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Behavior of the Hermite sheet with respect to the Hurst index

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Abstract

We consider a d-parameter Hermite process with Hurst index $\mathbf{H} = (H_1, .., H_d) \in \left(\frac{1}{2}, 1\right)^d$ and we study its limit behavior in distribution when the Hurst parameters $H_i, i = 1, .., d$ (or a part of them) converges to $\frac{1}{2}$ and/or 1. The limit obtained is Gaussian (when at least one parameter tends to $\frac{1}{2}$) and non-Gaussian (when at least one-parameter tends to 1 and none converges to $\frac{1}{2}$).

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1 Introduction

Several recent works investigated the behavior of some fractional processes, called the Hermite processes, with respect to the Hurst parameter (see [3], [19], [2]). The Hermite process of order $q \geq 1$ and with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$ lives in the qth Wiener chaos. It is defined as a multiple stochastic integral, i.e. for every $t \geq 0$

$$Z_H^q(t) = c(H,q) \int_{\mathbb{R}} dB(y_1) \dots \int_{\mathbb{R}} dB(y_q) \left(\int_0^t (s-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \dots (s-y_q)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \right) \quad (1)$$

where $x_+ = \max(x, 0)$, $c(H,q)$ is a normalizing positive constant chosen such that $\mathbf{E}\left(Z_H^q(1)\right)^2 = 1$ and $(B(y))_{y \in \mathbb{R}}$ is a Wiener process with time interval $\mathbb{R}$. The process (1) is H-self-similar and it has stationary increments and long memory.

The class of Hermite processes includes the fractional Brownian motion (fBm) which is obtained for $q=1$ and the Rosenblatt process ($q=2$). The fBm is the only Gaussian Hermite process. The Hermite process is non-Gaussian if $q \geq 2$. These processes have been widely studied since the seventies (see the monographs [14], [18] and the references therein).

Let us start by presenting some known facts concerning the behavior of the Hermite processes with respect to the Hurst parameter. If $q=1$ then there is not too much to discuss. It is well-known that Z_H^1 concides in distribution with the Brownian motion if $H=\frac{1}{2}$ and with the process $(tZ)_{t\geq 0}$ if $H=1$ where Z denotes (throughout the work) a standard normal random variable, i.e. $Z \sim N(0,1)$.

The case $q=2$ has been discussed in [19]. It has been shown that Z_H^2 converges weakly as $H \to \frac{1}{2}$, in the space of continuous functions $C([0,T])$ ($T>0$), to a Brownian motion while if $H \to 1$, it tends to $(t\frac{1}{\sqrt{2}}(Z^2-1))_{t\geq 0}$, Z^2-1 being a so-called *centered chi-square random variable*. The main argument of the proofs relies on the expression of the characteristic function of the Rosenblatt process.

In the case $q \geq 3$ we know from [3] that, if $H \to \frac{1}{2}$, then again the process Z_H^q defined by (1) converges weakly to a Brownian motion in $C([0,T])$. The proof is based on the Fourth Moment Theorem. Similar results for the generalized Hermite process can be found in [2] or [3], and for Rosenblatt Ornstein-Uhlenbeck process in [16].

In a first step, we discuss the unsolved case concerning the asymptotic behavior of the Hermite process Z_H^q when $H \to 1$. We show that it converges weakly in $C([0,T])$ to the stochastic process $(t\frac{1}{\sqrt{q!}}H_q(Z))_{t\in[0,T]}$ where H_q is the Hermite polynomial of degree q. Since the limit is not Gaussian and we have no tractable information on the characteristic function of (1) when $q \geq 3$, we will need a different argument from [3] or [19], based on the non-central limit theorem. Notice that there is an interesting contrast with the case of the generalized Hermite process treated in [2] and [3]. In these works, the limits is always non-Gaussian unless the parameters tend jointly to the boundary between long and short memory, while in our case we may have Gaussian limits.

Next, we consider the case of a d-parameter Hermite process (or *Hermite sheet*, denoted by $Z_{\mathbf{H}}^{q,d}$ in the sequel) with d-dimensional Hurst parameter $\mathbf{H}=(H_1,..,H_d) \in (\frac{1}{2},1)^d$. Given the results recalled above, it is natural and interesting to ask what happens when one or several components H_i converge to the boundary of the interval of definition. We found the following results:

- If at least one of the parameters H_i goes to $1/2$ (and the other parameters are fixed in $(\frac{1}{2},1)$ or converge to 1) then the Hermite sheet $Z_{\mathbf{H}}^{q,d}$ converges weakly in $C([0,T]^d)$ to a d-parameter Gaussian process.

- If $(H_{j_1},..,H_{j_k}) \to (1,..,1) \in \mathbb{R}^k$ ($1 \leq k < d$) where $A_k := \{j_1,..,j_k\} \subset \{1,2,..,d\}$ and the parameters $H_j, j \in \{1,2,..,d\} \setminus \{j_1,..,j_k\} := \overline{A}_k$ are fixed, then the process $Z_{\mathbf{H}}^{q,d}$ converges weakly in $C([0,T]^d)$ to the d-parameter stochastic process $(X_{\mathbf{t}})_{\mathbf{t}\geq 0}$ defined by $X_{\mathbf{t}} = \langle \mathbf{t} \rangle_{A_k} Z_{\mathbf{H}}^{q,d-k}(t_{\overline{A}_k})$ where $\left(Z_{\mathbf{H}}^{q,d-k}(\mathbf{t}_{\overline{A}_k})\right)_{\mathbf{t}_{\overline{A}_k}\in\mathbb{R}_+^{d-k}}$ is a $(d-k)$-parameter

Hermite process and $\langle \mathbf{t} \rangle_{A_k} = t^{(j_1)} \cdot t^{(j_2)} \cdots t^{(j_k)}$ if $\mathbf{t} = (t^{(1)}, .., t^{(d)})$.

- If $(H_1, H_2, ..., H_d) \to (1, 1, .., 1) \in \mathbb{R}^d$, then the process $Z^{q,d}_{\mathbf{H}}$ converges weakly in $C([0,T]^d)$ to the stochastic process $(\langle \mathbf{t} \rangle_d \frac{1}{\sqrt{q!}} H_q(Z))_{\mathbf{t} \geq 0}$ where $\langle \mathbf{t} \rangle_d = t^{(1)} ... t^{(d)}$ if $\mathbf{t} = (t^{(1)}, .., t^{(d)})$.

The first point is proved via the Fourth Moment Theorem while the proof of the second and third point are based on the non-central approximation of the Hermite sheet, see [13] or [15]. We also included a separate (easier) proof in the case $q = 2$, based on the cumulants, although this can also be obtained from the general case.

We organized our work as follows. Section 2 contains some preliminaries. We introduce the Hermite sheet and remember several of its properties, and we also give the basic tools of Malliavin calculus needed throughout the paper. In Section 3 we analyze the asymptotic behavior with respect to the Hurst parameter of the Rosenblatt sheet. As mentioned above, we provided a specific proof for this case based on the fact that the law of a multiple integral of order two is completely determined by the cumulants. Finally, Section 4 is devoted to the study of the behavior of the Hermite sheet of general order. The main argument of the proof relies on the non-central approximation of the Hermite sheet by some partial sums.

2 Preliminaries

In this section, we introduce the Hermite sheet and we present the tools from the stochastic calculus on Wiener space needed in the sequel.

2.1 The Hermite sheet

The Hermite sheet has been introduced in [7]. We recall its definition and its basic properties (see also [13], [15] or [18]).

Let us introduce some notation. For $d \in \mathbb{N} \setminus \{0\}$ we will work with multi-parametric processes indexed in $\mathbb{R}^d$. We shall use bold notation for multi-indexed quantities, i.e., $\mathbf{a} = (a_1, a_2, \ldots, a_d)$, $\mathbf{b} = (b_1, .., b_d)$, $\alpha = (\alpha_1, .., \alpha_d)$, $\mathbf{ab} = \prod_{i=1}^d a_i b_i$, $|\mathbf{a} - \mathbf{b}|^\alpha = \prod_{i=1}^d |a_1 - b_1|^{\alpha_i}$, $\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, \ldots, a_d/b_d)$, $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^d [a_i, b_i]$, $(\mathbf{a}, \mathbf{b}) = \prod_{i=1}^d (a_i, b_i)$, $\sum_{\mathbf{i}=0}^{\mathbf{N}} a_{\mathbf{i}} = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_d=0}^{N_d} a_{i_1, i_2, \ldots, i_d}$, $\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^d a_i^{b_i}$, and $\mathbf{a} < \mathbf{b}$ iff $a_1 < b_1, a_2 < b_2, \ldots, a_d < b_d$ (analogously for the other inequalities). Also

$$[\mathbf{s}]_d := ([s_1], ..., [s_d]) \in \mathbb{Z}^d \quad \text{and} \quad \langle \mathbf{s} \rangle_d := s_1 ... s_d \in \mathbb{R} \tag{2}$$

where $[\cdot]$ denotes the integer part.

Let $q \geq 1$ integer and the Hurst multi-index $\mathbf{H} = (H_1, H_2, \ldots, H_d) \in (\frac{1}{2}, 1)^d$. The *Hermite sheet of order q and with self-similarity index $\mathbf{H}$* is given by

$$\begin{aligned} Z^q_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{t^{(1)}} \cdots \int_0^{t^{(d)}} \left(\prod_{j=1}^q (s_1 - y_{1,j})_+^{-\left(\frac{1}{2} + \frac{1-H_1}{q}\right)} \cdots (s_d - y_{d,j})_+^{-\left(\frac{1}{2} + \frac{1-H_d}{q}\right)} \right) \\ & \quad ds_d \ldots ds_1 \quad dW(y_{1,1}, \ldots, y_{d,1}) \ldots dW(y_{1,q}, \ldots, y_{d,q}) \\ &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^q (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q) \end{aligned} \tag{3}$$

where $x_+ = \max(x, 0)$ and $\mathbf{t} = (t^{(1)}, .., t^{(d)}) \in \mathbb{R}^d_+$. The above stochastic integral is a multiple stochastic integral with respect to the Wiener sheet $(W(\mathbf{y}), \mathbf{y} \in \mathbb{R}^d)$, see the next section. The constant $c(\mathbf{H}, q)$ ensures that $\mathbf{E}\left(Z^q_{\mathbf{H}}(\mathbf{t})\right)^2 = \mathbf{t}^{\mathbf{2H}}$ for every $\mathbf{t} \in \mathbb{R}^d_+$. As pointed out before, when $q = 1$, (3) is the fractional Brownian sheet with Hurst multi-index $\mathbf{H} = (H_1, H_2, \ldots, H_d) \in (\frac{1}{2}, 1)^d$. For $q \geq 2$ the process $Z^{q,d}_{\mathbf{H}}$ is not Gaussian and for $q = 2$ we denominate it as the *Rosenblatt sheet*. The Hermite sheet is $(H_1, .., H_d)$ self-similar, i.e. for any $\mathbf{h} = (h_1, \ldots, h_d) > 0$ the stochastic process $\left(\hat{Z}^{q,d}_{\mathbf{H}}(\mathbf{t})\right)_{\mathbf{t} \in (\mathbb{R}^d_+)}$ given by

$$\hat{Z}^{q,d}_{\mathbf{H}}(\mathbf{t}) = \mathbf{h}^{\alpha} \hat{Z}^q_{\mathbf{H}}\left(\frac{\mathbf{t}}{\mathbf{h}}\right) = h_1^{\alpha_1} ... h_d^{\alpha_d} \hat{Z}^{q,d}_{\mathbf{H}}\left(\frac{t_1}{h_1}, \ldots, \frac{t_d}{h_d}\right) \tag{4}$$

has the same finite dimensional distributions as the process $Z^q_{\mathbf{H}}$.

The Hermite sheet also has stationnary increments. Let us recall that the increment of a d-parameter process X on a rectangle $[\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^d$, $\mathbf{s} = (s_1, \ldots, s_d), \mathbf{t} = (t_1, \ldots, t_d)$, with $\mathbf{s} \leq \mathbf{t}$ (denoted by $\Delta X([\mathbf{s}, \mathbf{t}])$) is given by

$$\Delta X([\mathbf{s}, \mathbf{t}]) = \sum_{r \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d r_i} X_{\mathbf{s} + \mathbf{r} \cdot (\mathbf{t} - \mathbf{s})}. \tag{5}$$

When $d = 1$ one obtains $\Delta X([\mathbf{s}, \mathbf{t}]) = X_t - X_s$ while for $d = 2$ one gets $\Delta X([\mathbf{s}, \mathbf{t}]) = X_{t_1,t_2} - X_{t_1,s_2} - X_{s_1,t_2} + X_{s_1,s_2}$.

The fact that the process $(Z^{q,d}_{\mathbf{H}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d)$ has stationary increments means that for every $\mathbf{h} > 0, \mathbf{h} \in \mathbb{R}^d$ the stochastic processes $(\Delta Z^{q,d}_{\mathbf{H}}([0, \mathbf{t}]), \mathbf{t} \in \mathbb{R}^d)$ and $(\Delta Z^{q,d}_{\mathbf{H}}([\mathbf{h}, \mathbf{h} + \mathbf{t}]), \mathbf{t} \in \mathbb{R}^d)$ have the same finite dimensional distributions.

Moreover, its covariance is the same for every $q \geq 1$ and it coincides with the covariance of the d-parameter fractional Brownian motion, i.e.

$$\mathbf{E} Z^{q,d}_{\mathbf{H}}(\mathbf{t}) Z^{q,d}_{\mathbf{H}}(\mathbf{s}) = \prod_{j=1}^d \left(\frac{1}{2} \left(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right) \right) \qquad t_i, s_i \geq 0.$$

The Hermite sheet is Hölder continuous of order $\delta = (\delta_1, .., \delta_d)$ for every $\delta \in (0, \mathbf{H})$, see [7], [18].

In the rest of this work, we will denote by $L_{t,\mathbf{H},q}$ its kernel given by

$$L_{\mathbf{t},\mathbf{H},q}(\mathbf{y}_1, .., \mathbf{y}_q) = c(\mathbf{H}, q) \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \tag{6}$$

for every $\mathbf{y}_1, .., \mathbf{y}_q \in \mathbb{R}^d, \mathbf{t} \in \mathbb{R}^d_+$.

2.2 Multiple stochastic integrals and the Fourth Moment Theorem

Here, we shall only recall some elementary facts; our main reference is [11]. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle ., . \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$, for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}} \| \cdot \|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by:

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q} \left(\exp\left(-\frac{x^2}{2}\right)\right), \; x \in \mathbb{R}. \tag{7}$$

The isometry of multiple integrals can be written as: for $p, \; q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

$$\mathbf{E}\Big(I_p(f) I_q(g)\Big) = \begin{cases} q! \langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{otherwie.} \end{cases} \tag{8}$$

It also holds that:

$$I_q(f) = I_q(\tilde{f}),$$

where $\tilde{f}$ denotes the canonical symmetrization of f and it is defined by:

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

in which the sum runs over all permutations σ of $\{1, \ldots, q\}$.

In the particular case when $\mathcal{H} = L^2(T, \mathcal{B}(T), \mu)$, the rth contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$, which is defined by:

$$\begin{aligned} &(f \otimes_r g)(s_1, \ldots, s_{p-r}, t_1, \ldots, t_{q-r}) \\ &= \textstyle\int_{T^r} \mathrm{d}u_1 \ldots \mathrm{d}u_r f(s_1, \ldots, s_{p-r}, u_1, \ldots, u_r) g(t_1, \ldots, t_{q-r}, u_1, \ldots, u_r), \end{aligned} \tag{9}$$

for every $f \in L^2([0,T]^p)$, $g \in L^2([0,T]^q)$ and $r = 1, \ldots, p \wedge q$.

We will use the following famous result initially proven in [12] that characterizes the convergence in distribution of a sequence of multiple integrals torward the Gaussian law.

Theorem 1 *Fix $n \geq 2$ and let $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ (with $f_k \in \mathcal{H}^{\odot n}$ for every $k \geq 1$), be a sequence of square-integrable random variables in the nth Wiener chaos such that $\mathbf{E}\left[F_k^2\right] \to 1$ as $k \to \infty$. The following are equivalent:*

1. *the sequence $(F_k)_{k\geq 0}$ converges in distribution to the normal law $\mathcal{N}(0,1)$;*
2. *$\mathbf{E}\left[F_k^4\right] = 3$ as $k \to \infty$;*
3. *for all $1 \leq l \leq n-1$, it holds that $\lim_{k\to\infty} \|f_k \otimes_l f_k\|_{\mathcal{H}^{\otimes 2(n-l)}} = 0$;*

Other equivalent condition can be stated in term of the Malliavin derivatives of F_k, see [10].

3 The Rosenblatt case

Let us first consider the case $q = 2$. In this situation, the process $\left(Z_{\mathbf{H}}^{2,d}(\mathbf{t})\right)_{\mathbf{t}\geq 0}$ given by (3) is called the Rosenblatt sheet (or the d-parameter Rosenblatt process) and lives in the second Wiener chaos. It is given by

$$Z_{\mathbf{H}}^{2,d}(\mathbf{t}) = c(\mathbf{H},2)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dW(\mathbf{y}_1)dW(\mathbf{y}_2)\int_{[0,\mathbf{t}]} d\mathbf{s}(\mathbf{s}-\mathbf{y}_1)_+^{\frac{\mathbf{H}}{2}-1}(\mathbf{s}-\mathbf{y}_2)_+^{\frac{\mathbf{H}}{2}-1} \tag{10}$$

where $(W(\mathbf{y}))_{\mathbf{y}\in\mathbb{R}^d}$ a d -parameter Wiener sheet. The normalizing constant $c(\mathbf{H},2)$, which ensures that $Z_{\mathbf{H}}^2(\mathbf{1})$ has unit variance, is given by (see e.g. [18], Proposition 3.1)

$$c(\mathbf{H},2)^2 = \frac{\mathbf{H}(2\mathbf{H}-1)}{2\beta\left(\frac{\mathbf{H}}{2}, 1-\mathbf{H}\right)^2}. \tag{11}$$

where β is Beta function $\beta(p,q) = \int_0^1 z^{p-1}(1-z)^{q-1}dz, p, q > 0$ and with the notation

$$\beta(\mathbf{a},\mathbf{b}) = \prod_{i=1}^{d} \beta\left(a^{(i)}, b^{(i)}\right)$$

if $\mathbf{a} = (a^{(1)}, .., a^{(d)})$ and $\mathbf{b} = (b^{(1)}, .., b^{(d)})$. This constant $c(\mathbf{H},2)$ plays an important role in our calculations since it determines the asymptotic behavior with respect to the Hurst parameter.

In order to understand the limit behavior in distribution with respect to $\mathbf{H}$ of the process $\left(Z_{\mathbf{H}}^{2,d}(\mathbf{t})\right)_{\mathbf{t}\geq 0}$, it suffices to analyze the behavior of its cumulants. This is because the distributions of random variables in the second Wiener chaos are entirely determined by their cumulants. This is the reason why we prefer to present a separate proof in the case $q = 2$, although it can be obtained from the results stated later.

Let us denote by $k_m(F)$, $m \geq 1$ the mth cumulant of a random variable F. It is defined as

$$k_m(F) = (-i)^n \frac{\partial^n}{\partial t^n} \ln \mathbf{E}(e^{itF})|_{t=0},$$

if $F \in L^m(\Omega)$. When $G = I_2(f)$ is a multiple integral of order 2 with respect to a Wiener sheet $(B(y))_{y\in\mathbb{R}^d}$, then its cumulants can be computed as

$$k_m(G) = 2^{m-1}(m-1)! \int_{(\mathbb{R}^d)^m} d\mathbf{u}_1 \dots d\mathbf{u}_m f(\mathbf{u}_1, \mathbf{u}_2) f(\mathbf{u}_2, \mathbf{u}_3) \dots f(\mathbf{u}_{m-1}, \mathbf{u}_m) f(\mathbf{u}_m, \mathbf{u}_1). \tag{12}$$

Let us compute the cumulants of the Rosenblatt sheet. We need the following formula (see [18] Lemma 3.1): if $a \in (0, \frac{1}{2})$

$$\int_{\mathbb{R}} (u-y)_+^{a-1} (v-y)_+^{a-1} dy \quad = \quad \beta(a, 1-2a)|u-v|^{2a-1}. \tag{13}$$

Proposition 1 *Consider the Rosenblatt sheet $Z^{2,d}_{\mathbf{H}}$ given by (10). Let $N \geq 1$, $\lambda_1, .., \lambda_N \in \mathbb{R}$ and $\mathbf{t}_1, .., \mathbf{t}_N \in \mathbb{R}^d_+$. Deenote by k_m the mth cumulant. Then*

$$k_1\left(\sum_{i=1}^N \lambda_j Z^{2,d}_{\mathbf{H}}(\mathbf{t}_j)\right) = 0, \qquad k_2\left(\sum_{i=1}^N \lambda_j Z^{2,d}_{\mathbf{H}}(\mathbf{t}_j)\right) = \sum_{i,j=1}^N \lambda_i \lambda_j R_{\mathbf{H}}(\mathbf{t}_i, \mathbf{t}_j) \tag{14}$$

and for $m \geq 3$

$$\begin{aligned} k_m\left(\sum_{i=1}^N \lambda_j Z^{2,d}_{\mathbf{H}}(\mathbf{t}_j)\right) \quad &= \quad 2^{\frac{m}{2}-1}(m-1)!(\mathbf{H}(2\mathbf{H}-1))^{\frac{m}{2}} \sum_{i_1,..,i_m=1}^N \lambda_{i_1} .... \lambda_{i_m} \int_{[0,\mathbf{t}_{i_1}]} ... \int_{[0,\mathbf{t}_{i_m}]} \\ & d\mathbf{s}_1 ... d\mathbf{s}_m |\mathbf{s}_1 - \mathbf{s}_2|^{\mathbf{H}-1} ... |\mathbf{s}_m - \mathbf{s}_1|^{\mathbf{H}-1}. \end{aligned} \tag{15}$$

Proof: Clearly, since the first cumulant is the expectation and the second cumulant is the variance, we have

$$k_1\left(\sum_{i=1}^N \lambda_j Z^{2,d}_{\mathbf{H}}(\mathbf{t}_j)\right) = \mathbf{E}\left(\sum_{i=1}^N \lambda_j Z^{2,d}_{\mathbf{H}}(\mathbf{t}_j)\right) = 0$$

and

$$k_2\left(\sum_{i=1}^N \lambda_j Z^{2,d}_{\mathbf{H}}(\mathbf{t}_j)\right) = \mathbf{E}\left(\sum_{i=1}^N \lambda_j Z^{2,d}_{\mathbf{H}}(\mathbf{t}_j)\right)^2 = \sum_{i,j=1}^N \lambda_i \lambda_j R_{\mathbf{H}}(\mathbf{t}_i, \mathbf{t}_j).$$

Consider the kernel $L_{\mathbf{t},\mathbf{H},2}$ of the Rosenblatt sheet given by (6). We compute the cumulants of order $m \geq 3$ via the formula (12). Since

$$\sum_{i=1}^N \lambda_j Z^{2,d}_{\mathbf{H}}(\mathbf{t}_j) = I_2\left(\sum_{i=1}^N \lambda_j L_{\mathbf{t}_i,\mathbf{H},2}\right)$$

we will have

$$
\begin{aligned}
& k_m\left(\sum_{i=1}^{N}\lambda_j Z_{\mathbf{H}}^{2,d}(\mathbf{t}_j)\right) \\
= \; & 2^{m-1}(m-1)! \sum_{i_1,..,i_m=1}^{N} \lambda_{i_1}....\lambda_{i_m}\int_{(\mathbb{R}^d)^m} d\mathbf{y}_1..d\mathbf{y}_m \\
& \times L_{\mathbf{t}_{i_1},\mathbf{H},2}(\mathbf{y}_1,\mathbf{y}_2)L_{\mathbf{t}_{i_2},\mathbf{H},2}(\mathbf{y}_2,\mathbf{y}_3)...L_{\mathbf{t}_{i_{m-1}},\mathbf{H},2}(\mathbf{y}_{m-1},\mathbf{y}_m)L_{\mathbf{t}_{i_m},\mathbf{H},2}(\mathbf{y}_m,\mathbf{y}_1) \\
= \; & 2^{m-1}(m-1)!c(\mathbf{H},2)^m \sum_{i_1,..,i_m=1}^{N} \lambda_{i_1}....\lambda_{i_m}\int_{(\mathbb{R}^d)^m} d\mathbf{y}_1..d\mathbf{y}_m \\
& \int_{[0,\mathbf{t}_{i_1}]} d\mathbf{s}_1(\mathbf{s}_1-\mathbf{y}_1)_+^{\frac{\mathbf{H}}{2}-1}(\mathbf{s}_1-\mathbf{y}_2)_+^{\frac{\mathbf{H}}{2}-1}\ldots\times\int_{[0,\mathbf{t}_{i_m}]} d\mathbf{s}_m(\mathbf{s}_m-\mathbf{y}_m)_+^{\frac{\mathbf{H}}{2}-1}(\mathbf{s}_m-\mathbf{y}_1)_+^{\frac{\mathbf{H}}{2}-1}
\end{aligned}
$$

and by interchanging the order of integration and by using the integrals $d\mathbf{y}_i$, $i=1,..,m$ via (13) we obtain

$$
\begin{aligned}
k_m\left(\sum_{i=1}^{N}\lambda_j Z_{\mathbf{H}}^{2,d}(\mathbf{t}_j)\right) &= 2^{m-1}(m-1)!c(\mathbf{H},2)^m \sum_{i_1,..,i_m=1}^{N} \lambda_{i_1}....\lambda_{i_m}\beta\left(\frac{\mathbf{H}}{2},1-\mathbf{H}\right)^m \\
& \int_{[0,\mathbf{t}_{i_1}]}...\int_{[0,\mathbf{t}_{i_m}]} d\mathbf{s}_1...d\mathbf{s}_m|\mathbf{s}_1-\mathbf{s}_2|^{\mathbf{H}-1}...|\mathbf{s}_m-\mathbf{s}_1|^{\mathbf{H}-1} \\
&= 2^{\frac{m}{2}-1}(m-1)!(2\mathbf{H}(2\mathbf{H}-1))^{\frac{m}{2}} \sum_{i_1,..,i_m=1}^{N} \lambda_{i_1}....\lambda_{i_m} \\
& \times\int_{[0,\mathbf{t}_{i_1}]}...\int_{[0,\mathbf{t}_{i_m}]} d\mathbf{s}_1...d\mathbf{s}_m|\mathbf{s}_1-\mathbf{s}_2|^{\mathbf{H}-1}...|\mathbf{s}_m-\mathbf{s}_1|^{\mathbf{H}-1}
\end{aligned}
$$

and this is the right-hand side of (15). ■

Remark 1 *In the one-parameter case ($d=1$) the formulas (14), (15) are well known, see e.g. [14] or [9].*

From Proposition 1, we immediately get the cumulants of the Rosenblatt process multiplied by a deterministic function.

Remark 2 *Assume $Y(\mathbf{t}) = g_{\mathbf{t}}Z_{\mathbf{H}}^{2,d}(\mathbf{t})$ where $g_{\mathbf{t}}$ is deterministic and $Z_{\mathbf{H}}^{2,d}(\mathbf{t})$ is the Rosenblatt sheet (10). Then*

$$
k_1\left(\sum_{i=1}^{N}\lambda_j Y(\mathbf{t}_j)\right) = 0, \qquad k_2\left(\sum_{i=1}^{N}\lambda_j Y(\mathbf{t}_j)\right) = \sum_{i,j=1}^{N}\lambda_i\lambda_j g_{\mathbf{t}_i}g_{\mathbf{t}_j}R_{\mathbf{H}}(\mathbf{t}_i,\mathbf{t}_j)
$$

and for $m \geq 3$

$$k_m\left(\sum_{i=1}^N \lambda_j Y(\mathbf{t}_j)\right) = 2^{\frac{m}{2}-1}(m-1)!(\mathbf{H}(2\mathbf{H}-1))^{\frac{m}{2}} \sum_{i_1,..,i_m=1}^N \lambda_{i_1}....\lambda_{i_m} g_{\mathbf{t}_{i_1}}...g_{\mathbf{t}_{i_m}} \int_{[0,\mathbf{t}_{i_1}]}...\int_{[0,\mathbf{t}_{i_m}]} d\mathbf{s}_1...d\mathbf{s}_m|\mathbf{s}_1-\mathbf{s}_2|^{\mathbf{H}-1}...|\mathbf{s}_m-\mathbf{s}_1|^{\mathbf{H}-1}.$$

This follows from (14) and (15) since $Y(\mathbf{t}) = I_2(g_{\mathbf{t}} L_{\mathbf{t},\mathbf{H},2})$ *with* $L_{\mathbf{t},\mathbf{H},2}$ *from (6).*

We now deduce the asymptotic behavior of the Rosenblatt sheet $Z^{2,d}_{\mathbf{H}}$ with respect to its Hurst parameter via the analysis of its cumulants. We have the following result.

Theorem 2 *Let* $\left(Z^{2,d}_{\mathbf{H}}(\mathbf{t})\right)_{\mathbf{t}\geq 0}$ *be given by (10). Consider* $A_k = \{j_1,..,j_k\} \subset \{1,..,d\}$ *such that* $1 \leq k < d$. *Let* $\overline{A}_k = \{1,..,d\} \setminus A_k$.

We introduce the following notation:

$$\mathbf{H}_{A_k} = (H_{j_1},..,H_{j_k}),\ \ \mathbf{t}_{A_k} = (t^{(j_1)},...,t^{(j_k)}) \text{ and } \langle\mathbf{t}\rangle_{A_k} = t^{(j_1)}...t^{(j_k)} \tag{16}$$

and

1. *Assume* $\mathbf{H}_{A_k} \to \left(\frac{1}{2},...,\frac{1}{2}\right) \in \mathbb{R}^k$. *Assume that the parameters* $H_j, j \in \overline{A}_k$ *are fixed. Then the process* $Z^{2,d}_{\mathbf{H}}$ *converges weakly in* $C([0,T]^d)$ *to a d-parameter centered Gaussian process* $(X(\mathbf{t}))_{\mathbf{t}\geq 0}$ *with covariance*
$$\mathbf{E}X_{\mathbf{t}}X_{\mathbf{s}} = \left(\prod_{a\in A_k}\left(t^{(a)} \wedge s^{(a)}\right)\right)\left(\prod_{b\in\overline{A}_k} R_{H_b}(t^{(b)}, s^{(b)})\right). \tag{17}$$

2. *Assume* $\mathbf{H}_{A_k} \to (1,1,..,1) \in \mathbb{R}^k$. *Assume that the parameters* $H_j, j \in \overline{A}_k$ *are fixed. Then the process* $Z^{2,d}_{\mathbf{H}}$ *converges weakly in* $(C[0,T]^d)$ *to the d-parameter stochastic process* $(X_{\mathbf{t}})_{\mathbf{t}\geq 0}$ *defined by*
$$X_{\mathbf{t}} = \langle\mathbf{t}\rangle_{A_k} Z^{2,d-k}_{\mathbf{H}}(\mathbf{t}_{\overline{A}_k}) \tag{18}$$
where $\left(Z^{2,d-k}_{\mathbf{H}}(t_{\overline{A}_k})\right)_{\mathbf{t}_{\overline{A}_k}\in\mathbb{R}^{d-k}_+}$ *is a* $(d-k)$*-parameter Rosenblatt process.*

3. *Assume* $\mathbf{H} = (H_1,..,H_d) \to (1,..,1) \in \mathbb{R}^d$. *Then the process* $Z^{2,d}_{\mathbf{H}}$ *converges weakly in* $C([0,T]^d)$ *to the d-parameter stochastic process* $(X_{\mathbf{t}})_{\mathbf{t}\geq 0}$ *defined by*
$$X_{\mathbf{t}} = \langle\mathbf{t}\rangle_d \frac{1}{\sqrt{2}}(Z^2-1) \tag{19}$$
where $Z \sim N(0,1)$ *and* $\langle\mathbf{t}\rangle_d$ *is defined by (2).*

4. *Denote by* $B_p = \{l_1, .., l_p\} \subset \{1, .., d\}$ *with* $p \geq 1$ *and* $p + k < d$. *Suppose* $B_p \cap A_k = \emptyset$. *Assume* $\mathbf{H}_{A_k} \to \left(\frac{1}{2}, .., \frac{1}{2}\right) \in \mathbb{R}^k$ *and* $\mathbf{H}_{B_p} \to (1, .., 1) \in \mathbb{R}^p$. *Assume that the* H_j *with* $j \in \{1, 2, .., d\} \setminus (A_k \cup B_p)$ *are fixed. Then the process* $Z^{2,d}_{\mathbf{H}}$ *converges weakly in* $C([0,T]^d)$ *to a* d*-parameter Gaussian process* $(X(\mathbf{t}))_{\mathbf{t}\geq 0}$ *with covariance*

$$\mathbf{E}X_{\mathbf{t}}X_{\mathbf{s}} = \left(\prod_{a\in A_k}(t^{(a)} \wedge s^{(a)})\right)\left(\prod_{b\in B_p} t^{(b)}s^{(b)}\right)\left(\prod_{c\in \overline{A_k\cup B_p}} R_{H_c}(t^{(c)}, s^{(c)})\right). \qquad (20)$$

Proof: We first prove the convergence of finite dimensional distributions for points 1. -4. and then we prove the tightness.

We start with the proof of point 1.. From (15), combined with Lemma 3.3 and Corollary 3.1 in [2], the cumulants of order bigger than of equal to 3 of the finite dimensional distributions of $Z^{2,d}_{\mathbf{H}}$ converge to zero if $2\mathbf{H} - 1 = \prod_{i=1}^d (2H_i - 1) \to 0$, i.e. if there exists $i \in 1, .., d$ such that $H_i \to 1/2$. This means that $Z^{2,d}_{\mathbf{H}}$ converges in the sense of finite dimensional distributions to a Gaussian process.

Let $\lambda_j \in \mathbb{R}, \mathbf{t}_j \in \mathbb{R}^d_+$ for $j = 1, ..., N$. From formula (14), we notice that first cumulant of $\sum_{i=1}^N \lambda_j Z^{2,d}_{\mathbf{H}}(\mathbf{t}_j)$ is zero while the second cumulant

$$\begin{aligned} k_2\left(\sum_{i=1}^N \lambda_j Z^2_{\mathbf{H}}(\mathbf{t}_j)\right) &= \sum_{i,j=1}^N \lambda_i\lambda_j R_{\mathbf{H}}(\mathbf{t}_i, \mathbf{t}_j) \\ &= \sum_{i,j=1}^N \lambda_i\lambda_j \left(\prod_{a\in A_k} R_{H_a}(t_i^{(a)}, t_j^{(a)})\right)\left(\prod_{b\in \overline{A}_k} R_{H_b}(t_i^{(b)}, t_j^{(b)})\right) \end{aligned}$$

tends, as $\mathbf{H}_{A_k} \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^k$ to

$$\sum_{i,j=1}^N \lambda_i\lambda_j \left(\prod_{a\in A_k}(t_i^{(a)} \wedge t_j^{(a)})\right)\left(\prod_{b\in \overline{A}_k} R_{H_b}(t_i^{(b)}, t_j^{(b)})\right)$$

which represents the second cumulant (or the variance) of $\sum_{i=1}^N \lambda_j X(\mathbf{t}_j)$ where X is the d-parameter centered Gaussian process with covariance (17).

Concerning the second point, we notice from (15) that for $m \geq 3$

$$\begin{aligned} & k_m\left(\sum_{i=1}^N \lambda_j Z^2_{\mathbf{H}}(\mathbf{t}_j)\right) \\ = \; & 2^{\frac{m}{2}-1}(m-1)! \sum_{i_1,..,i_m=1}^N \lambda_{i_1}....\lambda_{i_m} \end{aligned}$$

$$\prod_{a\in A_k}\left((H_a(2H_a-1))^{\frac{m}{2}}\int_{[0,t_{i_1}^{(a)}]}ds_1^{(a)}....\int_{[0,t_{im}^{(a)}]}ds_m^{(a)}|s_1^{(a)}-s_2^{(a)}|^{H_a-1}.....|s_m^{(a)}-s_1^{(a)}|^{H_a-1}\right)$$

$$\times\prod_{b\in\overline{A}_k}\left((H_b(2H_b-1))^{\frac{m}{2}}\int_{[0,t_{i_1}^{(b)}]}ds_1^{(b)}ds_1^{(a)}....\int_{[0,t_{im}^{(b)}]}ds_m^{(b)}ds_m^{(b)}|s_1^{(b)}-s_2^{(b)}|^{H_b-1}.....|s_m^{(b)}-s_1^{(b)}|^{H_b-1}\right).$$

Since if $\mathbf{H}_{A_k}\to(1,...,1)\in\mathbb{R}^k$, we have

$$\mathbf{H}_{A_k}(2\mathbf{H}_{A_k}-1)=\prod_{a\in A_k}(H_a(2H_a-1))^{\frac{m}{2}}\to 1$$

and

$$\prod_{a\in A_k}\left(\int_{[0,t_{i_1}^{(a)}]}ds_1^{(a)}ds_1^{(a)}....\int_{[0,t_{im}^{(a)}]}ds_m^{(a)}ds_m^{(a)}|s_1^{(a)}-s_2^{(a)}|^{H_a-1}.....|s_m^{(a)}-s_1^{(a)}|^{H_a-1}\right)\to\prod_{a\in A_k}\left(t_{i_1}^{(a)}...t_{i_m}^{(a)}\right)$$

we deduce that

$$\begin{aligned}
&k_m\left(\sum_{i=1}^N\lambda_j Z_{\mathbf{H}}^{2,d}(\mathbf{t}_j)\right)\\
\to\quad &2^{\frac{m}{2}-1}(m-1)!\sum_{i_1,..,i_m=1}^N\lambda_{i_1}....\lambda_{i_m}\left(\prod_{a\in A_k}t_{i_1}^{(a)}...t_{i_m}^{(a)}\right)\\
&\prod_{b\in\overline{A}_k}\left((H_b(2H_b-1))^{\frac{m}{2}}\int_{[0,t_{i_1}^{(b)}]}ds_1^{(b)}ds_1^{(a)}....\int_{[0,t_{im}^{(b)}]}ds_m^{(b)}ds_m^{(b)}|s_1^{(b)}-s_2^{(b)}|^{H_b-1}.....|s_m^{(b)}-s_1^{(b)}|^{H_b-1}\right)
\end{aligned}$$

which constitutes the mth cumulant of the finite dimensional distributions of the process (18), see Remark 2. The analysis of the first two cumulants k_1,k_2 is immediate.

Concerning 3., from (15) and following the proof of point 2., we see that the first cumulant is always zero and for $m\geq 2$, as $\mathbf{H}\to(1,..,1)\in\mathbb{R}^d$

$$k_m\left(\sum_{i=1}^N\lambda_j Z_{\mathbf{H}}^{2,d}(\mathbf{t}_j)\right)\quad\to\quad 2^{\frac{m}{2}-1}(m-1)!\sum_{i_1,..,i_m=1}^N\lambda_{i_1}....\lambda_{i_m}\mathbf{t}_{i_1}...\mathbf{t}_{i_m}.$$

On the other hand, the cumulants of the linear combinations of $X_{\mathbf{t}}=\langle\mathbf{t}\rangle_d\frac{1}{\sqrt{2}}(Z^2-1)=I_2\left(\frac{1}{\sqrt{2}}\langle\mathbf{t}\rangle_d 1_{[0,1]}^{\otimes 2}\right)$ are

$$k_m\left(\sum_{i=1}^N\lambda_j X_{\mathbf{t}_j}\right)\quad=\quad 2^{m-1}(m-1)!2^{-\frac{m}{2}}\sum_{i_1,..,i_m=1}^N\lambda_{i_1}....\lambda_{i_m}\langle\mathbf{t}_{i_1}\rangle...\langle\mathbf{t}_{i_m}\rangle$$

$$= 2^{\frac{m}{2}-1}(m-1)! \sum_{i_1,..,i_m=1}^{N} \lambda_{i_1}....\lambda_{i_m} \mathbf{t}_{i_1}...\mathbf{t}_{i_m}.$$

Point 4. is a slightly modification of point 1. From (15) we see that the cumulant of order bigger than or equal to 3 converges to zero if at least one of the Hurst parameters H_i, $i = 1, 2.., d$ converges to 1/2. It then suffices to look at the limit of (14) in order to conclude.

The tightness follows from the relation (see [7])

$$\mathbf{E}\left|\Delta Z^2_{\mathbf{H}}([\mathbf{s},\mathbf{t}])\right|^p = \mathbf{E}\,|Z|^p\left(|t_1 - s_1|\cdots|t_d - s_d|\right)^{p\mathbf{H}} \tag{21}$$

(recall that Δ the higher order increment defined by (5)) and the criterion stated in Theorem 4 in [4] by using the fact that the process $Z^{2,d}_{\mathbf{H}}(\mathbf{t})$ is almost surely equal to 0 when $t_i = 0$ (here $\mathbf{t} = (t_1, .., t_d)$). ■

Let us discuss some particular cases.

Remark 3 *If $H_i \to \frac{1}{2}$ for every $i = 1, .., d$, then $Z^{2,d}_{\mathbf{H}}$ converges weakly in $C([0,T]^d)$ to a d-parameter Brownian motion.*

If $H_1 \to \frac{1}{2}$ and $H_2, .., H_d$ are fixed in $(\frac{1}{2}, 1)$, then $Z^{2,d}_{\mathbf{H}}$ converges weakly in $C([0,T]^d)$ to a d-parameter Gaussian process $(X(\mathbf{t}))_{\mathbf{t}\geq 0}$ with covariance

$$\mathbf{E}X_{\mathbf{t}}X_{\mathbf{s}} = (t_1 \wedge s_1)\prod_{j=2}^{d} R_{H_j}(t_j, s_j).$$

If $H_1 \to 1$ and the other indices are fixed, then the Rosenblatt sheet (10) converges to the process

$$X_{(t^{(1)},..,t^{(d)})} = tZ^{2,d-1}_{\mathbf{H}}(t^{(2)}, ..., t^{(d)}).$$

4 The behavior of the Hermite sheet of arbitrary order

Assume $q \geq 2$, $\mathbf{H} = (H_1, .., H_d) \in \left(\frac{1}{2}, 1\right)^d$ and let $Z^{q,d}_{\mathbf{H}}$ be a Hermite sheet given by (3), We will analyze the behavior of $Z^{q,d}_{\mathbf{H}}$ where the Hurst parameters (or some of them) converge to $\frac{1}{2}$ or to 1.

From the result obtained in the previous section in Theorem 2. it would be natural to expect a central limit theorem when at least one of the parameters H_i,$i = 1, .., d$ converges to one half and a non-central limit convergence when at least one parameter converges to 1 (and none to $\frac{1}{2}$). This will be indeed the case. therefore, we separate this section into two parts: in the first we study the convergence for the Hurst index in the vicinity of $\frac{1}{2}$ and in the second part we regard the behavior when the Hurst index is close to 1.

4.1 Convergence when at least one Hurst parameter converges to $1/2$

As mentioned before, given Theorem 2, we would expect the convergence to a Gaussian limit if at least one H_i goes to one half. In this situation, since we deal with sequences of multiple Wiener-Itô integrals, an excellent tool to prove the convergence is the famous Fourth Moment Theorem (recalled in Theorem 1) by [12].

Recall that the constant from (3) $c(\mathbf{H}, q)$ is given by .

$$c(\mathbf{H}, q)^2 = \frac{\mathbf{H}(2\mathbf{H}-1)}{q!\beta\left(\frac{1}{2} - \frac{1-\mathbf{H}}{q}, \frac{2-2\mathbf{H}}{q}\right)^q}. \tag{22}$$

Let us state the main result from this section. The limit will be the same as in Theorem 2 but the proof is different.

Theorem 3 *Let $\left(Z^{q,d}_{\mathbf{H}}(\mathbf{t})\right)_{\mathbf{t}\geq 0}$ be given by (3). Let the notation from Theorem 2 prevail.*

1. *Assume $\mathbf{H}_{A_k} \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^k$. Assume that the parameters $H_j, j \in \overline{A}_k$ are fixed. Then the process $Z^{q,d}_{\mathbf{H}}$ converges weakly in $C([0,T]^d)$ to a d-parameter centered Gaussian process $(X(\mathbf{t}))_{\mathbf{t}\geq 0}$ with covariance (17).*

2. *Assume $\mathbf{H}_{A_k} \to \left(\frac{1}{2}, .., \frac{1}{2}\right) \in \mathbb{R}^k$ and $\mathbf{H}_{B_p} \to (1, .., 1) \in \mathbb{R}^p$. Assume that the H_j with $j \in \{1, 2, .., d\} \setminus (A_k \cup B_p)$ are fixed. Then the process $Z^{q,d}_{\mathbf{H}}$ converges weakly in $C([0,T]^d)$ to a d-parameter Gaussian process $(X(\mathbf{t}))_{\mathbf{t}\geq 0}$ with covariance (20).*

Proof: We will apply the Fourth Moment Theorem by proving that point 3. in Theorem 1) is satisfied. We need to calculate

$$L_{\mathbf{t},\mathbf{H},q} \otimes_r L_{\mathbf{t},\mathbf{H},q}$$

for every $r = 1, 2, .., q-1$ with $L_{\mathbf{t},\mathbf{H},q}$ from (6).

Note that $L_{t,\mathbf{H},q}$ is a symmetric function in $(\mathbb{R}^d)^q$. For every $\mathbf{y}_1, .., \mathbf{y}_{2q-2r} \in \mathbb{R}^d$

$$\begin{aligned}
&\left(L_{t,\mathbf{H},q} \otimes_r L_{t,\mathbf{H},q}\right)(\mathbf{y}_1, .., \mathbf{y}_{2q-2r}) \\
&= \int_{(\mathbb{R}^d)^r} L_{t,\mathbf{H},q}(\mathbf{y}_1, .., \mathbf{y}_{q-r}, \mathbf{u}_1, ..., \mathbf{u}_r) L_{t,\mathbf{H},q}(\mathbf{y}_{q-r+1}, .., \mathbf{y}_{2q-2r}, \mathbf{u}_1, ..., \mathbf{u}_r) d\mathbf{u}_1 ... d\mathbf{u}_r \\
&= c(\mathbf{H}, q)^2 \int_{(\mathbb{R}^d)^r} d\mathbf{u}_1 ... d\mathbf{u}_r \\
&\quad \int_0^{\mathbf{t}} \left(\prod_{j=1}^{q-r} (\mathbf{u} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right) \left(\prod_{j=1}^{r} (\mathbf{u} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right) d\mathbf{u} \\
&\quad \times \int_0^{\mathbf{t}} \left(\prod_{j=q-r+1}^{2q-2r} (\mathbf{v} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right) \left(\prod_{j=1}^{r} (\mathbf{v} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right) d\mathbf{v}
\end{aligned}$$

and by calculating the integrals $d\mathbf{u}_i$ through (13),

$$
\begin{aligned}
&\left(L_{t,\mathbf{H},q} \otimes_r L_{t,\mathbf{H},q}\right)(\mathbf{y}_1, .., \mathbf{y}_{2q-2r}) \\
= \;& c(\mathbf{H},q)^2 \beta\left(\frac{1}{2} - \frac{1-\mathbf{H}}{q}, \frac{2-2\mathbf{H}}{q}\right)^r \int_0^{\mathbf{t}}\int_0^{\mathbf{t}} d\mathbf{u}d\mathbf{v}|\mathbf{u}-\mathbf{v}|^{\frac{2(H-1)r}{q}} \\
& \left(\prod_{j=1}^{q-r} (\mathbf{u}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right)\left(\prod_{j=q-r+1}^{2q-2r} (\mathbf{v}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|L_{\mathbf{t},\mathbf{H},q} \otimes_r L_{\mathbf{t},\mathbf{H},q}\|^2_{L^2(\mathbb{R}^{d(2q-2r)})} \\
= \;& c(\mathbf{H},q)^4 \beta\left(\frac{1}{2} - \frac{1-\mathbf{H}}{q}, \frac{2-2\mathbf{H}}{q}\right)^{2r} \int_{(\mathbb{R}^d)^{2q-2r}} d\mathbf{y}_1 ... d\mathbf{y}_{2q-2r} \\
& \int_0^{\mathbf{t}}\int_0^{\mathbf{t}} d\mathbf{u}d\mathbf{v}\left(\prod_{j=1}^{q-r} (\mathbf{u}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right)\left(\prod_{j=q-r+1}^{2q-2r} (\mathbf{v}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}|\mathbf{u}-\mathbf{v}|^{\frac{2(\mathbf{H}-1)r}{q}}\right) \\
& \times \int_0^{\mathbf{t}}\int_0^{\mathbf{t}} d\mathbf{u}'d\mathbf{v}'\left(\prod_{j=1}^{q-r} (\mathbf{u}'-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right)\left(\prod_{j=q-r+1}^{2q-2r} (\mathbf{v}'-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}|\mathbf{u}'-\mathbf{v}'|^{\frac{2(\mathbf{H}-1)r}{q}}\right)
\end{aligned}
$$

and by Fubini and the identity (13) and (22)

$$
\begin{aligned}
&\|L_{\mathbf{t},\mathbf{H},q} \otimes_r L_{\mathbf{t},\mathbf{H},q}\|^2_{L^2(\mathbb{R}^{d(2q-2r)})} \\
= \;& c(\mathbf{H},q)^4 \beta\left(\frac{\mathbf{H}}{2}, 1-\mathbf{H}\right)^{2r} \beta\left(\frac{\mathbf{H}}{2}, 1-\mathbf{H}\right)^{2q-2r} \\
& \int_0^{\mathbf{t}}\int_0^{\mathbf{t}}\int_0^{\mathbf{t}}\int_0^{\mathbf{t}} d\mathbf{u}d\mathbf{v}d\mathbf{u}'d\mathbf{v}'|\mathbf{u}-\mathbf{v}|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}}|\mathbf{u}'-\mathbf{v}'|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}}|\mathbf{u}-\mathbf{u}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}}|\mathbf{v}-\mathbf{v}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}} \\
= \;& \frac{1}{q!^2}(\mathbf{H}(2\mathbf{H}-1))^2 \int_0^{\mathbf{t}}\int_0^{\mathbf{t}}\int_0^{\mathbf{t}}\int_0^{\mathbf{t}} d\mathbf{u}d\mathbf{v}d\mathbf{u}'d\mathbf{v}' \\
& \times|\mathbf{u}-\mathbf{v}|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}}|\mathbf{u}'-\mathbf{v}'|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}}|\mathbf{u}-\mathbf{u}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}}|\mathbf{v}-\mathbf{v}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}}.
\end{aligned}
$$

Going now to the finite dimensional distributions, notice that for every $\lambda_1, .., \lambda_N \in \mathbb{R}$ and $\mathbf{t}_1, .., \mathbf{t}_N \in \mathbb{R}^d_+$ we have

$$
\sum_{j=1}^N \lambda_j Z^q_{\mathbf{H}}(\mathbf{t}_j) = I_q\left(\sum_{j=1}^N \lambda_j L_{\mathbf{t}_j,\mathbf{H},q}\right)
$$

and we can similarly show that for every $r = 1, .., q-1$

$$\left(\sum_{j=1}^{N}\lambda_j L_{\mathbf{t}_j,\mathbf{H},q}\right)\otimes_r\left(\sum_{j=1}^{N}\lambda_j L_{\mathbf{t}_j,\mathbf{H},q}\right)=\sum_{k,j=1}^{N}\lambda_k\lambda_j\left(L_{\mathbf{t}_k,\mathbf{H},q}\right)\otimes_r\left(L_{\mathbf{t}_j,\mathbf{H},q}\right)$$

$$\begin{aligned}=&\sum_{k,j=1}^{N}\lambda_k\lambda_j c(\mathbf{H},q)^2\beta\left(\frac{1}{2}-\frac{1-\mathbf{H}}{q},\frac{2-2\mathbf{H}}{q}\right)^r\int_0^{\mathbf{t}}\int_0^{\mathbf{t}}d\mathbf{u}d\mathbf{v}|\mathbf{u}-\mathbf{v}|^{\frac{2(\mathbf{H}-1)r}{q}}\\&\left(\prod_{j=1}^{q-r}(\mathbf{u}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right)\left(\prod_{j=q-r+1}^{2q-2r}(\mathbf{v}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}\right).\end{aligned}$$

The above relation implies

$$\begin{aligned}&\left\|\left(\sum_{j=1}^{N}\lambda_j L_{\mathbf{t}_j,\mathbf{H},q}\right)\otimes_r\left(\sum_{j=1}^{N}\lambda_j L_{\mathbf{t}_j,\mathbf{H},q}\right)\right\|^2_{L^2(\mathbb{R}^{d(2q-2r)})}\\=&\frac{1}{q!^2}(\mathbf{H}(2\mathbf{H}-1))^2\sum_{j,k=1}^{N}\lambda_k\lambda_j\int_0^{\mathbf{t}_j}\int_0^{\mathbf{t}_k}\int_0^{\mathbf{t}_j}\int_0^{\mathbf{t}_k}d\mathbf{u}d\mathbf{v}d\mathbf{u}'d\mathbf{v}'\\&\times|\mathbf{u}-\mathbf{v}|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}}|\mathbf{u}'-\mathbf{v}'|^{\frac{2(\mathbf{H}-1)\mathbf{r}}{q}}|\mathbf{u}-\mathbf{u}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}}|\mathbf{v}-\mathbf{v}'|^{\frac{2(\mathbf{H}-1)(\mathbf{q}-\mathbf{r})}{q}}.\end{aligned}$$

Thus, due to Lemma 3.3 and Corollary 3.1 in [2] (which shows that the above integral $d\mathbf{u}d\mathbf{v}d\mathbf{u}'d\mathbf{v}'$ is finite), the quantity

$$\left\|\left(\sum_{j=1}^{N}\lambda_j L_{\mathbf{t}_j,\mathbf{H},q}\right)\otimes_r\left(\sum_{j=1}^{N}\lambda_j L_{\mathbf{t}_j,\mathbf{H},q}\right)\right\|^2_{L^2(\mathbb{R}^{d(2q-2r)})}$$

converges to zero for every $r=1,..,q-1$ and this implies that the random variable $\sum_{j=1}^{N}\lambda_j Z^{q,d}_{\mathbf{H}}(\mathbf{t}_j)$ converges in distribution to a centered Gaussian random variable with variance equal to

$$\begin{aligned}\lim_{(H_{j_1},..,H_{j_k})\to\left(\frac{1}{2},..,\frac{1}{2}\right)}\mathbf{E}\left(\sum_{j=1}^{N}\lambda_j Z^q_{\mathbf{H}}(\mathbf{t}_j)\right)^2&=\sum_{j,k=1}^{N}\lambda_j\lambda_k\prod_{a\in A_k}(t_j^{(a)}\wedge t_k^{(a)})\prod_{b\in\overline{A}_k}R_{H_b}(t_h^{(b)},t_k^{(b)})\\&=\mathbf{E}\left(\sum_{j=1}^{N}\lambda_j X(\mathbf{t}_j)\right)^2\end{aligned}$$

where X is the Gaussian field with covariance (17). The second point of the conclusion follows the same lines. ■

Remark 4 *In particular, if $H_i \to \frac{1}{2}$ for every $i = 1,..,d$, then $Z^{q,d}_{\mathbf{H}}$ converges weakly in $C([0,T]^d)$ to a Brownian sheet. We retrieve a result in [3].*

Obviously, the proof of Theorem 3 holds in particular for $q = 2$ by providing a different proof to Theorem 2.

4.2 Convergence when at least one Hurst parameter converges to 1

We now analyze the limit behavior of $Z^{q,d}_{\mathbf{H}}$ with $q \geq 3$ integer, when at least one of the Hurst indices $H_i, i = 1,..,d$ converges to 1 and none of them converges to zero. We already gained some intuition from the study of the case $q = 2$. We showed in Theorem 2 that, when k of the Hurst indices $H_1,..,H_d$ tend to 1 ($1 \leq k < d$), the weak limit of the d-parameter Rosenblatt process is a $(d-k)$-parameter Rosenblatt process multiplied by a deterministic function of k variables. When all the $H_i, i = 1,..,d$ converge to 1, then the weak limit of $Z^{2,d}_{\mathbf{H}}$ is $\langle \mathbf{t} \rangle_d \frac{1}{\sqrt{2}} H_2(Z)$ where H_2 is the Hermite polynomial of order 2 and Z is a standard normal random variable.

A similar phenomenon will happen when $q \geq 3$. Since we deal with elements of the Wiener chaos of order 3 of higher, we cannot use anymore, as in the proof of Theorem 2, the argument based on cumulants. Instead, we will use a proof based on the non-central limit theorem (see [6] or the monographs [10] and [18]).

We separate again the proof into two cases: $d = 1$ and $d \geq 2$.

4.2.1 Behavior of the Hermite sheet: case $d = 1$, $q > 2$.

In order to make our approach easier and to avoid the use of complicated vectorial notation, let us start with the case $d = 1$. Consider the Hermite process $(Z^{q,1}_H(t))_{t\geq 0} := (Z^q_H(t))_{t\geq 0}$ defined by (1). We will recall the following key argument for our proof: if H_q is the qth Hermite polynomial (7), the sequence

$$Z^q_{H,N}(t) := \frac{1}{N^H} \sum_{k=0}^{[Nt]-1} H_q \left(B^{H'}_{k+1} - B^{H'}_k \right)$$

converges in the sense of finite dimensional distributions to the Hermite process $d(H,q)(Z^q_H(t))_{t\geq 0}$. Above $B^{H'}$ is a fBm with index $H' \in (1 - \frac{1}{2q}, 1)$ and (see relation (2.11) in [13])

$$d(H,q)^2 = q! \frac{(H'(2H'-1))^q}{H(2H-1)} \text{ and } H' = 1 - \frac{1-H}{q}. \tag{23}$$

This is the well known non-central limit theorem (see [6], [8], [17], [10]).

The idea of the proof is simple: write formally

$$\lim_{H\to 1} Z^q_H(t) = \lim_{H\to 1} \lim_{N\to\infty} d(H,q)^{-1} Z^q_{H,N}(t) \tag{24}$$

and suppose that we can switch the two limits. Then

$$\lim_{H\to 1} Z^q_H(t) = \lim_{N\to\infty} \lim_{H\to 1} d(H,q)^{-1} Z^q_{H,N}(t)$$

Since obviosly the process B^H converges as $H \to 1$ in the sense of finite dimensional distributions to tZ, where Z is a standard normal random variable, we have for every $N \geq 1$

$$\lim_{H\to 1} Z^q_{H,N}(t) = \frac{1}{\sqrt{q!}}\frac{1}{N}\sum_{k=0}^{[Nt]-1} H_q(Z) = \frac{1}{\sqrt{q!}}\frac{[Nt]}{N}H_q(Z) \to_N t\frac{1}{\sqrt{q!}}H_q(Z).$$

Below, we will make this heuristics rigourous. The main difficulty is to interchange the two limits in (24) and to do this, we will need some uniform convergence (in the sense of characteristic functions, see Proposition 2 below) of $Z^q_{H,N}$ to Z^q_H.

For $\lambda_1,..,\lambda_p \in \mathbb{R}$ and $t_1,..,t_p \geq 0$, we let

$$V_{N,H,p} = \sum_{j=1}^p \lambda_j d(H,q)^{-1} Z^q_{H,N}(t_j) \text{ and } V_{H,p} = \sum_{j=1}^p \lambda_j Z^q_H(t_j). \tag{25}$$

Also, let for every $\alpha \in \mathbb{R}$

$$g_{N,\alpha}(H) = \mathbf{E}\left(e^{i\alpha V_{N,H,p}}\right) \text{ and } g_\alpha(H) = \mathbf{E}\left(e^{i\alpha V_{H,p}}\right) \tag{26}$$

the characteristic functions of the random variables $V_{N,H,p}$ and $V_{H,p}$ respectively. We show below the uniform convergence with respect to H of the characteristic function of $V_{N,H,p}$ to the characteristic function of $V_{H,p}$.

Proposition 2 *For every $0 < \varepsilon < \frac{1}{2q}$, we have for every $\alpha \in \mathbb{R}$*

$$\sup_{H\in[\frac{1}{2}+\varepsilon,1)} |g_{N,\alpha}(H) - g_\alpha(H)| = \sup_{H'\in[1-\frac{1}{2q}+\varepsilon,1)} |g_{N,\alpha}(H) - g_\alpha(H)| \to_N 0.$$

Proof: Since $|e^{i\alpha x} - e^{i\alpha y}| \leq |\alpha||x-y|$ for every $x,y,\alpha \in \mathbb{R}$, we have

$$|g_{N,\alpha}(H) - g_\alpha(H)| \leq |\alpha|\mathbf{E}\,|V_{N,H,p} - V_{H,p}| \leq |\alpha|\left(\mathbf{E}\,|V_{N,H,p} - V_{H,p}|^2\right)^{\frac{1}{2}}.$$

Thus, it we need to prove

$$\sup_{H'\in[1-\frac{1}{2q}+\varepsilon,1)} \mathbf{E}\,|V_{N,H,p} - V_{H,p}|^2 \to_{N\to\infty} 0.$$

It suffices to show that for every fixed $t \geq 0$

$$\sup_{H'\in[1-\frac{1}{2q}+\varepsilon,1)} \mathbf{E}\left|d(H,q)^{-1}Z^q_{H,N}(t) - Z^q_H(t)\right|^2 \to_{N\to\infty} 0$$

with $d(H,q)$ from (23).

Now, from the proof of Proposition 3.1 in [5], we have the following estimate for the mean square of the difference $Z^q_{H,N}(t) - Z^q_H(t)$

$$
\begin{aligned}
& \mathbf{E}\left| Z^q_{H,N}(t) - d(H, Z^q_H(t) \right|^2 \\
= \;& d(H,q)^{-2}(H'(2H'-1))^q N^{2q-2-2qH'} \sum_{k,l=0}^{N-1} \left[\left(\int_0^1 du \int_0^1 dv |k-l+u-v|^{2H'-2} \right)^q \right. \\
& \left. -2 \int_0^1 dv \left(\int_0^1 du |k-l+u-v|^{2H'-2} \right)^q + \int_0^1 du \int_0^1 dv |k-l+u-v|^{q(2H'-2)} \right] \\
\leq \;& d(H,q)^{-2}(H'(2H'-1))^q N^{2q-1-2qH'} \sum_{r\in\mathbb{Z}} \left[\left(\int_0^1 du \int_0^1 dv |r+u-v|^{2H'-2} \right)^q \right. \\
& \left. -2 \int_0^1 dv \left(\int_0^1 du |r+u-v|^{2H'-2} \right)^q + \int_0^1 du \int_0^1 dv |r+u-v|^{q(2H'-2)} \right] := A(H')
\end{aligned}
$$

Notice that

$$
\sup_{H' \in [1-\frac{1}{2q}+\varepsilon, 1)} A(H') = \sup_{H' \in [1-\frac{1}{2q}+\varepsilon, 1]} A(H')
$$

because $A(1) = 0$. On the other hand, A is clearly continuous on $[1-\frac{1}{2q}+\varepsilon, 1]$ so it is bounded above and there exists $H_0 \in [1-\frac{1}{2q}+\varepsilon, 1]$ such that

$$
\sup_{H' \in [1-\frac{1}{2q}+\varepsilon, 1)} A(H') = A(H_0).
$$

We have $H_0 \in [1-\frac{1}{2q}+\varepsilon, 1)$ because $A(1) = 0$. Consequently,

$$
\sup_{H \in [1-\frac{1}{2q}+\varepsilon, 1)} \mathbf{E}\left| d(H,q)^{-1} Z^q_{H,N}(t) - Z^q_H(t) \right|^2 = A(H_0) \leq C N^{2q-1-2qH_0}
$$

where the last inequality has been showed in Proposition 3.1 in [5]. Since $H_0 \in [1-\frac{1}{2q}+\varepsilon, 1)$, $N^{2q-1-2qH_0} \leq N^{-2q\varepsilon} \to_{N\to\infty} 0$. This concludes the proof. ■

Theorem 4 *Assume Z^q_H is a Hermite process. Then as $H \to 1$, Z^q_H converges weakly in $C([0,T])$ to the stochastic process $(t\frac{1}{\sqrt{q!}}H_q(Z))_{t\geq 0}$.*

Proof: We will prove the convergence of finite dimensional distributions, since the tightness follows from the relation (21). Consider the linear combinations $V_{N,H,p}$ and $V_{H,p}$ given by (25) and their characteristic function $g_{N,\alpha}(H), g_\alpha(H)$ defined in (26). We can write

$$
\begin{aligned}
\lim_{H\to 1} \mathbf{E}\left(e^{i\alpha \sum_{j=1}^p \lambda_j d(H,q) Z^q_H(t_j)} \right) &= \lim_{H\to 1} g_\alpha(H) \\
&= \lim_{H\to 1} \lim_{N\to\infty} g_{N,\alpha}(H).
\end{aligned}
$$

Recall that if $f_j, j \geq 1$ is a sequence of functions on $D \subset \mathbb{R}$ converging uniformly to f on D and if a is a limit point for D, then $\lim_{j\to\infty} \lim_{x\to a} f_j(x) = \lim_{x\to a} f(x)$ provided that $\lim_{x\to a} f(x), \lim_{x\to a} f_j(x)$ exist.

Using the uniform convergence from Proposition 2, we have, since $Z^q_{H,N}$ converges in the sense of finite dimensional distributions to $\frac{[Nt]}{N}\frac{1}{\sqrt{q!}}H_q(Z)$

$$\begin{aligned}
\lim_{H\to 1} \mathbf{E}\left(e^{i\alpha\sum_{j=1}^{p}\lambda_j Z^q_H(t_j)}\right) &= \lim_{H\to 1} g_\alpha(H) = \lim_{N\to\infty}\lim_{H\to 1} g_{N,\alpha}(H) \\
&= \lim_{N\to\infty} \mathbf{E}\left(e^{i\alpha\sum_{j=1}^{p}\lambda_j \frac{[Nt_j]}{N}\frac{1}{\sqrt{q!}}H_q(Z)}\right) \\
&= \mathbf{E}\left(e^{i\alpha\sum_{j=1}^{p}\lambda_j t_j \frac{1}{\sqrt{q!}}H_q(Z)}\right)
\end{aligned}$$

and this gives the conclusion of the theorem. ■

4.2.2 Behavior of the Hermite sheet: case $d > 1$, $q > 2$.

In the multiparameter case $d > 1$, we will use the same idea related to the approximation of the Hermite sheet by some partial sums, but the situation is more complex and the limit depends on the number of Hurst parameters that converges to 1. Let us define, for $\mathbf{N} = (N^{(1)}, .., N^{(d)}) := (N, .., N) \in \mathbb{Z}^d_+$,

$$Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t}) = \mathbf{N}^{-\mathbf{H}} \sum_{\mathbf{i}=\mathbf{1}}^{[\mathbf{N}\mathbf{t}]} H_q\left(\mathbf{N}^{\mathbf{H}} \Delta B^{\mathbf{H}}\left(\left[\frac{\mathbf{i}-1}{\mathbf{N}}, \frac{\mathbf{i}}{\mathbf{N}}\right]\right)\right) \tag{27}$$

where H_q is Hermite polynomial of degree q (7), $B^{\mathbf{H}} = Z^{1,q}_{\mathbf{H}}$ is the d-parameter fractional Brownian sheet, Δ is its high-order increment (5) and

$$\mathbf{H}' = 1 - \frac{1-\mathbf{H}}{q}. \tag{28}$$

It has been shown in [13], Proposition 3.5 (see also [15]) that the sequence $Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t})$ (27) converges in sense of finite dimensional distributions to $d(\mathbf{H}, q)Z^{q,d}_{\mathbf{H}}$ with

$$d(\mathbf{H}, q)^2 = q!\frac{(\mathbf{H}'(2\mathbf{H}'-1))^q}{\mathbf{H}(2\mathbf{H}-1)}. \tag{29}$$

Let us start again by explaining the heuristic idea of the proof. Assume first that $\mathbf{H} = (H_1, .., H_d) \to (1, .., 1) \in \mathbb{R}^d$. As before we write

$$\lim_{\mathbf{H}\to 1} Z^{q,d}_{\mathbf{H}}(\mathbf{t}) = \lim_{\mathbf{H}\to 1}\lim_{\mathbf{N}\to\infty} d(\mathbf{H}, q)^{-1} Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t})$$

and suppose again that we can switch the two limits. Then

$$\lim_{\mathbf{H}\to 1} Z^{q,d}_{\mathbf{H}}(\mathbf{t}) = \lim_{\mathbf{N}\to\infty}\lim_{\mathbf{H}\to 1} d(\mathbf{H}, q)^{-1} Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t}).$$

By the definition of generalized increments (5), the self-similarity of $B^{\mathbf{H}}$ (see (4)) and the fact that the process $B^{\mathbf{H}}$ converges in the sense of finite dimensional distributions to $\langle \mathbf{t}\rangle_d Z$ when $\mathbf{H}$ goes to $\mathbf{1}$, where Z is a standard normal random variable, we will have for every $N \geq 1$

$$\lim_{\mathbf{H}\to 1} Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t}) = \frac{1}{\langle \mathbf{N}\rangle_d} \sum_{\mathbf{i}=\mathbf{1}}^{[\mathbf{Nt}]} \frac{1}{\sqrt{q!}} H_q(Z) = \frac{\langle [\mathbf{Nt}]\rangle_d}{\langle \mathbf{N}\rangle_d} \frac{1}{\sqrt{q!}} H_q(Z) \to_{\mathbf{N}} \langle \mathbf{t}\rangle_d \frac{1}{\sqrt{q!}} H_q(Z)$$

since $d(\mathbf{H},q) \to_{\mathbf{H}\to(1,..,1)\in\mathbb{R}^d} \sqrt{q!}$.

A more interesting case is when $(H_{j_1},..,H_{j_k}) \to (1,..,1) \in \mathbb{R}^k$ $(1 \leq k \leq d)$ where $\{j_1,..,j_k\} \subset \{1,2,..,d\}$ and the parameters $H_j, j \in \{1,2,..,d\} \setminus \{j_1,..,j_k\}$ are fixed. Recall the notation $A_k, \mathbf{H}_{A_k}, \mathbf{t}_{A_k}, \langle \mathbf{t}\rangle_{A_k}$ from (16). Notice that we have the following convergence in the sense of finite dimensional distributions

$$B^{\mathbf{H}'}(\mathbf{t}) \to_{\mathbf{H}_{A_k}\to(1,..,1)\in\mathbb{R}^k} X_{\mathbf{t}} = \langle \mathbf{t}\rangle_{A_k} B^{\mathbf{H}'_{\overline{A}_k}}(\mathbf{t}_{\overline{A}_k}) \tag{30}$$

and also

$$d(\mathbf{H},q) \to_{\mathbf{H}_{A_k}\to(1,..,1)\in\mathbb{R}^k} d(\mathbf{H}_{\overline{A}_k}, q-k).$$

In this case, under the same heuristics as in the one dimensional case and by using the convergence of (27) to the Hermite sheet $d(\mathbf{H},q)Z^{q,d}_{\mathbf{H}}$, we formally get, by assuming that we can invert the limits over $\mathbf{H}_{A_k}$ and over $\mathbf{N}$ and via the self-similarity property (4),

$$\begin{aligned}
&\lim_{(H_{j_1},...,H_{j_k})\to(1,...,1)\in\mathbb{R}^k} Z^{q,d}_{\mathbf{H}}(\mathbf{t}) = \lim_{\mathbf{N}\to\infty} \lim_{\mathbf{H}_{A_k}\to(1,...,1)\in\mathbb{R}^k} d(\mathbf{H},q)^{-1} Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t}) \\
= &\lim_{\mathbf{N}\to\infty} \lim_{\mathbf{H}_{A_k}\to(1,...,1)\in\mathbb{R}^k} N^{-H_{j_1}} \cdots N^{-H_{j_d}} d(\mathbf{H},q)^{-1} \sum_{i_1=1}^{[Nt^{(j_1)}]} \sum_{i_2=1}^{[Nt^{(j_2)}]} \cdots \sum_{i_d=1}^{[Nt^{(j_d)}]} H_q\left(\Delta B^{\mathbf{H}'}([\mathbf{i}-1,\mathbf{i}])\right) \\
= &\lim_{\mathbf{N}\to\infty} \lim_{\mathbf{H}_{A_k}\to(1,..,1)\in\mathbb{R}^k} \left(\prod_{a\in A_k} N^{-H_a}\right)\left(\prod_{b\in\overline{A}_k} N^{-H_b}\right) \\
&d(\mathbf{H},q)^{-1} \sum_{i_1=1}^{[Nt^{(j_1)}]} \sum_{i_2=1}^{[Nt^{(j_2)}]} \cdots \sum_{i_d=1}^{[Nt^{(j_d)}]} H_q\left(\Delta B^{\mathbf{H}'}([\mathbf{i}-1,\mathbf{i}])\right) \\
= &\lim_{\mathbf{N}\to\infty} N^{-k}\left(\prod_{b\in\overline{A}_k} N^{-H_b}\right) d(\mathbf{H}_{\overline{A}_k}, q-k)^{-1} \sum_{i_1=1}^{[Nt^{(j_1)}]} \sum_{i_2=1}^{[Nt^{(j_2)}]} \cdots \sum_{i_d=1}^{[Nt^{(j_d)}]} H_q\left(\Delta X([\mathbf{i}-1,\mathbf{i}])\right)
\end{aligned}$$

with X the d-parameter Gaussian process (30). Notice that

$$\Delta X([\mathbf{i}-1,\mathbf{i}]) = \Delta\langle \mathbf{t}\rangle_{A_k}([\mathbf{i}-1,\mathbf{i}]) \Delta B^{\mathbf{H}_{\overline{A}_k}}_{\mathbf{t}_{\overline{A}_k}}([\mathbf{i}-1,\mathbf{i}]) = \Delta B^{\mathbf{H}'_{\overline{A}_k}}_{\mathbf{t}_{\overline{A}_k}}([\mathbf{i}-1,\mathbf{i}]).$$

Therefore

$$
\begin{aligned}
& \lim_{\mathbf{H}_{A_k}\to(1,..,1)} Z^{q,d}_{\mathbf{H}}(\mathbf{t}) \\
= & \lim_{\mathbf{N}\to\infty} N^{-k}\left(\prod_{b\in\overline{A}_k} N^{-H_b}\right)[Nt^{(j_1)}]...[Nt^{(j_k)}]d(\mathbf{H}_{\overline{A}_k}, q-k)^{-1} \\
& \sum_{b\in\overline{A}_k}\sum_{i_b=1}^{[Nt^{(b)}]} H_q\left(\Delta B^{\mathbf{H}_{\overline{A}_k}}_{\mathbf{t}_{\overline{A}_k}}([\mathbf{i}-1,\mathbf{i}])\right) \\
= & \langle \mathbf{t}\rangle_{A_k} Z^{q,d-k}_{\mathbf{H}}(\mathbf{t})
\end{aligned}
$$

where $Z^{q,(d-k)}_{\mathbf{H}}(\mathbf{t})$ is the $(d-k)$- parameter Hermite sheet . We used the convergence of (27) for in the $d-k$ parameter case and the fact that

$$N^{-k}[Nt^{(j_1)}]...[Nt^{(j_k)}]\to_{N\to\infty}\langle \mathbf{t}\rangle_{A_k}.$$

As before, we need to make this heuristics rigourous. First we have to noticed that the partial sum $Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t})$ can be written as a multiple stochastic integral with respect to d-parameter Wiener process as follows (see formula (3.15) in [13])

$$Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t}) = I_q\left(F^{\mathbf{N}}(\mathbf{t},\cdot)\right)$$

where for every $\mathbf{t}\in\mathbb{R}^d_+$

$$F^{\mathbf{N}}(\mathbf{t},\cdot) = \mathbf{N}^{q(1-\mathbf{H}')-1}\sum_{\mathbf{i}=1}^{[\mathbf{N}\mathbf{t}]}\left(h^{\mathbf{N}}_{\mathbf{i}}\right)^{\otimes q}, \tag{31}$$

and

$$h^{\mathbf{N}}_{\mathbf{i}} := \mathbf{N}^{\mathbf{H}'} L_{\left[\frac{\mathbf{i}-1}{\mathbf{N}},\frac{\mathbf{i}}{\mathbf{N}}\right],H,q}$$

where $L_{\mathbf{t},\mathbf{H},q}$ is given by (6) and $L_{\left[\frac{\mathbf{i}-1}{\mathbf{N}},\frac{\mathbf{i}}{\mathbf{N}}\right],H,q}$ means the high-order increment (5) of $L_{\mathbf{t},\mathbf{H},q}$ over $\left[\frac{\mathbf{i}-1}{\mathbf{N}},\frac{\mathbf{i}}{\mathbf{N}}\right]$. We know from [13], [15] that $F^{\mathbf{N}}(\mathbf{t},\cdot)$ is Cauchy sequence in $\mathcal{H}^{\otimes q}$ where $\mathcal{H} = L^2(\mathbb{R}^d)$ converging to $L_{\mathbf{t},\mathbf{H},q}$ in $\mathcal{H}^{\otimes q}$.

We will need to estimate the mean square of the increment $Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t}) - Z^{q,d}_{\mathbf{H}}(\mathbf{t})$. For every $\mathbf{t}$, we have the isometry

$$
\begin{aligned}
\mathbf{E}\left|d(\mathbf{H},q)^{-1}Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t}) - Z^{q,d}_{\mathbf{H}}(\mathbf{t})\right|^2 &= \mathbf{E}\left|I_q\left(d(\mathbf{H},q)^{-1}F^{\mathbf{N}}(\mathbf{t},\cdot)\right) - I_q\left(L_{\mathbf{t},\mathbf{H},q}\right)\right|^2 \qquad (32) \\
&= = q!\left\|d(\mathbf{H},q)^{-1}F^{\mathbf{N}}(\mathbf{t},\cdot) - L_{\mathbf{t},\mathbf{H},q}\right\|^2_{\mathcal{H}^{\otimes q}}
\end{aligned}
$$

so we need to estimate $\left\|d(\mathbf{H},q)^{-1}F^{\mathbf{N}}(\mathbf{t},\cdot) - L_{\mathbf{t},\mathbf{H},q}\right\|^2_{\mathcal{H}^{\otimes q}}$. This is done below. The proof is a multidimensional extension of the proof of Proposition 3.1 in [5] combined with the proof of Proposition 3.3 in [13]. Its proof is postponed to the Appendix.

Proposition 3 *If $F^{\mathbf{N}}$ be given by (31) and let $L_{\mathbf{t},\mathbf{H},q}$ be given by (6). Then for every $\mathbf{t} \geq 0$*

$$
\begin{aligned}
&\left\| d(\mathbf{H},q)^{-1} F^{\boldsymbol{N}}(\mathbf{t},\cdot) - L_{\mathbf{t},\mathbf{H},q} \right\|^2_{\mathcal{H}^{\otimes q}} \\
= \; & d(\mathbf{H},q)^{-2} \boldsymbol{H}^q (2\boldsymbol{H}-1)^q \boldsymbol{N}^{2q-2-2q\boldsymbol{H}'} \sum_{\boldsymbol{i}^{(1)}=\boldsymbol{1}}^{[\boldsymbol{N}\boldsymbol{t}]} \sum_{\boldsymbol{i}^{(2)}=\boldsymbol{1}}^{[\boldsymbol{N}\boldsymbol{t}]} \left[\left(\int_{\boldsymbol{0}}^{\boldsymbol{1}} d\boldsymbol{u} \int_{\boldsymbol{0}}^{\boldsymbol{1}} d\boldsymbol{v} \left| \boldsymbol{u} - \boldsymbol{v} + \boldsymbol{i}^{(1)} - \boldsymbol{i}^{(2)} \right|^{-2(1-\boldsymbol{H}')} \right)^q \right. \\
- \; & \left. 2 \int_{\boldsymbol{0}}^{\boldsymbol{1}} d\boldsymbol{v} \left(\int_{\boldsymbol{0}}^{\boldsymbol{1}} d\boldsymbol{u} |\boldsymbol{i}^{(1)} - \boldsymbol{i}^{(2)} + \boldsymbol{u} - \boldsymbol{v}|^{2\boldsymbol{H}'-2} \right)^q + \int_{\boldsymbol{0}}^{\boldsymbol{1}} d\boldsymbol{u} \int_{\boldsymbol{0}}^{\boldsymbol{1}} d\boldsymbol{v} \left| \boldsymbol{i}^{(1)} - \boldsymbol{i}^{(2)} + \boldsymbol{u} - \boldsymbol{v} \right|^{-2q(1-\boldsymbol{H}')} \right].
\end{aligned}
$$

Now, for $\lambda_1, .., \lambda_p \in \mathbb{R}$ and $\mathbf{t}_1, .., \mathbf{t}_p \in \mathbb{R}^d_+$, we consider the linear combinations

$$
V_{\mathbf{N},\mathbf{H},p} = \sum_{j=1}^{p} \lambda_j d(\mathbf{H},q)^{-1} Z^q_{\mathbf{H},\mathbf{N}}(\mathbf{t}_j) \text{ and } V_{\mathbf{H},p} = \sum_{j=1}^{p} \lambda_j Z^q_{\mathbf{H}}(\mathbf{t}_j). \tag{33}
$$

Also, let for every $u \in \mathbb{R}$

$$
g_{\mathbf{N},u}(\mathbf{H}) = \mathbf{E}\left(e^{iuV_{\mathbf{N},\mathbf{H},p}}\right) \text{ and } g_u(\mathbf{H}) = \mathbf{E}\left(e^{iuV_{\mathbf{H},p}}\right), \tag{34}
$$

the characteristic functions of the d-dimensional random variables $V_{\mathbf{N},\mathbf{H},p}$ and $V_{\mathbf{H},p}$ respectively. We need the uniform convergence of the charactertistic functions in order two invert the order of the limits (with respect to $\mathbf{N}$(and with respect to the Hurst parameter)). Recall that $\mathbf{H}' = (H'_1, .., H'_d)$ is constructed from $\mathbf{H} = (H_1, .., H_d)$ via (28).

Proposition 4 *If $1 \leq k \leq d$, then For every $0 < \varepsilon < \frac{1}{2q}$, we have for every $u \in \mathbb{R}$*

$$
\sup_{H_1,..,H_k \in [\frac{1}{2}+\varepsilon, 1]} |g_{\mathbf{N},u}(\mathbf{H}) - g_u(\mathbf{H})| = \sup_{H'_1,..,H'_k \in [1-\frac{1}{2q}+\varepsilon,1]^k} |g_{\mathbf{N},u}(\mathbf{H}) - g_u(\mathbf{H})| \to_{N\to\infty} 0
$$

Proof: Since for every $u \in \mathbb{R}$

$$
|g_{\mathbf{N},u}(\mathbf{H}) - g_u(\mathbf{H})| \leq |u| \mathbf{E} |V_{\mathbf{N},\mathbf{H},p} - V_{\mathbf{H},p}| \leq |u| \left(\mathbf{E} |V_{\mathbf{N},\mathbf{H},p} - V_{\mathbf{H},p}|^2 \right)^{\frac{1}{2}}.
$$

Thus, it we need to prove that for every fixed $\mathbf{t} \geq 0$

$$
\sup_{H_1,..,H_k \in [1-\frac{1}{2q}+\varepsilon,1]^k} \mathbf{E} \left| Z^{q,d}_{\mathbf{H},\mathbf{N}}(\mathbf{t}) - Z^{q,d}_{\mathbf{H}}(\mathbf{t}) \right|^2 \to_{\mathbf{N}\to\infty} 0.
$$

By Proposition 3, we need to show that

$$
\sup_{H'_1,..,H'_k \in [1-\frac{1}{2q}+\varepsilon,1]^k} A(\mathbf{H}') \to_{N\to\infty} 0 \tag{35}
$$

where

$$
A(\mathbf{H}') = d(\mathbf{H},q)^{-2} (\mathbf{H}')^q (2\mathbf{H}'-1)^q \mathbf{N}^{2q-2-2q\mathbf{H}'} \sum_{\mathbf{r}\in\mathbb{Z}^d} \left[\left(\int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{u} \int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{v} \, |\mathbf{u} - \mathbf{v} + \mathbf{r}|^{-2(1-\mathbf{H}')} \right)^q \right.
$$

$$- \quad 2\int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{v} \left(\int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{u} |\mathbf{u}-\mathbf{v}+\mathbf{r}|^{2\mathbf{H}'-2} \right)^q + \int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{u} \int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{v} \, |\mathbf{u}-\mathbf{v}+\mathbf{r}|^{-2q(1-\mathbf{H}')} \Bigg]$$

From the proof of Proposition 3.1 in [5], we can show that

$$A(\mathbf{H}') \le C\mathbf{N}^{2q-1-2q\mathbf{H}'} = CN^{2q-1-2qH'_1} \ldots N^{2q-1-2qH'_d} \tag{36}$$

if $\mathbf{N} = (N, N, .., N) \in \mathbb{Z}_+^d$. Indeed, it follows from (a slightly adaptation of) relations (3.6), (3.7) and (3.8) in [5] that the term after $\sum_{\mathbf{r}\in\mathbb{Z}^d}$ above is less that a constant times $|\mathbf{r}|^{2qH'-2q-1}$ which implies that the series is convergent, since $H'_i > 1 - \frac{1}{2q}$.

Since the function $(H'_1, .., H'_k) \to A(\mathbf{H}')$ is obviously continuous on the compact set $[1 - \frac{1}{2q} + \varepsilon, 1]^k$, there exists a multi-index

$$\mathbf{H}_0 = (H_{0,1}, H_{0,2}, .., H_{0,k}) \in [1 - \frac{1}{2q} + \varepsilon, 1]^k$$

such that

$$\sup_{H_1,..,H_k \in [1-\frac{1}{2q}+\varepsilon,1]^k} A(\mathbf{H}') = A(H_{0,1}, .., H_{0,k}, H_{k+1}, .., H_d).$$

Assume $k < d$. From the inequality (36) (recall that $H'_{k+1}, ..., H'_d$ are fixed in $[1 - \frac{1}{2q} + \varepsilon, 1)$ since $H_{k+1}, .., H_d$ are fixed in $(\frac{1}{2}, 1)$),

$$\begin{aligned} A(H_{0,1}, .., H_{0,k}, H_{k+1}, .., H_d) &\le CN^{2q-1-2qH_{0,1}} \ldots N^{2q-1-2qH_{0,k}} N^{2q-1-2qH_{k+1}} \ldots . N^{2q-1-2qH_d} \\ &\le CN^{2q-1-2qH_{k+1}} \ldots . N^{2q-1-2qH_d} \to_{\mathbf{N}\to\infty} 0. \end{aligned}$$

If $k = d$, then $(H_{0,1}, H_{0,2}, .., H_{0,k})$ should be different from $(1, ..., 1) \in \mathbb{R}^k$ since $A(1, 1, ..., 1) = 0$. Thus, there exists $i_0 \in \{1, 2, ..., d\}$ such that $H_{0,i_0} \neq 1$ and then from (36), we have

$$A(H_{0,1}, .., H_{0,k}, H_{k+1}, .., H_d) \le CN_{i_0}^{2q-1-2H_{0,i_0}} \to_{\mathbf{N}\to\infty} 0.$$

This proves (35) and gives the conclusion of the theorem. ■

The main result on the behavior of the Hermite sheet when the parameters are near 1 states as follows.

Theorem 5 *Let $\left(Z_{\mathbf{H}}^{q,d}(\mathbf{t})\right)_{\mathbf{t}\ge 0}$ be given by (3). Let the notation from Theorem 2 prevail.*

1. *Assume $H_{A_k} \to \left(\frac{1}{2}, ..., \frac{1}{2}\right) \in \mathbb{R}^k$. Assume that the parameters $H_j, j \in \overline{A}_k$ are fixed. Then the process $Z_{\mathbf{H}}^{q,d}$ converges weakly in $C([0, T]^d)$ to the d-parameter stochastic process $(X_{\mathbf{t}})_{\mathbf{t}\ge 0}$ defined by*

$$X_{\mathbf{t}} = \langle \mathbf{t} \rangle_{A_k} Z_{\mathbf{H}}^{q,d-k}(\mathbf{t}_{\overline{A}_k}) \tag{37}$$

where $\left(Z_{\mathbf{H}}^{q,d-k}(\mathbf{t}_{\overline{A}_k})\right)_{\mathbf{t}_{\overline{A}_k} \in \mathbb{R}_+^{d-k}}$ is a $(d-k)$-parameter Hermite process.

2. *Assume* $(H_1,..,H_d) \to (1,..,1) \in \mathbb{R}^d$. *Then the process* $Z^{q,d}_{\mathbf{H}}$ *converges weakly in* $C([0,T]^d)$ *to the d-parameter stochastic process* $(X_{\mathbf{t}})_{\mathbf{t}\geq 0}$ *defined by*

$$X_{\mathbf{t}} = \langle \mathbf{t} \rangle_d \frac{1}{\sqrt{q!}} H_q(Z) \tag{38}$$

where $Z \sim N(0,1)$.

Proof: Tightness follows from (21), so we need to prove the convergence of finite dimensional distributions. Let us recall that the linear combinations $V_{\mathbf{N},\mathbf{H},p}$ and $V_{\mathbf{H},p}$ are given by (33) and their characteristic function $g_{\mathbf{N},\mathbf{u}}(\mathbf{H}), g_{\mathbf{u}}(\mathbf{H})$ (34). In a similar way to the one dimensional case, we can write

$$\begin{aligned}\lim_{(H_{j_1},\ldots,H_{j_k})\to(1,\ldots,1)} \mathbf{E}\left(e^{iu\sum_{j=1}^{p}\lambda_j Z^{q,d}_{\mathbf{H}}(\mathbf{t}_j)}\right) &= \lim_{(H_{j_1},\ldots,H_{j_k})\to(1,\ldots,1)} g_{\mathbf{u}}(\mathbf{H}) \\ &= \lim_{(H_{j_1},\ldots,H_{j_k})\to(1,\ldots,1)} \lim_{\mathbf{N}\to\infty} g_{\mathbf{N},\mathbf{u}}(\mathbf{H}).\end{aligned}$$

and the calculations at the beginning of this section will give the conclusion of the theorem, similarly to the proof of Theorem 4. ■

Remark 5 *Theorem covers also the case* $q=2$. *In this case we retrieve the result from Theorem 2. If* $q=1$, *then the limit (37) is the Gaussian sheet* $\langle \mathbf{t} \rangle_{A_k} Z^{1,d-k}(t_{\overline{A_k}})$.

5 Appendix

Proof of Proposition 3: We can write

$$\begin{aligned}\|F^{\mathbf{N}}(\mathbf{t},\cdot)\|^2_{\mathcal{H}^{\otimes q}} &= \mathbf{N}^{2q(1-\mathbf{H}')-2} \sum_{\mathbf{i}^{(1)}=1}^{[\mathbf{Nt}]} \sum_{\mathbf{i}^{(2)}=1}^{[\mathbf{Nt}]} \langle \left(h^{\mathbf{N}}_{\mathbf{i}^{(1)}}\right)^{\otimes q}, \left(h^{\mathbf{N}}_{\mathbf{i}^{(2)}}\right)^{\otimes q} \rangle_{\mathcal{H}^{\otimes q}} \\ &= \mathbf{N}^{2q(1-\mathbf{H}')-2} \sum_{\mathbf{i}^{(1)}=1}^{[\mathbf{Nt}]} \sum_{\mathbf{i}^{(2)}=1}^{[\mathbf{Nt}]} \langle \left(h^{\mathbf{N}}_{\mathbf{i}^{(1)}}\right), \left(h^{\mathbf{N}}_{\mathbf{i}^{(2)}}\right) \rangle^q_{\mathcal{H}} \\ &= \prod_{j=1}^{d} N^{2q(1-H'_j)-2} \sum_{k_1=1}^{[Nt_j]} \sum_{k_2=1}^{[Nt_j]} r_{H'_j}(k_1-k_2)^q,\end{aligned}$$

where $\mathbf{H}'$ is given by (28) and

$$r_{H'_j}(k_1-k_2) = N^{2H'_j}\mathbf{E}\left[B^{H_j}\left(\left[\frac{k_1-1}{N},\frac{k_1}{N}\right]\right)B^{H'_j}\left(\left[\frac{k_2-1}{N},\frac{k_2}{N}\right]\right)\right]$$

$$= N^{2H'_j} H'_j(2H'_j - 1)\int_{(k_1-1)/N}^{k_1/N}\int_{(k_2-1)/N}^{k_2/N} |u-v|^{-2(1-H'_j)}\,dudv.$$

From the above relation,

$$\begin{aligned}
\|F^{\mathbf{N}}(\mathbf{t},\cdot)\|^2_{\mathcal{H}^{\otimes q}} &= \prod_{j=1}^{d}(H'_j)^q(2H'_j-1)^q N^{2q-2}\sum_{k_1=1}^{[Nt_j]}\sum_{k_2=1}^{[Nt_j]}\left(\int_{(k_1-1)/N}^{k_1/N}\int_{(k_2-1)/N}^{k_2/N}|u-v|^{-2(1-H'_j)}\,dudv\right)^q \\
&= (\mathbf{H}')^q(2\mathbf{H}'-1)^q\mathbf{N}^{2q-2}\sum_{\mathbf{i}^{(1)}=1}^{[\mathbf{N}\mathbf{t}]}\sum_{\mathbf{i}^{(2)}=1}^{[\mathbf{N}\mathbf{t}]}\left(\int_{(\mathbf{i}^{(1)}-1)/\mathbf{N}}^{\mathbf{i}^{(1)}/\mathbf{N}}\int_{(\mathbf{i}^{(2)}-1)/\mathbf{N}}^{\mathbf{i}^{(2)}/\mathbf{N}}|\mathbf{u}-\mathbf{v}|^{-2(1-\mathbf{H}')}\,d\mathbf{u}d\mathbf{v}\right)^q
\end{aligned}$$

Now, by letting $\mathbf{N}$ goes to infinity, we have, if $F(\mathbf{t},\cdot) = L_{\mathbf{t},\mathbf{H},q}$

$$\begin{aligned}
\|F(\mathbf{t},\cdot)\|^2_{\mathcal{H}^{\otimes q}} &= \prod_{j=1}^{d}(H'_j)^q(2H'_j-1)^q\int_0^{t_j}\int_0^{t_j}|u-v|^{-2q(1-H'_j)}\,dudv \\
&= \prod_{j=1}^{d}(H'_j)^q(2H'_j-1)^q\sum_{k_1=1}^{[Nt_j]}\sum_{k_2=1}^{[Nt_j]}\int_{(k_1-1)/N}^{k_1/N}\int_{(k_2-1)/N}^{k_2/N}|u-v|^{-2q(1-H'_j)}\,dudv \\
&= (\mathbf{H}')^q(2\mathbf{H}'-1)^q\sum_{\mathbf{i}^{(1)}=1}^{[\mathbf{N}\mathbf{t}]}\sum_{\mathbf{i}^{(2)}=1}^{[\mathbf{N}\mathbf{t}]}\int_{(\mathbf{i}^{(1)}-1)/\mathbf{N}}^{\mathbf{i}^{(1)}/\mathbf{N}}\int_{(\mathbf{i}^{(2)}-1)/\mathbf{N}}^{\mathbf{i}^{(2)}/\mathbf{N}}|\mathbf{u}-\mathbf{v}|^{-2q(1-\mathbf{H}')}\,d\mathbf{u}d\mathbf{v}
\end{aligned}$$

Now, if $\phi \in \mathcal{H}$ is smooth

$$\begin{aligned}
\langle F^{\mathbf{N}}(\mathbf{t},\cdot),\phi^{\otimes q}\rangle_{\mathcal{H}^{\otimes q}} &= \mathbf{N}^{q(1-\mathbf{H}')-1}\sum_{\mathbf{i}^{(1)}=1}^{[\mathbf{N}\mathbf{t}]}\langle h^{\mathbf{N}}_{\mathbf{i}^{(1)}},\phi\rangle^q_{\mathcal{H}} \\
&= \prod_{j=1}^{d}(H'_j)^q(2H'_j-1)^q N^{q-1}\sum_{k=1}^{\lfloor Nt_j\rfloor}\left(\int_{(k-1)/N}^{k/N}dv\int_0^1 du\phi(u)|u-v|^{2H'_j-2}\right)^q.
\end{aligned}$$

By letting again $\mathbf{N}$ goes to infinity, we have

$$\langle F(\mathbf{t},\cdot),\phi^{\otimes q}\rangle_{\mathcal{H}^{\otimes q}} = \prod_{j=1}^{d}(H'_j)^q(2H'_j-1)^q\int_0^1 dv\left(\int_0^1 du\phi(u)|u-v|^{2H'_j-2}\right)^q.$$

Therefore, we have

$$\langle F(\mathbf{t},\cdot),F^{\mathbf{N}}(\mathbf{t},\cdot)\rangle_{\mathcal{H}^{\otimes q}} = \prod_{j=1}^{d}(H'_j)^q(2H'_j-1)^q N^{q-1}\sum_{k=1}^{\lfloor Nt_j\rfloor}\int_0^1 dv\left(\int_{(k-1)/N}^{k/N}du|u-v|^{2H'_j-2}\right)^q.$$

$$= \prod_{j=1}^{d}(H_j')^q(2H_j'-1)^q N^{q-1}\sum_{k,l=1}^{\lfloor Nt_j\rfloor}\int_{(l-1)/N}^{l/N} dv\left(\int_{(k-1)/N}^{k/N} du|u-v|^{2H_j-2}\right)^q.$$

$$= (\mathbf{H}')^q(2\mathbf{H}'-1)^q\mathbf{N}^{q-1}\sum_{\mathbf{i}^{(1)}=1}^{[\mathbf{Nt}]}\sum_{\mathbf{i}^{(2)}=1}^{[\mathbf{Nt}]}\int_{(\mathbf{i}^{(2)}-1)/\mathbf{N}}^{\mathbf{i}^{(2)}/\mathbf{N}} d\mathbf{v}\left(\int_{(\mathbf{i}^{(1)}-1)/\mathbf{N}}^{\mathbf{i}^{(1)}/\mathbf{N}} d\mathbf{u}|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}'-2}\right)^q.$$

Taking into account all the previous calculations, and by some elementary change of variables, we have

$$\begin{aligned}
&\left\|F^N(\mathbf{t},\cdot)-F(\mathbf{t},\cdot)\right\|^2_{\mathcal{H}^{\otimes q}}\\
=\;& (\mathbf{H}')^q(2\mathbf{H}'-1)^q\sum_{\mathbf{i}^{(1)}=1}^{[\mathbf{Nt}]}\sum_{\mathbf{i}^{(2)}=1}^{[\mathbf{Nt}]}\Bigg[\mathbf{N}^{2q-2}\left(\int_{(\mathbf{i}^{(1)}-1)/\mathbf{N}}^{\mathbf{i}^{(1)}/\mathbf{N}}\int_{(\mathbf{i}^{(2)}-1)/\mathbf{N}}^{\mathbf{i}^{(2)}/\mathbf{N}}|\mathbf{u}-\mathbf{v}|^{-2(1-\mathbf{H}')}\,d\mathbf{u}d\mathbf{v}\right)^q\\
-\;& 2\mathbf{N}^{q-1}\int_{(\mathbf{i}^{(2)}-1)/\mathbf{N}}^{\mathbf{i}^{(2)}/\mathbf{N}} d\mathbf{v}\left(\int_{(\mathbf{i}^{(1)}-1)/\mathbf{N}}^{\mathbf{i}^{(1)}/\mathbf{N}} d\mathbf{u}|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}'-2}\right)^q\\
+\;& \int_{(\mathbf{i}^{(1)}-1)/\mathbf{N}}^{\mathbf{i}^{(1)}/\mathbf{N}}\int_{(\mathbf{i}^{(2)}-1)/\mathbf{N}}^{\mathbf{i}^{(2)}/\mathbf{N}}|\mathbf{u}-\mathbf{v}|^{-2q(1-\mathbf{H}')}\,d\mathbf{u}d\mathbf{v}\Bigg]\\
=\;& (\mathbf{H}')^q(2\mathbf{H}'-1)^q\mathbf{N}^{2q-2-2q\mathbf{H}'}\sum_{\mathbf{i}^{(1)}=1}^{[\mathbf{Nt}]}\sum_{\mathbf{i}^{(2)}=1}^{[\mathbf{Nt}]}\Bigg[\left(\int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{u}\int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{v}\left|\mathbf{u}-\mathbf{v}+\mathbf{i}^{(1)}-\mathbf{i}^{(2)}\right|^{-2(1-\mathbf{H}')}\right)^q\\
-\;& 2\int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{v}\left(\int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{u}|\mathbf{i}^{(1)}-\mathbf{i}^{(2)}+\mathbf{u}-\mathbf{v}|^{2\mathbf{H}'-2}\right)^q+\int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{u}\int_{\mathbf{0}}^{\mathbf{1}} d\mathbf{v}\left|\mathbf{i}^{(1)}-\mathbf{i}^{(2)}+\mathbf{u}-\mathbf{v}\right|^{-2q(1-\mathbf{H}')}\Bigg]
\end{aligned}$$

and with this, the result is achieved. ∎

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