w=0}^{t} \int_{u=0}^{\infty} v(dy_1, dw; u)\,\delta_u(dy_2)\, du K \\ &\quad \times ((-\infty, x_1 - y_1], [t-w, \infty))\,\delta_0(dx_2 - y_2) \\ &= \int_{y_1\in\mathbb{R}} \int_{w=0}^{t} \int_{u=0}^{\infty} v(dy_1, dw; u)\, du K((-\infty, x_1 - y_1], [t-w,\infty))\,\delta_u(dx_2). \end{aligned} \tag{3.18}$$

Integrating w.r.t. x_2 on $[0, q]$ for some $q > 0$ we have

$$\begin{aligned} &P(X_t \in (-\infty, x_1], E_t \le q) \\ &= \int_{y_1\in\mathbb{R}} \int_{w=0}^{t} \int_{u=0}^{\infty} v(dy_1, dw; u)\, du K((-\infty, x_1 - y_1], [t-w,\infty))\, 1_{\{u\le q\}} \\ &= \int_{y_1\in\mathbb{R}} \int_{w=0}^{t} \int_{u=0}^{q} v(dy_1, dw; u)\, du K((-\infty, x_1 - y_1], [t-w,\infty)). \end{aligned} \tag{3.19}$$

Taking derivative w.r.t. q we have

$$\begin{aligned} &\frac{\partial}{\partial q} P(X_t \in (-\infty, x_1], E_t \le q) \\ &= \int_{y_1\in\mathbb{R}} \int_{w=0}^{t} v(dy_1, dw; q)\, du K((-\infty, x_1 - y_1], [t-w,\infty)). \end{aligned}$$

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The measure $K(dx_1, dw)$ is continuous because $K(\mathbb{R}, dw) = K_2(dw)$ is continuous. Letting $q \to 0$ we see that $\upsilon(dy_1, dw; q) \overset{w}{\to} \delta_{(0,0)}(dy_1, dw)$ and hence $\frac{\partial}{\partial q} P(X_t \in (-\infty, x_1], E_t \le q) \to K((-\infty, x_1], [t, \infty))$ as $q \to 0$ which is equivalent to (3.15). □

Proposition 1 sheds light on the "remainder" term that appears in Theorems 2 and 3. It appears that using a so-called multidimensional RL PDO in the finite dimensional FFPE one is left with a term that accounts for the portion of particles that have not been mobilized since $t = 0$. More philosophically, if we think of the value of E_t as the number of mobilizations of the process by time t, then $\frac{\partial}{\partial u} P(E_t \le u)$ is the ratio between the portion of particles Δm that experienced between u and $u + \Delta u$ mobilizations up to time t. Evaluating $\frac{\partial}{\partial u} P(E_t \le u)$ at $u = 0$ is then the ratio between the portion of particles Δm that experienced an infinitesimal number of mobilizations Δu by time t and Δu. If $\frac{\partial}{\partial u} P(E_t \le u)|_{u=0}$ is big then the diffusion becomes very dynamic at time t as many particles get loose and "take part" in the diffusion. Considering now $\frac{\partial}{\partial u} P(X_t \in dx, E_t \le u)|_{u=0}$ we see that since X_t is the limit of the Overshooting CTRW, where a jump precedes a waiting time the position of the particle that has been "stuck" until time t depends on that first jump in space. In that context it is worthwhile to compare this to [8, Equation 4.2], the dynamics of the coupled CTRWL where the jump in space succeeds that in time. The "remainder" term therefore accounts for the Finite dimensional dynamics that only "kick in" at time t.

4. Examples

Theorem 1 as well as Theorems 2 and 3 should be compared with their one-dimensional counterparts to gain a better understanding of the dynamics of the processes whose distributions govern the FFPE. We start with a specific case of the one dimensional analogue of Theorem 1.

Example 1. Let D_t be a standard stable subordinator of index $0 < \alpha < 1$, i.e. $E\left(e^{-sD_t}\right) = e^{t(-s^\alpha)}$. Its inverse E_t has a distribution $h(x, t)$ which satisfies [10, Equation 5.5]

$$\mathfrak{D}_t^\alpha h(x, t) = -\mathbb{D}_x h(x, t),$$

on $x, t > 0$. Since here $\phi(s) = -s^\alpha$, we see that $\Phi_t = \mathfrak{D}_t^\alpha$.

Next we look at the one dimensional analogue of Theorem 2.

Example 2. Again we let D_t be a standard stable subordinator of index $0 < \alpha < 1$, and A_t be a Lévy process s.t. $E\left(e^{ikA_t}\right) = e^{t\psi(k)}$. Then the distribution $p(dx, t)$ of A_{E_t} satisfies [10, Equation 5.6]

$$\mathfrak{D}_t^\alpha p(dx, t) = \Psi_x p(dx, t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \delta_0(dx). \tag{4.1}$$

To see why (3.8) can be thought of as a generalization of (4.1) note that $h\left(0^+, t\right) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ [10, Equation 4.3] and rewrite (4.1) as

$$\mathfrak{D}_t^\alpha p(x, t) = \Psi_x p(x, t) + \delta_0(dx)\, h\left(0^+, t\right).$$

Our last example concerns the one dimensional analogue of Theorem 3.

Example 3. Let (A_t, D_t) be a Lévy process as in Theorem 3. Then its one dimensional distribution $p(dx, t)$ satisfies

$$\Xi_{x,t}\, p(dx, t) = \int_{r=0}^{\infty} K(dx, dr + t). \tag{4.2}$$

This was shown in [8, Theorem 4.1].

Remark 4. In [3, Equation 5.9], using different methods, Baule and Friedrich essentially obtained Eq. (3.2) for the case where D_t is a standard stable subordinator on $x_1 < x_2 < \cdots < x_n$. In [4, Equation 14] Baule and Friedrich used the results in [3] to obtain Eq. (3.8) (uncoupled case) for the two dimensional case where D_t is a standard stable subordinator. The methods used in [3] can be used to find the finite dimensional FFPEs of the inverse of any subordinator [7], however, cannot be used to find the equations on $x_1 \le x_2 \le \cdots \le x_n$. Moreover, these methods are ill-suited for the coupled case. To see this, we outline the proof in [3] for the two dimensional case. Since E_t is the inverse of the subordinator D_t we see that

$$P\left(E_{t_1} \le x_1, E_{t_2} \le x_n\right) = P\left(D_{x_1} \ge t_1, D_{x_2} \ge t_2\right) \qquad x_1 \le x_2. \tag{4.3}$$

Taking the LT of both sides of the equation w.r.t. t_1 and t_2 and derivatives w.r.t. x_1 and x_2 we see that on $x_1 < x_2$

$$h(dx_1, dx_2; s_1, s_2) = \frac{s_2^{\alpha}\left((s_1+s_2)^{\alpha} - s_2^{\alpha}\right)}{s_1 s_2} e^{-x_1(s_1+s_2)^{\alpha} - (x_2-x_1)s_1^{\alpha}}, \tag{4.4}$$

and it follows that

$$\mathfrak{D}_{\mathbf{t}}^{\alpha} h(dx_1, dx_2; t_1, t_2) = -\mathbb{D}_{\mathbf{x}} h(dx_1, dx_2; t_1, t_2). \tag{4.5}$$

However, since $P\left(D_{x_1} \ge t_1, D_{x_2} \ge t_2\right)$ is not differentiable on $x_1 = x_2$ we cannot obtain Eq. (4.5) on $x_1 = x_2$ through Eq. (4.3). In [4], the authors used the independence of the outer process A_t and the inverse subordinator E_t to obtain the two dimensional FFPE of A_{E_t} through integration by parts, hence, this method cannot be used for the coupled CTRWL. The Markov embedding of CTRWL enables us to find the dynamics of the inverse subordinator on $x_1 \le x_2 \le \cdots \le x_n$ and the coupled and uncoupled CTRWL.

5. Directional pseudo-differential operators

In this section we wish to give a meaning to the PDO $\Psi_{\mathbf{x}}$, $\Phi_{\mathbf{t}}$ and $\Xi_{\mathbf{x},\mathbf{t}}$ discussed earlier. We shall see that they are directional versions of their one-dimensional counterparts Ψ_x, Φ_t and $\Xi_{x,t}$. We shall focus on $\Xi_{\mathbf{x},\mathbf{t}}$ as it is a generalization of $\Psi_{\mathbf{x}}$, $\Phi_{\mathbf{t}}$. To illustrate the kind of results we are looking for, let us look at the next simple example. Assume we have the following equation in $\mathbb{R}^2$

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) f(x_1, x_2) = h(x_1, x_2). \tag{5.1}$$

By using the change of variables $(x_1, x_2)^T = \mathcal{T}\left(x_1', x_2'\right)^T$ where $\mathcal{T} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ we can rewrite Eq. (5.1) as

$$\frac{\partial}{\partial x_1'} f\left(\mathcal{T}\mathbf{x}'\right) = h\left(\mathcal{T}\mathbf{x}'\right).$$

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If we think of the change of variables $\mathcal{T}$ as an operator on functions, i.e. $\mathcal{T} f := f(\mathcal{T}\mathbf{x})$ we see that

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) = \mathcal{T}^{-1} \frac{\partial}{\partial x_1'} \mathcal{T}.$$

Since the operator $\mathbb{D}_{\mathbf{x}}$ is the classic one-dimensional derivative under the change of variables $\mathcal{T}$ we say that it is a directional version of the classic derivative. We wish to show a similar result, i.e. that if $\xi(-k, s)$ is a Lévy symbol, and therefore a symbol of a one-dimensional PDO $\Xi_{x,t}$, then $\xi\left(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i\right)$ is the symbol of a PDO $\Xi_{\mathbf{x},\mathbf{t}}$ that is a directional version of $\Xi_{x,t}$.

In [1], the authors showed that if $\xi(k, s)$ is a Lévy symbol then it is the symbol of a PDO on a Banach space. More precisely, let $X = L^1_\omega(\mathbb{R} \times \mathbb{R}_+)$ be the space of measurable functions s.t. $\|f\|_\omega = \int_{\mathbb{R}} \int_{\mathbb{R}_+} |f(x, t)| e^{-\omega t} dt dx < \infty$ where $\omega > 0$ is fixed. With this norm, the space X is a Banach space and the FLT is defined for each $f \in X$ for $k \in \mathbb{R}$, $s \in (\omega, \infty)$. Let $\xi(k, s)$ be a Lévy symbol, then it was shown that $\xi(-k, s)$ is the symbol of the generator L of a Feller semigroup on X. Moreover, the domain of L is given by

$$D(L) = \left\{ f \in X : \xi(-k, s) \overline{f}(k, s) = \overline{h}(k, s), \exists h \in X \right\}.$$

Let $L^1_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^n)$ denote the space of measurable functions that are defined on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\mathbf{x}, \mathbf{t})| e^{-\langle \boldsymbol{\omega}, \mathbf{t} \rangle} d\mathbf{x} d\mathbf{t} < \infty,$$

for some $\boldsymbol{\omega} \in \mathbb{R}^n_+$. Let $\mathcal{A}_n \subset L^1_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^n)$ be the set of functions that vanish outside $\mathbb{R}^n \times A^n$ where $A^n = \{\mathbf{t} : 0 \le t_1 \le t_2 \le \cdots \le t_n < \infty\}$. Note that $\mathcal{A}_n$ is itself a Banach space and that the FLT of $f \in \mathcal{A}_n$ is defined for $\mathbf{k} \in \mathbb{R}^n$, $\mathbf{s} \in \hat{A}^n := \{\mathbf{s} : \omega_i < s_i\}$. Let $\mathcal{T}$ be the element of GL_n ($n \times n$ invertible matrices) s.t. its first column is $(1, 1, \ldots, 1)^T$ and its i'th column $c_i(j)$ is $\delta_i(j)$ (1 if $i = j$ and zero otherwise) for $1 < i \le n$. Note that $\det(\mathcal{T}) = 1$ so that $\mathcal{T}$ is a bijection from the set $B^n = \{(x_1, x_2, \ldots, x_n) : x_1 \ge 0, 0 \le x_2 \le x_3 \le \cdots \le x_n\}$ onto A^n. We introduce the change of variables $\mathcal{T}\mathbf{x}' = \mathbf{x}$ and $\mathcal{T}\mathbf{t}' = \mathbf{t}$ and see that for $f \in \mathcal{A}_n$

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n_+} |f(\mathbf{x}, \mathbf{t})| e^{-\langle \boldsymbol{\omega}, \mathbf{t} \rangle} d\mathbf{x} d\mathbf{t} &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n_+} \left| f\left(\mathcal{T}\mathbf{x}', \boldsymbol{\mathcal{T}}\mathbf{t}'\right) \right| e^{-\langle \boldsymbol{\omega}, \mathcal{T}\mathbf{t}' \rangle} det(\mathcal{T})^2 d\mathbf{x}' d\mathbf{t}' \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n_+} \left| f\left(\mathcal{T}\mathbf{x}', \boldsymbol{\mathcal{T}}\mathbf{t}'\right) \right| e^{-\langle \mathcal{T}^* \boldsymbol{\omega}, \mathbf{t}' \rangle} d\mathbf{x}' d\mathbf{t}', \end{aligned} \tag{5.2}$$

where $\mathcal{T}^*$ is the adjoint of $\mathcal{T}$. Note that $\mathcal{T}^*$ is the matrix whose first row is $(1, 1, \ldots, 1)$ and its i'th row $r_i(j)$ is $\delta_i(j)$ for $1 < i \le n$. Let $\mathcal{B}_n \subset L^1_{\mathcal{T}^*\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^n)$ be the subspace of functions that vanish outside $\mathbb{R}^n \times B^n$ and note that it is a Banach space w.r.t. the norm $\|f\|_{\mathcal{T}^*\boldsymbol{\omega}}$ and that its FLT is defined for $\left(\mathbf{k}'; \mathbf{s}'\right) \in \mathbb{R}^n \times \hat{B}^n$ where $\hat{B}^n := \mathcal{T}^* \hat{A}^n$. We can now define the operator $\mathcal{T} : \mathcal{A}_n \to \mathcal{B}_n$ by $(\mathcal{T} f)\left(\mathbf{x}', \mathbf{t}'\right) = f\left(\mathcal{T}\mathbf{x}', \mathcal{T}\mathbf{t}'\right)$. Note that by (5.2) $\mathcal{T}$ is an isometric isomorphism from $\mathcal{A}_n$ to $\mathcal{B}_n$ and that if $fg \in \mathcal{A}_n$ then $\mathcal{T}(fg) = \mathcal{T}(f)\mathcal{T}(g) \in \mathcal{B}_n$. We abuse notation and define the operator $\Xi_{x',t'} : \mathcal{B}^n \to \mathcal{B}^n$ by $f\left(\mathbf{x}'; \mathbf{t}'\right) \mapsto \Xi_{x',t'} f\left(\cdot, x_2', \ldots, x_n'; \cdot, t_2', \ldots, t_n'\right)$. Finally, we define the operator $\Xi_{\mathbf{x},\mathbf{t}} : \mathcal{A}^n \to \mathcal{A}^n$ by $\Xi_{\mathbf{x},\mathbf{t}} = \mathcal{T}^{-1} \Xi_{x',t'} \mathcal{T}$.

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Proposition 2. *Let $\xi(-k, s)$ be a Lévy symbol of the one dimensional PDO $\Xi_{x',t'}$, then $\Xi_{\mathbf{x},\mathbf{t}}$ is a PDO with symbol $\xi\left(-\sum_{i=1}^{n} k_i, \sum_{i=1}^{n} s_i\right)$ and its domain is*

$$D\left(\Xi_{\mathbf{x},\mathbf{t}}\right) = \left\{ f \in \mathcal{A}_n : \xi\left(-\sum_{i=1}^{n} k_i, \sum_{i=1}^{n} s_i\right) \overline{f}(\mathbf{k}, \mathbf{s}) = \overline{h}(\mathbf{k}, \mathbf{s}), \exists h \in \mathcal{A}_n \right\}. \tag{5.3}$$

Proof. By (5.2) and Fubini's Theorem we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} \left| \mathcal{T} f\left(x_1', x_2', \dots, x_n'; t_1', t_2' \dots, t_n'\right)\right| e^{-\langle \mathcal{T}^* \boldsymbol{\omega}, (t_1', t_2', \dots, t_n')\rangle} dx_1' dt_1' < \infty, \tag{5.4}$$

and we see that $\mathcal{T} f\left(\cdot, x_2', \dots, x_n'; \cdot, t_2', \dots, t_n'\right) \in L^1_{\sum_{i=1}^{n} \omega_i}(\mathbb{R} \times \mathbb{R})$ for almost every $\left(x_2', \dots, x_n'; t_2', \dots, t_n'\right) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. On the other hand, introducing $\mathbf{k} = (\mathcal{T}^*)^{-1} \mathbf{k}'$ and $\mathbf{s} = (\mathcal{T}^*)^{-1} \mathbf{s}'$ on $\left(\mathbf{k}'; \mathbf{s}'\right) \in \mathbb{R}^n \times \hat{B}^n$ and $f \in \mathcal{A}_n$ we have,

$$\begin{aligned}
\overline{f}\left(\left(\mathcal{T}^*\right)^{-1} \mathbf{k}', \left(\mathcal{T}^*\right)^{-1} \mathbf{s}'\right) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n_+} e^{-i\left\langle (\mathcal{T}^*)^{-1} \mathbf{k}', \mathbf{x}\right\rangle - \left\langle (\mathcal{T}^*)^{-1} \mathbf{s}', \mathbf{t}\right\rangle} f(\mathbf{x}, \mathbf{t})\, d\mathbf{x} d\mathbf{t} \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n_+} e^{-i\langle \mathbf{k}', (\mathcal{T}^{-1})\mathbf{x}\rangle - \langle \mathbf{s}', (\mathcal{T}^{-1})\mathbf{t}\rangle} f(\mathbf{x}, \mathbf{t})\, d\mathbf{x} d\mathbf{t} \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n_+} e^{-i\langle \mathbf{k}', \mathbf{x}'\rangle - \langle \mathbf{s}', \mathbf{t}'\rangle} f\left(\mathcal{T}\mathbf{x}', \mathcal{T}\mathbf{t}'\right) det(\mathcal{T}^{-1})^2 d\mathbf{x}' d\mathbf{t}' \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n_+} e^{-i\langle \mathbf{k}', \mathbf{x}'\rangle - \langle \mathbf{s}', \mathbf{t}'\rangle} \mathcal{T} f\left(\mathbf{x}', \mathbf{t}'\right) d\mathbf{x}' d\mathbf{t}'.
\end{aligned}$$

It follows that for $f \in \mathcal{A}_n$ on $\mathbb{R}^n \times \hat{B}^n$ we have

$$\left(\mathcal{T}^*\right)^{-1} \overline{f} = \overline{\mathcal{T} f}.$$

To summarize, we have shown that the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{A}_n \arrow[rr, bend right, "\mathcal{T}"'] \arrow[dd, bend right, "FLT"'] & & \mathcal{B}_n \arrow[ll, bend right, "\mathcal{T}^{-1}"'] \arrow[dd, bend right, "FLT"'] \\
& & \\
\overline{\mathcal{A}}_n \arrow[uu, bend right, "IFLT"'] \arrow[rr, bend right, "(\mathcal{T}^*)^{-1}"'] & & \overline{\mathcal{B}}_n \arrow[uu, bend right, "IFLT"'] \arrow[ll, bend right, "\mathcal{T}^*"']
\end{tikzcd}$$

Here $\overline{\mathcal{A}}_n$ and $\overline{\mathcal{B}}_n$ denote the image of $\mathcal{A}_n$ and $\mathcal{B}_n$ respectively under the FLT map and IFLT is the inverse FLT. Note that $\overline{\mathcal{A}}_n$ is defined on $\mathbb{R}^n \times \hat{A}^n$ while $\overline{\mathcal{B}}_n$ is defined on $\mathbb{R}^n \times \hat{B}^n$. Next, we note that the domain of $\Xi_{x',t'}$ on $\mathcal{B}_n$ is

$$D\left(\Xi_{x',t'}\right) = \left\{ f : \xi\left(-k_1', s_1'\right) f\left(\mathbf{k}', \mathbf{s}'\right) = h\left(\mathbf{k}', \mathbf{s}'\right), \exists h \in \mathcal{B}_n \right\}. \tag{5.5}$$

Indeed, if $f \in \mathcal{B}_n$ satisfies $\xi\left(-k_1', s_1'\right) f\left(\mathbf{k}', \mathbf{s}'\right) = h\left(\mathbf{k}', \mathbf{s}'\right)$ for some $h \in \mathcal{B}_n$ then by (5.4) and the results in [1] we see that $f \in D\left(\Xi_{x',t'}\right)$. Next we show that $\Xi_{\mathbf{x},\mathbf{t}}$ is a PDO with symbol

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$\xi\left(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i\right)$. Suppose $f \in D\left(\Xi_{\mathbf{x},\mathbf{t}}\right)$, then

$$\Xi_{x',t'}\mathcal{T}f\left(\mathbf{x}',\mathbf{t}'\right) = \mathcal{T}h\left(\mathbf{x}',\mathbf{t}'\right), \tag{5.6}$$

for some $h \in \mathcal{A}_n$. Applying FLT on both sides we obtain

$$\xi\left(-k_1', s_1'\right)\left(\mathcal{T}^*\right)^{-1}\overline{f}\left(\mathbf{k}',\mathbf{s}'\right) = \left(\mathcal{T}^*\right)^{-1}\overline{h}\left(\mathbf{k}',\mathbf{s}'\right), \tag{5.7}$$

multiplying both sides by $\mathcal{T}^*$ we have

$$\xi\left(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i\right)\left[\mathcal{T}^*\left(\mathcal{T}^*\right)^{-1}\overline{f}\left(\mathbf{k},\mathbf{s}\right)\right] = \overline{h}\left(\mathbf{k},\mathbf{s}\right). \tag{5.8}$$

This can be seen to be true since $\mathcal{T}^*(fg) = \mathcal{T}^*(f)\mathcal{T}^*(g)$ while recalling the presentation of $\mathcal{T}^*$ as a matrix. Hence, $\Xi_{\mathbf{x},\mathbf{t}}$ is a PDO with symbol $\xi\left(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i\right)$. It is left to show that the domain of $\Xi_{\mathbf{x},\mathbf{t}}$ is as in (5.3). Since $\mathcal{T}$ is a bijection it is clear that $\mathcal{T}D\left(\Xi_{\mathbf{x},\mathbf{t}}\right) = D\left(\Xi_{x',t'}\right)$ and the claim can be seen to be true through Eq. (5.5). □

In order to give a meaning to Eqs. (3.8) and (3.13) through Proposition 2 we define the function

$$f(\mathbf{x};\mathbf{t}) = \int_{\mathbb{R}^n} g(\mathbf{x}-\mathbf{y})\, p(d\mathbf{y};\mathbf{t}),$$

where g is a smooth function with compact support in $\mathbb{R}^n$ and $p(d\mathbf{x};\mathbf{t})$ is a parameterized distribution as in Section 2.1. It follows that $f(\mathbf{x};\mathbf{t})$ is smooth in $\mathbb{R}^n$ for every $\mathbf{t} \in \mathbb{R}_+^n$. Multiply both sides of Eq. (3.14) by $\widetilde{g}(\mathbf{k})$ and use the convolution–multiplication property of the FT to obtain

$$\Xi_{\mathbf{x},\mathbf{t}}f(\mathbf{x};\mathbf{t}) = \int_{\mathbb{R}^n} g(\mathbf{x}-\mathbf{y})\, p_0(d\mathbf{y};\mathbf{t}),$$

where

$$p_0(d\mathbf{x};\mathbf{t}) = \int_{r_1=0}^{\infty} K\left(dx_1, dr_1 + t_1\right) \\ \times Q_{t_2-t_1}\left(x_1, r_1; dx_2, dr_2\right) \circ \cdots Q_{t_n-t_{n-1}}\left(x_{n-1}, r_{n-1}; dx_n, dr_n\right) \circ .$$

This interpretation of (3.8) and (3.13) is in the spirit of that in [12, Chapter 4] and was used in [6, p. 15].

6. Conclusions

In this paper we find the FFPEs of the FDDs of the process A_{E_t} where (A_t, D_t) is a Lévy process, D_t is a strictly increasing subordinator with no drift and E_t is the inverse of D_t. The general form of these FFPEs (Eq. (3.13)) is a PDO in time and space variables applied to the distribution of the process on one side of the equation while on the other we have a term that accounts for the portion of particles that yet to be mobilized. Moreover, considering the difference between the RL derivative and that of Caputo's in the one dimensional case, and compared to the finite dimensional one, it seems that the RL derivative is more suitable in the context of CTRWL (similar conclusions were obtained in [4] where a generalized Caputo derivative was suggested). We also showed that the PDOs which appear in Theorem 3 are indeed bona fide PDO and in fact a directional version of their one dimensional counterparts.

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