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Gaussian limit theorems for diffusion processes and an application

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Abstract

Suppose that $L = \sum_{i,j=1}^{d} a_{ij}(x)\partial^2/\partial x_i \partial x_j$ is uniformly elliptic. We use $X_L(t)$ to denote the diffusion associated with L. In this paper we show that, if the dimension of the set $\{x\colon [a_{ij}(x)] \neq \frac{1}{2}I\}$ is strictly less than d, the random variable $(X_L(T) - X_L(0))/\sqrt{T}$ converges in distribution to a standard Gaussian random variable. In fact, we also provide rates of convergence. As an application, these results are used to study a problem of a random walk in a random environment.

Keywords: Random walks; Diffusions; Random environments

1. Introduction

Let $X(t)$, $t \geqslant 0$, denote Brownian motion in $\mathbb{R}^d$ $(d \geqslant 1)$. Hence, for $x \in \mathbb{R}^d$ and reasonably behaved functions $g\colon \mathbb{R}^d \to \mathbb{R}$ one has

$$E_x[g(X(t) - x)] = \int_{\mathbb{R}^d} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{y^2}{2t}\right) g(y)\,\mathrm{d}y. \tag{1}$$

Here E_x denotes the expectation with respect to the Brownian motion starting from x. We can rescale Eq. (1) to yield

$$E_x\left[f\left(\frac{X(t) - x}{\sqrt{t}}\right)\right] = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{z^2}{2}\right) f(z)\,\mathrm{d}z,$$

where $f(z) = g(z\sqrt{t})$. This is the same as

$$E_x\left[f\left(\frac{X(t) - x}{\sqrt{t}}\right)\right] = E[f(Y)], \tag{2}$$

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where Y is a d dimensional Gaussian variable with covariance matrix equal to the identity. In this paper we shall be concerned with showing that identity (2) holds approximately at large time for diffusions $X_L(t)$ associated with certain elliptic operators of the form

$$L = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{3}$$

where $x \in \mathbb{R}^d$ has coordinates $(x_1, \ldots, x_d)$. We assume that the matrix $A(x) = [a_{ij}(x)]$, $x \in \mathbb{R}^d$, is symmetric and that there are constants $\lambda, \Lambda > 0$ such that

$$\lambda I \leqslant A(x) \leqslant \Lambda I, \quad x \in \mathbb{R}^d, \tag{4}$$

where I is the identity matrix. Therefore L is uniformly elliptic. The operator L generates a diffusion process which we denote by $X_L(t)$, $t \geqslant 0$. If $A(x) = \frac{1}{2}I$, $x \in \mathbb{R}^d$, then $X_L(t)$ is just Brownian motion.

Suppose that the operator L satisfies Eq. (4). We expect that Eq. (2) holds approximately at large time for $X_L(t)$ provided the dimension of the set $\{x\colon A(x) \neq \frac{1}{2}I\}$ is strictly less than d. For $0 \leqslant \alpha \leqslant d$ we shall say that a set U has dimension less than or equal to α if there exists a constant C such that for all balls B_R of radius R,

$$|U \cap B_R| \leqslant C R^{\alpha}, \quad R \geqslant 1,$$

where $|\cdot|$ denotes the Lebesgue measure. This notion of dimension is very different from the Hausdorff dimension. For instance, a line in $\mathbb{R}^d$ has dimension 0, and so does any hypersurface in $\mathbb{R}^d$. From the definition above, one can check that any bounded set in $\mathbb{R}^d$ has dimension 0. So our notion of dimension measures how big the set is near infinity. The following theorem is proved in Section 2.

Theorem 1.1. *Suppose $\{x\colon A(x) \neq \frac{1}{2}I\}$ has dimension less than or equal to $\alpha < d$. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a function such that*

$$|f(x)| + \sum_{i=1}^{d} \left| \frac{\partial f}{\partial x_i}(x) \right| + \sum_{i,j=1}^{d} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right| \leqslant A e^{M|x|}, \quad x \in \mathbb{R}^d,$$

for some constants A, M. Then for any $x_0 \in \mathbb{R}^d$, $T \geqslant 1$, there is a constant C depending only on A, M, λ and Λ such that

$$\left| E_{x_0}\left[f\left(\frac{X_L(T) - x_0}{\sqrt{T}} \right) \right] - E[f(Y)] \right| \leqslant \frac{C}{T^{(1-\alpha/d)/2}}. \tag{5}$$

We prove Theorem 1.1 by using the Alexander–Bakelman–Pucci (ABP) inequality. For a statement of the ABP inequality, see Gilbarg and Trudinger (1983). The estimate $C/T^{(1-\alpha/d)/2}$ given by the ABP inequality is not sharp. Consider the situation where $\{x\colon A(x) \neq \frac{1}{2}I\} \subset \{x\colon |x_1| < 1\}$. In this case $\alpha = d-1$ and Theorem 1.1 yields an estimate $C/T^{1/(2d)}$. The best possible estimate is $C/\sqrt{T}$ for this case. We shall also prove this in Section 2 and a corresponding result for the case when $\{x\colon A(x) \neq \frac{1}{2}I\} \subset \{x\colon |x_1| < 1,\ |x_2| < 1\}$.

Theorem 1.2. *Let* $f\colon \mathbb{R}^d \to \mathbb{R}$ *satisfy the conditions of Theorem* 1.1 *and* $x_0 \in \mathbb{R}^d$. *If* $\{x\colon A(x) \neq \frac{1}{2}I\} \subset \{x\colon |x_1| < 1\}$, *then*

$$\left| E_{x_0}\left[f\left(\frac{X_L(T) - x_0}{\sqrt{T}} \right) \right] - E[f(Y)] \right| \leqslant \frac{C}{\sqrt{T}}, \quad T \geqslant 1, \tag{6}$$

where C *depends only on* A, M, λ *and* Λ. *If* $\{x\colon A(x) \neq \frac{1}{2}I\} \subset \{x\colon |x_1| < 1,\ |x_2| < 1\}$, *then*

$$\left| E_{x_0}\left[f\left(\frac{X_L(T) - x_0}{\sqrt{T}} \right) \right] - E[f(Y)] \right| \leqslant \frac{C\,|\log T|}{T}, \quad T \geqslant 2, \tag{7}$$

where C *depends only on* A, M, λ *and* Λ.

Observe that Theorems 1.1 and 1.2 not only prove convergence in distribution. They also give us a rate of convergence. This rate of convergence will become important in Section 3 when we consider a problem of a random walk in a random environment. The walk consists of a random walk with a drift which is random in both space and time. To specify the walk let $\boldsymbol{b}(i,x)$, $i = 1,2,\ldots,$ $x \in \mathbb{Z}^d$, be a vector field on $\mathbb{Z}^d$ such that (1) the components of the vector $\boldsymbol{b}(i,x)$ are independent symmetric Bernoulli random variables, i.e., they only take on the values ± 1 and the probability that they take the value ± 1 are both 1/2 and (2) the collection $\{\boldsymbol{b}(i,x),\ i = 1,2,\ldots,\ x \in \mathbb{Z}^d\}$ of random vectors are independent. Suppose $\xi(j)$, $j = 1,2,\ldots,T$, is a standard random walk on $\mathbb{Z}^d$ with measure $\mathrm{d}W_T(\xi)$. For any $i = 1,2,\ldots,$ let $\Delta\xi_i = \xi_i - \xi_{i-1}$. Then for any $\beta \in \mathbb{R}^d$, the measure $P^{\beta,\boldsymbol{b}}$ defined by

$$\frac{\exp[\beta \sum_{i=1}^{T} \boldsymbol{b}(i, \xi(i-1)) \cdot \Delta\xi_i] \mathrm{d}W_T(\xi)}{E[\exp[\beta \sum_{i=1}^{T} \boldsymbol{b}(i, \xi(i-1)) \cdot \Delta\xi_i]]} = \frac{\exp[\beta \sum_{i=1}^{T} \boldsymbol{b}(i, \xi(i-1)) \cdot \Delta\xi_i] \mathrm{d}W_T(\xi)}{(\cosh\beta)^T} \tag{8}$$

gives us a new measure on the walks ξ. This measure is also Markovian. Indeed, one can easily check that

$$\begin{aligned} P^{\beta,\boldsymbol{b}}(\xi_{T+1} = x_{T+1} \mid \xi_0 = x_0, \ldots, \xi_T = x_T) &= \frac{P^{\beta,\boldsymbol{b}}(\xi_0 = x_0, \ldots, \xi_T = x_T, \xi_{T+1} = x_{T+1})}{P^{\beta,\boldsymbol{b}}(\xi_0 = x_0, \ldots, \xi_T = x_T)} \\ &= \frac{1}{\cosh\beta} E_{x_T}[\exp[\beta \boldsymbol{b}(1, \xi_0) \cdot \Delta\xi_1]]. \end{aligned}$$

This measure corresponds to a random walk with a drift. We denote the random walk with measure (9) by $\xi_{\boldsymbol{b}}$.

Theorem 1.3. *Let* $\boldsymbol{b}(i,x) = (b^{(1)}(i,x), \ldots, b^{(d)}(i,x))$, *where* $b^{(j)}(i,x)$, $j = 1,\ldots,d$, $i = 1,2,\ldots,$ $x \in \mathbb{Z}^d$, *are independent random variables with mean zero*, *taking values* ± 1. *Suppose* $f\colon \mathbb{R}^d \to \mathbb{R}$ *satisfies the conditions of Theorem* 1.1. *Then*
(i) *when* $d \geqslant 3$,

$$\lim_{N \to \infty} E_{x_0}\left[f\left(\frac{\xi_{\boldsymbol{b}}(N) - x_0}{\sqrt{N/d}} \right) \right] = E[f(Y)], \tag{9}$$

with probability 1 *in* $\boldsymbol{b}$;

(ii) *when* $d=2$, *Eq.* (9) *holds for* $N=a_n\in\mathbb{Z}$, $a_n\geqslant 1$, $n=1,2,\dots$, *with*

$$\sum_{n=1}^{\infty}\frac{\log a_n}{a_n}<\infty.$$

(iii) *when* $d=1$, *Eq.* (9) *holds for* $N=a_n\in\mathbb{Z}$, $a_n\geqslant 1$, $n=1,2,\dots$, *with*

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{a_n}}<\infty.$$

The relation between Theorem 1.3 and Theorems 1.1 and 1.2 is as follows. To prove Theorem 1.3 we estimate the mean square fluctuation

$$E_{\boldsymbol{b}}\left[\left\{E_{x_0}\left[f\left(\frac{\xi_{\boldsymbol{b}}(N)-x_0}{\sqrt{N/d}}\right)\right]-E[f(Y)]\right\}^2\right], \tag{10}$$

where $E_{\boldsymbol{b}}$ denotes expectation is taken with respect to $\boldsymbol{b}$. This quantity turns out to be a discrete version of the LHS of Eq. (5). Now the dimension of the space is $2d$ and $\alpha=d$ in Theorem 1.1. Theorem 1.3 for $d=1,2$ follows then by establishing discrete versions of inequalities (6), (7), respectively. To prove Theorem 1.3 for $d\geqslant 3$ we use an argument from Conlon and Olsen (1996) which enables us to exploit the fact that random walk in $\mathbb{Z}^{2d}$ is non-recurrent to a set of dimension d provided $d\geqslant 3$.

In this paper we only estimate mean square fluctuations similar to Eq. (10). It seems likely that one could prove expression (9) converges along the entire integer sequence in dimensions $d=1,2$ by estimating moments higher than the mean square fluctuation. This is considerably more difficult since to do this one must compare two different non-standard random walks. Mean square fluctuations can be estimated by comparing a non-standard random walk to the standard random walk.

The results in this paper should be compared to the problem of random walk with a drift which is random only in space. It has been shown in Sinai (1982) that for $d=1$ this walk is strongly sub-diffusive and in Bricmont and Kupiainen (1991) that for $d\geqslant 3$ it is diffusive with a renormalized diffusion constant provided the noise – corresponding to β in Eq. (9) – is small. It is also interesting to compare the results here to the problem of random walk with a random potential (Bolthausen, 1989; Imbrie and Spencer, 1988; Olsen and Song, 1996; Song and Zhou, 1996). In that case it has been proven that for $d\geqslant 3$ and small noise the walk is diffusive. Numerical evidence, (Kardar, 1985; Kardar and Zhang, 1987) suggests the walk is super-diffusive at large noise if $d\geqslant 3$ and for any noise if $d=1,2$. Some rigorous results have been established for a first passage percolation problem which is closely related to the $d=1$ large noise problem, see Licea and Newman (1996), Licea et al. (1995) and Newman and Piza (1995).

2. Convergence in distribution to a Gaussian

We turn to the proof of Theorem 1.1. Let g: $\mathbb{R}^d \to \mathbb{R}$ and

$$w(x,t) = E_x[g(X(t))] = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left[-\frac{(x-y)^2}{2t}\right] g(y)\,\mathrm{d}y.$$

Then w satisfies the initial value problem

$$\frac{\partial w}{\partial t} = \frac{1}{2}\Delta w(x,t), \quad x \in \mathbb{R}^d, \ t > 0,$$

$$w(x,0) = g(x).$$

We can rewrite the heat equation above as

$$\frac{\partial w}{\partial t} = Lw + \left(\frac{1}{2}\Delta - L\right) w.$$

Now let v be the solution of the initial value problem

$$\frac{\partial v}{\partial t} = Lv + \left(\frac{1}{2}\Delta - L\right) w, \quad x \in \mathbb{R}^d, \ t > 0,$$

$$v(x,0) = 0.$$

Then it is clear that

$$E_x[g(X_L(t))] = w(x,t) - v(x,t).$$

If we write now

$$h(x,t) = (\tfrac{1}{2}\Delta - L)w(x,t),$$

then v has the probabilistic representation

$$v(x,t) = E_x\left[\int_0^t h(X_L(t-s),s)\,\mathrm{d}s\right]. \tag{11}$$

Now suppose f: $\mathbb{R}^d \to \mathbb{R}$ and define g: $\mathbb{R}^d \to \mathbb{R}$ by

$$g(x) = f\left(\frac{x - x_0}{\sqrt{T}}\right),$$

where $x_0 \in \mathbb{R}^d$ is some arbitrary fixed point. Then

$$w(x,t) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left[-\frac{(x-x_0-z)^2}{2t}\right] f\left(\frac{z}{\sqrt{T}}\right) \mathrm{d}z. \tag{12}$$

It is easy to see from this formula that if f is of exponential growth,

$$|f(y)| \leqslant A\mathrm{e}^{M|y|},$$

then there are constants A_1 and M_1 depending only on A and M such that

$$|w(x,t)| \leqslant A_1 \mathrm{e}^{M_1|x-x_0|/\sqrt{T}}, \quad 0 < t < T.$$

To get an estimate on $h(x,t)$, we interchange x differentiation with z differentiation in Eq. (12) and integrate by parts. Hence if the second-order partial derivatives of f have exponential growth

$$\sum_{i,j=1}^{d} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right| \leqslant A\mathrm{e}^{M|x|},$$

there are constants A_1 and M_1 depending only on A and M such that

$$|h(x,t)| \leqslant \frac{A_1}{T} e^{M_1|x-x_0|/\sqrt{T}}, \quad 0<t<T. \tag{13}$$

Let $U=\{z\colon A(z)\neq \frac{1}{2}I\}$ and 1_U be the indicator function of U. Then from Eq. (13), $v(x,t)$ is bounded by

$$|v(x,t)| \leqslant E_x\left[\int_0^t 1_U(X_L(t-s))\frac{A_1}{T}\exp[M_1|X_L(t-s)-x_0|/\sqrt{T}]\,ds\right], \quad t<T.$$

Hence

$$\left|E_{x_0}\left[f\left(\frac{X_L(T)-x_0}{\sqrt{T}}\right)\right] - E_{x_0}\left[f\left(\frac{X(T)-x_0}{\sqrt{T}}\right)\right]\right|$$

$$\leqslant E_{x_0}\left[\int_0^T \frac{A_1}{T}1_U(X_L(T-s))e^{M_1|X_L(t-s)-x_0|/\sqrt{T}}\,ds\right]. \tag{14}$$

Inequality (14) is basic to the argument of this section, since from here on we shall estimate the RHS of this inequality. To do this we need the following:

Lemma 2.1. *Let τ_R be the time for the diffusion process X_L, started from $x\in\mathbb{R}^d$, to go a distance R. Then there is a constant $\gamma>0$ depending only on λ and Λ such that*

$$P(\tau_R<t)\leqslant e^{-\gamma R^2/t}, \quad R\geqslant\sqrt{t}.$$

Proof. Let $r>0$ be arbitrary and τ_1 be the time taken for the process to exit the strip $\{y\colon |y_1-x_1|<r\}$. If we put

$$u(y)=E_y[e^{-\eta\tau_1}],$$

then $u(y)$ satisfies the boundary value problem

$$Lu(y)=\eta u(y), \quad |y_1-x_1|<r,$$
$$u(y)=1, \quad |y_1-x_1|=r.$$

Now let $w(z)$, $-r<z<r$, satisfy the boundary value problem

$$\frac{d^2w}{dz^2}=\frac{\eta}{\Lambda}w(z), \quad -r<z<r,$$
$$w(z)=1, \quad z=\pm r.$$

The function w is given explicitly by the formula

$$w(z)=\cosh(z(\eta/\Lambda)^{1/2})/\cosh(r(\eta/\Lambda)^{1/2}).$$

If we put $\bar{u}(y)=w(y_1-x_1)$, it is clear from Eq. (4) that

$$L\bar{u}(y)\leqslant\eta\bar{u}(y), \quad |y_1-x_1|<r,$$
$$\bar{u}(y)=1, \quad |y_1-x_1|=r.$$

Hence by the maximum principle we have $u(y)\leqslant\bar{u}(y)$ when $|y_1-x_1|<r$. Thus

$$E_x[e^{-\eta\tau_1}]\leqslant 1/\cosh(r(\eta/\Lambda)^{1/2}).$$

We can similarly define τ_j, $j=2,3,\dots,d$, to be the time for the process X_L to exit the strip $\{y\colon |y_j-x_j|<r\}$. Taking $r=R/\sqrt{d}$ it is clear that

$$\begin{aligned}P(\tau_R<t) &\leqslant \sum_{j=1}^{d} P(\tau_j<t) \leqslant \sum_{j=1}^{d} \mathrm{e}^{\eta t} E_x[\mathrm{e}^{-\eta\tau_j}]\\ &\leqslant d\mathrm{e}^{\eta t}/\cosh(R\eta^{1/2}/\Lambda^{1/2}d^{1/2})\\ &\leqslant 2d\exp[\eta t-\eta^{1/2}R/\Lambda^{1/2}d^{1/2}].\end{aligned}$$

If we optimize the last inequality with respect to η we obtain

$$P(\tau_R<t)\leqslant 2d\mathrm{e}^{-R^2/4\Lambda td}$$

which yields the result. $\square$

Lemma 2.2. *Suppose the set $U\subset\mathbb{R}^d$ has dimension less than or equal to $\alpha<d$. Then for any $x\in\mathbb{R}^d$,*

$$E_x\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right]\leqslant CT^{(1+\alpha/d)/2}, \tag{15}$$

$$E_x\left[\left\{\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right\}^2\right]\leqslant CT^{1+\alpha/d}, \tag{16}$$

where the constant C depends only on λ and Λ.

Proof. Let τ_R be the time for X_L started at x to go a distance R. Then we can write

$$\begin{aligned}E_x\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right] &= \sum_{n=0}^{\infty} E_x\left[\int_{T\wedge\tau_{n\sqrt{T}}}^{T\wedge\tau_{(n+1)\sqrt{T}}} 1_U(X_L(s))\,\mathrm{d}s\right]\\ &\leqslant \sum_{n=0}^{\infty} E_x\left[\int_{\tau_{n\sqrt{T}}}^{\tau_{(n+1)\sqrt{T}}} 1_U(X_L(s))\,\mathrm{d}s;\tau_{n\sqrt{T}}<T\right].\end{aligned}$$

Thus

$$E_x\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right]\leqslant \sum_{n=0}^{\infty} P(\tau_{n\sqrt{T}}<T)E_x\left[\int_{\tau_{n\sqrt{T}}}^{\tau_{(n+1)\sqrt{T}}} 1_U(X_L(s))\,\mathrm{d}s|\tau_{n\sqrt{T}}<T\right]. \tag{17}$$

We can bound the expectation in the last sum by using the ABP inequality. Thus

$$E_x\left[\int_{\tau_{n\sqrt{T}}}^{\tau_{(n+1)\sqrt{T}}} 1_U(X_L(s))\,\mathrm{d}s|\tau_{n\sqrt{T}}<T\right]\leqslant \sup_{y:|y-x|=n\sqrt{T}} E_y\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right], \tag{18}$$

where τ is the time taken for the diffusion process started at y to exit the ball $\{z\colon |z-x|\leqslant(n+1)\sqrt{T}\}$. It follows from the ABP inequality that

$$\begin{aligned}E_y\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right] &\leqslant C(n+1)\sqrt{T}\left[\int_{|z-x|<(n+1)\sqrt{T}} 1_U(z)\,\mathrm{d}z\right]^{1/d}\\ &\leqslant C(n+1)^{1+\alpha/d}T^{(1+\alpha/d)/2},\end{aligned}$$

where the constant C depends only on λ and Λ. Hence we have the inequality

$$E_x\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right] \leqslant CT^{(1+\alpha/d)/2}\sum_{n=0}^{\infty} P(\tau_{n\sqrt{T}} < T)(n+1)^{1+\alpha/d}.$$

From Lemma 2.1 it easily follows that

$$\sum_{n=0}^{\infty} P(\tau_{n\sqrt{T}} < T)(n+1)^{1+\alpha/d} \leqslant C,$$

where C depends only on λ and Λ. Inequality (15) follows from this and the previous inequality.

Next we turn to the proof of Eq. (16). We write the left-hand side of Eq. (16) as

$$2E_x\left[\int_{0<s<s'<T} 1_U(X_L(s))1_U(X_L(s'))\,\mathrm{d}s\,\mathrm{d}s'\right]$$

$$=2E_x\left[\int_0^T \mathrm{d}s\,1_U(X_L(s))E\left[\int_s^T 1_U(X_L(s'))\,\mathrm{d}s' \,|\, X_L(s)\right]\right].$$

From Eq. (15) it follows that

$$E\left[\int_s^T 1_U(X_L(s'))\,\mathrm{d}s' \,|\, X_L(s)\right] \leqslant CT^{(1+\alpha/d)/2},$$

whence

$$\begin{aligned} E_x\left[\left\{\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right\}^2\right] &\leqslant 2CT^{(1+\alpha/d)/2}E_x\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right] \\ &\leqslant 2CT^{(1+\alpha/d)/2}CT^{(1+\alpha/d)/2} \\ &= 2C^2T^{1+\alpha/d}, \end{aligned}$$

again by using Eq. (15). □

Proof of Theorem 1.1. We estimate the term on the RHS of Eq. (14). Thus

$$E_{x_0}\left[\int_0^T 1_U(X_L(T-s))\exp[M_1|X_L(T-s)-x_0|/\sqrt{T}]\right]$$

$$\leqslant \sum_{n=0}^{\infty} E_{x_0}\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\exp[M_1(n+1)];\ \sup_{0<s<T}|X_L(s)-x_0| \geqslant n\sqrt{T}\right]$$

$$\leqslant \sum_{n=0}^{\infty}\exp[M_1(n+1)]E_{x_0}\left[\left\{\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right\}^2\right]^{1/2}$$

$$\times P_{x_0}\left(\sup\nolimits_{0<s<T}|X_L(s)-x_0| \geqslant n\sqrt{T}\right)^{1/2}$$

$$\leqslant CT^{(1+\alpha/d)/2}\sum_{n=0}^{\infty}\exp[M_1(n+1)]P_{x_0}\left(\sup_{0<s<T}|X_L(s)-x_0| \geqslant n\sqrt{T}\right)^{1/2},$$

by Lemma 2.2. Now from Lemma 2.1 it follows that

$$P_{x_0}\left(\sup_{0<s<T}|X_L(s)-x_0|\geqslant n\sqrt{T}\right)\leqslant\exp[-\gamma n^2].$$

We conclude then from Eq. (14) that Eq. (5) holds. □

We turn to the proof of Theorem 1.2. Evidently the theorem will follow if we can obtain an improvement of inequality (15) of Lemma 2.2. Estimate (6) is therefore a consequence of the following lemma.

Lemma 2.3. *Let $U=\{x: |x_1|<1\}$. Then there is a constant C depending only on λ and Λ such that*

$$E_x\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right]\leqslant C\sqrt{T},\quad T\geqslant 1.$$

Proof. From Eqs. (17) and (18) it will be sufficient to show that

$$u(y)=E_y\left[\int_0^{\tau_R} 1_U(X_L(s))\,\mathrm{d}s\right]\leqslant CR,\quad y\in B_R, \tag{19}$$

where B_R is an arbitrary ball of radius $R\geqslant 1$ and τ_R is the time for the diffusion started at $y\in B_R$ to hit the boundary ∂B_R. We can prove this using the maximum principle since u is a solution of the boundary value problem

$$-Lu(y)=1_U(y),\quad y\in B_R,$$
$$u(y)=0,\quad y\in\partial B_R.$$

Suppose B_R is centered at the point $x=(x_1,\dots,x_d)$. If $|x_1|>R+1$ then $u\equiv 0$ so we shall assume $|x_1|<R+1$. Let $v(z)$ be the solution of the one dimensional problem

$$-v''(z)=1_{(-1,1)}(z),\quad |z|<2R+1,$$
$$v(z)=0,\quad |z|=2R+1.$$

One can explicitly solve this problem and see that $0\leqslant v(z)\leqslant CR$, $|z|<2R+1$, for some universal constant C. By the maximum principle one has

$$u(y)\leqslant\lambda^{-1}v(y_1),\quad y=(y_1,\dots,y_d)\in B_R,$$

where λ is given by Eq. (4). Inequality (19) follows from this and hence the result. □

Estimate (7) is a consequence of the following lemma.

Lemma 2.4. *Let $U=\{x: A(x)\neq\frac{1}{2}I\}\subset\{x: |x_1|<1,\ |x_2|<1\}$. Then there is a constant C depending only on λ and Λ such that*

$$E_x\left[\int_0^T 1_U(X_L(s))\,\mathrm{d}s\right]\leqslant C|\log T|,\quad T\geqslant 2.$$

Proof. Again from Eqs. (17) and (18) it will be sufficient to show that

$$u(y)=E_y\left[\int_0^{\tau_R} 1_U(X_L(s))\,\mathrm{d}s\right]\leqslant C|\log R|,\quad R\geqslant 2. \tag{20}$$

To prove this observe that the diffusion process is just regular Brownian motion outside the set U. Consider the cylinders S_n, $n=0,1,2,\dots$, defined by

$$S_n=\{x=(x_1,\dots,x_d):\ (x_1^2+x_2^2)^{1/2}=2^{n+1}\}.$$

We view the diffusion as a random walk on the cylinders S_n. Observe that U is contained completely inside S_0. Hence the transition probabilities for the walk are determined by the Brownian motion probabilities. Let the walk be denoted by $Y(i)$, $i=0,1,2,\dots$, where i is an integer time variable. Then $Y(i)$ will take one of the values $0,1,2,\dots$, denoting which cylinders the walk is on. In particular we have

$$\begin{aligned}&P(Y(i+1)=1\mid Y(i)=0)=1,\\&P(Y(i+1)=n+1\mid Y(i)=n)=\tfrac{1}{2},\quad n\geqslant 1,\\&P(Y(i+1)=n-1\mid Y(i)=n)=\tfrac{1}{2},\quad n\geqslant 1.\end{aligned}$$

This follows from the fact that for $n\geqslant 1$, the probability that Brownian motion started on S_n exits the region between S_{n-1} and S_{n+1} through S_{n+1} is $\frac{1}{2}$. Let $N\geqslant 3$ be an arbitrary integer. We wish to estimate the number of times the walk hits S_0 before exiting through S_N. If the walk starts at n, $0\leqslant n\leqslant N$, the expectation of this quantity is

$$w(n)=E_n\left[\sum_{i=0}^{\tau_N}\delta_0(Y(i))\right],$$

where τ_N is the exit time to the cylinder S_N and δ_0 is the Kronecker δ, $\delta_0(k)=0$ if $k\neq 0$, $\delta_0(0)=1$. The function w satisfies the finite difference equation

$$\begin{aligned}&w(n)=\tfrac{1}{2}w(n+1)+\tfrac{1}{2}w(n-1),\quad 1\leqslant n\leqslant N-1,\\&w(0)=1+w(1),\\&w(N)=0.\end{aligned}$$

The solution to this is evidently given by the formula $w(n)=N-n$, $0\leqslant n\leqslant N$.

We can use the function w to estimate the function $u(y)$ as in Eq. (20). In fact let N be the smallest integer such that $2^{N+1}\geqslant R$. For z inside S_0 let τ be the time for the diffusion process started at z to hit S_1. Then we have for y outside S_0, the inequality

$$E_y\left[\int_0^{\tau_R}1_U(X_L(s))\mathrm{d}s\right]\leqslant E\left[E_{n(y)}\left[\sum_{i=0}^{\tau_N}\delta_0(Y(i))\sup_{z\in S_0}E_z\left[\int_0^{\tau}1_U(X_L(s))\mathrm{d}s\right]\right]\right],$$

where $S_{n(y)}$ denotes the first cylinder hit by the diffusion process started at y. By the same maximum principle argument that we had in Lemma 2.3 it follows that

$$\sup_{z\in S_0}E_z\left[\int_0^{\tau}1_U(X_L(s))\mathrm{d}s\right]\leqslant C,$$

for some constant C depending only on λ and Λ. Hence

$$u(y)\leqslant CE[w(n(y))]\leqslant cN,$$

whence Eq. (20) follows. $\square$

3. Random walk in random environment

We turn to the proof of Theorem 1.3. We consider the expression

$$E_{\boldsymbol{b}}\left\{E_{x_0}\left[f\left(\frac{\xi_{\boldsymbol{b}}(N)-x_0}{\sqrt{N}}\right)\right]^2\right\}$$
$$=\int f\left(\frac{\xi(N)-x_0}{\sqrt{N}}\right)f\left(\frac{\zeta(N)-x_0}{\sqrt{N}}\right)$$
$$\times E_{\boldsymbol{b}}\left[\exp\left[\beta\sum_{i=1}^{N}\boldsymbol{b}(i,\xi(i-1))\cdot\Delta\xi_i+\boldsymbol{b}(i,\zeta(i-1))\cdot\Delta\zeta_i\right]\right]\frac{\mathrm{d}W_N(\xi)\mathrm{d}W_N(\zeta)}{(\cosh\beta)^{2N}},$$

where ξ and ζ are independent standard random walks on $\mathbb{Z}^d$. Define a random walk X_L on $\mathbb{Z}^{2d}$ by $X_L(i)=(\xi(i),\zeta(i))$, $i=0,1,2,\ldots$. Then if we let g: $\mathbb{R}^{2d}\to\mathbb{R}$ be given by $g(x,y)=f(x)f(y)$, it follows that

$$E_{\boldsymbol{b}}\left\{E_{x_0}\left[f\left(\frac{\xi_{\boldsymbol{b}}(N)-x_0}{\sqrt{N}}\right)\right]^2\right\}=E_{(x_0,x_0)}\left[g\left(\frac{X_L(N)-(x_0,x_0)}{\sqrt{N}}\right)\right],$$

where $X_L(i)$, $i=0,1,\ldots,N$, has measure

$$E_{\boldsymbol{b}}\left[\exp\left[\beta\sum_{i=1}^{N}\boldsymbol{b}(i,\xi(i-1))\cdot\Delta\xi_i+\boldsymbol{b}(i,\zeta(i-1))\cdot\Delta\zeta_i\right]\right]\frac{\mathrm{d}W_N(\xi)\mathrm{d}W_N(\zeta)}{(\cosh\beta)^{2N}}. \quad (21)$$

It is clear that the measure (21) is Markovian. We can compute the transition probabilities for X_L. In fact

$$P(\Delta X_L(i)=(\delta x,\delta y)\,|\,X_L(i-1)=(x,y))=\left(\frac{1}{2d}\right)^2,$$

for vectors $\delta x,\delta y\in\mathbb{Z}^d$ of length 1 provided $x\neq y$. We also have

$$P(\Delta X_L(i)=(\delta x,\delta y)\,|\,X_L(i-1)=(x,x))=\left(\frac{1}{2d}\right)^2, \quad (22)$$

provided $\delta x\neq\pm\delta y$. If $\delta x=\pm\delta y$, then the transition probability depends on β. We have

$$P(\Delta X_L(i)=(\delta x,\delta x)\,|\,X_L(i-1)=(x,x))=\left(\frac{1}{2d}\right)^2\frac{\cosh(2\beta)}{(\cosh\beta)^2}, \quad (23)$$

$$P(\Delta X_L(i)=(\delta x,-\delta x)\,|\,X_L(i-1)=(x,x))=\left(\frac{1}{2d}\right)^2\frac{1}{(\cosh\beta)^2}. \quad (24)$$

Let $X(i)$, $i=0,1,2,\ldots$, denote the random walk in $\mathbb{Z}^{2d}$ with transition probabilities

$$P(\Delta X(i)=(\delta x,\delta y)\,|\,X(i-1)=(x,y))=\left(\frac{1}{2d}\right)^2, \quad (25)$$

for vectors δx, $\delta y\in\mathbb{Z}^d$ of length 1, $x,y\in\mathbb{Z}^d$. Then X is a translation invariant random walk in $\mathbb{Z}^{2d}$, whence it converges to Brownian motion in a large time limit. The walk X_L is like X except on the diagonal set $\{(x,x)\colon x\in\mathbb{Z}^d\}$ where the transition probabilities are different. If we take the scaling limit then the generator of X_L converges to an elliptic operator of form (3) where the coefficients differ from the Laplacian coefficients only along the diagonal set, that is a set of co-dimension d. Hence we should be able to prove a theorem analogous to Theorem 1.2.

Theorem 3.1. *Let g: $\mathbb{R}^{2d} \to \mathbb{R}$ be a function such that*

$$|g(x,y)| + \sum_{i=1}^{2d} \left|\frac{\partial g}{\partial z_i}(z)\right| + \sum_{i,j=1}^{2d} \left|\frac{\partial^2 g}{\partial z_i \partial z_j}(z)\right| \leqslant A \exp[M|z|], \quad z \in \mathbb{R}^{2d}, \tag{26}$$

for some constants A and M. Then for any $x_0, y_0 \in \mathbb{Z}^d$, there is a constant C depending only on A, M, β such that

$$\left| E_{x_0,y_0}\left[g\left(\frac{X_L(N) - (x_0,y_0)}{\sqrt{N}}\right)\right] - E_{x_0,y_0}\left[g\left(\frac{X(N) - (x_0,y_0)}{\sqrt{N}}\right)\right]\right| \leqslant C/\gamma_d(N),$$

where

$$\gamma_1(N) = \sqrt{N},$$
$$\gamma_2(N) = N/|\log N|, \quad N \geqslant 2,$$
$$\gamma_d(N) = N[1 + |x_0 - y_0|^{d-2}] \quad \textit{if } d \geqslant 3.$$

To prove Theorem 3.1 we shall do a discrete version of the argument of Section 2. Our first goal is to establish the analogue of Eq. (14). To do this let q: $\mathbb{R}^{2d} \to \mathbb{R}$ be an arbitrary function and $w(x,y,t)$ be defined by

$$w(x,y,t) = E_{x,y}[q(X(t))], \quad x,y \in \mathbb{Z}^d, \ t = 0,1,\dots \tag{27}$$

Then

$$w(x,y,t+1) - w(x,y,t) = \tfrac{1}{2}\Delta w(x,y,t),$$

where

$$\frac{1}{2}\Delta w(x,y,t) = \left(\frac{1}{2d}\right)^2 \sum_{\delta x, \delta y} [w(x+\delta x, y+\delta y, t) - w(x,y,t)],$$

where the sum is over $\delta x, \delta y \in \mathbb{Z}^d$ of length 1. Next consider $u(x,y,t)$ defined by

$$u(x,y,t) = E_{x,y}[q(X_L(t))], \quad x,y \in \mathbb{Z}^d, \ t = 0,1,\dots$$

Then

$$u(x,y,t+1) - u(x,y,t) = Lu(x,y,t),$$

where

$$Lu(x,y,t) = \tfrac{1}{2}\Delta u(x,y,t), \quad x \neq y,$$

and

$$Lu(x,x,t) = \left(\frac{1}{2d}\right)^2 \sum_{\delta x \neq \pm\delta y} [u(x+\delta x, x+\delta y, t) - u(x,x,t)]$$
$$+ \left(\frac{1}{2d}\right)^2 \sum_{\delta x} \left[\frac{\cosh(2\beta)}{(\cosh\beta)^2} u(x+\delta x, x+\delta x, t)\right.$$
$$\left. + \frac{1}{(\cosh\beta)^2} u(x+\delta x, x-\delta x, t) - 2u(x,x,t)\right],$$

where again the sum is over $\delta x, \delta y \in \mathbb{Z}^d$ of length 1. If we put $v(x,y,t) = w(x,y,t) - u(x,y,t)$, then

$$v(x,y,t+1) - v(x,y,t) = Lv(x,y,t) + h(x,y,t),$$

where

$$h(x,y,t) = (\tfrac{1}{2}\Delta - L)w(x,y,t).$$

Evidently $v(x,y,0) = 0$. Hence $v(x,y,t)$ is given by the formula

$$v(x,y,t) = \sum_{s=1}^{t} (I+L)^{t-s} h(x,y,s-1),$$

where I is the identity matrix. We can write this as an expectation value, namely

$$v(x,y,t) = E_{x,y}\left[\sum_{s=1}^{t} h(X_L(t-s), s-1)\right].$$

This is the analogue of formula (11). Next we wish to prove the analogue of Eq. (13).

Lemma 3.1. *Suppose g: $\mathbb{R}^{2d} \to \mathbb{R}$ satisfies the conditions of Theorem* 3.1. *Let $x_0, y_0 \in \mathbb{Z}^d$ and q: $\mathbb{R}^{2d} \in \mathbb{R}$ be defined by*

$$q(x,y) = g\left(\frac{x-x_0}{\sqrt{N}}, \frac{y-y_0}{\sqrt{N}}\right), \quad x, y \in \mathbb{R}^d,$$

and w be given by Eq. (27), *with $h = (\frac{1}{2}\Delta - L)w$. Then there are constants A_1 and M_1 depending only on A and M such that*

$$|h(x,y,t)| \leqslant \frac{A_1}{N} \exp[M_1\{|x-x_0| + |y-y_0|\}/\sqrt{N}], \quad 0 < t \leqslant N.$$

Proof. By translation invariance we have

$$w(x,y,t) = E_{x,y}\left[g\left(\frac{X(t)-(x_0,y_0)}{\sqrt{N}}\right)\right] = E_{0,0}\left[g\left(\frac{X(t)-(x-x_0,y-y_0)}{\sqrt{N}}\right)\right].$$

If we use now the fact that Δ and L are given by second-order differences and the bound (26) on g we can conclude that

$$h(x,y,t) \leqslant \frac{A}{N} E_{0,0}[\exp[M|\xi(t) + x - x_0|/\sqrt{N} + M|\zeta(t) + y - y_0|\sqrt{N}]]$$

where ξ and ζ are independent standard random walks on $\mathbb{Z}^d$. Hence we need to show that

$$E_0[\exp[M|\xi(t) + z_0|/\sqrt{N}]] \leqslant A_1 \exp[M_1|z_0|/\sqrt{N}], \quad 0 < t < N,$$

for constants A_1 and M_1 depending only on M. This follows from the standard estimate

$$P_0(\xi(t)=m) \leqslant \frac{C_\varepsilon}{t^{d/2}} \exp[-\varepsilon|m|^2/t],$$

where $\varepsilon > 0$ is sufficiently small and C_ε is a constant depending only on ε. □

From Lemma 3.1 it follows that

$$\begin{aligned} &\left| E_{x_0,y_0}\left[g\left(\frac{X_L(N)-(x_0,y_0)}{\sqrt{N}} \right)\right] - E_{x_0,y_0}\left[g\left(\frac{X(N)-(x_0,y_0)}{\sqrt{N}} \right)\right] \right| \\ &\quad = |v(x_0,y_0,N)| \\ &\quad \leqslant \frac{A_1}{N} E_{x_0,y_0}\left[\sum_{s=1}^{N} 1_U(X_L(N-s)) \exp[M_1|X_L(N-s)-(x_0,y_0|/\sqrt{N}] \right], \end{aligned} \tag{28}$$

where U is the set $U=\{(x,x) \in \mathbb{Z}^{2d}\colon x \in \mathbb{Z}^d\}$. Inequality (28) corresponds to relation (14). It is clear now that Theorem 3.1 will follow by the argument of Section 2 provided we can establish the analogues of Lemmas 2.1 and 2.2. The analogue of Lemma 2.1 is as follows.

Lemma 3.2. *Let τ_R be the time for the random walk X_L started at $(x_0,y_0) \in \mathbb{Z}^{2d}$ to go a distance R. Then there is a constant $\gamma > 0$ depending only on d such that*

$$P(\tau_R < t) \leqslant e^{-\gamma R^2/t}, \quad R \geqslant \sqrt{t}.$$

Proof. Let us write $x_0=(x_0^{(1)},\ldots,x_0^{(d)})$ and $y_0=(y_0^{(1)},\ldots,y_0^{(d)})$. For $(m,n)\in\mathbb{Z}^{2d}$, $m=(m^{(1)},\ldots,m^{(d)})$, $n=(n^{(1)},\ldots,n^{(d)})$ let τ_1 be the time taken for the random walk to exit the strip $\{(m,n)\in\mathbb{Z}^{2d}\colon |m^{(1)}-x_0^{(1)}|<r\}$, where r is an arbitrary positive integer. Consider the function $u(m,n)$ defined by

$$\begin{aligned} &Lu(m,n)=\eta u(m,n), \quad |m^{(1)}-x_0^{(1)}|<r, \\ &u(m,n)=1, \quad |m^{(1)}-x_0^{(1)}|=r. \end{aligned}$$

Then $u(m,n)$ is given as an expectation value by

$$u(m,n)=E_{(m,n)}[(1+\eta)^{-\tau_1}].$$

We can obtain an explicit formula for $u(m,n)$. To see this let $w(j)$, $j\in\mathbb{Z}$ satisfy

$$\begin{aligned} &\tfrac{1}{2}[w(j+1)+w(j-1)-2w(j)]=d\eta w(j), \quad |j-x_0^{(1)}|<r, \\ &w(j)=1, \quad |j-x_0^{(1)}|=r. \end{aligned}$$

It is easy to see that $u(m,n)=w(m^{(1)})$. We also have that

$$w(j)=\cosh((j-x_0^{(1)})k)/\cosh(rk), \quad |j-x_0^{(1)}|<r,$$

where

$$k=\cosh^{-1}(1+d\eta).$$

Hence

$$E_{x_0,y_0}[(1+\eta)^{-\tau_1}] \leqslant 1/\cosh(rk).$$

Arguing now as in Lemma 2.1 we conclude that

$$P(\tau_R < t) \leqslant 2d(1+\eta)^t/\cosh(Rk/(2d)^{1/2}).$$

If we use the fact that

$$\cosh(Rk/(2d)^{1/2}) \geqslant \tfrac{1}{2}\exp[Rk/(2d)^{1/2}], \quad 1+\eta \leqslant e^{\eta},$$

we have that

$$P(\tau_R < t) \leqslant 4d\ \exp[\eta t - Rk/(2d)^{1/2}]. \tag{29}$$

Evidently there is a constant c depending only on d such that if $R > ct$ then $P(\tau_R < t) = 0$. Hence we can assume that $\sqrt{t} \leqslant R \leqslant ct$. Observe also that for any $\eta_0 \geqslant 1$ there is a constant $c_0 > 0$ depending only on η_0, d such that $k \geqslant c_0\sqrt{\eta}$, $0 < \eta \leqslant \eta_0$. Hence relation (29) yields the inequality

$$P(\tau_R < t) \leqslant 4d\exp[\eta t - Rc_0\sqrt{\eta}/(2d)^{1/2}].$$

Optimizing this inequality with respect to η yields

$$P(\tau_R < t) \leqslant 4d\exp[-R^2c_0^2/(8dt)],$$

where the optimizing η is given by

$$\eta = \frac{c_0^2}{8d}\left(\frac{R}{t}\right)^2.$$

Since we are assuming $R \leqslant ct$ we can choose an appropriate η_0. $\square$

Next we prove the analogue of Lemma 2.3 in the case $d = 1$.

Lemma 3.3. *Let $U = \{(x,x) \in \mathbb{Z}^2 \colon x \in \mathbb{Z}\}$. Then for any $x_0, y_0 \in \mathbb{Z}$, one has*

$$E_{x_0,y_0}\left[\sum_{s=0}^{N-1} 1_U(X_L(s))\right] \leqslant C\sqrt{N}, \quad N \geqslant 1,$$

where the constant C depends only on β.

Proof. In view of Lemma 3.2 it will be sufficient to show that

$$E_{x_0,y_0}\left[\sum_{s=0}^{\tau_R} 1_U(X_L(s))\right] \leqslant CR, \tag{30}$$

where τ_R is the time taken for the random walk to exit the disc of radius R centered at (x_0, y_0). To estimate this we consider X_L as a random walk on the lines $\mathscr{H}_n = \{(x-n, x+n) \colon x \in \mathbb{Z}\}$, $n \in \mathbb{Z}$. Clearly $U = \mathscr{H}_0$. Denote this random walk by Y. Then for $i = 0, 1, 2, \ldots, Y(i) = n$ if and only if $X_L(i) \in \mathscr{H}_n$. The walk Y is Markovian and we can easily compute the transition probabilities. We have

$$P(\Delta Y(i) = 0 \mid Y(i-1) = n) = \tfrac{1}{2},$$

$$P(\Delta Y(i) = 1 \mid Y(i-1) = n) = \tfrac{1}{4},$$

$$P(\Delta Y(i) = -1 \mid Y(i-1) = n) = \tfrac{1}{4},$$

provided $n \neq 0$.

$$P(\Delta Y(i)=0 \mid Y(i-1)=0)=\cosh(2\beta)/(2(\cosh\beta)^2),$$

$$P(\Delta Y(i)=1 \mid Y(i-1)=0)=1/(4(\cosh\beta)^2),$$

$$P(\Delta Y(i)=-1 \mid Y(i-1)=0)=1/(4(\cosh\beta)^2).$$

Suppose now that the random walk Y starts at a point $n \in \mathbb{Z}$ inside the interval $\{n \in \mathbb{Z}\colon n_0-r \leqslant n \leqslant n_0+r\}$. Let τ_r be the first time it exits this interval and put

$$w(n)=E_n\left[\sum_{i=0}^{\tau_r}\delta_0(Y(i))\right],$$

where δ_0 is the Kronecker δ, $\delta_0(j)=0$ if $j \neq 0$, $\delta_0(0)=1$. Evidently if y_0-x_0 is even and we put $n_0=(y_0-x_0)/2$ then

$$E_{x_0,y_0}\left[\sum_{s=0}^{\tau_R}1_U(X_L(s))\right] \leqslant E_{n_0}\left[\sum_{i=0}^{\tau_r}\delta_0(Y(i))\right],$$

for any r satisfying $r \geqslant R/\sqrt{2}$. The function $w(n)$ satisfies a finite difference equation

$$\begin{aligned}
&w(n)=\tfrac{1}{2}w(n-1)+\tfrac{1}{2}w(n+1), \quad n_0-r<n<n_0+r,\ n\neq 0,\\
&w(0)=1+\frac{\cosh(2\beta)}{2(\cosh\beta)^2}w(0)+\frac{1}{4(\cosh\beta)^2}[w(1)+w(-1)],\\
&w(n_0-r)=w(n_0+r)=0.
\end{aligned} \tag{31}$$

Here we are assuming that $n_0-r,\ n_0+r \neq 0$. Let us assume that $n_0-r<0<n_0+r$, for otherwise w is identically zero. We can then solve problem (31) by a piecewise linear function. We put

$$w(n)=\alpha(n_0+r-n)/(n_0+r), \quad 0<n\leqslant n_0+r,$$

$$w(n)=\alpha(n-n_0+r)/(r-n_0), \quad n_0-r\leqslant n<0.$$

Solving the second equation in relation (31) for α we obtain

$$\alpha=2(r^2-n_0^2)(\cosh\beta)^2/r.$$

Since $n_0-r<0<n_0+r$ it follows that $\alpha>0$. Hence there is a constant C depending only on β such that

$$w(n)\leqslant Cr, \quad n_0-r<n<n_0+r.$$

We conclude that Eq. (30) holds and hence the result. $\square$

Theorem 3.1 for the case of $d=1$ follows from Lemma 3.3. To establish Theorem 3.1 for $d>1$ we need to make a comparison between random walk probabilities and Brownian motion probabilities. For $R>0$, $d\geqslant 0$, let $A_{R,d}$ denote the annulus,

$$A_{R,d}=\{(x,y)\in\mathbb{R}^{2d}\colon x,y\in\mathbb{R}^d,\ R<|y-x|<4R\}.$$

Let $w_{R,d}(x,y)$ be the probability that Brownian motion started at $(x,y)\in A_{R,d}$ exits $A_{R,d}$ through the boundary $\{(x',y'):\ |y'-x'|=R\}$. Thus $w_{R,d}$ satisfies the Dirichlet problem

$$\Delta w_{R,d}(x,y)=0,\quad (x,y)\in A_{R,d},$$

$$w_{R,d}(x,y)=1,\quad |y-x|=R,$$

$$w_{R,d}(x,y)=0,\quad |y-x|=4R.$$

It is easy to see that $w_{R,d}$ is given by the formulas

$$w_{R,2}(x,y)=\log(4R/|y-x|)/\log 4,$$

$$w_{R,d}(x,y)=\left[\left(\frac{4R}{|y-x|}\right)^{d-2}-1\right]\Big/[4^{d-2}-1],\quad d\geqslant 3.$$

Lemma 3.4. *For* $(x,y)\in A_{R,d}\cap\mathbb{Z}^{2d}$ *let* $u_{R,d}(x,y)$ *be the probability that the random walk* X *defined by Eq.* (25) *exits* $A_{R,d}$ *through the boundary* $\{(x',y'):\ |y'-x'|=R\}$. *Then there is a constant* C *depending only on* d *such that*

$$|u_{R,d}(x,y)-w_{R,d}(x,y)|\leqslant C/R,\quad (x,y)\in A_{R,d}\cap\mathbb{Z}^{2d}.$$

Proof. The function $u_{R,d}(x,y)$ satisfies the boundary value problem

$$\sum_{\delta x,\delta y}[u_{R,d}(x+\delta x,y+\delta y)-u_{R,d}(x,y)]=0,\quad (x,y)\in \mathrm{Int}(A_{R,d})\cap\mathbb{Z}^{2d},$$

$$u_{R,d}(x,y)=0,\quad (x,y)\in\mathbb{Z}^{2d},|y-x|\geqslant 4R,$$

$$u_{R,d}(x,y)=1,\quad (x,y)\in\mathbb{Z}^{2d},|y-x|\leqslant R,$$

where the sum is over all vectors $\delta x,\delta y\in\mathbb{Z}^d$ of length 1. We use the Taylor expansion

$$\begin{aligned}w_{R,d}((x,y)+v)&=w_{R,d}(x,y)+v\cdot\nabla w_{R,d}(x,y)+\tfrac12(v\cdot\nabla)^2w_{R,d}(x,y)\\&\quad+\tfrac16(v\cdot\nabla)^3w_{R,d}((x,y)+\theta v),\end{aligned}$$

for some, θ, $0<\theta<1$. It is clear that for $(x,y)\in\mathrm{Int}(A_{R,d})\cap\mathbb{Z}^{2d}$, we have

$$\sum_{\delta x,\delta y}[w_{R,d}(x+\delta x,y+\delta y)-w_{R,d}(x,y)]$$

$$=\tfrac12\Delta_x w_{R,d}(x,y)+\tfrac12\Delta_y w_{R,d}(x,y)+\mathrm{O}(1/R^3),$$

whence

$$\left|\sum_{\delta x,\delta y}[w_{R,d}(x+\delta x,y+\delta y)-w_{R,d}(x,y)]\right|\leqslant C/R^3,\quad (x,y)\in\mathrm{Int}(A_{R,d})\cap\mathbb{Z}^{2d}.$$

Next let us extend $w_{R,d}$ by $w_{R,d}(x,y)=0, |y-x|\geqslant 4R+3$, $w_{R,d}$ by $w_{R,d}(x,y)=1$, $|y-x|\leqslant R-3$. Since $|\nabla w_{R,d}|=\mathrm{O}(1/R)$, it follows that the function $v=u_{R,d}-w_{R,d}$ satisfies

$$\left|\sum_{\delta x,\delta y}[v_{R,d}(x+\delta x,y+\delta y)-v_{R,d}(x,y)]\right|\leqslant C/R^3,\quad (x,y)\in\mathrm{Int}(A_{R,d})\cap\mathbb{Z}^{2d},$$

$$|v(x,y)|\leqslant C/R,\quad |y-x|\leqslant R \quad \text{or} \quad |y-x|\geqslant 4R,$$

where C is a constant depending only on d. Hence

$$|v(x,y)|\leqslant \frac{C}{R}+\frac{C}{(2d)^2R^3}E_{x,y}[\tau],$$

where τ is the time taken for the random walk X started at (x,y) to exit $A_{R,d}$. It is easy to see that

$$E_{x,y}[\tau]\leqslant \tfrac{1}{2}[(4R+\sqrt{2d})^2-|y-x|^2],$$

whence the result follows. $\square$

Lemma 3.5. *Let* $U=\{(x,x)\in\mathbb{Z}^4\colon x\in\mathbb{Z}^2\}$. *Then for any* $(x_0,y_0)\in\mathbb{Z}^2$, *one has*

$$E_{x_0,y_0}\left[\sum_{s=0}^{N-1}1_U(X_L(s))\right]\leqslant C|\log N|,\quad N\geqslant 2,$$

where the constant C *depends only on* β.

Proof. We follow the lines of the proof of Lemma 2.4. From Lemma 3.2 it will be sufficient to show that

$$E_{x_0,y_0}\left[\sum_{s=0}^{\tau_R}1_U(X_L(s))\right]\leqslant C|\log R|,\quad R\geqslant 2, \tag{32}$$

where τ_R is the time taken for the random walk to exit the cylinder $\{(x,y)\in\mathbb{Z}^4\colon |y-x|<R\}$. For $n=0,1,\ldots,$ let Γ_n be the cylinder

$$\Gamma_n=\{(x,y)\in\mathbb{Z}^4\colon 2^{n+1}-2\leqslant|x-y|<2^{n+1}\}.$$

Evidently the Γ_n are disjoint and $U\subset\Gamma_0$. Further, the random walk X_L hits a point in Γ_n for each n. Hence we may regard X_L as a random walk on the cylinders Γ_n.

We can estimate the expected number of times X_L visits Γ_0 before hitting Γ_M, $M\geqslant 1$, by using Lemma 3.4 and the argument of Lemma 3.17 of Conlon and Olsen (1997). Thus for $j=1,\ldots,M-1$ let $p_j(m,m')$, $m\in\Gamma_j$, $m'\in\Gamma_{j+1}$ be the probability that the random walk X_L started at m exits the region between Γ_{j-1} and Γ_{j+1} through the point m'. Let us put

$$p_j=\inf_{m\in\Gamma_j}\sum_{m'\in\Gamma_{j+1}}p_j(m,m').$$

From Lemma 3.4 it follows that there is a constant C such that

$$p_j\geqslant\tfrac{1}{2}\exp[-C2^{-j}],\quad j=1,2,\ldots. \tag{33}$$

Putting $q_j=1-p_j$, $j=1,2,\ldots,$ it follows from the argument of Lemma 3.17 of Conlon and Olsen (1997) that

$$E_m[N_0]\leqslant 1+\sum_{j=1}^{M-1}\prod_{i=1}^{j}\frac{q_i}{p_i},\quad m\in\Gamma_0, \tag{34}$$

where N_0 is the number of times the walk hits Γ_0 before hitting Γ_M. In view of Eq. (33) it follows that

$$E_m[N_0]\leqslant CM$$

for some constant C. If we take now M to be the smallest integer such that $2^{M+1} > R+2$, then

$$E_{x_0,y_0}\left[\sum_{s=0}^{\tau_R} 1_U(X_L(s))\right] \leqslant CM \sup_{(x_0,y_0)\in\Gamma_0} E_{x_0,y_0}\left[\sum_{s=0}^{\tau_1} 1_U(X_L(s))\right],$$

where τ_1 is the first time the walk X_L hits Γ_1. Inequality (32) will follow now if we can establish that there is a constant $C(\beta)$ depending on β such that

$$\sup_{(x_0,y_0)\in\Gamma_0} E_{x_0,y_0}[\tau_1] \leqslant C(\beta).$$

It is a well known fact that if we put

$$\gamma(\beta) = \sup_{(x_0,y_0)\in\Gamma_0} P_{x_0,y_0}(\tau_1 > 10),$$

then

$$\sup_{(x_0,y_0)\in\Gamma_0} E_{x_0,y_0}[\tau_1] \leqslant \sum_{n=1}^{\infty} 10n\gamma(\beta)^{n-1} = 10/[1-\gamma(\beta)]^2,$$

provided $\gamma(\beta) < 1$. Since we can easily construct a path from any point in Γ_0 to Γ_1 in less than 10 steps it follows that $\gamma(\beta) < 1$. □

Theorem 3.1 for the case of $d = 2$ follows from Lemma 3.5. To establish Theorem 3.1 for the case of $d \geqslant 3$ we prove the following:

Lemma 3.6. *Let* $U = \{(x,x) \in \mathbb{Z}^{2d}\colon x \in \mathbb{Z}^d\}$. *Then for any* $(x_0,y_0) \in \mathbb{Z}^{2d}$, $d \geqslant 3$, *one has*

$$E_{x_0,y_0}\left[\sum_{s=0}^{N-1} 1_U(X_L(s))\right] \leqslant C,$$

where C *depends only on* d *and* β.

Proof. We proceed exactly as in Lemma 3.5. Thus for $n = 0,1,\ldots,$ let Γ_n be the cylinder

$$\Gamma_n = \{(x,y) \in \mathbb{Z}^{2d}\colon 2^{n+1} - 2 \leqslant |x-y| < 2^{n+1}\}. \tag{35}$$

Defining $p_j(m,m')$ and p_j as in Lemma 3.5 we can conclude from Lemma 3.4 that

$$p_j \geqslant \frac{2^{d-2}}{2^{d-2}+1}\exp[-C2^{-j}], \quad j = 1,2,\ldots. \tag{36}$$

Hence from Eq. (34) it follows that $E_m[N_0] \leqslant C$, $m \in \Gamma_0$, for some constant C depending only on d. The result follows now by continuing the argument exactly as in Lemma 3.5. □

Lemma 3.7. *Let* $U = \{(x,x) \in \mathbb{Z}^{2d}\colon x \in \mathbb{Z}^d\}$. *For* $(x_0,y_0) \in \mathbb{Z}^{2d}$ *let* $P_{x_0,y_0}(U)$ *be the probability that the walk* X_L, *started at* (x_0,y_0), *hits* U *before exiting to infinity. Then there is a constant* C *depending only on* $d \geqslant 3$ *such that*

$$P_{x_0,y_0}(U) \leqslant C/[1+|x_0-y_0|^{d-2}].$$

Proof. We use the notations of Lemma 3.6. For $M \geqslant 1$ let $P_{x_0,y_0,M}$ be the probability of hitting Γ_0 before Γ_M for the random walk X_L started at (x_0, y_0). Then by Lemma 6.3 of Conlon and Redondo (1995) we have, for any $(x_0, y_0) \in \Gamma_{M'}$, $0 < M' < M$,

$$P_{x_0,y_0,M} \leqslant \frac{1 + \frac{p_{M-1}}{q_{M-1}} + \frac{p_{M-1}p_{M-2}}{q_{M-1}q_{M-2}} + \cdots + \prod_{j=1}^{M-M'-1} \frac{p_{M-j}}{q_{M-j}}}{1 + \frac{p_{M-1}}{q_{M-1}} + \frac{p_{M-1}p_{M-2}}{q_{M-1}q_{M-2}} + \cdots + \prod_{j=1}^{M-1} \frac{p_{M-j}}{q_{M-j}}}.$$

From Eq. (36) and the above inequality we can conclude that

$$P_{x_0,y_0,M} \leqslant C/2^{M'(d-2)}, \quad (x_0, y_0) \in \Gamma_{M'},\ 0 < M' < M,$$

for some constant C depending only on d. Letting $M \to \infty$ and using the fact that $U \in \Gamma_0$ it follows that

$$P_{x_0,y_0}(U) \leqslant C/2^{M'(d-2)}, \quad (x_0, y_0) \in \Gamma_{M'},\ 0 < M' < M.$$

The result follows from this last inequality. $\square$

Theorem 3.1 for $d \geqslant 3$ is a consequence now of Lemmas 3.6, 3.7 and the argument of Section 2.

4. Convergence with probability one of RWRE to a Gaussian

Here we shall use Theorem 3.1 to prove Theorem 1.3. First we give a proof of the central limit theorem for the standard random walk in $\mathbb{Z}^d$, $d \geqslant 1$.

Lemma 4.1. *Let $f: \mathbb{R}^d \to \mathbb{R}$ be a function such that*

$$|f(y)| + \sum_{i=1}^{d} \left| \frac{\partial f}{\partial x_i}(y) \right| \leqslant A \exp[M|y|], \quad y \in \mathbb{R}^d, \tag{37}$$

for some constant A and M. Let $\xi(t)$, $t = 0, 1, 2, \ldots$, be the standard random walk on $\mathbb{Z}^d$. Then for any $x_0 \in \mathbb{Z}^d$,

$$\left| E_{x_0}\left[f\left(\frac{\xi(N) - x_0}{\sqrt{N/d}} \right) \right] - E[f(Y)] \right| \leqslant C/\sqrt{N},$$

where the constant C depends only on A, M, d.

Proof. We have

$$\begin{aligned} E[f(Y)] &= \frac{1}{(\frac{2\pi N}{d})^{d/2}} \int_{\mathbb{R}^d} \exp\left[-\frac{y^2}{2N/d} \right] f\left(\frac{y}{\sqrt{N/d}} \right) \mathrm{d}y \\ &= \sum_{m \in \mathbb{Z}^d} \frac{1}{(2\pi N/d)^{d/2}} \int_{Q_d} \exp\left[-\frac{(z+m)^2}{2N/d} \right] f\left(\frac{z+m}{\sqrt{N/d}} \right) \mathrm{d}z, \end{aligned} \tag{38}$$

where Q_d is the unit cube in $\mathbb{R}^d$ centered at the origin. Since ξ is the standard random walk in $\mathbb{Z}^d$ it is well known that there exists $\varepsilon > 0$ and a constant C_ε such that

$$P(\xi(N) - \xi(0) = m) \leqslant \frac{C_\varepsilon}{N^{d/2}} \exp[-\varepsilon m^2/N], \quad m \in \mathbb{Z}^d. \tag{39}$$

We also have that

$$P(\xi(N)-\xi(0)=m)=\frac{1}{(2\pi N/d)^{d/2}}\exp\left[-\frac{m^2}{2N/d}\right][1+\mathrm{O}(1/N^{1-4\nu})], \tag{40}$$

provided $|m|\leqslant N^{1/2+\nu}$ and ν satisfies $0<\nu<1/4$. We have now

$$E_{x_0}\left[f\left(\frac{\xi(N)-x_0}{\sqrt{N/d}}\right)\right]=\sum_{m\in\mathbb{Z}^d}P(\xi(N)-\xi(0)=m)f\left(\frac{m}{\sqrt{N/d}}\right).$$

From Eq. (37), it follows that for any $\varepsilon>0$,

$$\sum_{m\in\mathbb{Z}^d,|m|>N^{1/2+\nu}}\frac{1}{N^{d/2}}\exp\left[-\frac{\varepsilon m^2}{N}\right]\left|f\left(\frac{m}{\sqrt{N/d}}\right)\right|\leqslant C\exp\left[-\frac{\varepsilon N^{2\nu}}{2}\right],$$

where C depends only on A, M and d. Hence from Eqs. (39) and (40) it follows that

$$\left|E_{x_0}\left[f\left(\frac{\xi(N)-x_0}{\sqrt{N/d}}\right)\right]-\sum_{m\in\mathbb{Z}^d}\frac{1}{(2\pi N/d)^{d/2}}\exp\left[-\frac{m^2}{2N/d}\right]f\left(\frac{m}{\sqrt{N/d}}\right)\right|\leqslant C/N^{1-4\nu}, \tag{41}$$

where C depends only on A, M and d. Observe next that for any $z\in Q_d$, $m\in\mathbb{Z}^d$,

$$\left|\exp\left[-\frac{m^2}{2N/d}\right]f\left(\frac{m}{\sqrt{N/d}}\right)-\exp\left[-\frac{(z+m)^2}{2N/d}\right]f\left(\frac{z+m}{\sqrt{N/d}}\right)\right|\leqslant\frac{C}{\sqrt{N}}\exp\left[-\frac{m^2}{4N/d}\right],$$

where the constant C depends only on A, M and d. The result follows from this last inequality and Eqs. (38) and (41) provided we take $\nu<1/8$. $\square$

Proof of Theorem 1.3. Case $d=1$: Let Z_N be the random variable

$$Z_N=E_{x_0}\left[f\left(\frac{\xi_{\boldsymbol{b}}(N)-x_0}{\sqrt{N/d}}\right)\right]-E_{x_0}\left[f\left(\frac{\xi(N)-x_0}{\sqrt{N/d}}\right)\right]. \tag{42}$$

Then from Theorem 3.1 we have that

$$E[Z_N^2]\leqslant C/\sqrt{N},$$

where C depends only on A, M and β. Hence by standard argument

$$\lim_{N\to\infty}Z_N=0,\quad \text{with probability } 1, \tag{43}$$

provided N goes to infinity along a sequence $N=a_n$, where

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{a_n}}<\infty.$$

Theorem 3.1 for $d=1$ follows now from Eq. (43) and Lemma 4.1. $\square$

Case $d=2$: By Theorem 3.1 the random variable Z_N satisfies the inequality

$$E[Z_N^2]\leqslant C(\log N)/N,\quad N\geqslant 3.$$

Hence the result follows just as in the case $d=1$.

Case $d \geqslant 3$: Observe that just as in the $d = 1,2$ cases the result would hold if we let $N = a_n$, where

$$\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty.$$

To get convergence through the full integer sequence we follow the strategy in Conlon and Olsen (1996). Thus, let $\rho_{\boldsymbol{b}}(t,x)$, $t = 0,1,\ldots$, $x \in \mathbb{Z}^d$, be the probability density for $\xi_{\boldsymbol{b}}(t)$,

$$\rho_{\boldsymbol{b}}(t,x) = P(\xi_{\boldsymbol{b}}(t) = x | \xi_{\boldsymbol{b}}(0) = x_0).$$

Let $0 < \gamma < 1$ and consider the random variable

$$W_N = E_{x_0}\left[f\left(\frac{\xi_{\boldsymbol{b}}(N) - x_0}{\sqrt{N/d}} \right) \right] - \sum_{x \in \mathbb{Z}^d} \rho_{\boldsymbol{b}}(N^\gamma, x) E_x \left[f\left(\frac{\xi(N - N^\gamma) - x_0}{\sqrt{N/d}} \right) \right].$$

Lemma 4.2. *For any $p > 1$, there is a constant C depending only on p, A, M, d, γ and β such that*

$$E[W_N^2] \leqslant C/N^{1-\gamma/p+\gamma d/(2p)}.$$

Proof. Observe that if we condition on the variables $\boldsymbol{b}(t,z)$, $t \leqslant N^\gamma$, $z \in \mathbb{Z}^d$, then

$$\begin{aligned} E[W_N^2 | \boldsymbol{b}(t,z),\ t \leqslant N^\gamma,\ z \in \mathbb{Z}^d] = \sum_{x,y \in \mathbb{Z}^d} & \rho_{\boldsymbol{b}}(N^\gamma, x)\rho_{\boldsymbol{b}}(N^\gamma, y) \\ & \times \left\{ E_{x,y}\left[g\left(\frac{X_L(N - N^\gamma) - (x_0, x_0)}{\sqrt{N}} \right) \right] \right. \\ & \left. - E_{x,y}\left[g\left(\frac{X(N - N^\gamma) - (x_0, x_0)}{\sqrt{N}} \right) \right] \right\}, \end{aligned}$$

where X, X_L are the random variables in $\mathbb{Z}^{2d}$ defined in Section 3 and the function g is given in terms of f by

$$g(x,y) = f(x\sqrt{d}) f(y\sqrt{d}), \quad x, y, \in \mathbb{R}^d.$$

It follows now from Theorem 3.1 that

$$\begin{aligned} & E[W_N^2 | \boldsymbol{b}(t,z),\ t \leqslant N^\gamma,\ z \in \mathbb{Z}^d] \\ & \leqslant \sum_{x,y \in \mathbb{Z}^d} \rho_{\boldsymbol{b}}(N^\gamma, x)\rho_{\boldsymbol{b}}(N^\gamma, y) \frac{C}{N[1 + |x - y|^{d-2}]} \exp\left[\frac{M|x - x_0|}{\sqrt{N/d}} + \frac{M|y - x_0|}{\sqrt{N/d}} \right], \end{aligned}$$

where M is the constant of Theorem 1.1 and C depends only on A, M, d, γ and β.

Observing next that

$$E_{\boldsymbol{b}}[\rho_{\boldsymbol{b}}(N^\gamma, x)\rho_{\boldsymbol{b}}(N^\gamma, y)] = P(X_L(N^\gamma) = (x,y) \,|\, X_L(0) = (x_0, x_0))],$$

we conclude that

$$E[W_N^2] \leqslant \frac{C}{N} E_{x_0,x_0}[h(X_L(N^\gamma)) \exp[\sqrt{d/N} M |X_L(N^\gamma) - (x_0, x_0)|]],$$

where the function h is given by

$$h(x,y) = 1/[1 + |x - y|^{d-2}], \quad x, y \in \mathbb{R}^d.$$

Let p satisfy the inequality $1 < p < d/(d-2)$. Then Holder's inequality yields

$$E[W_N^2] \leqslant \frac{C}{N} E_{x_0,x_0}[h(X_L(N^\gamma))^p]^{1/p} E_{x_0,x_0}[\exp[p' \sqrt{d/N} M |X_L(N^\gamma) - (x_0,x_0)|]]^{1/p'}$$

where $1/p + 1/p' = 1$. It follows easily from Lemma 3.2 that

$$E_{x_0,x_0}[\exp[p' \sqrt{d/N} M |X_L(N^\gamma) - (x_0,x_0)|]]^{1/p'} \leqslant C,$$

where the constant C depends only on p', d, γ. Hence we conclude that there is a constant C such that

$$E[W_N^2] \leqslant \frac{C}{N} E_{x_0,x_0}[h(X_L(N^\gamma))^p]^{1/p}. \tag{44}$$

We bound the RHS of Eq. (44) by using the methodology of Lemma 3.6. We define regions U_n, $n = 0, 1, \ldots$ by $U_0 = \Gamma_0$, U_n is the region bounded by Γ_{n+1} and disjoint from U_{n-1}, $n = 1, 2, \ldots$, where the sets Γ_n are defined by Eq. (35). Thus

$$U_n = \{(x,y) \in \mathbb{Z}^{2d}\colon 2^{n+1} - 2 \leqslant |x-y| < 2^{n+2} - 2\}, \quad n \geqslant 1.$$

Hence

$$E_{x_0,x_0}[h(X_L(N^\gamma))^p] \leqslant \sum_{n=0}^{\infty} \frac{P(X_L(N^\gamma) \in U_n)}{[1 + (2^{n+1} - 2)^{d-2}]^p}.$$

We define the random variable n^* by

$$n^* = \sup\{n \in \mathbb{Z}\colon X_L(s) \in U_n, \text{ for some } s,\ 0 \leqslant s < N^\gamma\}.$$

From the argument of Lemma 3.7 we can conclude that there is a constant C depending only on d such that

$$P(X_L(N^\gamma) \in U_n) \leqslant P(n^* < n) + C \sum_{m=n}^{\infty} \frac{P(n^* = m)}{2^{(m-n)(d-2)}}.$$

Since $p > 1$ it follows that

$$\sum_{n=0}^{m} \frac{1}{2^{(m-n)(d-2)}} \frac{1}{[1 + (2^{n+1} - 2)^{d-2}]^p} \leqslant \frac{C}{2^{m(d-2)}},$$

for some constant C depending only on p and d. Hence we have the inequality

$$E_{x_0,x_0}[h(X_L(N^\gamma))^p] \leqslant C \sum_{m=0}^{\infty} \frac{P(n^* \leqslant m)}{2^{m(d-2)}},$$

for some constant C depending only on p and d. Define m_0 as the smallest integer such that $2^{m_0} \geqslant N^{\gamma/2}$. We shall see that there are constants C, $\delta > 0$ depending only on d and β such that

$$P(n^* \leqslant m) \leqslant C \exp[-\delta 2^{2(m_0 - m)}], \quad m \leqslant m_0. \tag{45}$$

It follows from this that

$$\begin{aligned} E_{x_0,x_0}[h(X_L(N^\gamma))^p] &\leqslant C \sum_{m=m_0+1}^{\infty} \frac{1}{2^{m(d-2)}} + C \sum_{m=0}^{m_0} \exp[-\delta 2^{2(m_0-m)}] \frac{1}{2^{m(d-2)}} \\ &\leqslant \frac{C_1}{2^{m_0(d-2)}} \leqslant \frac{C_1}{N^{\gamma(d-2)/2}}, \end{aligned}$$

for some constant C_1 depending only on d and β. The result follows from this last inequality and Eq. (44) provided we can prove Eq. (45).

To see this we define regions W_R by

$$W_R = \{(x, y) \in \mathbb{Z}^{2d}: |x-y| < R\}.$$

Let τ_R be the time taken for the walk X_L to exit the region W_R. Then

$$P(n^* \leqslant m) \leqslant P_{x_0,x_0}(\tau_{2^{m+2}} > N^{\gamma}).$$

Inequality (45) follows from this if we can show that

$$P_{x,y}(\tau_R > t) \leqslant C \exp[-\delta t/R^2], \quad t \geqslant 0, \tag{46}$$

for any $(x, y) \in W_R$. It is well known that inequality (46) is a consequence of the inequality

$$u(x, y) = E_{x,y}[\tau_R] \leqslant C_1 R^2, \quad (x, y) \in W_R, \tag{47}$$

where C_1 depends only on d and β. In view of Eqs. (22)–(24) it follows that

$$u(x, y) = 1 + \left(\frac{1}{2d}\right)^2 \sum_{\delta x, \delta y} u(x + \delta x, y + \delta y), \quad x \neq y,$$

and

$$\begin{aligned} u(x,x) = 1 &+ \left(\frac{1}{2d}\right)^2 \sum_{\delta x \neq \pm \delta y} u(x + \delta x, x + \delta y) \\ &+ \left(\frac{1}{2d}\right)^2 \sum_{\delta x} \frac{\cosh(2\beta)}{(\cosh \beta)^2} u(x + \delta x, x + \delta x) \\ &+ \left(\frac{1}{2d}\right)^2 \sum_{\delta x} \frac{1}{(\cosh \beta)^2} u(x + \delta x, x - \delta x), \end{aligned}$$

where $\delta x, \delta y$ range over vectors in $\mathbb{Z}^d$ of length 1. The boundary condition on u is $u(x, y) = 0$, $(x, y) \notin W_R$. Let w be the function

$$w(x, y) = (R + \sqrt{2d})^2 - |y - x|^2.$$

Then one has

$$w(x, y) = 2 + \left(\frac{1}{2d}\right)^2 \sum_{\delta x, \delta y} w(x + \delta x, y + \delta y), \quad x \neq y,$$

and

$$\begin{aligned} w(x,x) = \frac{2d-2}{d} + \frac{2}{d}\frac{1}{(\cosh \beta)^2} &+ \left(\frac{1}{2d}\right)^2 \sum_{\delta x \neq \pm \delta y} w(x + \delta x, x + \delta y) \\ &+ \left(\frac{1}{2d}\right)^2 \sum_{\delta x} \frac{\cosh(2\beta)}{(\cosh \beta)^2} w(x + \delta x, x + \delta x) \\ &+ \left(\frac{1}{2d}\right)^2 \sum_{\delta x} \frac{1}{(\cosh \beta)^2} w(x + \delta x, x - \delta x). \end{aligned}$$

If we extend w by zero for $|y-x|>R+\sqrt{2d}$ then it is clear that $w(x,y)\geqslant 0$, $(x,y)\notin W_R$. It follows then from the last two equations that

$$u(x,y)\leqslant \left[\frac{2d-2}{d}+\frac{2}{d}\frac{1}{(\cosh\beta)^2}\right]^{-1} w(x,y), \quad (x,y)\in W_R.$$

Inequality (47) follows from this last inequality. □

For any γ, $0<\gamma<1$, one can find $p>1$ sufficiently small such that $1-\gamma/p+\gamma d/(2p)>1$. Hence

$$\lim_{N\to\infty} W_N=0 \quad \text{with probability 1} \tag{48}$$

as N goes to infinity through the whole integer sequence. Now with Z_N given by Eq. (42) one has

$$W_N-Z_N=\sum_{x\in\mathbb{Z}^d}\rho_{\boldsymbol{b}}(N^\gamma,x)\left\{E_{x_0}\left[f\left(\frac{\xi(N)-x_0}{\sqrt{N/d}}\right)\right]-E_x\left[f\left(\frac{\xi(N-N^\gamma)-x_0}{\sqrt{N/d}}\right)\right]\right\}.$$

Lemma 4.3. *As N goes to infinity through the whole integer sequence,*

$$\lim_{N\to\infty}(W_N-Z_N)=0 \quad \textit{with probability } 1.$$

Proof. It is easy to see that there is a constant C depending only on d and the constants A,M in Theorem 1.1 such that for any $x\in\mathbb{Z}^d$,

$$\left|E_{x_0}\left[f\left(\frac{\xi(N)-x_0}{\sqrt{N/d}}\right)\right]-E_x\left[f\left(\frac{\xi(N-N^\gamma)-x_0}{\sqrt{N/d}}\right)\right]\right|\leqslant C\exp[M|x-x_0|(d/N)^{1/2}].$$

Let us define the random variable Y_N by

$$Y_N=\sum_{|x-x_0|>N^{(1+\gamma)/4}}\rho_{\boldsymbol{b}}(N^\gamma,x)\exp[M|x-x_0|(d/N)^{1/2}].$$

Then

$$E[Y_N]=E[\exp[M|\xi(N^\gamma)-\xi(0)|(d/N)^{1/2}];|\xi(N^\gamma)-\xi(0)|>N^{(1+\gamma)/4}],$$

where ξ is the standard random walk on $\mathbb{Z}^d$. In view of Eq. (39) it follows that

$$E[Y_N]\leqslant C_\varepsilon\exp[-\varepsilon N^{(1-\gamma)/2}],$$

for some $\varepsilon>0$ and constant C_ε depending only on ε,γ,d. It follows that $\lim_{N\to\infty}Y_N=0$ with probability 1 as N goes to infinity through the integer sequence.

Evidently we have

$$|W_N-Z_N|\leqslant|Y_N|+\sup_{|x-x_0|<N^{(1+\gamma)/4}}\left|E_{x_0}\left[f\left(\frac{\xi(N)-x_0}{\sqrt{N/d}}\right)\right]\right.$$
$$\left.-E_x\left[f\left(\frac{\xi(N-N^\gamma)-x_0}{\sqrt{N/d}}\right)\right]\right|.$$

It follows now from Lemma 4.1 that if $\gamma<1/2$,

$$\left|E_{x_0}\left[f\left(\frac{\xi(N)-x_0}{\sqrt{N/d}}\right)\right]-E[f(Y)]\right|\leqslant C/\sqrt{N},$$

and for any $x \in \mathbb{Z}^d$,

$$\left| E_x \left[f\left(\frac{\xi(N - N^\gamma) - x_0}{\sqrt{N/d}} \right) \right] - E\left[f\left(Y + \frac{x - x_0}{\sqrt{N/d}} \right) \right] \right|$$
$$\leqslant C \exp[M|x - x_0|(d/N)^{1/2}]/\sqrt{N},$$

where the constant C depends only on d and A, M of Theorem 1.1. It is easy to see that

$$\left| E\left[f\left(Y + \frac{x - x_0}{\sqrt{N/d}} \right) \right] - E[f(Y)] \right| \leqslant C/N^{(1-\gamma)/4}, \quad |x - x_0| < N^{(1+\gamma)/4},$$

where C depends only on d, A, M. We conclude then that

$$|W_N - Z_N| \leqslant |Y_N| + C/N^{(1-\gamma)/4},$$

whence the result follows. □

Theorem 1.3 follows now from Lemma 4.1, Lemma 4.3 and Eq. (48).

References

Bolthausen, E., 1989. A note on the diffusion of directed polymers in a random environment. Commun. Math. Phys. 123, 529–534.

Bricmont, J., Kupiainen, A., 1991. Random walks in asymmetric random environments. Commun. Math. Phys. 142, 345–420.

Conlon, J., Olsen, P., 1996. A Brownian motion version of the Directed polymer problem. J. Statist. Phys. 84, 415–454.

Conlon, J., Olsen, P., 1997. Estimates on the solution of an elliptic equation related to Brownian motion with drift, II. Rev. Mat. Iberoamericana 13, 567–711.

Conlon, J., Redondo, J., 1995. Estimates on the solution of an elliptic equation related to Brownian motion with drift. Rev. Mat. Iberoamericana 11, 1–65.

Gilbarg, D., Trudinger, N.S., 1983. Elliptic Partial Differential Equations of Second Order. Springer, Berlin.

Imbrie, J., Spencer, T., 1988. Diffusion of directed polymers in a random environment. J. Statist. Phys. 52, 609–626.

Kardar, M., 1985. Roughening by impurities at finite temperature. Phys. Rev. Lett. 55, 2923.

Kardar, M., Zhang, Y.-C., 1987. Scaling of directed polymers in random media. Phys. Rev. Lett. 58, 2087–2090.

Licea, C., Newman, C.M., 1996. Geodesics in two-dimensional first-passage percolation. Ann. Probab. 24, 399–410.

Licea, C., Newman, C.M., Piza, M.S.T., 1995. Superdiffusivity in first-passage percolation. Preprint.

Newman, C.M., Piza, M.S.T., 1995. Divergence of shape fluctuations in two dimensions. Ann. Probab. 23, 977–1005.

Olsen, P., Song, R., 1996. Diffusion of directed polymers in a strong random environment. J. Statist. Phys. 83, 727–738.

Sinai, Y., 1982. The limiting behavior of a one-dimensional random walk in a random medium. Theory Probab. Appl. 27, 256–268.

Song, R., Zhou, X.Y., 1996. A remark on diffusion of directed polymers in random environments. J. Statist. Phys. 85, 277–289.