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Relation between the rate of convergence of strong law of large numbers and the rate of concentration of Bayesian prior in game-theoretic probability

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Abstract

We study the behavior of the capital process of a continuous Bayesian mixture of fixed proportion betting strategies in the one-sided unbounded forecasting game in game-theoretic probability. We establish the relation between the rate of convergence of the strong law of large numbers in the self-normalized form and the rate of divergence to infinity of the prior density around the origin. In particular we present prior densities ensuring the validity of Erdős–Feller–Kolmogorov–Petrowsky law of the iterated logarithm.

Keywords: Constant-proportion betting strategy; Erdős–Feller–Kolmogorov–Petrowsky law of the iterated logarithm; One-sided unbounded game; Self-normalized processes; Upper class

1. Introduction

The most basic proof of the strong law of large numbers (SLLN) in Chapter 3 of [15] uses a discrete mixture of fixed proportion betting strategies. The capital process of the strategy is

$$\sum_{j=1}^{\infty} 2^{-j-1} \prod_{i=1}^{n} (1 + 2^{-j} x_i) \; + \; \sum_{j=1}^{\infty} 2^{-j-1} \prod_{i=1}^{n} (1 - 2^{-j} x_i) \tag{1}$$

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and is nonnegative since $|x_i|$ is supposed to be bounded by 1. The mixture puts the weight 2^{-j-1} on the points $\pm 2^{-j}$, $j = 1, 2, \ldots$. The sum of weights on the interval $[0, 2^{-k}]$, $k \geq 1$, is

$$\sum_{j=k}^{\infty} 2^{-j-1} = 2^{-k},$$

which is equal to the length of the interval $[0, 2^{-k}]$. Hence this mixture can be understood as the discrete approximation to the continuous uniform distribution over the interval $[-1/2, 1/2]$. In Chapter 3 of [15] this mixture is used only to prove the usual form of SLLN, and any discrete distribution having the origin as an accumulation point of mixture weights serves the same purpose. As we present in Section 4, the concentration of the mixture weights around the origin has the direct implication on the rate of convergence of SLLN forced by the mixture.

In this paper we consider a continuous mixture in the integral form. Although a proof based on a discrete mixture is conceptually simpler, the integral in a continuous mixture is more convenient for analytic treatment. In fact in this paper we give a unified treatment covering the usual SLLN, the validity (i.e., the upper bound) of the usual law of iterated logarithm (LIL), and finally the validity of Erdős–Feller–Kolmogorov–Petrowsky (EFKP) LIL [13, Chapter 5.2].

Another feature of this paper is the self-normalized form in which we consider SLLN. Let

$$S_n = X_1 + \cdots + X_n, \qquad A_n^2 = X_1^2 + \cdots + X_n^2$$

denote the sum of n random variables and the sum of their squares, respectively. We compare S_n to A_n^2, rather than to n. For example $S_n/n \to 0$ is the usual non-self-normalized SLLN, whereas $S_n/A_n^2 \to 0$ is the SLLN in the self-normalized form. We often obtain cleaner statements and proofs in the self-normalized form. For instance, in measure-theoretic probability, Griffin and Kuelbs [7] established the EFKP-LIL in the self-normalized form in the i.i.d. case with some additional conditions. Later, Wang [19] and Csörgő et al. [3] eliminated some of the conditions. A similar argument in the self-normalized form can also be seen in game-theoretic probability as shown in [14]. Self-normalized processes including the t-statistic are known for their statistical applications. In studying self-normalized processes, self-normalized sums S_n/A_n are regarded as important and one of the reasons is that they have close relations to the t-statistic; see, e.g., [5,6]. For a survey of results concerning self-normalized processes, especially self-normalized sums, see [4], [16], and references in [14]. In the self-normalized form of SLLN we only consider paths of Reality's moves such that $A_\infty = \lim_n A_n = \infty$. Hence the events considered in this paper are conditional events given $A_\infty = \infty$. In Section 6 we give a non-self-normalized result on the behavior of S_n relative to deterministic sequence b_n of n. The result is very different from the LIL which holds for the self-normalized case.

Yet another feature of this paper is that we mainly consider the one-sided unbounded forecasting game. We notice that the results on the bounded forecasting game can be derived from the results on the one-sided unbounded forecasting game. Also it is an interesting fact that in the one-sided unbounded forecasting game the usual non-self-normalized SLLN does not hold. We discuss this fact in Section 6. The importance of this game will be explained in Sections 2 and 7.

We remark on the terminology used in this paper. We use "increasing" and "decreasing" in the weak sense, i.e. we simply write "increasing" instead of more accurate "(monotone) nondecreasing".

The organization of this paper is as follows. In Section 2 we introduce the protocol of the one-sided unbounded forecasting game and relate it to betting on positive price processes. We also define Bayesian strategies with prior densities. In Section 3 we establish some preliminary results on the capital process of a Bayesian strategy. In Section 4 we prove a general inequality for the capital process of a Bayesian strategy and apply it to prove SLLN and the validity of LIL in the self-normalized form based on Bayesian strategies. In Section 5 we define two functionals which connect prior densities and functions of the upper class for EFKP-LIL. Based on these results, we give a proof of the validity of EFKP-LIL using a corresponding Bayesian strategy. In Section 6 we prove another basic property of the one-sided unbounded forecasting game, including the rate of SLLN in the usual non-self-normalized form and the Reality's deterministic strategy against the usual form of SLLN. We end the paper with some discussion in Section 7.

2. The one-sided unbounded forecasting game and the Bayesian strategy

In this paper we mainly consider the one-sided unbounded forecasting game defined as follows.

THE ONE-SIDED UNBOUNDED FORECASTING GAME (OUFG)
Players: Skeptic, Reality
Protocol:
Skeptic starts with the initial capital $\mathcal{K}_0 > 0$.
FOR $n = 1, 2, \ldots$:
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [-1, \infty)$.
$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n$.
Collateral Duties: Skeptic must keep $\mathcal{K}_n$ nonnegative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

In this protocol one player loses when he/she cannot keep the collateral duty. Hence Skeptic wins if he keeps $\mathcal{K}_n$ nonnegative and make $\mathcal{K}_n \to \infty$ $(n \to \infty)$. Otherwise Reality wins. Although there is a distinction between "weak forcing" ($\limsup_n \mathcal{K}_n = \infty$) and "forcing" ($\lim \mathcal{K}_n = \infty$), from the viewpoint of winning they are equivalent by Lemma 3.1 of [15]. Note that Skeptic has to announce $M_n \geq 0$, because if $M_n < 0$ then Reality can choose sufficiently large x_n such that $\mathcal{K}_n < 0$. In game-theoretic probability the initial capital $\mathcal{K}_0$ is usually standardized to be 1. However the value of the initial capital is not relevant in discussing whether Skeptic can force an event E or not. Also notation is sometimes simpler if we allow an arbitrary positive value for $\mathcal{K}_0$.

In OUFG, Reality's move x_n is unbounded on the positive side. On the other hand, in the bounded forecasting game (BFG) of Chapter 3 of [15], x_n is restricted as $|x_n| \leq 1$. Since Reality's move space is smaller in BFG than in OUFG, Reality is weaker against Skeptic in BFG than in OUFG. This implies that if Skeptic can force an event E in OUFG, then he can force E in BFG. In this sense the one-sided version of SLLN (cf. (3.9) in Lemma 3.3 of [15]) in OUFG is stronger than that in BFG. This is one of the reasons why we mainly consider OUFG in this paper.

Let $\epsilon_n = M_n/\mathcal{K}_{n-1}$ denote the proportion of the capital Skeptic bets on the round n. The collateral duty of Skeptic is $\epsilon_n \in [0, 1]$ in OUFG, whereas it is $\epsilon_n \in [-1, 1]$ in BFG. This again reflects the fact that Skeptic is stronger in BFG than in OUFG.

OUFG is a natural protocol, when we consider betting on a positive price process $\{p_n\}$ of some risky financial asset. Let

$$x_n = \frac{p_n}{p_{n-1}} - 1$$

denote the return of the price process. Then $x_n \geq -1$ and there is no upper bound for the return x_n if p_n is allowed to take an arbitrary large value. Furthermore we can write $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n = \mathcal{K}_{n-1}(1 + \epsilon_n x_n)$ as

$$\mathcal{K}_n = \mathcal{K}_{n-1}\left(1 + \epsilon_n\left(\frac{p_n}{p_{n-1}} - 1\right)\right) = \mathcal{K}_{n-1}\left((1 - \epsilon_n) + \epsilon_n \frac{p_n}{p_{n-1}}\right). \tag{2}$$

Hence $(1-\epsilon_n)$ is the proportion of the current capital Skeptic keeps as cash, which does not change in value from round to round, and ϵ_n is the proportion of the current capital Skeptic bets on the risky asset.

When $\epsilon_n \equiv \epsilon$ is a constant, we call the strategy the constant-proportion betting strategy or the ϵ-strategy. The capital process of the ϵ-strategy is written as

$$\mathcal{K}_n = \mathcal{K}_0 \prod_{i=1}^{n} (1+\epsilon x_i).$$

We now consider a continuous mixture of ϵ-strategies ("Bayesian strategy") where the initial capital is distributed according to the (unnormalized) prior density $\pi \geq 0$ of an absolutely continuous finite measure on the unit interval [0, 1]:

$$\int_0^1 \pi(\epsilon) d\epsilon < \infty.$$

We use the term "Bayesian", because the role of the prior density π is formally the same as the prior density in Bayesian statistics. Bayesian priors are important intuitively and historically; they were used in their continuous form already by Ville [17] in his classic study of the validity part of EFKP-LIL for coin tossing. The betting proportion ϵ_n of the Bayesian strategy with the prior density π is defined as

$$\epsilon_n = \frac{\int_0^1 \epsilon \prod_{i=1}^{n-1}(1+\epsilon x_i)\pi(\epsilon) d\epsilon}{\int_0^1 \prod_{i=1}^{n-1}(1+\epsilon x_i)\pi(\epsilon) d\epsilon}. \tag{3}$$

Then the capital process $\mathcal{K}_n^\pi$ of this strategy is written as

$$\mathcal{K}_n^\pi = \int_0^1 \prod_{i=1}^{n} (1+\epsilon x_i)\pi(\epsilon) d\epsilon \tag{4}$$

with the initial capital $\mathcal{K}_0^\pi = \int_0^1 \pi(\epsilon) d\epsilon$. Eq. (4) is checked by induction on n, since with ϵ_n given in (3), we have

$$\begin{aligned} \mathcal{K}_{n-1}(1+\epsilon_n x_n) &= \int_0^1 \prod_{i=1}^{n-1}(1+\epsilon x_i)\pi(\epsilon) d\epsilon \left(1 + \frac{\int_0^1 \epsilon \prod_{i=1}^{n-1}(1+\epsilon x_i)\pi(\epsilon) d\epsilon}{\int_0^1 \prod_{i=1}^{n-1}(1+\epsilon x_i)\pi(\epsilon) d\epsilon} x_n\right) \\ &= \int_0^1 (1+\epsilon x_n) \prod_{i=1}^{n-1}(1+\epsilon x_i)\pi(\epsilon) d\epsilon = \int_0^1 \prod_{i=1}^{n}(1+\epsilon x_i)\pi(\epsilon) d\epsilon. \end{aligned}$$

A similar argument is also used in [14]. One should notice that (1) can be seen as one discretization of (4), where π is the uniform density on $[-1/2, 1/2]$.

We are concerned with the rate of increase of $\pi(\epsilon)$ as $\epsilon \downarrow 0$. Hence we require some regularity conditions on the behavior of $\pi(\epsilon)$ around the origin. The following condition on π is convenient in discussing EFKP-LIL in Section 5.

Assumption 2.1. There exist $\epsilon_\pi \in (0, 1)$ and $\delta_\pi > 0$ such that

1. $\pi(\epsilon)$ is a prior density, that is, nonnegative and integrable on [0, 1],
2. $\pi(\epsilon) \geq \delta_\pi$ on $(0, \epsilon_\pi)$, and
3. $\epsilon\pi(\epsilon)$ is increasing on $(0, \epsilon_\pi)$.

For simplicity, we allow the case that the integral of π on [0, 1] is not 1, which does not cause a problem when considering limit theorems. Note that we are allowing $\infty = \lim_{\epsilon\downarrow 0}\pi(\epsilon)$, but

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by the monotonicity and the integrability we are assuming $0 = \lim_{\epsilon \downarrow 0} \epsilon \pi(\epsilon)$. Assumption 2.1 holds for particular examples discussed in the next section. Concerning the condition on δ_π in Assumption 2.1, Skeptic can always allocate the initial capital of δ_π to the uniform prior of Section 4.2 to satisfy this condition.

3. Preliminary results

In this section we see the self-normalized form of SLLN in the one-sided unbounded forecasting game.

Proposition 3.1. *In OUFG, by any Bayesian strategy with π satisfying Assumption 2.1, Skeptic weakly forces*

$$A_\infty = \infty \quad \Rightarrow \quad \limsup_n \frac{S_n}{A_n^2} \le 0. \tag{5}$$

We can prove this proposition by more or less the same way as in Lemma 3.3 of [15]. The difference is that the mixture we use here is not discrete but continuous. We will see later that we can establish stronger forms with similar arguments.

We begin with the following simple lemma. The particular case $C = 1$ is used in Lemma 3.3 of [15]. We prove this lemma separately in a stronger form for later use.

Lemma 3.2. *For any $C > 0$*

$$\ln(1+t) \ge t - \frac{1+C}{2} t^2 \qquad \textit{for} \quad t \ge -\frac{C}{1+C}. \tag{6}$$

Proof. Let

$$g(t) = \ln(1+t) - t + \frac{1+C}{2} t^2, \quad t > -1.$$

Then $g(0) = 0$ and

$$g'(t) = \frac{1}{1+t} - 1 + (1+C)t = \frac{t(C + (1+C)t)}{1+t}.$$

Hence $g'(t) = 0$ for $t = 0$ or $t = -C/(1+C)$, and $g'(t) < 0$ only for $-C/(1+C) < t < 0$. This implies the lemma. □

Proof of Proposition 3.1. We use Lemma 3.2 with $C = 1$. Suppose that Reality has chosen a path such that $A_\infty = \infty$ and $\limsup_n S_n / A_n^2 > 0$. Then there exists a sufficiently small $0 < \delta < \min(\epsilon_\pi, 1/2)$ such that $S_n > \delta A_n^2$ for infinitely many n. For such an n,

$$\begin{aligned}
\mathcal{K}_n^\pi &\ge \int_{\delta/3}^{2\delta/3} \exp\left(\sum_{i=1}^n \ln(1 + \epsilon x_i) \right) \pi(\epsilon) d\epsilon \\
&\ge \int_{\delta/3}^{2\delta/3} \exp(\epsilon S_n - \epsilon^2 A_n^2) \frac{1}{\epsilon} \epsilon \pi(\epsilon) d\epsilon \\
&\ge \frac{3}{2\delta} \frac{\delta}{3} \pi\left(\frac{\delta}{3}\right) \int_{\delta/3}^{2\delta/3} \exp(A_n^2 \epsilon(\delta - \epsilon)) d\epsilon \\
&\ge \frac{1}{2} \pi\left(\frac{\delta}{3}\right) \frac{\delta}{3} \exp\left(A_n^2 \frac{\delta^2}{9}\right).
\end{aligned}$$

Notice that the strategy and the capital process $\mathcal{K}_n^\pi$ do not depend on δ. Because $\lim_n A_n = \infty$, we have $\limsup_n \mathcal{K}_n^\pi = \infty$. $\square$

Remark 3.3. Skeptic cannot force the event $A_\infty = \infty \Rightarrow \liminf_n \frac{S_n}{A_n^2} \geq 0$. Reality can announce $x_n = -1$ for all n, and $A_\infty = \infty$ and $\frac{S_n}{A_n^2} = -1$ for all n.

4. Rate of convergence of SLLN implied by a Bayesian strategy

In this section we present results on the rate of convergence of SLLN by a Bayesian strategy, by establishing some lower bounds for the capital process. Note that since the Bayesian strategy already weakly forces (5), we only need to consider lower bounds when S_n/A_n^2 is sufficiently small.

In Section 4.1 we present our first inequality, which will be used to evaluate the rate of convergence of SLLN for the uniform prior in Section 4.2. Then we give a more refined inequality in Section 4.3, which will be used to evaluate the rate of convergence of SLLN for the power prior (Section 4.4), for the prior ensuring the validity in the usual form of LIL (Section 4.5) and for the prior ensuring the validity of a typical form of EFKP-LIL (Section 4.6).

4.1. A lower bound for the capital process of a Bayesian strategy

Our first theorem of this paper bounds $\mathcal{K}_n^\pi$ from below as follows.

Theorem 4.1. *In OUFG, for any $C \in (0, \min(\epsilon_\pi, 1/2))$ and $0 < S_n/A_n^2 < C/2$,*

$$\mathcal{K}_n^\pi \geq \frac{\sqrt{C}}{6}\frac{S_n}{A_n^2}\pi\left(\frac{S_n}{A_n^2}\right)\exp\left(\frac{(1-2C)S_n^2}{2A_n^2}\right). \tag{7}$$

Note that Eq. (7) is satisfied on the rounds when the condition $0 < S_n/A_n^2 < C/2$ happens to be true and this condition can be assumed by Proposition 3.1. We now prove Theorem 4.1.

Proof. Let $\delta = C/(1+C)$. Then $\delta < C < \epsilon_\pi$. Also

$$\frac{1+\sqrt{C}}{1+C}\frac{S_n}{A_n^2} < \frac{1+\sqrt{C}}{2}\frac{C}{1+C} < \frac{C}{1+C} = \delta < \epsilon_\pi.$$

Then, by Lemma 3.2

$$\begin{aligned}
\mathcal{K}_n^\pi &\geq \int_0^\delta \exp\left(\sum_{i=1}^n \ln(1+\epsilon x_i)\right)\pi(\epsilon)d\epsilon \\
&\geq \int_0^\delta \exp\left(\epsilon S_n - \frac{1+C}{2}\epsilon^2 A_n^2\right)\pi(\epsilon)d\epsilon \\
&= \exp\left(\frac{S_n^2}{2(1+C)A_n^2}\right)\int_0^\delta \exp\left(-\frac{(1+C)A_n^2}{2}\left(\epsilon - \frac{S_n}{(1+C)A_n^2}\right)^2\right)\pi(\epsilon)d\epsilon \\
&\geq \exp\left(\frac{S_n^2}{2(1+C)A_n^2}\right)\int_{\frac{S_n}{A_n^2}}^{\frac{1+\sqrt{C}}{1+C}\frac{S_n}{A_n^2}} \exp\left(-\frac{(1+C)A_n^2}{2}\left(\epsilon - \frac{S_n}{(1+C)A_n^2}\right)^2\right)\frac{1}{\epsilon}\epsilon\pi(\epsilon)d\epsilon
\end{aligned}$$

$$
\begin{aligned}
&\geq \exp\left(\frac{S_n^2}{2(1+C)A_n^2}\right) \frac{(1+C)A_n^2}{(1+\sqrt{C})S_n} \frac{S_n}{A_n^2}\pi\left(\frac{S_n}{A_n^2}\right) \\
&\quad \times \int_{\frac{S_n}{A_n^2}}^{\frac{1+\sqrt{C}}{1+C}\frac{S_n}{A_n^2}} \exp\left(-\frac{(1+C)A_n^2}{2}\left(\epsilon - \frac{S_n}{(1+C)A_n^2}\right)^2\right) d\epsilon \\
&\geq \exp\left(\frac{S_n^2}{2(1+C)A_n^2}\right) \frac{1+C}{1+\sqrt{C}}\pi\left(\frac{S_n}{A_n^2}\right) \frac{\sqrt{C}(1-\sqrt{C})S_n}{(1+C)A_n^2} \\
&\quad \times \exp\left(-\frac{(1+C)S_n^2}{2A_n^2}\left(\frac{1+\sqrt{C}}{1+C} - \frac{1}{1+C}\right)^2\right) \\
&= \exp\left(\frac{S_n^2}{2(1+C)A_n^2}\right) \frac{1-\sqrt{C}}{1+\sqrt{C}}\sqrt{C}\,\frac{S_n}{A_n^2}\pi\left(\frac{S_n}{A_n^2}\right) \exp\left(-\frac{CS_n^2}{2(1+C)A_n^2}\right) \\
&= \frac{1-\sqrt{C}}{1+\sqrt{C}}\sqrt{C}\,\frac{S_n}{A_n^2}\pi\left(\frac{S_n}{A_n^2}\right) \exp\left(\frac{(1-C)S_n^2}{2(1+C)A_n^2}\right). \qquad (8)
\end{aligned}
$$

Now

$$
\frac{1-C}{1+C} > (1-C)^2 > 1-2C
$$

and for $C < 1/2$

$$
\frac{1-\sqrt{C}}{1+\sqrt{C}} > \frac{1-\sqrt{1/2}}{1+\sqrt{1/2}} > 0.171 > \frac{1}{6}.
$$

Hence we obtain (7). □

4.2. *Uniform prior*

Let $\pi(\epsilon) = 1$, $\epsilon \in (0, 1)$, be the uniform prior. By Theorem 4.1 we obtain the following result.

Proposition 4.2. *In OUFG, by the uniform prior, Skeptic weakly forces*

$$
A_\infty = \infty \quad \Rightarrow \quad \limsup_n \frac{S_n}{\sqrt{A_n^2 \ln A_n^2}} \leq 1. \qquad (9)
$$

Proof. By Proposition 3.1 it suffices to consider a path satisfying $\limsup_n S_n/A_n^2 \leq 0$. Suppose that Reality chooses a path such that $A_\infty = \infty$ and

$$
\limsup_n \frac{S_n}{\sqrt{A_n^2 \ln A_n^2}} > 1.
$$

Then for some small positive C, we have

$$
\limsup_n \frac{S_n}{\sqrt{A_n^2 \ln A_n^2}} > \frac{1}{\sqrt{1-2C}}
$$

and

$$
\frac{S_n^2}{A_n^2} \geq \frac{\ln A_n^2}{1-2C}
$$

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for infinitely many n. For such an n,

$$
\begin{aligned}
\frac{\sqrt{C}}{6}\frac{S_n}{A_n^2}\pi\left(\frac{S_n}{A_n^2}\right)\exp\left(\frac{(1-2C)S_n^2}{2A_n^2}\right) &\geq \frac{\sqrt{C}}{6}\exp\left(\frac{(1-2C)S_n^2}{2A_n^2}+\ln\frac{S_n}{A_n^2}\right)\\
&\geq \frac{\sqrt{C}}{6}\exp\left(\frac{\ln A_n^2}{2}+\ln\frac{S_n}{A_n}+\ln\frac{1}{A_n}\right)\\
&= \frac{\sqrt{C}}{6}\exp\left(\ln\frac{S_n}{A_n}\right)=\frac{\sqrt{C}}{6}\frac{S_n}{A_n}\\
&\geq \frac{\sqrt{C}}{6}\frac{1}{\sqrt{1-2C}}\sqrt{\ln A_n^2}.
\end{aligned}
$$

Since $\lim_n A_n = \infty$, we have $\limsup_n \mathcal{K}_n^\pi = \infty$. □

In Proposition 4.2, we considered continuous uniform mixture. However the same proof can be applied to discrete mixture setting. In particular the discrete mixture in the basic proof of SLLN in Chapter 3 of [15] achieves the same rate $\sqrt{A_n^2 \ln A_n^2}$ of (9). Note also that this rate is the same as the rate of SLLN of the dynamic strategies studied in [9] and [8].

4.3. A refined lower bound for the capital process

By Assumption 2.1, $\pi(\epsilon) \geq \delta_\pi > 0$ holds around the origin. This implies that for any prior satisfying Assumption 2.1, the rate of convergence $S_n = O(\sqrt{A_n^2 \ln A_n^2})$ already holds. In fact, for $\epsilon < \epsilon_\pi$ write

$$\pi(\epsilon) = \delta_\pi + (\pi(\epsilon) - \delta_\pi).$$

Then

$$\mathcal{K}_n^\pi \geq \delta_\pi \int_0^{\epsilon_\pi} \prod_{i=1}^n (1+\epsilon x_i) d\epsilon$$

and the result for the uniform prior applies to the right-hand side. Hence in considering other priors satisfying Assumption 2.1, we can only consider paths such that $\limsup_n S_n/\sqrt{A_n^2 \ln A_n^2} \leq 1$. In particular $S_n^3 = O((A_n^2 \ln A_n^2)^{3/2})$ and $S_n^3/A_n^4 \to 0$ as $n \to \infty$. For such paths, we consider maximizing the right-hand side of (7) with respect to C. For the case S_n/A_n is large, we put $C = A_n^2/S_n^2$ and obtain the following theorem.

Theorem 4.3. *In OUFG, if* $S_n > 0$, $S_n^2/A_n^2 > \max(2, 1/\epsilon_\pi)$ *and* $S_n^3/A_n^4 < 1/2$, *then*

$$\mathcal{K}_n^\pi \geq \frac{1}{6e}\frac{1}{A_n}\pi\left(\frac{S_n}{A_n^2}\right)\exp\left(\frac{S_n^2}{2A_n^2}\right). \tag{10}$$

Again Eq. (10) is satisfied on the rounds when the condition happens to be true and the condition $S_n^3/A_n^4 < 1/2$ can be assumed by Proposition 4.2. We now prove Theorem 4.3.

Proof. Let $C = A_n^2/S_n^2$. By the assumption $C < \min(1/2, \epsilon_\pi)$. Also

$$\frac{S_n}{A_n^2} < \frac{A_n^2}{2S_n^2} = \frac{C}{2}.$$

Hence the conditions of Theorem 4.1 are satisfied. Furthermore with this C

$$\exp\left(\frac{(1-2C)S_n^2}{2A_n^2}\right) = \exp\left(\frac{S_n^2}{2A_n^2} - 1\right).$$

Substituting the above inequalities into (7) we obtain (10). □

Remark 4.4. In Section 5 we will multiply $\pi(\epsilon)$ by a decreasing function $c(\epsilon)$ and use $c(\epsilon)\pi(\epsilon)$ as the prior. Then $\epsilon c(\epsilon)\pi(\epsilon)$ may not be increasing. However in view of the range of integration in (8), we can still bound the capital process by

$$\mathcal{K}_n^{\pi} \geq c(u_n)\frac{1}{6e}\frac{1}{A_n}\pi\left(\frac{S_n}{A_n^2}\right)\exp\left(\frac{S_n^2}{2A_n^2}\right), \qquad u_n = \frac{1 + A_n/S_n}{1 + A_n^2/S_n^2}\frac{S_n}{A_n^2}. \tag{11}$$

For the rest of this section we apply Theorem 4.3 to some typical nonuniform priors.

4.4. Power prior

For $0 < a < 1$, let

$$\pi(\epsilon) = \epsilon^{-a}, \quad \epsilon \in (0, 1). \tag{12}$$

For this power prior, the following result holds.

Proposition 4.5. *In OUFG, by the power prior, Skeptic weakly forces*

$$A_\infty = \infty \quad \Rightarrow \quad \limsup_n \frac{S_n}{\sqrt{(1-a)A_n^2 \ln A_n^2}} \leq 1.$$

Proof. Consider the case $S_n > \sqrt{(1-a+\delta)A_n^2 \ln A_n^2}$ for some $\delta > 0$ for infinitely many n. Note that the function $f(x) = x^{-a/2}e^{x/2}$ is increasing for sufficiently large x. Then for sufficiently large n,

$$\begin{aligned}
\frac{1}{A_n}\left(\frac{S_n}{A_n^2}\right)^{-a}\exp\left(\frac{S_n^2}{2A_n^2}\right) &= A_n^{-(1-a)}\left(\frac{S_n}{A_n}\right)^{-a}\exp\left(\frac{S_n^2}{2A_n^2}\right) \\
&\geq A_n^{-(1-a)}\left((1-a+\delta)\ln A_n^2\right)^{-a/2}\exp\left(\frac{(1-a+\delta)}{2}\ln A_n^2\right) \\
&= \frac{1}{(1-a+\delta)^{a/2}}\frac{A_n^\delta}{(\ln A_n^2)^{a/2}} \uparrow \infty \quad (n \to \infty). \qquad \square
\end{aligned}$$

4.5. Prior for the validity of LIL

Here we present a prior for the validity of LIL. Let $\epsilon_0 > 0$ sufficiently small such that $\ln\ln\frac{1}{\epsilon}$ is positive for $\epsilon \leq \epsilon_0$. Define

$$\pi(\epsilon) = \frac{1}{\epsilon\,\ln\frac{1}{\epsilon}\left(\ln\ln\frac{1}{\epsilon}\right)^2}, \quad \epsilon \in (0, \epsilon_0). \tag{13}$$

This π is integrable around the origin.

Proposition 4.6. *In OUFG, by the prior (13), Skeptic weakly forces*

$$A_\infty = \infty \quad \Rightarrow \quad \limsup_n \frac{S_n}{\sqrt{2A_n^2 \ln\ln A_n^2}} \leq 1. \tag{14}$$

Proof. Again we use the fact that $f(x) = x^{-1/2}e^{x/2}$ is increasing for sufficiently large x. For $S_n > \max(\sqrt{2(1+\delta)A_n^2 \ln\ln A_n^2}, 1)$ for some $\delta > 0$ and sufficiently large n, we have

$$\begin{aligned}
\frac{1}{A_n}\pi\left(\frac{S_n}{A_n^2}\right)\exp\left(\frac{S_n^2}{2A_n^2}\right) &= \frac{S_n}{A_n^2}\pi\left(\frac{S_n}{A_n^2}\right)\left(\frac{S_n}{A_n}\right)^{-1}\exp\left(\frac{S_n^2}{2A_n^2}\right) \\
&= \frac{1}{\ln\frac{A_n^2}{S_n}\left(\ln\ln\frac{A_n^2}{S_n}\right)^2}\left(\frac{S_n}{A_n}\right)^{-1}\exp\left(\frac{S_n^2}{2A_n^2}\right) \\
&\geq \frac{1}{\left(2(1+\delta)\ln\ln A_n^2\right)^{1/2}}\exp\left((1+\delta)\ln\ln A_n^2 - \ln\ln\frac{A_n^2}{S_n} - 2\ln\ln\ln\frac{A_n^2}{S_n}\right) \\
&\geq \frac{1}{\left(2(1+\delta)\right)^{1/2}}\exp\left((1+\delta)\ln\ln A_n^2 - \ln\ln A_n^2 - \frac{5}{2}\ln\ln\ln A_n^2\right) \\
&= \frac{1}{\left(2(1+\delta)\right)^{1/2}}\exp\left(\delta\ln\ln A_n^2 - \frac{5}{2}\ln\ln\ln A_n^2\right) \uparrow \infty \quad (n\to\infty). \qquad \square
\end{aligned}$$

4.6. Priors for the validity of typical EFKP-LIL

Here we generalize the prior (14) in view of EFKP-LIL. We give a more general and complete treatment of EFKP-LIL in Section 5.

Write

$$\ln_b x = \underbrace{\ln\ln\ldots\ln}_{b \text{ times}} x.$$

The following prior density

$$\pi(\epsilon) = \frac{1}{\epsilon\left(\ln\frac{1}{\epsilon}\right)\left(\ln_2\frac{1}{\epsilon}\right)\cdots\left(\ln_{b-1}\frac{1}{\epsilon}\right)\left(\ln_b\frac{1}{\epsilon}\right)^{1+\gamma}}, \qquad \gamma > 0 \tag{15}$$

is integrable around the origin. We compare this with the bound $A_n\psi(A_n^2)$, where

$$\psi(A_n^2) = \sqrt{2\ln_2 A_n^2 + 3\ln_3 A_n^2 + 2\ln_4 A_n^2 + \cdots + 2\ln_b A_n^2 + 2(1+2\gamma)\ln_{b+1} A_n^2}. \tag{16}$$

For notational simplicity we take $b \geq 4$ in (15), since the coefficient of $\ln_k$ is 2 except for $k = 3$ in (16). Note that γ in (15) is multiplied by 2 in (16). This is needed, because unlike the usual LIL in (14), where we considered the ratio of S_n to $\sqrt{2A_n^2 \ln\ln A_n^2}$, in EFKP-LIL we need to consider the difference $S_n - A_n\psi(A_n^2)$. This difference corresponds to multiplication of $\pi(\epsilon)$ by $c(\epsilon)$ in the proof of Theorem 5.4.

Proposition 4.7. *In OUFG, by the prior (15), Skeptic weakly forces*

$$A_\infty = \infty \;\;\Rightarrow\;\; S_n - A_n\psi(A_n^2) \leq 0 \quad a.a.,$$

where $\psi(A_n^2)$ is defined in (16), and $a.a.$ means almost always, or except for a finite number of n.

Proof. For $S_n > \max(A_n\psi(A_n^2), 1)$ and sufficiently large n, we have

$$\left(\frac{S_n^2}{A_n^2}\right)^{-1/2}\exp\left(\frac{S_n^2}{2A_n^2}\right) \geq \frac{1}{\psi(A_n^2)}\exp\left(\frac{\psi(A_n^2)^2}{2}\right) = \exp\left(\frac{\psi(A_n^2)^2}{2} - \ln\psi(A_n^2)\right).$$

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Here

$$\begin{aligned}\ln \psi(A_n^2) &= \frac{1}{2}\ln(2\ln_2 A_n^2)\\ &\quad + \frac{1}{2}\ln\left(1+\frac{3\ln_3 A_n^2 + 2\ln_4 A_n^2 + \cdots + 2\ln_b A_n^2 + 2(1+2\gamma)\ln_{b+1} A_n^2}{2\ln_2 A_n^2}\right)\\ &= \frac{1}{2}\ln_3 A_n^2 + \frac{1}{2}\ln 2 + o(1).\end{aligned}$$

Combining this with

$$\begin{aligned}\ln\left[\frac{S_n}{A_n^2}\pi\left(\frac{S_n}{A_n^2}\right)\right] &= -\ln_2\frac{A_n^2}{S_n} - \ln_3\frac{A_n^2}{S_n} - \cdots - \ln_b\frac{A_n^2}{S_n} - (1+\gamma)\ln_{b+1}\frac{A_n^2}{S_n}\\ &\geq -\ln_2 A_n^2 - \ln_3 A_n^2 - \cdots - \ln_b A_n^2 - (1+\gamma)\ln_{b+1} A_n^2,\end{aligned}$$

we have

$$\frac{1}{A_n}\pi\left(\frac{S_n}{A_n^2}\right)\exp\left(\frac{S_n^2}{2A_n^2}\right) \geq \exp\left(\gamma\ln_{b+1} A_n^2 - \frac{1}{2}\ln 2 - o(1)\right) \uparrow \infty \quad (n\to\infty). \qquad \square$$

5. Equivalence of prior densities and the upper class

The typical priors of the previous section suggest that higher concentration of the prior around the origin corresponds to a tighter convergence rate of SLLN. In particular, from the viewpoint of EFKP-LIL, it is of interest to establish the relation between priors and the function of the upper class.

Also it is natural to conjecture that the rate of SLLN implied by a Bayesian strategy only depends on the rate of concentration of the prior around the origin. This idea can be clarified if we classify priors with the same rate of concentration into the same class. Let Π_0 denote the set of priors π satisfying Assumption 2.1 by

$$\Pi_0 = \{\pi \mid \pi \text{ satisfies Assumption 2.1}\}. \tag{17}$$

We use the subscript 0, because Assumption 2.1 is on the behavior of π around the origin.

5.1. The upper class

In studying the validity of EFKP-LIL, the notion of upper class of functions is essential. A positive function $\psi(\lambda)$ defined for $\lambda > M_\psi > 0$ is said to *belong to the upper class in OUFG* if Skeptic can force the event

$$A_\infty = \infty \Rightarrow S_n - A_n\psi(A_n^2) \leq 0 \quad a.a.$$

For the terminology, see [13]. We characterize the upper class by an integral test:

$$\int_{M_\psi}^\infty \psi(\lambda)e^{-\psi(\lambda)^2/2}\frac{d\lambda}{\lambda} < \infty. \tag{18}$$

A typical function in the upper class is ψ given in (16). For convenience, we put the following regularity conditions on ψ.

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Assumption 5.1. There exist some $M_\psi > 0$ and $\delta_\psi > 0$, such that

1. $\psi(\lambda)$ is a positive increasing function on (M_ψ, ∞),
2. the integral in (18) is finite, and
3. for $\lambda > M_\psi$, we have

$$\lambda\psi(\lambda)e^{-\psi(\lambda)^2/2} > \delta_\psi. \tag{19}$$

If $\lim_{\lambda\uparrow\infty}\psi(\lambda)$ is finite, then the integral in (18) diverges, since $\int_{M_\psi}^\infty d\lambda/\lambda = \infty$. Hence $\lim_{\lambda\uparrow\infty}\psi(\lambda) = \infty$ for ψ satisfying Assumption 5.1. Furthermore the function $xe^{-x^2/2}$ is decreasing for $x > 1$ and converges to zero as $x \to \infty$. Hence $\psi(\lambda)e^{-\psi(\lambda)^2/2}$ is eventually decreasing to zero. For EFKP-LIL, as the typical ψ in (16), we are mainly concerned with functions ψ, for which $\psi(\lambda)e^{-\psi(\lambda)^2/2}$ decreases to zero slower than any negative power of λ. Hence (19) is a mild regularity condition. We denote the set of functions ψ satisfying Assumption 5.1 by

$$\Psi_\infty = \{\psi \mid \psi \text{ satisfies Assumption 5.1}\}.$$

We use the subscript ∞ because Assumption 5.1 is on the behavior of ψ around the infinity.

The goal is, when a $\psi \in \Psi_\infty$ is given, to find a $\pi \in \Pi_0$ that is a witness of the ψ in the upper class. Now we define two functionals $F : \Psi_\infty \to \Pi_0$ and $G : \Pi_0 \to \Psi_\infty$ as follows.

Definition 5.2.

$$F[\psi](\epsilon) = \frac{\psi(\frac{1}{\epsilon})}{\epsilon}\exp\left(-\psi\left(\frac{1}{\epsilon}\right)^2/2\right), \tag{20}$$

$$G[\pi](\lambda) = \sqrt{\beta\left(\frac{1}{\lambda}\right) + \ln\beta\left(\frac{1}{\lambda}\right)}, \qquad \beta(\epsilon) = \max(-2\ln(\epsilon\pi(\epsilon)), 1). \tag{21}$$

See Fig. 1. These two functionals are "asymptotic inverse functionals" as we clarify in (27) and Theorem 5.8. Since we are interested in the case that ϵ is sufficiently small and λ is sufficiently large, for the rest of this paper we only consider λ that fulfills the condition $-2\ln(\frac{1}{\lambda}\pi(\frac{1}{\lambda})) \geq 1$. In fact, this condition holds true for sufficiently large λ because of the monotonicity of $\frac{1}{\lambda}\pi(\frac{1}{\lambda})$ and $\lim_{\lambda\uparrow\infty}\frac{1}{\lambda}\pi(\frac{1}{\lambda}) = 0$. First we need to check that $F[\psi] \in \Pi_0$ and $G[\pi] \in \Psi_\infty$. For checking $G[\pi] \in \Psi_\infty$, we use the following relation.

$$\begin{aligned} G[\pi](\lambda)\,\exp\left(-\frac{(G[\pi](\lambda))^2}{2}\right) &= \sqrt{\beta(1/\lambda) + \ln\beta(1/\lambda)}\,\frac{\exp\left(-\beta(1/\lambda)/2\right)}{\sqrt{\beta(1/\lambda)}} \\ &= \sqrt{1 + \frac{\ln\beta(1/\lambda)}{\beta(1/\lambda)}}\,\frac{1}{\lambda}\pi\left(\frac{1}{\lambda}\right). \end{aligned} \tag{22}$$

Lemma 5.3. *$F[\psi] \in \Pi_0$ for $\psi \in \Psi_\infty$. $G[\pi] \in \Psi_\infty$ for $\pi \in \Pi_0$.*

Proof. For the first part, the monotonicity of $\epsilon F[\psi](\epsilon) \in \Pi_0$ for sufficiently small ϵ holds because $\psi(\lambda)e^{-\psi(\lambda)^2/2}$ is eventually decreasing. The integrability of $F[\psi]$ follows directly from the change of variables $\epsilon = 1/\lambda$. Finally $F[\psi](\epsilon) > \delta_\psi$ for $\epsilon < 1/M_\psi$ by (19) and (20).

For the second part, the monotonicity of $G[\pi]$ is obvious. We note the fact that $\lim_{\epsilon\downarrow 0}\epsilon\pi(\epsilon) = 0$, because otherwise $\pi(\epsilon)$ is not integrable around the origin. Hence

$$\beta(\epsilon) \uparrow \infty \quad \text{as} \quad \epsilon \downarrow 0. \tag{23}$$

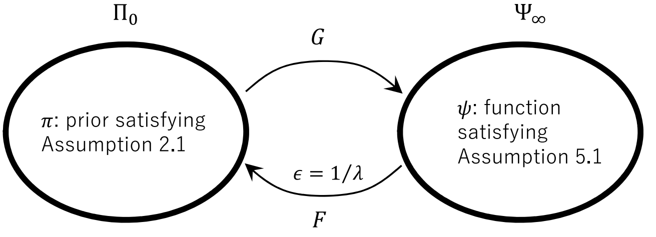

Fig. 1. Functionals F and G.

Then $G[\pi](\lambda) = \sqrt{\beta(1/\lambda)}(1 + o(1))$ as $\lambda \to \infty$. Then as $\lambda \to \infty$, by (22),

$$G[\pi](\lambda)\ \exp\left(-\frac{(G[\pi](\lambda))^2}{2}\right)\frac{1}{\lambda} = \sqrt{1 + \frac{\ln\beta(1/\lambda)}{\beta(1/\lambda)}}\frac{1}{\lambda^2}\pi\left(\frac{1}{\lambda}\right)$$
$$= \frac{1}{\lambda^2}\pi\left(\frac{1}{\lambda}\right)(1 + o(1)). \tag{24}$$

By the change of variables $\epsilon = 1/\lambda$, the integrability of the left-hand side of (24) reduces to the integrability of $\pi(\epsilon)$ around the origin. Finally

$$\lambda G[\pi](\lambda)\ \exp\left(-\frac{(G[\pi](\lambda))^2}{2}\right) = \pi\left(\frac{1}{\lambda}\right)(1 + o(1)),$$

which is bounded away from zero for large λ. $\square$

Based on this lemma, we can consider the compositions $F \circ G : \Pi_0 \to \Pi_0$ and $G \circ F : \Psi_\infty \to \Psi_\infty$. By direct computation we obtain

$$F[G[\pi]](\epsilon) = \pi(\epsilon)\sqrt{\frac{\beta(\epsilon) + \ln\beta(\epsilon)}{\beta(\epsilon)}}, \qquad \beta(\epsilon) = -2\ln(\epsilon\pi(\epsilon)), \tag{25}$$

$$G[F[\psi]](\lambda) = \sqrt{\psi(\lambda)^2 - 2\ln\psi(\lambda) + \ln\left(\psi(\lambda)^2 - 2\ln\psi(\lambda)\right)}. \tag{26}$$

In preparation for the subsequent arguments in this section, we introduce equivalence relations in Π_0 and Ψ_∞ to identify functions with the same rate of growth. For two functions $\pi_1, \pi_2 \in \Pi_0$ we write $\pi_1 \sim_0 \pi_2$ if

$$0 < \liminf_{\epsilon\downarrow 0}\frac{\pi_2(\epsilon)}{\pi_1(\epsilon)} \le \limsup_{\epsilon\downarrow 0}\frac{\pi_2(\epsilon)}{\pi_1(\epsilon)} < \infty.$$

It is easy to check that $\sim_0$ is an equivalence relation. For two functions $\psi_1, \psi_2 \in \Psi_\infty$ we write $\psi_1 \sim_\infty \psi_2$ if

$$0 < \liminf_{\lambda\uparrow\infty}\frac{\exp(\psi_2(\lambda)^2/2)}{\exp(\psi_1(\lambda)^2/2)} \le \limsup_{\lambda\uparrow\infty}\frac{\exp(\psi_2(\lambda)^2/2)}{\exp(\psi_1(\lambda)^2/2)} < \infty.$$

Again it is easy to check that $\sim_\infty$ is an equivalence relation.

5.2. Validity of EFKP-LIL via Bayesian strategy

Here we establish the validity of EFKP-LIL via a Bayesian strategy.

Theorem 5.4. *In OUFG, by a prior $\pi \in \Pi_0$, Skeptic weakly forces*

$$A_\infty = \infty \quad \Rightarrow \quad S_n - A_n \psi(A_n^2) \le 0 \quad a.a.$$

for all $\psi \sim_\infty G[\pi]$.

Proof. Given $\pi \in \Pi_0$, there exists a decreasing function $c(\epsilon)$, such that $\lim_{\epsilon \downarrow 0} c(\epsilon) = \infty$ and $\int_0^1 c(\epsilon)\pi(\epsilon) < \infty$. Indeed we can define

$$c(\epsilon) = k \ \text{ for } \ \epsilon \in [\epsilon_{k+1}, \epsilon_k), \ k = 1, 2, \dots,$$

where ϵ_k is defined by

$$\int_0^{\epsilon_k} \pi(\epsilon) d\epsilon = \frac{1}{2^k} \int_0^1 \pi(\epsilon) d\epsilon.$$

We use the Bayesian strategy with the prior $\tilde{\pi}(\epsilon) = c(\epsilon)\pi(\epsilon)$. For sufficiently large n such that $S_n \ge \max(A_n \psi(A_n^2), 1)$ and S_n / A_n^2 is sufficiently small, by (11) in Remark 4.4, we have

$$\begin{aligned}
\mathcal{K}_n^\pi &\ge c(u_n) \frac{1}{A_n} \pi\left(\frac{S_n}{A_n^2}\right) \exp\left(\frac{S_n^2}{2A_n^2}\right) \\
&= c(u_n) \frac{S_n}{A_n^2} \pi\left(\frac{S_n}{A_n^2}\right) \left(\frac{S_n^2}{A_n^2}\right)^{-1/2} \exp\left(\frac{S_n^2}{2A_n^2}\right) \\
&\ge c(u_n) \frac{S_n}{A_n^2} \pi\left(\frac{S_n}{A_n^2}\right) \exp\left(\frac{\psi_n(A_n^2)^2}{2} - \ln \psi(A_n^2)\right) \\
&\ge c(u_n) \frac{1}{A_n^2} \pi\left(\frac{1}{A_n^2}\right) \exp\left(\frac{\psi(A_n^2)^2}{2} - \ln \psi(A_n^2)\right).
\end{aligned}$$

Since $\psi \sim_\infty G[\pi]$, there exist positive κ_1, κ_2 such that

$$\kappa_1 \exp(\psi(A_n^2)^2/2) \le \exp(G[\pi](A_n^2)^2/2) \le \kappa_2 \exp(\psi(A_n^2)^2/2)$$

for sufficiently large n. Then

$$\sqrt{2 \ln \kappa_1 + \psi(A_n^2)^2} \le G[\pi](A_n^2) \le \sqrt{2 \ln \kappa_2 + \psi(A_n^2)^2}.$$

Also we obtain

$$\frac{1}{2} \psi(A_n^2) \le \sqrt{2 \ln \kappa_1 + \psi(A_n^2)^2}$$

for sufficiently large n. Combining these inequalities we have

$$\frac{\exp\left(\frac{\psi(A_n^2)^2}{2} - \ln \psi(A_n^2)\right)}{\exp\left(\frac{G[\pi](A_n^2)^2}{2} - \ln G[\pi](A_n^2)\right)} = \frac{\psi(A_n^2)^{-1} \exp\left(\frac{\psi(A_n^2)^2}{2}\right)}{G[\pi](A_n^2)^{-1} \exp\left(\frac{G[\pi](A_n^2)^2}{2}\right)} \ge \frac{1}{2\kappa_2}.$$

Now

$$\begin{aligned}
\exp\left(\frac{G[\pi](A_n^2)^2}{2}\right) &= \exp\left(\frac{1}{2}\beta\left(\frac{1}{A_n^2}\right) + \frac{1}{2} \ln \beta\left(\frac{1}{A_n^2}\right)\right) \\
&= \left(\frac{1}{A_n^2} \pi\left(\frac{1}{A_n^2}\right)\right)^{-1} \exp\left(\frac{1}{2} \ln \beta\left(\frac{1}{A_n^2}\right)\right).
\end{aligned}$$

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Hence

$$\mathcal{K}_n^\pi \geq \frac{c(u_n)}{2\kappa_2} \frac{1}{A_n^2} \pi \left(\frac{1}{A_n^2} \right) \exp \left(\frac{G[\pi](A_n^2)^2}{2} - \ln G[\pi](A_n^2) \right)$$
$$= \frac{c(u_n)}{2\kappa_2} \exp \left(\frac{1}{2} \ln \beta \left(\frac{1}{A_n^2} \right) - \ln G[\pi](A_n^2) \right).$$

But by (21),

$$\ln G[\pi](A_n^2) = \frac{1}{2} \ln \beta \left(\frac{1}{A_n^2} \right) + o(1)$$

and we have

$$\mathcal{K}_n^\pi \geq \frac{c(u_n)}{2\kappa_2} \exp(o(1)) \uparrow \infty \qquad (n \to \infty),$$

because $u_n \to 0$ as $n \to \infty$. □

As a corollary to this theorem, we state the following statement of the validity of EFKP-LIL.

Corollary 5.5. *Let ψ be a positive increasing function defined for $\lambda > M_\psi > 0$. Then, ψ belongs to the upper class in OUFG if and only if*

$$I(\psi) = \int_{M_\psi}^{\infty} \psi(\lambda) e^{-\psi(\lambda)^2/2} \frac{d\lambda}{\lambda} < \infty.$$

Proof. First suppose that $\psi \in \Psi_\infty$. Let $\pi = F[\psi]$ and apply Theorem 5.4. Then Skeptic can weakly force

$$A_\infty = \infty \;\Rightarrow\; S_n - A_n G[F[\psi]](A_n^2) \leq 0 \quad a.a.$$

But in (26) we have

$$G[F[\psi]](\lambda) < \psi(\lambda)$$

for all sufficiently large λ. Hence $S_n - A_n \psi(A_n^2) \leq 0 \quad a.a.$

Next let ψ be a positive increasing function such that $I(\psi) < \infty$. Let $\widetilde{\psi} = \min\{\psi, \sqrt{2\ln_2\lambda + 4\ln_3\lambda}\}$. Notice that $\sqrt{2\ln_2\lambda + 4\ln_3\lambda}$ belongs to the upper class. Since $xe^{-x^2/2}$ is decreasing for $x > 1$ and $\widetilde{\psi} \leq \sqrt{2\ln_2\lambda + 4\ln_3\lambda}$, we have

$$\lambda\widetilde{\psi}(\lambda)e^{-\widetilde{\psi}(\lambda)^2/2} \geq \lambda \frac{\sqrt{2\ln_2\lambda + 4\ln_3\lambda}}{(\ln\lambda)(\ln_2\lambda)^2}.$$

Then, $\widetilde{\psi}$ satisfies Assumption 5.1. Hence, $\widetilde{\psi}$ belongs to the upper class and so does ψ.

Finally, suppose that ψ is a positive increasing function, but $I(\psi) = \infty$. Then, in the fair-coin tossing game, Skeptic can force $S_n - \sqrt{n}\psi(n) \geq 0$ for infinitely many n. This is a corollary from the classical EFKP-LIL. This fact also follows from the main theorem in [14]. Thus, Reality can comply with this event with this restriction, which means that ψ does not belong to the upper class in OUFG. □

5.3. Equivalence

Consider $\psi \in \Psi_\infty$ such that the convergence rate of $I(\psi)$ is very slow. Then we obtain $G[F[\psi]] \in \Psi_\infty$ and $G[F[\psi]] < \psi$ for sufficiently large λ by (26). Thus the two functions ψ

and $G[F[\psi]]$ should have similar growing rates. One should also notice that, by (23),

$$\lim_{\epsilon\downarrow 0}\frac{F[G[\pi]](\epsilon)}{\pi(\epsilon)}=1. \tag{27}$$

In this asymptotic sense, F and G are inverse functionals. We clarify this point further below in Theorem 5.8. In the following two lemmas we prove that the functionals F and G preserve the equivalence relations.

Lemma 5.6. *Let $\psi_1,\psi_2\in\Psi_\infty$. If $\psi_1\sim_\infty\psi_2$, then $F[\psi_1]\sim_0 F[\psi_2]$.*

Proof. Since $\psi_1\sim_\infty\psi_2$, there exist positive $\kappa_1,\kappa_2,\lambda_1$ such that

$$\kappa_1\exp(\psi_1(\lambda)^2/2)\le\exp(\psi_2(\lambda)^2/2)\le\kappa_2\exp(\psi_1(\lambda)^2/2),\qquad\forall\lambda>\lambda_1.$$

Then

$$\sqrt{2\ln\kappa_1+\psi_1(\lambda)^2}\le\psi_2(\lambda)\le\sqrt{2\ln\kappa_2+\psi_1(\lambda)^2}.$$

Since $\psi_1(\lambda)$ and $\psi_2(\lambda)$ diverge to ∞ as $\lambda\to\infty$, for sufficiently small ϵ we have

$$\begin{aligned}\frac{\kappa_1}{2}\psi_1\left(\frac{1}{\epsilon}\right)\exp\left(\psi_1\left(\frac{1}{\epsilon}\right)^2/2\right)&\le\psi_2\left(\frac{1}{\epsilon}\right)\exp\left(\psi_2\left(\frac{1}{\epsilon}\right)^2/2\right)\\&\le 2\kappa_2\psi_1\left(\frac{1}{\epsilon}\right)\exp\left(\psi_1\left(\frac{1}{\epsilon}\right)^2/2\right).\end{aligned}$$

Dividing this by ϵ we obtain the lemma. $\square$

Lemma 5.7. *Let $\pi_1,\pi_2\in\Pi_0$. If $\pi_1\sim_0\pi_2$, then $G[\pi_1]\sim_\infty G[\pi_2]$.*

Proof. There exist positive $\kappa_1,\kappa_2,\epsilon_1$ such that

$$\kappa_1\pi_1(\epsilon)<\pi_2(\epsilon)<\kappa_2\pi_1(\epsilon),\qquad\forall\epsilon\in(0,\epsilon_1). \tag{28}$$

Then

$$-2\ln(\epsilon\pi_1(\epsilon))-2\ln\kappa_2\le-2\ln(\epsilon\pi_2(\epsilon))\le-2\ln(\epsilon\pi_1(\epsilon))-2\ln\kappa_1.$$

Since $-\ln(\epsilon\pi_i(\epsilon))$, $i=1,2$ diverge to ∞ as $\epsilon\downarrow 0$, for sufficiently small ϵ we have

$$\frac{1}{2}\sqrt{-2\ln(\epsilon\pi_1(\epsilon))}\le\sqrt{-2\ln(\epsilon\pi_2(\epsilon))}\le 2\sqrt{-2\ln(\epsilon\pi_1(\epsilon))}. \tag{29}$$

By (28) and (29)

$$\frac{1}{2\kappa_2}\frac{\sqrt{-2\ln(\epsilon\pi_1(\epsilon))}}{\epsilon\pi_1(\epsilon)}\le\frac{\sqrt{-2\ln(\epsilon\pi_2(\epsilon))}}{\epsilon\pi_2(\epsilon)}\le\frac{2}{\kappa_1}\frac{\sqrt{-2\ln(\epsilon\pi_1(\epsilon))}}{\epsilon\pi_1(\epsilon)}.$$

Now

$$\begin{aligned}\frac{\sqrt{-2\ln(\epsilon\pi_i(\epsilon))}}{\epsilon\pi_i(\epsilon)}&=\exp\left(\frac{\left(\sqrt{-2\ln(\epsilon\pi_i(\epsilon))+\ln(-2\ln(\epsilon\pi_i(\epsilon)))}\right)^2}{2}\right)\\&=\exp\left(\frac{G[\pi_i]\left(\frac{1}{\epsilon}\right)^2}{2}\right),\qquad i=1,2.\end{aligned}$$

Hence for sufficiently large λ

$$\frac{1}{2\kappa_2}\exp\left(\frac{G[\pi_1](\lambda)^2}{2}\right) \le \exp\left(\frac{G[\pi_2](\lambda)^2}{2}\right) \le \frac{2}{\kappa_1}\exp\left(\frac{G[\pi_1](\lambda)^2}{2}\right). \quad \square$$

Let $\Pi_0/\sim_0$ denote the set of equivalence classes in Π_0 with respect to $\sim_0$ and define $\Psi_\infty/\sim_\infty$ similarly. Based on the above two lemmas and (25), (26) we have the following theorem.

Theorem 5.8. *The functionals F and G give bijections between $\Pi_0/\sim_0$ and $\Psi_\infty/\sim_\infty$. Furthermore they are inverse functionals to each other.*

Proof is obvious and omitted. The theorem says that functions in the upper class correspond to the prior densities, and the integrabilities of $I(\psi)$ correspond to the integrabilities of densities.

6. Another property of the one-sided unbounded forecasting game

In this paper we have been considering SLLN in the self-normalized form. However SLLN for OUFG in non-self-normalized form exhibits an interesting property, which we present in the following proposition. The definition of *compliance* can be found in [10] and [11].

Proposition 6.1. *Let b_n be a sequence of increasing positive reals such that $\lim_n b_n = \infty$. In OUFG Skeptic can weakly force $S_n/b_n \to 0$ if and only if $\sum_n 1/b_n < \infty$.*

Proof. Suppose that $Z = \sum_n 1/b_n < \infty$. Consider Skeptic's strategy $M_n = 1/b_n$. Then $Y_n = Z + \sum_{i=1}^n M_i x_i$ is a nonnegative martingale and by the game-theoretic martingale convergence theorem (Lemma 4.5 of [15]) Skeptic can weakly force that Y_n converges to a finite value. Then by Kronecker's lemma Skeptic can weakly force $\lim_n \frac{S_n}{b_n} = 0$.

The converse part can be proved by a deterministic strategy of Reality complying with $\limsup_n |S_n|/b_n \ge 1$. Suppose $\sum_n 1/b_n = \infty$. If $b_n < n-1$ for infinitely many n, then Reality chooses $x_n = -1$ for all n, which complies with the desired property.

Now, we assume that $b_n \ge n-1$ for all but finitely many n. In each round Reality chooses either $x_n = -1$ or $x_n = 2b_n$. Let $p_n = 1/(1+2b_n)$, which corresponds to the probability of $x_n = 2b_n$ as we discuss after the proof. Let

$$c_n = \begin{cases} \dfrac{1}{2(1-p_n)} & \text{if } x_n = -1 \\ \dfrac{1}{2p_n} & \text{if } x_n = 2b_n \end{cases}$$

and

$$\mathcal{L}_n = \mathcal{K}_n + \prod_{k=1}^n c_k, \qquad \mathcal{K}_0 = 1,\ \mathcal{L}_0 = 2.$$

We show that, for each n, at least one of $x_n \in \{-1, 2b_n\}$ satisfies $\mathcal{L}_n \le \mathcal{L}_{n-1}$. Suppose otherwise. Then

$$\mathcal{K}_{n-1} - M_n + \left(\prod_{k=1}^{n-1} c_k\right)\cdot\frac{1}{2(1-p_n)} > \mathcal{K}_{n-1} + \left(\prod_{k=1}^{n-1} c_k\right), \tag{30}$$

$$\mathcal{K}_{n-1} + 2b_n M_n + \left(\prod_{k=1}^{n-1} c_k\right)\cdot\frac{1}{2p_n} > \mathcal{K}_{n-1} + \left(\prod_{k=1}^{n-1} c_k\right). \tag{31}$$

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Thus by $(1 - p_n) \times (30) + p_n \times (31)$ we have

$$- (1 - p_n) M_n + p_n 2 b_n M_n > 0,$$

which implies

$$p_n > \frac{1}{1 + 2b_n}.$$

This is a contradiction.

Reality chooses her move so that $\mathcal{L}_n \le \mathcal{L}_{n-1}$ for all n. Then $\sup_n \mathcal{K}_n \le 2$. Furthermore, $\sup_n \prod_{k=1}^n c_k \le 2$. If $x_n = 2b_n$ for at most finitely many n, then there exists m such that $x_n = -1$ for all $n \ge m$, whence $\prod_{k=1}^n c_k = \prod_{k=1}^{m-1} c_k \cdot \prod_{k=m}^n \frac{1}{2(1-p_n)} \to \infty$ because $\sum_n p_n = \infty$. Hence, $x_n = 2b_n$ for infinitely many n. For such an n that is large enough, we have

$$S_n = S_{n-1} + 2b_n \ge -(n-1) + 2b_n,$$
$$\frac{S_n}{b_n} \ge 2 - \frac{n-1}{b_n} \ge 1,$$

which implies $\limsup_n S_n / b_n \ge 1$. □

The intuition of p_n is the probability of $x_n = 2b_n$ so that the expectation is $(-1) \cdot (1 - p_n) + 2b_n p_n = 0$. Then, $\prod_{k=1}^n c_k$ can be seen as a capital process in a game, and so is $\mathcal{L}$.

In this proof we constructed a particular deterministic strategy of Reality. General theory of constructing Reality's deterministic strategy complying with certain events is developed in [10] and [11].

7. Discussion

In this paper we gave a unified treatment of the rate of convergence of SLLN in terms of Bayesian strategies, including the validity of LIL. On the other hand it is not known whether the rate can be improved further. For example, can $(1 - a)$ in the denominator be replaced by a smaller number in Proposition 4.5? Does π force only the corresponding ψ in the upper class in Theorem 5.4? To answer these questions we need to give good upper bounds for our capital processes. Hence we leave them as open problems.

Concerning LIL we did not discuss the sharpness of the bound. In OUFG the sharpness does not hold, because Reality can simply take $x_n \equiv -1$. For the sharpness we need some boundedness condition, such as the simplified predictably unbounded forecasting game (Section 5.1 of [15], [14]). Even with some boundedness conditions, current game-theoretic proofs of the sharpness are still complicated and the nature of the weights, involving also negative ones due to short selling of a strategy, does not seem to be clear. We hope that the unified treatment of this paper also helps to streamline proofs of the sharpness.

As we discussed in (2), a Bayesian strategy for OUFG can be understood as the portfolio of cash and one risky asset. In the literature on universal portfolio by Thomas Cover and other researchers (Chapter 16 of [2], [1,12,18]), Bayesian strategies for many risky assets are considered. They recommend power priors such as the Dirichlet prior, which corresponds to priors in Section 4.4. From the viewpoint of the rate of convergence of SLLN, we have shown that we should take $a = 1$ in (12) with additional multiplicative logarithmic terms. Hence our recommendations and those in universal portfolio literature seem to be different. This may be due to the difference of criteria for evaluating strategies. It would be interesting to clarify these differences.

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