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Stochastic Processes and their Applications 116 (2006) 1377–1408
stochastic processes and their applications
www.elsevier.com/locate/spa

On the concentration of Sinai's walk

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Received 18 March 2004; received in revised form 19 November 2004; accepted 22 December 2004
Available online 20 March 2006

Abstract

We consider Sinai's random walk in a random environment. We prove that for an interval of time $[1, n]$ Sinai's walk sojourns in a small neighborhood of the point of localization for the quasi-totality of this amount of time. Moreover the local time at the point of localization normalized by n converges in probability to a well defined random variable of the environment. From these results we get applications to the favorite sites of the walk and to the maximum of the local time.

MSC: 60G50; 60J55

Keywords: Random environment; Random walk; Sinai's regime; Markov chain; Local time

1. Introduction

Random Walks in a Random Environment (R.W.R.E.) are basic processes in random media. The one dimensional case with nearest neighbor jumps, introduced by Solomon [1], was first studied by Kesten et al. [2], Sinai [3], Golosov [4,6] and Kesten [6]; all these works show the diversity of the possible behaviors of such walks depending on the hypothesis assumed for the environment. At the end of the 1980s Deheuvels and Révész [7] and Révész [8] gave the first almost sure behavior of the R.W.R.E. in the recurrent case. Then we had to wait until the middle of the 1990s to see new results. An important part of these new results concerns the problem of

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doi:10.1016/j.spa.2004.12.008

large deviations first studied by Greven and Hollander [9] and then by Zeitouni and Gantert [10], Pisztora and Povel [11], Zeitouni et al. [12] and Comets et al. [13] (see [14] for a review). In the same period using the stochastic calculus for the recurrent case Shi [15], Hu and Shi [16], Hu and Shi [17], Hu [18], Hu [19] and Hu and Shi [20] followed the works of Schumacher [21] and Brox [22] to give very precise results on the random walk and its local time (see [23] for an introduction). Moreover recent results on the problem of aging are given in [24], on the moderate deviations in [25] for the recurrent case, and on the local time in [26] for the transient case. In parallel to all these results a continuous time model has been studied, see for example [21,22], the works of Tanaka [27], Mathieu [28], Tanaka [29], Tanaka and Kawazu [30], Mathieu [31] and Taleb [32].

Since the beginning of the 1980s the delicate case of R.W.R.E. in dimension larger than 2 has been studied a lot. For recent reviews (before 2002) on this topic see the papers of Sznitman [33] and Zeitouni [14]. See also [34–37].

In this paper we are interested in Sinai's walk, i.e., the one dimensional random walk in a random environment with three conditions on the random environment: two hypotheses necessary for getting a recurrent process (see [1]) which is not a simple random walk and a hypothesis of regularity which allows us to have a good control of the fluctuations of the random environment.

The asymptotic behavior of such a walk was discovered by Sinai [3], who showed that this process is sub-diffusive and that at time n it is localized in the neighborhood of a well defined point of the lattice. This *point of localization* is a random variable depending only on the random environment and n, its explicit limit distribution was given, independently, by Kesten [6] and Golosov [5].

We prove, with a probability very near one, that this process is concentrated in a small neighborhood of the point of localization, this means that for an interval of time $[1, n]$ Sinai's walk spends the quasi-totality of this amount of time in the neighborhood of the point of localization. The size of this neighborhood $\approx(\log\log n)^2$ is negligible compared to the typical range $(\log n)^2$ of Sinai's walk. Extending this result to a neighborhood of arbitrary size we get that, with a strong probability, the size of the interval where the walk spends more than a half of its time is smaller than every positive strictly increasing sequence. We also prove that the local time of this random walk at the point of localization normalized by n converges in probability to a random variable depending only on n and on the random environment. This random variable is the inverse of the mean of the local time at the valley where the walk is trapped within a time of return to the point of localization, we prove that the mean with respect to the environment of this mean is bounded. We generalize this result for neighboring points of the point of localization. All our results are "quenched" results, this means that we work with a fixed environment that belongs to a probability subset of the random environment that has a probability that goes to one as n diverges. We give some consequences for the maximum of the local time and the favorite sites of Sinai's walk.

This paper is organized as follows. In Section 2 we describe the model and recall Sinai's results. In Section 2 we present our main results. In Sections 4 and 5 we give the proof of these results.

2. Description of the model and Sinai's results

2.1. Sinai's random walk definition

Let $\alpha = (\alpha_i, i \in \mathbb{Z})$ be a sequence of i.i.d. random variables taking values in $(0, 1)$ defined on the probability space $(\Omega_1, \mathcal{F}_1, Q)$, this sequence will be called a random environment. A random walk in a random environment (denoted as R.W.R.E.) $(X_n, n \in \mathbb{N})$ is a sequence of random variables taking values in $\mathbb{Z}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- for every fixed environment α, $(X_n, n \in \mathbb{N})$ is a Markov chain with the following transition probabilities, for all $n \geq 1$ and $i \in \mathbb{Z}$
$$\begin{aligned} &\mathbb{P}^\alpha[X_n = i+1 | X_{n-1} = i] = \alpha_i, \\ &\mathbb{P}^\alpha[X_n = i-1 | X_{n-1} = i] = 1 - \alpha_i \equiv \beta_i. \end{aligned} \tag{2.1}$$
We denote as $(\Omega_2, \mathcal{F}_2, \mathbb{P}^\alpha)$ the probability space associated with this Markov chain.
- $\Omega = \Omega_1 \times \Omega_2$, $\forall A_1 \in \mathcal{F}_1$ and $\forall A_2 \in \mathcal{F}_2$, $\mathbb{P}[A_1 \times A_2] = \int_{A_1} Q(\mathrm{d}w_1) \int_{A_2} \mathbb{P}^{\alpha(w_1)}(\mathrm{d}w_2)$.

The probability measure $\mathbb{P}^\alpha[.|X_0 = a]$ will be denoted as $\mathbb{P}^\alpha_a[.]$, the expectation associated with $\mathbb{P}^\alpha_a$: $\mathbb{E}^\alpha_a$, and the expectation associated with Q: $\mathbb{E}_Q$.

Now we introduce the hypothesis we will use throughout this work. The following two hypotheses are the necessary hypotheses

$$\mathbb{E}_Q\left[\log \frac{1-\alpha_0}{\alpha_0}\right] = 0, \tag{2.2}$$

$$\mathrm{Var}_Q\left[\log \frac{1-\alpha_0}{\alpha_0}\right] \equiv \sigma^2 > 0. \tag{2.3}$$

Solomon [1] shows that under (2.2) the process $(X_n, n \in \mathbb{N})$ is $\mathbb{P}$ almost surely recurrent and (2.3) implies that the model is not reduced to the simple random walk. In addition to (2.2) and (2.3) we will consider the following hypothesis of regularity, there exists $0 < \eta_0 < 1/2$ such that

$$\sup\{x, Q[\alpha_0 \geq x] = 1\} = \sup\{x, Q[\alpha_0 \leq 1 - x] = 1\} \geq \eta_0. \tag{2.4}$$

We call the random walk in a random environment previously defined with the three hypotheses (2.2)–(2.4) *Sinai's random walk.*

2.2. *The random potential and the valleys*

Let

$$\epsilon_i \equiv \log \frac{1-\alpha_i}{\alpha_i}, \quad i \in \mathbb{Z}, \tag{2.5}$$

define:

Definition 2.1. The random potential $(S_m, m \in \mathbb{Z})$ associated with the random environment α is defined in the following way: for all k and j, if $k > j$

$$S_k - S_j = \begin{cases} \sum_{j+1 \leq i \leq k} \epsilon_i, & k \neq 0, \\ -\sum_{j \leq i \leq -1} \epsilon_i, & k = 0, \end{cases}$$
$$S_0 = 0,$$

and symmetrically if $k < j$.

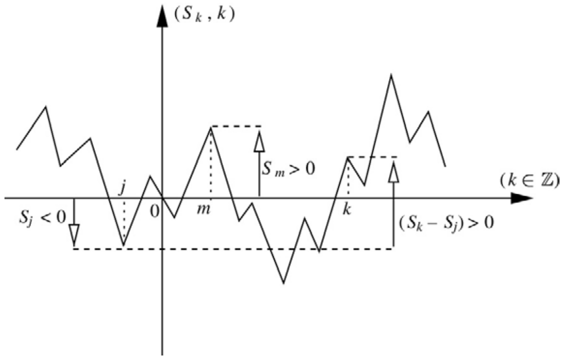

Fig. 1. Trajectory of the random potential.

Remark 2.2. Using Definition 2.1 we have:

$$S_k = \begin{cases} \sum_{1 \le i \le k} \epsilon_i, & k = 1, 2, \ldots, \\ \sum_{k \le i \le -1} \epsilon_i, & k = -1, -2, \ldots, \end{cases} \tag{2.6}$$

however, if we use (2.6) for the definition of (S_k, k), ϵ_0 does not appear in this definition and moreover it is not clear, when $j < 0 < k$, what the difference $S_k - S_j$ means (see Fig. 1).

Definition 2.3. We will say that the triplet $\{M', m, M''\}$ is a *valley* if

$$S_{M'} = \max_{M' \le t \le m} S_t, \tag{2.7}$$

$$S_{M''} = \max_{m \le t \le \tilde{M}''} S_t, \tag{2.8}$$

$$S_m = \min_{M' \le t \le M''} S_t. \tag{2.9}$$

If m is not unique we choose the one with the smallest absolute value.

Definition 2.4. We will define as the *depth of the valley* $\{M', m, M''\}$, denoting it by $d([M', M''])$, the quantity

$$\min(S_{M'} - S_m, S_{M''} - S_m). \tag{2.10}$$

Now we define the operation of *refinement*.

Definition 2.5. Let $\{M', m, M''\}$ be a valley and let M_1 and m_1 be such that $m \le M_1 < m_1 \le M''$ and

$$S_{M_1} - S_{m_1} = \max_{m \le t' \le t'' \le M''} (S_{t'} - S_{t''}). \tag{2.11}$$

We say that the couple (m_1, M_1) is obtained by a *right refinement* of $\{M', m, M''\}$. If the couple (m_1, M_1) is not unique, we will take the one such that m_1 and M_1 have the smallest absolute value. In a similar way we define the *left refinement* operation (see Fig. 2).

We denote the p iterated logarithm as $\log_p$ with $p \ge 2$. Throughout this work we will suppose that n is large enough that $\log_p n$ is positive.

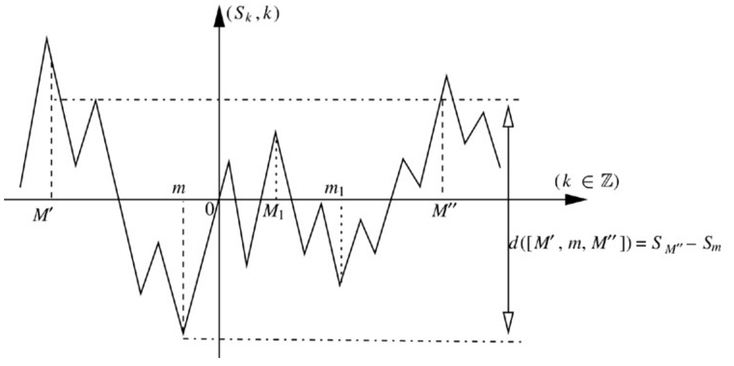

Fig. 2. Depth of a valley and refinement operation.

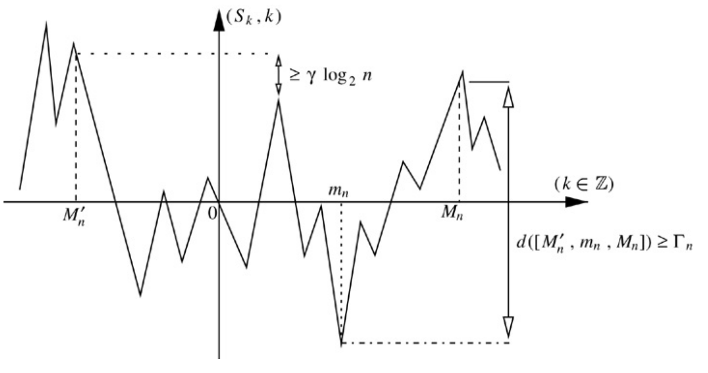

Fig. 3. Basic valley, case $m_n > 0$.

Definition 2.6. For $\gamma > 0$, $n > 3$ and $\Gamma_n \equiv \log n + \gamma \log_2 n$, we say that a valley $\{M', m, M''\}$ contains 0 and is of depth larger than Γ_n if and only if

1. $0 \in [M', M'']$,
2. $d([M', M'']) \geq \Gamma_n$,
3. if $m < 0$, $S_{M''} - \max_{m \leq t \leq 0}(S_t) \geq \gamma \log_2 n$,
   if $m > 0$, $S_{M'} - \max_{0 \leq t \leq m}(S_t) \geq \gamma \log_2 n$.

γ is a free parameter.

2.3. *The basic valley* $\{M_n{}', m_n, M_n\}$

We recall the notion of the *basic valley* introduced by Sinai and denoted here as $\{M_n{}', m_n, M_n\}$. The definition we give is inspired by the work of Kesten [6]. First let $\{M', m_n, M''\}$ be the smallest *valley that contains 0 and of depth larger than* Γ_n. Here smallest means that if we construct, with the operation of refinement, other valleys in $\{M', m_n, M''\}$ such valleys will not satisfy one of the properties of Definition 2.6 (see Fig. 3). $M_n{}'$ and M_n are defined from m_n in the following way: if $m_n > 0$

$$M_n{}' = \sup\left\{ l \in \mathbb{Z}_-,\ l < m_n,\ S_l - S_{m_n} \geq \Gamma_n, S_l - \max_{0 \leq k \leq m_n} S_k \geq \gamma \log_2 n \right\}, \tag{2.12}$$

$$M_n = \inf\{l \in \mathbb{Z}_+,\ l > m_n,\ S_l - S_{m_n} \geq \Gamma_n\} \tag{2.13}$$

if $m_n < 0$

$$M_n{}' = \sup\{l \in \mathbb{Z}_-,\ l < m_n,\ S_l - S_{m_n} \geq \Gamma_n\}, \tag{2.14}$$

$$M_n = \inf\left\{l \in \mathbb{Z}_+,\ l > m_n,\ S_l - S_{m_n} \geq \Gamma_n, S_l - \max_{m_n \leq k \leq 0} S_k \geq \gamma \log_2 n\right\} \tag{2.15}$$

if $m_n = 0$

$$M_n{}' = \sup\{l \in \mathbb{Z}_-, l < 0, S_l - S_{m_n} \geq \Gamma_n\}, \tag{2.16}$$

$$M_n = \inf\{l \in \mathbb{Z}_+, l > 0, S_l - S_{m_n} \geq \Gamma_n\}. \tag{2.17}$$

$\{M_n{}', m_n, M_n\}$ exists with a Q probability as close to one as we need. In fact it is not difficult to prove the following lemma (see Section 5.2 for the ideas of the proof).

Lemma 2.7. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $\gamma > 0$ there exists $n_0 \equiv n_0(\gamma, Q)$ such that for all $n > n_0$*

$$Q\left[\{M_n{}', m_n, M_n\} \neq \varnothing\right] \geq 1 - \frac{c\gamma \log_2 n}{\log n}. \tag{2.18}$$

Throughout this paper we use the same notation n_0 for an integer that could change from line to line. Moreover in the rest of the paper we do not always make explicit the dependence on the free parameter γ and on the distribution Q of all those n_0 even if Lemma 2.7 is constantly used.

2.4. Localization phenomena [3]

Theorem 2.8. *Assume (2.2)–(2.4) hold. For all $\gamma > 6$, $\epsilon > 0$ and $\delta > 0$ there exists n_0 such that for all $n > n_0$, there exists $G_n \subset \Omega_1$ with $Q[G_n] \geq 1 - \epsilon$ such that*

$$\lim_{n \to +\infty} \sup_{\alpha \in G_n} \mathbb{P}_0^\alpha[|X_n - m_n| > \delta(\log n)^2] = 0. \tag{2.19}$$

This result shows that with a Q and $\mathbb{P}^\alpha$ probability as close to 1 as we want, at time n the R.W.R.E. is localized in a small neighborhood of m_n. The parameter γ comes from the definition of $m_n \equiv m_n(\gamma)$. Ya. G. Sinai shows also that, with a probability close to one, for a given time interval $[0, n]$, the R.W.R.E. is trapped in the basic valley and therefore is sub-diffusive. In fact if we define

$$W_n = \{M_n{}',\ M_n{}' + 1, \ldots, m_n, \ldots, M_n - 1, M_n\}, \tag{2.20}$$

we have

Proposition 2.9. *Assume (2.2)–(2.4) hold. For all $\gamma > 6$, $\epsilon > 0$ there exists $E \equiv E(\epsilon)$ and n_0 such that for all $n > n_0$, there exists $G_n \subset \Omega_1$ with $Q[G_n] \geq 1 - \epsilon$ such that*

$$\lim_{n \to +\infty} \inf_{\alpha \in G_n} \mathbb{P}_0^\alpha\left[\bigcap_{m=0}^{n} \{X_m \in W_n\}\right] = 1, \tag{2.21}$$

moreover

$$\lim_{n \to +\infty} \inf_{\alpha \in G_n} \mathbb{P}_0^\alpha\left[\bigcap_{m=0}^{n} \{X_m \in [-E(\sigma^{-1} \log n)^2, E(\sigma^{-1} \log n)^2]\}\right] = 1. \tag{2.22}$$

See for example [14] or [38] for alternative proofs of these results.

3. Main results: Concentration phenomena

Let us define the local time at k ($k \in \mathbb{Z}$) within the interval of time $[1, T]$ ($T \in \mathbb{N}^*$) of $(X_n, n \in \mathbb{N})$

$$\mathcal{L}(k, T) \equiv \sum_{i=1}^{T} \mathbb{I}_{\{X_i = k\}}. \tag{3.1}$$

$\mathbb{I}$ is the indicator function (k and T can be deterministic or random variables). Let $V \subset \mathbb{Z}$, we define

$$\mathcal{L}(V, T) \equiv \sum_{j \in V} \mathcal{L}(j, T) = \sum_{i=1}^{T} \sum_{j \in V} \mathbb{I}_{\{X_i = j\}}. \tag{3.2}$$

3.1. Local time in a neighborhood of m_n

Let us define the following sequences, let $p \geq 2$ and n be large enough, define:

$$f_p(n) = [(\log_2 n \log_p n)^2]([-] \text{ is the integer part of } -), \tag{3.3}$$

$$R_p(n) = (\log_{p+1} n)^{1/2} (\log_p n)^{-1/2}. \tag{3.4}$$

For all $n \geq 1$ define the set:

$$\mathbb{F}_p(n) = \{m_n - f_p(n), m_n - f_p(n) + 1, \ldots, m_n, m_n + 1, \ldots, m_n + f_p(n)\}. \tag{3.5}$$

Theorem 3.1. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $p \geq 2$, $0 < \rho < 2$ and $\gamma \geq 11$, there exists $c_1 \equiv c_1(Q, \gamma) > 0$ and n_0 such that for all $n > n_0$ there exists $G_n' \subset \Omega_1$ with $Q[G_n'] \geq 1 - cR_p(n)$ and*

$$\inf_{\alpha \in G_n'} \left\{ \mathbb{P}_0^\alpha \left[\frac{\mathcal{L}(\mathbb{F}_p(n), n)}{n} \geq \left(1 - \frac{1}{(f_p(n))^{\rho/2}}\right) \right] \right\} \geq 1 - \frac{c_1}{(f_p(n))^{1-\rho/2}}. \tag{3.6}$$

Remark 3.2. The set G_n' called the *set of good environments* and will be defined in Section 4.1.

In fact n_0 and c_1 depend only on Q through $I_{\eta_0} \equiv \log[(1 - \eta_0)/(\eta_0)]$, $Q[\epsilon_0 > I_{\eta_0}/2]$, $Q[\epsilon_0 < -I_{\eta_0}/2]$, σ and $\mathbb{E}_Q[|\epsilon_0|^3]$, however to simplify the exposition we do not make this explicit. Throughout this work we denote as c a strictly positive numerical constant that can grow from line to line if needed.

If we choose $p = 2$ in Theorem 3.1 we get better rates for the convergence of the probabilities, however using (3.5) we get that $|\mathbb{F}_p(n)| = (\log_2 n)^4$ ($|A|$ denotes the cardinal of $A \subset \mathbb{Z}$), whereas for $p > 2$, $|\mathbb{F}_p(n)| = (\log_2 n)^2 (\log_p n)^2$. Recall that γ comes from the definition of the point $m_n \equiv m_n(\gamma)$, the condition $\gamma \geq 11$ will become clear in Section 4.3 when we prove this theorem.

In addition to Sinai's results on localization, with a $\mathbb{P}^\alpha$ and Q probability very near one, in a time interval $[0, n]$ the R.W.R.E. does not spend a finite proportion of this time interval outside

$\mathbb{F}_p(n)$. In fact the time spent outside $\mathbb{F}_p(n)$ does not exceed $n/(\log_2 n \log_p n)^{\rho}$. Moreover, notice that $|\mathbb{F}_p(n)|/|W_n| \approx (\log_2 n \log_p n)^2 (\log n)^{-2} \searrow 0$, that is the subset $\mathbb{F}_p(n)$ of W_n where the R.W.R.E. stays a time greater than $n(1 - 1/(\log_2 n \log_p n)^{\rho})$ has no density inside W_n in the limit when n goes to infinity.

Remark 3.3. Notice that if we look for an annealed result, that means a result in $\mathbb{P}$ probability, with the same condition as for Theorem 3.1 we get

$$\mathbb{P}\left[\frac{\mathcal{L}(\mathbb{F}_p(n), n)}{n} \geq \left(1 - \frac{1}{(f_p(n))^{\rho/2}}\right)\right] \geq 1 - \frac{c_1}{(f_p(n))^{1-\rho/2}} - cR_p(n). \tag{3.7}$$

By definition $\mathcal{L}(\mathbb{F}_p(n), n) \leq 1$, therefore we also have

$$\mathbb{P}\left[\left|\frac{\mathcal{L}(\mathbb{F}_p(n), n)}{n} - 1\right| \leq \frac{1}{(f_p(n))^{\rho/2}}\right] \geq 1 - \frac{c_1}{(f_p(n))^{1-\rho/2}} - cR_p(n). \tag{3.8}$$

To prove this theorem we will prove the following key result on the environment, define

$$T_x = \begin{cases} \inf\{k \in \mathbb{N}^*, X_k = x\} \\ +\infty, \quad \text{if such } k \text{ does not exist.} \end{cases} \tag{3.9}$$

Proposition 3.4. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $p \geq 2$ there exists n_0 such that for all $n > n_0$*

$$Q\left[\mathbb{E}^{\alpha}_{m_n}\left[\mathcal{L}(\bar{\mathbb{F}}_p(n), T_{m_n})\right] > \frac{2}{\eta_0}\frac{1}{f_p(n)+1}\right] \leq cR_p(n) \tag{3.10}$$

where $\bar{\mathbb{F}}_p(n)$ is the complementary of $\mathbb{F}_p(n)$ in W_n.

This implies that with a Q probability going to one when n goes to infinity, in $\mathbb{E}^{\alpha}$ mean, the R.W.R.E. will never reach a point $k \in \bar{\mathbb{F}}_p(n)$ in a time of return to m_n in spite of the fact that $|\bar{\mathbb{F}}_p(n)|/|W_n| \approx 1$. The proof of this proposition is given in Section 5.4.

If we want to replace in Theorem 3.1 the neighborhood $\mathbb{F}_p(n)$ by an *arbitrary small* neighborhood (Θ_n, n) but such that $\mathcal{L}(\Theta_n, n)/n$ converges in probability to one, we get with a similar method the following result:

Theorem 3.5. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $0 < \rho < 1/4$, all strictly positive increasing sequences $(\theta(n), n \geq 1)$ and $\gamma \geq 11$, there exists $c_1 \equiv c_1(Q, \gamma)$ and n_0 such that for all $n > n_0$ there exists $G'_n \subset \Omega_1$ with $Q[G'_n] \geq 1 - c(\theta(n))^{-\rho}$ and*

$$\inf_{\alpha \in G'_n}\left\{\mathbb{P}^{\alpha}_0\left[\mathcal{L}(\Theta(n), n) \geq n\left(1 - \frac{1}{(\theta(n))^{\rho}}\right)\right]\right\} \geq 1 - \frac{c_1}{(\theta(n))^{1/2-2\rho}}, \tag{3.11}$$

where $\Theta(n) = \{m_n - \theta(n), m_n - \theta(n) + 1, \ldots, m_n, m_{n+1}, \ldots, m_n + \theta(n)\}$.

The key result on the environment for the proof of this result is the following, assume that $(\theta(n), n)$ is a strictly positive increasing sequence.

Proposition 3.6. *Assume (2.2)–(2.4) hold, for all* $0 < \rho < 1/2$ *there exists* $c_1 \equiv c_1(Q)$ *and* n_0 *such that for all* $n > n_0$

$$Q\left[\mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(\bar{\Theta}(n), T_{m_n})] > \frac{1}{(\theta(n))^{1/2-\rho}}\right] \leq \frac{c_1}{(\theta(n))^{\rho}} \tag{3.12}$$

where $\bar{\Theta}(n)$ *is the complementary of* $\Theta(n)$ *in* W_n.

We call these (Theorems 3.1 and 3.5) *concentrations*, even if it could be more appropriate to define the concentration in terms of the random variable

$$Y_n = \inf_{x\in\mathbb{Z}} \min\{k > 0 : \mathcal{L}([x-k, x+k], n) > n/2\}. \tag{3.13}$$

In fact Theorem 3.5 implies the following result.

Theorem 3.7. *There exists* $c > 0$ *such that if (2.2)–(2.4) hold, for all strictly positive increasing sequences* $(\theta(n), n)$*, there exists* $c_1 \equiv c_1(Q) > 0$ *and* n_0 *such that for all* $n > n_0$ *there exists* $G_n' \subset \Omega_1$ *with* $Q[G_n'] \geq 1 - c(\theta(n))^{-1/4}$ *and*

$$\inf_{\alpha\in G_n'}\{\mathbb{P}_0^{\alpha}[Y_n \leq \theta(n)]\} \geq 1 - \frac{c_1}{(\theta(n))^{1/4}}. \tag{3.14}$$

Notice that the annealed result is given by

$$\mathbb{P}[Y_n \leq \theta(n)] \geq 1 - \frac{c_2}{(\theta(n))^{1/4}}, \tag{3.15}$$

with $c_2 \equiv c_2(Q) = c + c_1$.

It would now be interesting to study the $\mathbb{P}$ almost sure behavior of Y_n, these questions are beyond the scope of this paper, see [39] for a first approach.

3.2. Local time on m_n *and on its neighboring points*

The following theorem is a result of weak law of large number type for the local time of the R.W.R.E. at m_n. We obtain the convergence of the local time at m_n normalized by n to a Q random variable as Ya. Sinai obtained the convergence of X_n to m_n, which is also a Q random variable.

Theorem 3.8. *There exists* $c > 0$ *such that if (2.2)–(2.4) hold, for all* $\gamma \geq 11$*, there exists* $c_1 \equiv c_1(Q, \gamma)$ *and* n_0 *such that for all* $n > n_0$ *there exists* $G_n' \subset \Omega_1$ *with* $Q[G_n'] \geq 1 - cR_2(n)$ *and*

$$\sup_{\alpha\in G_n'}\left\{\mathbb{P}_0^{\alpha}\left[\left|\frac{\mathcal{L}(m_n, n)}{n} - \frac{1}{\mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(W_n, T_{m_n})]}\right| > \frac{1}{(\log_2 n)^2}\right]\right\} \leq \frac{c_1}{(\log_2 n)^2}. \tag{3.16}$$

We can easily give an intuitive idea of this result. From Proposition 2.9 we know that the R.W.R.E., within an interval of time $[1, n]$, spends all its time in the valley $\{M_n{}', m_n, M_n\}$ with a probability very near one. So, in spite of the fact that $\mathbb{E}_{m_n}^{\alpha}[T_{m_n}] = +\infty$ Q. a.s. as is easy to check, until the instant n, the mean of the time of return to m_n is, heuristically, of order $\mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(W_n, T_{m_n})]$. So we can chop the interval $[1, n]$ into $\mathcal{L}(m_n, n)$ pieces of length $\mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(W_n, T_{m_n})]$, therefore

$$n \approx \mathcal{L}(m_n, n)\mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(W_n, T_{m_n})] \Leftrightarrow \frac{\mathcal{L}(m_n, n)}{n} \approx (\mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(W_n, T_{m_n})])^{-1}.$$

Now let us give some precision on the random variable $\mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})]$, first for all fixed environments we have the following explicit formula:

Proposition 3.9. *For all fixed α and n, we have*

$$\mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})] = 1 + \sum_{k=m_n+1}^{M_n} \frac{\alpha_{m_n}}{\beta_k} \frac{\sum_{j=m_n+1}^{k-1} \exp(S_j - S_k)}{\sum_{j=m_n+1}^{k-1} \exp(S_j - S_{m_n})} + \sum_{k=M'_n}^{m_n-1} \frac{\beta_{m_n}}{\alpha_k} \frac{\sum_{j=k-1}^{m_n+1} \exp(S_j - S_k)}{\sum_{j=k-1}^{m_n+1} \exp(S_j - S_{m_n})}.$$

Remark 3.10. Trivially we have $\mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})] \geq 1$ because $m_n \in W_n$ and $\mathcal{L}(m_n, T_{m_n}) = 1$.

We see that $\mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})]$ depends on the random environment in a complicated way, however using the hypothesis (2.4) we easily prove the following result:

Proposition 3.11. *With a Q probability equal to one, for all n we have:*

$$\frac{\eta_0}{1-\eta_0} \sum_{k\in W_n, k\neq m_n} \frac{1}{\exp(S_k - S_{m_n})} \leq \mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})] - 1 \leq \frac{1}{\eta_0} \sum_{k\in W_n, k\neq m_n} \frac{1}{\exp(S_k - S_{m_n})}.$$

So we can have a good idea of the fluctuations of $\mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})]$ only by studying the random variable

$$\sum_{k\in W_n, k\neq m_n} \frac{1}{\exp(S_k - S_{m_n})}. \tag{3.17}$$

The following result is a key result for the random environment:

Proposition 3.12. *There exists $c_1 \equiv c_1(Q) > 0$ such that for all $n \geq 1$*

$$1 \leq \mathbb{E}_Q[\mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})]] \leq c_1. \tag{3.18}$$

In particular, using the Markov inequality, we get that for all positive increasing sequences $(\theta(n), n)$:

$$Q[\mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})] > \theta(n)] \leq \frac{c_1}{\theta(n)}. \tag{3.19}$$

Remark 3.13. c_1 depends only on Q through η_0 and $Q[\epsilon_0 < -I_{\eta_0}/2]$ but for simplicity we do not make this dependence explicit, see Section 5.3 for the proof of Proposition 3.12.

One can ask oneself about the convergence in law of $\mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})]$; this problem is not an easy consequence of our computations and can be by itself an independent work.

One can notice the specificity of (3.16), however easy modifications of our method give the following generalization.

Theorem 3.14. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $\gamma \geq 11$, there exists $c_1 \equiv c_1(Q, \gamma)$ and n_0 such that for all $n > n_0$ there exists $G'_n \subset \Omega_1$ with $Q[G'_n] \geq 1 - cR_2(n)$ and*

$$\sup_{\alpha \in G'_n} \left\{ \mathbb{P}_0^\alpha \left[\bigcup_{k \in \mathbb{L}(n)} \left\{ \left| \frac{\mathcal{L}(k,n)}{n} - \frac{1}{\mathbb{E}_k^\alpha[\mathcal{L}(W_n, T_k)]} \right| > \frac{1}{(\log_2 n)^2} \right\} \right] \right\} \leq \frac{c_1 \log_3 n}{\log_2 n} \tag{3.20}$$

where $\mathbb{L}(n) = \{m_n - l(n), m_n - l(n) + 1, \ldots, m_n, m_{n+1}, \ldots, m_n + l(n)\}$, $l(n) = \log_3(n)/I_{\eta_0}$ and I_{η_0} is given by Remark 3.2.

Remark 3.15. We deduce this result from the computations we made to prove Theorem 3.8, the point is that, under the hypothesis of Propositions 3.4 and 3.12, we have the following results, for all $k \in \mathbb{L}(n)$:

$$Q\left[\mathbb{E}_k^\alpha[\mathcal{L}(\bar{\mathbb{F}}_p(n), T_k)] > \frac{2}{\eta_0} \frac{\log_2 n}{f_p(n) + 1}\right] \leq cR_p(n), \tag{3.21}$$

$$Q[\mathbb{E}_k^\alpha[\mathcal{L}(\mathbb{F}_p(n), T_k)] > c_1 (\log_2 n) \log_{p+1} n] \leq cR_p(n). \tag{3.22}$$

However we think that Theorem 3.14 is certainly true for a neighborhood $\mathcal{V}_n$ of m_n ($\mathbb{F}_p(n) \subset \mathcal{V}_n$) of size larger than $(\log \log n)^a$ for some $a > 0$, but it does not seem to be a simple extension of our computations.

The following corollary is a simple consequence of (3.19) and Theorem 3.8.

Corollary 3.16. *There exists $c > 0$ such if (2.2)–(2.4) hold, for all $p \geq 2$, $\gamma \geq 11$, there exists $c_1 \equiv c_1(Q)$ and n_0 such that for all $n > n_0$ there exists $c_2 \equiv c_2(Q) > 0$ and $G'_n \subset \Omega_1$ with $Q[G'_n] \geq 1 - cR_p(n)$ and*

$$\inf_{\alpha \in G'_n} \left\{ \mathbb{P}_0^\alpha \left[\mathcal{L}(m_n, n) \geq \frac{n}{2c_2 \log_{p+1} n} \right] \right\} \geq 1 - \frac{c_1}{\log_2 n \log_p n}, \tag{3.23}$$

$$\inf_{\alpha \in G'_n} \left\{ \mathbb{P}_0^\alpha \left[\bigcap_{k \in \mathbb{F}_p(n)} \{\mathcal{L}(k,n)\} \geq \frac{n}{2c_2 (\log_2 n) \log_{p+1} n} \right] \right\} \geq 1 - \frac{c_1}{\log_p n}. \tag{3.24}$$

All these results show that in addition to being *localized* the R.W.R.E. is *concentrated* and the region of concentration and localization are extremely linked together.

In the following subsection we give some simple consequences for the maximum of the local time and for the favorite sites of Sinai's walk.

3.3. Simple consequences for the maximum of the local time and for the favorite site of the R.W.R.E.

Let us introduce the following random variables

$$\mathcal{L}^*(n) = \max_{k \in \mathbb{Z}} (\mathcal{L}(k,n)), \qquad \tilde{\mathbb{F}}(n) = \{k \in \mathbb{Z},\ \mathcal{L}(k,n) = \mathcal{L}^*(n)\}. \tag{3.25}$$

$\tilde{\mathbb{F}}(n)$ is the set of all the *favorite sites* and $\mathcal{L}^*(n)$ is the maximum of the local times (for a given instant n). The following corollary is a simple consequence (just by inspection) of (3.19), Theorems 3.1 and 3.8.

Corollary 3.17. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $p \geq 2$, and $\gamma \geq 11$, there exists $c_1 \equiv c_1(Q, \gamma)$ and n_0 such that for all $n > n_0$ there exists $c_2 \equiv c_2(Q) > 0$ and $G_n' \subset \Omega_1$ with $Q[G_n'] \geq 1 - cR_p(n)$ and*

$$\inf_{\alpha \in G_n'} \left\{ \mathbb{P}_0^\alpha \left[\mathcal{L}^*(n) \geq \frac{n}{2c_2 \log_{p+1} n} \right] \right\} \geq 1 - \frac{c_1}{\log_2 n \log_p n}, \tag{3.26}$$

$$\inf_{\alpha \in G_n'} \left\{ \mathbb{P}_0^\alpha \left[\tilde{\mathbb{F}}(n) \subset \mathbb{F}_p(n) \right] \right\} \geq 1 - \frac{c_1}{\log_2 n \log_p n}. \tag{3.27}$$

To finish we give an interesting application (just by inspection) of (3.26), Theorems 3.5 and 3.14:

Corollary 3.18. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $\gamma \geq 11$, and all strictly positive increasing sequences $(\theta(n), n)$ there exists n_0 and $c_1 \equiv c_1(Q)$ such that for all $n > n_0$ there exists $G_n' \subset \Omega_1$ with $Q[G_n'] \geq 1 - c(\theta(n))^{-1/4}$ and*

$$\sup_{\alpha \in G_n'} \left\{ \mathbb{P}_0^\alpha \left[\left| \frac{\mathcal{L}^*(n)}{n} - \max_{k \in \Theta(n)} \left\{ \frac{1}{\mathbb{E}_k^\alpha \left[\mathcal{L}(W_n, T_k) \right]} \right\} \right| > \frac{1}{(\log_2 n)^2} \right] \right\} \leq \frac{c_1 \theta(n)}{\log_2 n}, \tag{3.28}$$

recalling that $\Theta(n) = \{m_n - \theta(n), m_n - \theta(n) + 1, \ldots, m_n, m_{n+1}, \ldots, m_n + \theta(n)\}$.

Notice that if we are interested in an annealed result Corollary 3.18 implies:

$$\mathbb{P} \left[\left| \frac{\mathcal{L}^*(n)}{n} - \max_{k \in \Theta(n)} \left\{ \frac{1}{\mathbb{E}_k^\alpha [\mathcal{L}(W_n, T_k)]} \right\} \right| > \frac{1}{(\log_2 n)^2} \right] \leq \frac{c_1 \theta(n)}{\log_2 n} + \frac{c}{(\theta(n))^{1/4}}. \tag{3.29}$$

The delicate problem of the limit distribution of $\mathcal{L}^*(n)/n$ considered by Révész [8] cannot be directly deduced from these results, however they are a good starting point for further investigations on this topic. We think that this limit distribution is very much linked to the Q limit distribution of the random variable $\mathbb{E}_{m_n}^\alpha[\mathcal{L}(W_n, T_{m_n})]$, recall moreover that we have good knowledge of W_n thanks to the works of Kesten [6] and Golosov [5].

4. Proof of the main results

First we define what we call a *good environment* and the *set of good environments*.

4.1. Good properties and the set of good environments

Definition 4.1. Let $p \geq 2$, $\gamma > 0$, $c_1 > 0$ and $\omega \in \Omega_1$, we will say that $\alpha \equiv \alpha(\omega)$ is a *good environment* if there exists n_0 such that for all $n \geq n_0$ the sequence $(\alpha_i, i \in \mathbb{Z}) = (\alpha_i(\omega), i \in \mathbb{Z})$ satisfies the properties (4.1)–(4.13).

- The valley $\{M_n{}', m_n, M_n\}$ exists, in particular: (4.1)

$$0 \in [M_n{}', M_n], \tag{4.2}$$

$$\text{If } m_n > 0, \quad S_{M_n{}'} - \max_{0 \leq m \leq m_n} (S_m) \geq \gamma \log_2 n, \tag{4.3}$$

$$\text{if } m_n < 0, \quad S_{M_n} - \max_{m_n \le m \le 0} (S_m) \ge \gamma \log_2 n, \tag{4.4}$$

$$S_{M_n'} - S_{m_n} \ge \log n + \gamma \log_2 n, \tag{4.5}$$

$$S_{M_n} - S_{m_n} \ge \log n + \gamma \log_2 n. \tag{4.6}$$

- $M_n{}' \ge (\sigma^{-1} \log n)^2 \log_p n, \quad M_n \le (\sigma^{-1} \log n)^2 \log_p n.$ (4.7)

Define M_1' and m_1', respectively, as the maximizer and minimizer obtained by the first *left refinement* of the valley $\{M_n{}', m_n, M_n\}$ and in the same way M_1 and m_1, respectively, as the maximizer and minimizer obtained by the first *right refinement* of the valley $\{M_n{}', m_n, M_n\}$.

- $S_{M_1'} - S_{m_1'} \le \log n - \gamma \log_2 n,$ (4.8)

$$S_{M_1} - S_{m_1} \le \log n - \gamma \log_2 n. \tag{4.9}$$

Define $\mathbb{F}_p^+(n) = \{m_n + 1, \ldots, m_n + f_p(n)\}, \mathbb{F}_p^-(n) = \{m_n - f_p(n), \ldots, m_n - 1\}$ where $f_p(n)$ is given by (3.3).

- $\min_{k \in \mathbb{F}_p(n)^+} (\beta_k \mathbb{P}_{k-1}^\alpha[T_k > T_{m_n}]) \ge (g_1(n))^{-1},$ (4.10)

$$\min_{k \in \mathbb{F}_p(n)^-} (\alpha_k \mathbb{P}_{k+1}^\alpha[T_k > T_{m_n}]) \ge (g_1(n))^{-1}, \tag{4.11}$$

where $g_1(n) = \exp(((4\sqrt{3}\sigma f_p(n))^2 \log_3(n))^{1/2})$.

- $\mathbb{E}_{m_n}^\alpha[\mathcal{L}(\mathbb{F}_p(n), T_{m_n})] \le c_1 \log_{p+1} n.$ (4.12)
- $\mathbb{E}_{m_n}^\alpha[\mathcal{L}(\bar{\mathbb{F}}_p(n), T_{m_n})] \le 2(\eta_0(f_p(n) + 1))^{-1}.$ (4.13)

See (3.5) for the definition of $\mathbb{F}_p(n)$, and we recall that $\bar{\mathbb{F}}_p(n)$ is the complementary of $\mathbb{F}_p(n)$ in W_n.

Define the *set of good environments*

$$G_n' = \{\omega \in \Omega_1,\ \alpha(\omega) \text{ is a } \textit{good environment}\}. \tag{4.14}$$

G_n' depends on p, γ, c_1 and n, however we do not make explicit its p, γ and c_1 dependence.

Proposition 4.2. *There exists $c > 0$ such that if (2.2)–(2.4) hold, there exists $c_1 > 0$ such that for all $p \ge 2$ and $\gamma > 0$, there exists $n_0 \equiv n_0(\gamma)$ such that for all $n > n_0$*

$$Q[G_n'] \ge 1 - cR_p(n). \tag{4.15}$$

Proof. The main ideas of the proof are the subject of Section 5. □

For completeness, we recall some results of Chung [40] and Révész [8] on inhomogeneous discrete time birth and death processes.

4.2. Basic results on the random walk in a fixed environment

We will always assume that α is fixed (denoted as $\alpha \in \Omega_1$ in this work). Let $x \in \mathbb{Z}$, assume $a < x < b$, the two following lemmata can be found in [40] (pages 73–76), the proof follows from the method of difference equations.

Lemma 4.3. *Recalling (3.9), we have*

$$\mathbb{P}_x^\alpha[T_a > T_b] = \frac{\sum_{i=a+1}^{x-1} \exp(S_i - S_a) + 1}{\sum_{i=a+1}^{b-1} \exp(S_i - S_a) + 1}, \tag{4.16}$$

$$\mathbb{P}_x^\alpha[T_a < T_b] = \frac{\sum_{i=x+1}^{b-1} \exp(S_i - S_b) + 1}{\sum_{i=a+1}^{b-1} \exp(S_i - S_b) + 1}. \tag{4.17}$$

Let $T_a \wedge T_b$ be the minimum between T_a and T_b.

Lemma 4.4. *We have*

$$\mathbb{E}_{a+1}^\alpha[T_a \wedge T_b] = \frac{\sum_{l=a+1}^{b-1} \sum_{j=l}^{b-1} \frac{1}{\alpha_l} F(j,l)}{\sum_{j=a+1}^{b-1} F(j,a) + 1}, \tag{4.18}$$

$$\mathbb{E}_x^\alpha[T_a \wedge T_b] = \mathbb{E}_{a+1}^\alpha[T_a \wedge T_b]\left(1 + \sum_{j=a+1}^{x-1} F(j,a)\right) - \sum_{l=a+1}^{x-1} \sum_{j=l}^{x-1} \frac{1}{\alpha_l} F(j,l), \tag{4.19}$$

where $F(j,l) = \exp(S_j - S_l)$.

Now we give some explicit expressions for the local times that can be found in [8] (page 279).

Lemma 4.5. *Under $\mathbb{P}_x^\alpha$, $\mathcal{L}(x, T_b \wedge T_a)$ is a geometric random variable with parameter*

$$p = \alpha_x \mathbb{P}_{x+1}^\alpha[T_x < T_b] + \beta_x \mathbb{P}_{x-1}^\alpha[T_x < T_a], \tag{4.20}$$

that is for all $l \geq 0$, $\mathbb{P}_x^\alpha[\mathcal{L}(x, T_b \wedge T_a) = l] = p^l(1-p)$.

Lemma 4.6. *For all $i \in \mathbb{Z}$, we have, if $x > i$*

$$\mathbb{E}_i^\alpha[\mathcal{L}(x, T_i)] = \frac{\alpha_i \mathbb{P}_{i+1}^\alpha[T_x < T_i]}{\beta_x \mathbb{P}_{x-1}^\alpha[T_x > T_i]}, \tag{4.21}$$

if $x < i$

$$\mathbb{E}_i^\alpha[\mathcal{L}(x, T_i)] = \frac{\beta_i \mathbb{P}_{i-1}^\alpha[T_x < T_i]}{\alpha_x \mathbb{P}_{x+1}^\alpha[T_x > T_i]}. \tag{4.22}$$

4.3. Proof of Theorems 3.1, 3.5 and 3.7

First we recall the following two elementary results.

Proposition 4.7. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $p \geq 2$ and $\gamma > 2$ there exists $n_0 \equiv n_0(\gamma)$ such that for all $n > n_0$, $Q[G'_n] \geq 1 - cR_p(n)$ and for all $\alpha \in G'_n$*

$$\mathbb{P}_0^\alpha\left[\bigcup_{m=0}^{n}\{X_m \notin W_n\}\right] \leq \frac{2\log_p n}{\sigma^2(\log n)^{\gamma-2}}. \tag{4.23}$$

Proof. We use the properties (4.7)–(4.9) and make computations similar from those done by Sinai [3]. The constraint $\gamma > 2$ is here to get a useful result. □

Lemma 4.8. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $p \geq 2$ and $\gamma \geq 11$, there exists $n_0 \equiv n_0(\gamma)$ such that for all $n > n_0$, $Q[G'_n] \geq 1 - cR_p(n)$ and for all $\alpha \in G'_n$*

$$\mathbb{P}_0^\alpha\left[T_{m_n} > \frac{n}{(\log n)^4}\right] \leq \frac{2(\log_p n)^3}{\eta_0\sigma^6(\log n)^{\gamma-10}}. \tag{4.24}$$

Remark 4.9. The constraint $\gamma \geq 11$ that appears in the Theorems 3.1, 3.5, 3.8 and 3.14 and in their corollaries comes from (4.24).

Proof. Assume $m_n > 0$, first we remark that

$$\mathbb{P}_0^\alpha\left[T_{m_n} > \frac{n}{(\log n)^4}\right] \leq \mathbb{P}_0^\alpha[T_{m_n} > T_{M'_n-1}] + \mathbb{P}_0^\alpha\left[T_{m_n} \wedge T_{M'_n-1} > \frac{n}{(\log n)^4}\right]. \tag{4.25}$$

Using (4.7) and (4.8) and making computations similar from those done by Sinai [3] we get that, for all $\gamma > 0$, there exists $n_0 \equiv n_0(\gamma)$ such that for all $n > n_0$ and $\alpha \in G'_n$

$$\mathbb{P}_0^\alpha[T_{m_n} > T_{M'_n-1}] \leq \frac{\log_p n}{\sigma^2(\log n)^\gamma}. \tag{4.26}$$

Using the Markov inequality, Lemma 4.4, the properties (4.7) and (4.8) we obtain for all $\gamma \geq 11$, $n > n_0$ and $\alpha \in G'_n$

$$\mathbb{P}_0^\alpha\left[T_{m_n} \wedge T_{M'_n-1} > \frac{n}{(\log n)^4}\right] \leq \frac{(\log_p n)^3}{\eta_0\sigma^6(\log n)^{\gamma-10}}. \tag{4.27}$$

Collecting (4.27), (4.26) and (4.25) ends the proof of the lemma. □

Proof of Theorem 3.1. Let $\gamma \geq 11$ and $p \geq 2$, using Proposition 4.2 we take $c > 0$ and n_1 such that for all $n \geq n_1$, $Q[G'_n] \geq 1 - cR_p(n)$. Let $0 < \rho < 2$, define $\delta_n = (f_p(n))^{-\rho/2}$, to prove Theorem 3.1 we need to give an upper bound of the probability $\mathbb{P}_0^\alpha[\mathcal{L}(\mathbb{F}_p^c(n), n) \geq \delta_n n]$ where $\mathbb{F}_p^c(n)$ is the complementary of $\mathbb{F}_p(n)$ in $\mathbb{Z}$.

First we use Proposition 4.7 to reduce the set $\mathbb{F}_p^c(n)$: there exists n_2 such that for all $n > n_2$ and $\alpha \in G'_n$

$$\mathbb{P}_0^\alpha[\mathcal{L}(\mathbb{F}_p^c(n), n) > \delta_n n] \leq \mathbb{P}_0^\alpha[\mathcal{L}(\bar{\mathbb{F}}_p(n), n) \geq \delta_n n] + \frac{2\log_p n}{\sigma^2(\log n)^{\gamma-2}} \tag{4.28}$$

recalling that $\bar{\mathbb{F}}_p(n)$ is the complementary of $\mathbb{F}_p(n)$ in W_n. By Lemma 4.8, there exists n_3 such that for all $n > n_3$ and $\alpha \in G'_n$

$$\mathbb{P}_0^\alpha[\mathcal{L}(\bar{\mathbb{F}}_p(n), n) \geq \delta_n n] \leq \mathbb{P}_0^\alpha\left[\mathcal{L}(\bar{\mathbb{F}}_p(n), n) \geq \delta_n n, T_{m_n} \leq \frac{n}{(\log n)^4}\right]$$

$$+\frac{2(\log_p n)^6}{\eta_0\sigma^6(\log n)^{\gamma-10}}. \tag{4.29}$$

Let us define $N_0 = [n(\log n)^{-4}] + 1$ and $\delta_n' = \delta_n - N_0/n$. By the Markov property and the homogeneity of the Markov chain we get

$$\mathbb{P}_0^\alpha\left[\mathcal{L}(\bar{\mathbb{F}}_p(n), n) \geq \delta_n n, T_{m_n} \leq \frac{n}{(\log n)^4}\right] \leq \mathbb{P}_{m_n}^\alpha\left[\sum_{k=1}^{n}\mathbb{I}_{\{X_k\in\bar{\mathbb{F}}_p(n)\}} \geq \delta_n' n\right]. \tag{4.30}$$

Let $j \geq 2$, define the following return times

$$T_{m_n,j} \equiv \begin{cases}\inf\{k > T_{m_n,j-1}, X_k = m_n\}, \\ +\infty, \quad \text{if such } k \text{ does not exist.}\end{cases}$$
$$T_{m_n,1} \equiv T_{m_n} (\text{see } (3.9)).$$

Since by definition $T_{m_n,n} > n$, we have

$$\mathbb{P}_{m_n}^\alpha\left[\sum_{k=1}^{n}\mathbb{I}_{\{X_k\in\bar{\mathbb{F}}_p(n)\}} \geq \delta_n' n\right] \leq \mathbb{P}_{m_n}^\alpha\left[\sum_{k=1}^{T_{m_n,n}}\mathbb{I}_{\{X_k\in\bar{\mathbb{F}}_p(n)\}} \geq \delta_n' n\right]. \tag{4.31}$$

By definition of the local time and the Markov inequality we get

$$\mathbb{P}_{m_n}^\alpha\left[\sum_{k=1}^{T_{m_n,n}}\mathbb{I}_{\{X_k\in\bar{\mathbb{F}}_p(n)\}} \geq \delta_n' n\right]$$
$$\leq \left(\sum_{s_1=m_n+f_p(n)+1}^{M_n}\mathbb{E}_{m_n}^\alpha[\mathcal{L}(s_1, T_{m_n,n})] + \sum_{s_2=M_n'}^{m_n-f_p(n)-1}\mathbb{E}_{m_n}^\alpha[\mathcal{L}(s_2, T_{m_n,n})]\right)(\delta_n' n)^{-1}. \tag{4.32}$$

Now we use the fact that, by the strong Markov property, the random variables $\mathcal{L}(s, T_{m_n,i+1} - T_{m_n,i})(0 \leq i \leq n-1)$ are i.i.d., therefore the right hand side of (4.32) is equal to

$$\mathbb{E}_{m_n}^\alpha[\mathcal{L}(\bar{\mathbb{F}}_p(n), T_{m_n})](\delta_n')^{-1}. \tag{4.33}$$

Using the property (4.13), for all $n > n_1$ and all $\alpha \in G_n'$

$$\mathbb{E}_{m_n}^\alpha\left[\mathcal{L}\left(\bar{\mathbb{F}}_p(n), T_{m_n}\right)\right] \leq \frac{2}{\eta_0}\frac{1}{f_p(n)+1}, \tag{4.34}$$

recalling $f_p(n) = [(\log_2 n\log_p n)^2]$. Collecting what we did above, we obtain for all $n \geq n_1 \vee n_2 \vee n_3$ ($a \vee b = \max(a, b)$)

$$\mathbb{P}_0^\alpha[\mathcal{L}(\mathbb{F}_p^c(n), n) > \delta_n n] \leq \frac{2}{\eta_0}\frac{1}{(\log_2 n\log_p n)^2\delta_n'} + \frac{2\log_p n}{\sigma^2(\log n)^{\gamma-2}}$$
$$+\frac{2(\log_p n)^3}{\eta_0\sigma^6(\log n)^{\gamma-10}}. \tag{4.35}$$

Taking $\gamma \geq 11$ and choosing $n_0 \geq n_1 \vee n_2 \vee n_3$ ends the proof of the theorem. $\square$

In a similar way, we can prove the following proposition that will be used in the next section.

Proposition 4.10. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $p \geq 2$, $0 < \rho < 1$ and $0 < \xi' \leq 1$, there exists n_0 such that for all $n > n_0$, $Q[G'_n] \geq 1 - cR_p(n)$ and for all $\alpha \in G'_n$*

$$\mathbb{P}^\alpha_{m_n}[\mathcal{L}(\mathbb{F}_p(n), [\xi' n]) \geq [n\xi'](1 - (\log_2 n (\log_p n)^{2-\rho})^{-1})] \geq 1 - \frac{2}{\eta_0} \frac{1}{(\log_2 n)(\log_p n)^\rho}. \tag{4.36}$$

Proof of Theorems 3.5 and 3.7. First, to prove Theorem 3.5 we use exactly the same method as in the proof of Theorem 3.1 but instead of using Proposition 3.4 we use Proposition 3.6. Now, to get Theorem 3.7 it is enough to find a point x such that $\mathcal{L}([x - \theta(n), x + \theta(n)], n) \geq n/2$ in probability, so choosing $x = m_n$ and using Theorem 3.5 we get the result. □

4.4. *Proof of Theorem 3.8*

Let $p \geq 2$, $\gamma \geq 11$, throughout this proof we take $c > 0$, c_1 and $n_1 \equiv n_1(\gamma)$ such that for all $n > n_1$, $Q[G'_n] \geq 1 - cR_p(n)$. Let $0 < \rho < 1$, define $\nu_n = (\log_2 n)^{-1}(\log_p n)^{-\rho}$, $x_1 = \mathbb{E}^\alpha_{m_n}[\mathcal{L}(\mathbb{F}_p(n), T_{m_n})]$ and $x_2 = \mathbb{E}^\alpha_{m_n}[\mathcal{L}(\bar{\mathbb{F}}_p(n), T_{m_n})]$, by definition we have

$$\mathbb{E}^\alpha_{m_n}[\mathcal{L}(W_n, T_{m_n})] = x_1 + x_2. \tag{4.37}$$

Notice that $m_n \in \mathbb{F}_p(n)$ so $x_1 \geq 1$. Moreover the property (4.13) implies that for all $\alpha \in G'_n$, $x_2 \leq 2(\eta_0(f_p(n) + 1))^{-1}$. We have $(x_1 + x_2)^{-1} = (x_1)^{-1} - x_2(x_1(x_1 + x_2))^{-1}$ and one can choose n_2 such that for all $n > n_2$, $x_2(x_1(x_1 + x_2))^{-1} \leq \nu_n/2$. Therefore, for all $n > n_1 \vee n_2$ and all $\alpha \in G'_n$, we get

$$\mathbb{P}^\alpha_0\left[\left|\frac{\mathcal{L}(m_n, n)}{n} - \frac{1}{x_1 + x_2}\right| > \nu_n\right] \leq \mathbb{P}^\alpha_0\left[\left|\frac{\mathcal{L}(m_n, n)}{n} - \frac{1}{x_1}\right| > \frac{\nu_n}{2}\right]. \tag{4.38}$$

We are left to estimate the right hand side of (4.38), we have

$$\mathbb{P}^\alpha_0\left[\left|\frac{\mathcal{L}(m_n, n)}{n} - \frac{1}{x_1}\right| > \nu_n/2\right] \leq \mathbb{P}^\alpha_0[\mathcal{L}(m_n, n) < n\eta_1] + \mathbb{P}^\alpha_0[\mathcal{L}(m_n, n) > n\eta_2] \tag{4.39}$$

where $\eta_1 \equiv \eta_1(\rho) = (x_1)^{-1} - \nu_n/2$, $\eta_2 \equiv \eta_2(\rho) = (x_1)^{-1} + \nu_n/2$.

We give an estimate for each of the terms in the right hand side of (4.39) in the following proposition.

Proposition 4.11. *There exists n'_1 such that for all $n > n'_1$ and $\alpha \in G'_n$*

$$\mathbb{P}^\alpha_0[\mathcal{L}(m_n, n) < \eta_1 n] \leq \frac{4}{\eta_0} \frac{1}{\log_2 n (\log_p n)^\rho}, \tag{4.40}$$

$$\mathbb{P}^\alpha_0[\mathcal{L}(m_n, n) > \eta_2 n] \leq \frac{32(\log_p n)^3}{\eta_0^2 \sigma^6 (\log n)^{\frac{1}{2}}}. \tag{4.41}$$

Proof. We will only prove (4.40), the proof of (4.41) is easier, one can check it with a similar method. By Lemma 4.8, there exists n_2 such that for all $n > n_2$ and $\alpha \in G_n'$

$$\mathbb{P}_0^\alpha[\mathcal{L}(m_n, n) < \eta_1 n] \le \mathbb{P}_0^\alpha\left[\mathcal{L}(m_n, n) < \eta_1 n,\ T_{m_n} \le \frac{n}{(\log n)^4}\right] + \frac{2(\log_p(n))^3}{\eta_0\sigma^6(\log n)^{\gamma-10}}. \tag{4.42}$$

Using the strong Markov property and the fact that $\sum_{j=1}^{T_{m_n}} \mathbb{I}_{X_j=m_n} = 1$ we get that

$$\mathbb{P}_0^\alpha\left[\mathcal{L}(m_n, n) < \eta_1 n,\ T_{m_n} \le \frac{n}{(\log n)^4}\right] \le \mathbb{P}_{m_n}^\alpha[\mathcal{L}(m_n, (1-\zeta_n)n) < \eta_1 n], \tag{4.43}$$

where $\zeta_n = N_0/n$, with $N_0 = [n(\log n)^{-4}] + 1$. Using Proposition 4.10 with $\xi' = 1 - \xi_n$, there exists n_3 such that for all $n > n_3$ and all $\alpha \in G_n'$

$$\begin{aligned}&\mathbb{P}_{m_n}^\alpha[\mathcal{L}(m_n, (1-\zeta_n)n) < \eta_1 n] \\ &\quad \le \mathbb{P}_{m_n}^\alpha[\mathcal{L}(m_n, (1-\zeta_n)n) < \eta_1 n, \mathcal{L}(\mathbb{F}_p(n), (1-\zeta_n)n) \ge (1-\delta_n'')(1-\zeta_n)n] \\ &\qquad + \frac{2(\eta_0)^{-1}}{(\log_2 n)(\log_p n)^\rho}.\end{aligned} \tag{4.44}$$

where $\delta_n'' = (\log_2 n \log_{p+1})^{-1}$. Let us define $\eta_1' \equiv \eta_1((1-\delta_n'')(1-\zeta_n))^{-1}$, we have

$$\begin{cases}\mathcal{L}(m_n, n(1-\zeta_n)) < \eta_1 n, \quad \text{and} \\ \mathcal{L}(\mathbb{F}_p(n), n(1-\zeta_n)) \ge (1-\delta_n'')(1-\zeta_n)n\end{cases} \Rightarrow \mathcal{L}(m_n, n(1-\zeta_n)) < \eta_1'\mathcal{L}(\mathbb{F}_p(n), n(1-\zeta_n)), \tag{4.45}$$

therefore

$$\begin{aligned}&\mathbb{P}_{m_n}^\alpha[\mathcal{L}(m_n, n(1-\zeta_n)) < \eta_1 n, \mathcal{L}(\mathbb{F}_p(n), (1-\zeta_n)n) \ge (1-\delta_n'')(1-\zeta_n)n] \\ &\quad \le \mathbb{P}_{m_n}^\alpha[\mathcal{L}(m_n, n(1-\zeta_n)) < \eta_1'\mathcal{L}(\mathbb{F}_p(n), (1-\zeta_n)n)].\end{aligned} \tag{4.46}$$

To estimate the right hand side of (4.46), first we prove the following lemma.

Lemma 4.12. *For all $0 < \xi \le 1$ there exists $n_4 \ge n_1$ such that for all $n > n_4$ and $\alpha \in G_n'$*

$$\mathbb{P}_{m_n}^\alpha\left[\mathcal{L}(m_n, \xi n) \ge \frac{n}{(\log n)^7}\right] \ge 1 - \frac{16(\log_p n)^3}{\eta_0^2\sigma^6(\log n)^{\frac{1}{2}}\xi}.$$

Proof. Let us define the two points $M_< \in [M_n{}', m_n]$ and $M_> \in [m_n, M_n]$ by

$$M_< = \sup\{t, 0 > t > M_n{}', S_t - S_{m_n} \ge \log n - (6 + 1/2)\log_2 n\}, \tag{4.47}$$
$$M_> = \inf\{t,\ 0 < t < M_n, S_t - S_{m_n} \ge \log n - (6 + 1/2)\log_2 n\}. \tag{4.48}$$

Using (2.4), (4.5) and (4.6) it is easy to show that for all $n > n_1$ and all $\alpha \in G_n'$ these two points exist. By the Markov inequality and using (4.19), we obtain that

$$\mathbb{P}_{m_n}^\alpha[T_{M_<-1} \wedge T_{M_>+1} > \xi n] \le |M_< - M_>|^3 \exp[(S_{M_<} - S_{m_n}) \vee (S_{M_>} - S_{m_n})](\xi n)^{-1}.$$

Using (4.47) and (4.48) and the property (4.7) we get that for all $n > n_1$ and all $\alpha \in G'_n$

$$\mathbb{P}^{\alpha}_{m_n}[T_{M_<-1} \wedge T_{M_>+1} > \xi n] \leq \frac{8(\log_p n)^3}{\eta_0^2 \sigma^6 (\log n)^{\frac{1}{2}} \xi}.$$

Therefore, for all $n > n_1$ and all $\alpha \in G'_n$

$$\mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(m_n, \xi n) \geq \frac{n}{(\log n)^7}\right] \geq \mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(m_n, T_{M_<-1} \wedge T_{M_>+1}) \geq \frac{n}{(\log n)^7}\right] - \frac{8(\log_p n)^3}{\eta_0^2 \sigma^6 (\log n)^{\frac{1}{2}} \xi}. \tag{4.49}$$

By Lemma 4.5

$$\mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(m_n, T_{M_<-1} \wedge T_{M_>+1}) \geq \frac{n}{(\log n)^7}\right] = (1 - \alpha_{m_n}\mathbb{P}^{\alpha}_{m_n+1}[T_{m_n} \geq T_{M_>+1}] - \beta_{m_n}\mathbb{P}^{\alpha}_{m_n-1}[T_{m_n} \geq T_{M_<-1}])^{[n(\log n)^{-7}]+1}. \tag{4.50}$$

Using (4.16), (4.17) and the definition of $M_>$ and $M_<$ we have

$$1 - \alpha_{m_n}\mathbb{P}^{\alpha}_{m_n+1}[T_{m_n} > T_{M_>}] - \beta_{m_n}\mathbb{P}^{\alpha}_{m_n-1}[T_{m_n} > T_{M_<}] \geq 1 - \frac{(\log n)^{6+\frac{1}{2}}}{n}. \tag{4.51}$$

Now replacing (4.51) in (4.50) and noticing that $(1-x)^m \geq 1 - mx$ for all $0 \leq x \leq 1$ and $m \geq 1$ we have

$$\mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}\left(m_n, T_{M_<} \wedge T_{M_>}\right) \geq \frac{n}{(\log n)^7}\right] \geq 1 - \frac{1}{(\log n)^{1/2}}, \tag{4.52}$$

and inserting (4.52) in (4.49) we get the lemma. □

Coming back to (4.46), using Lemma 4.12 with $\xi = 1 - \zeta_n$, for all $n > n_4$ and all $\alpha \in G'_n$

$$\mathbb{P}^{\alpha}_{m_n}[\mathcal{L}(m_n, n(1-\zeta_n)) < \eta'_1 \mathcal{L}(\mathbb{F}_p(n), (1-\zeta_n)n)] \leq \mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(m_n, n(1-\zeta_n)) < \eta'_1 \mathcal{L}(\mathbb{F}_p(n), (1-\zeta_n)n), \mathcal{L}(m_n, n(1-\zeta_n)) \geq \frac{n}{(\log n)^7}\right] + \frac{16(\log_p n)^3}{\eta_0^2 \sigma^6 (\log n)^{\frac{1}{2}}(1-\zeta_n)}. \tag{4.53}$$

Let us define $N_1 = [n(\log n)^{-7}] + 1$, we have

$$\mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(m_n, n(1-\zeta_n)) < \eta'_1 \mathcal{L}(\mathbb{F}_p(n), (1-\zeta_n)n), \mathcal{L}(m_n, n(1-\zeta_n)) \geq \frac{n}{(\log n)^7}\right] \leq \sum_{l=N_1}^{n} \mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(\mathbb{F}_p(n), T_{m_n,l+1}) > \frac{l}{\eta'_1}\right], \tag{4.54}$$

recalling $T_{m_n,l+1} = \inf\{k > T_{m_n,l}, X_k = m_n\}$ for all $l \geq 1$ and $T_{m_n,1} \equiv Tm_n$ (see (3.9)).

To estimate the probability in (4.54) we want to use the exponential Markov inequality. We need the following lemma which is easy to prove by elementary computations.

Lemma 4.13. *Let* $(\rho_n, n \in \mathbb{N})$ *be a positive decreasing sequence such that* $\lim_{n\to+\infty} \rho_n(\log_2 n)^2 = 0$*, there exists* $n_5 \geq n_1$ *such that for all* $n > n_5$ *and all* $\alpha \in G'_n$

$$\frac{1}{\eta'_1} - x_1 - \rho_n \geq (\nu_n x_1^2)/4 > 0$$

where η'_1 *is defined just before (4.45),* x_1 *and* ν_n *just before (4.37).*

Coming back to (4.54), we have

$$\sum_{l=N_1}^{n} \mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(\mathbb{F}_p(n), T_{m_n,l+1}) > \frac{l}{\eta'_1}\right]$$
$$\leq \sum_{l=N_1}^{n} \mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(\mathbb{F}_p(n),\ T_{m_n,l+1}) - (l+1)x_1 > l\left(\frac{1}{\eta'_1} - x_1\left(1 + \frac{1}{N_1}\right)\right)\right]. \tag{4.55}$$

Using Lemma 4.13, with $\rho_n = \frac{x_1}{N_1} \equiv \frac{x_1}{[n/(\log n)^{-7}]}$ and property (4.12), for all $n > n_5$ and $\alpha \in G'_n$

$$\frac{1}{\eta'_1} - x_1\left(1 + \frac{1}{N_1}\right) \equiv \frac{1}{\eta'_1} - x_1 - \frac{x_1}{N_1} \geq (\nu_n x_1^2)/4 > 0.$$

So for all $n > n_5$ and all $\alpha \in G'_n$ we use the exponential Markov inequality to estimate the probability in the right hand side of (4.55). Let $\lambda > 0$ for all $n > n_5$ and all $\alpha \in G'_n$

$$\mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(\mathbb{F}_p(n),\ T_{m_n,l+1}) - (l+1)x_1 > l\left(\frac{1}{\eta'_1} - x_1\left(1 + \frac{1}{N_1}\right)\right)\right]$$
$$\leq \mathbb{E}^{\alpha}_{m_n}[\exp\{\lambda(\mathcal{L}(\mathbb{F}_p(n), T_{m_n,l+1}) - (l+1)x_1)\}]$$
$$\times \exp\left[-\lambda l\left(\frac{1}{\eta'_1} - x_1\left(1 + \frac{1}{N_1}\right)\right)\right]. \tag{4.56}$$

By the strong Markov property we have

$$\mathbb{E}^{\alpha}_{m_n}[\exp\{\lambda(\mathcal{L}(\mathbb{F}_p(n),\ T_{m_n,l+1}) - (l+1)x_1)\}]$$
$$= (\mathbb{E}^{\alpha}_{m_n}[\exp\{\lambda(\mathcal{L}(\mathbb{F}_p(n), T_{m_n}) - x_1)\}])^{l+1}. \tag{4.57}$$

To estimate the Laplace transform on the right hand side of (4.57) we use the Hölder inequality and the results of M. Csörgö L. Horváth and P. Révész (see [8] pages 279–280): choosing

$$\lambda = \frac{(u_n^+ \wedge u_n^-)^2}{(|\mathbb{F}_p^-(n)| + |\mathbb{F}_p^+(n)|)^3}, \tag{4.58}$$

where

$$u_n^+ = \min_{q\in\mathbb{F}_p^+(n)} (\beta_q \mathbb{P}^{\alpha}_{q-1}[T_q > T_{m_n}]),$$
$$u_n^- = \min_{q\in\mathbb{F}_p^-(n)} (\alpha_q \mathbb{P}^{\alpha}_{q+1}[T_q > T_{m_n}]),$$

$\mathbb{F}_p^-(n)$ and $\mathbb{F}_p^+(n)$ have been defined just before (4.10) and $a \wedge b = \min(a, b)$, we get

$$\mathbb{E}^{\alpha}_{m_n}[\exp\{\lambda(\mathcal{L}(\mathbb{F}_p(n), T_{m_n}) - \mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(\mathbb{F}_p(n), T_{m_n})])\}] \leq \exp\left[\frac{2\lambda}{|\mathbb{F}_p(n)|}\right]. \tag{4.59}$$

Now using (4.56), for all $n > n_5$, all $\alpha \in G'_n$ and all $l \geq N_1$

$$\mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(\mathbb{F}_p(n),\ T_{m_n,l}) - (l+1)x_1 > l\left(\frac{1}{\eta'_1} - x_1\left(1+\frac{1}{N_1}\right)\right)\right]$$
$$\leq \exp\left[-\lambda l\left\{\frac{1}{\eta'_1} - \left(x_1 + \frac{2}{|\mathbb{F}_p(n)|}\right)\left(1+\frac{1}{N_1}\right)\right\}\right]. \tag{4.60}$$

Inserting (4.60) in (4.55) and using (4.54), we deduce that for all $n > n_5$ and $\alpha \in G'_n$

$$\mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(m_n, n(1-\zeta_n)) < \eta'_1\mathcal{L}(\mathbb{F}_p(n), (1-\zeta_n)n), \mathcal{L}(m_n, n(1-\zeta_n)) \geq \frac{n}{(\log n)^7}\right]$$
$$\leq \frac{\exp\left[-N_1\lambda\left(\frac{1}{\eta'_1} - \left(x_1 + \frac{2}{|\mathbb{F}_p(n)|}\right)(1+1/N_1)\right)\right]}{1-\exp\left[-\lambda\left(\frac{1}{\eta'_1} - \left(x_1 + \frac{2}{|\mathbb{F}_p(n)|}\right)(1+1/N_1)\right)\right]}. \tag{4.61}$$

Using Lemma 4.13 with $\rho_n = x_1/N_1 + (2/|\mathbb{F}_p(n)|)(1+1/N_1)$ we have for all $n > n_5$ and all $\alpha \in G'_n$

$$\frac{1}{\eta'_1} - \left(x_1 + \frac{2}{|\mathbb{F}_p(n)|}\right)(1+1/N_1) \geq (\nu_n x_1^2)/4. \tag{4.62}$$

Using (4.62) and (4.61) we obtain, after an easy computation, that for all $n > n_5$ and all $\alpha \in G'_n$

$$\mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(m_n, n(1-\zeta_n)) < \eta'_1\mathcal{L}(\mathbb{F}_p(n), (1-\zeta_n)n), \mathcal{L}(m_n, n(1-\zeta_n)) \geq \frac{n}{(\log n)^7}\right]$$
$$\leq \frac{8\exp(-(N_1\lambda\nu_n x_1^2)/4)}{\lambda\nu_n x_1^2}. \tag{4.63}$$

Now we need a lower and an upper bound for λ, using (4.58) and the properties (4.10) and (4.11) we deduce that for all $n > n_1$ and all $\alpha \in G'_n$ we have

$$\frac{1}{(g_1(n))^2(\log_2 n \log_p n)^6} \leq \lambda \leq \frac{2}{(\log_2 n \log_p n)^6}, \tag{4.64}$$

with $g_1(n) = \exp[((4\sqrt{3}\sigma f_p(n))^2 \log_3(n))^{1/2}]$. We deduce that there exists $n_6 \geq n_5$ such that for all $n \geq n_6$ and all $\alpha \in G'_n$

$$\mathbb{P}^{\alpha}_{m_n}\left[\mathcal{L}(m_n, n(1-\zeta_n)) < \eta'_1\mathcal{L}(\mathbb{F}_p(n), (1-\zeta_n)n), \mathcal{L}(m_n, n(1-\zeta_n)) \geq \frac{n}{(\log n)^7}\right]$$
$$\leq \exp(-n^{1/2}). \tag{4.65}$$

Collecting (4.65), (4.53), (4.46), (4.44) and (4.42) and taking $p = 2$, $n'_1 = n_2 \vee n_3 \vee n_4 \vee n_6$ we get (4.40). □

4.5. *Proof of Theorem 3.14*

Clearly the probability in (3.28) is bounded from above by:

$$|\mathbb{L}(n)| \max_{k\in\mathbb{L}(n)}\left\{\mathbb{P}^{\alpha}_0\left[\left|\frac{\mathcal{L}(k,n)}{n} - \frac{1}{\mathbb{E}^{\alpha}_k\left[\mathcal{L}(W_n, T_k)\right]}\right| > \frac{1}{(\log_2 n)^{1+\rho}}\right]\right\}.$$

We are left to give an upper bound of the probability into the bracket, uniformly in $k \in \mathbb{F}_p(n)$. Since the method we use is exactly the same as the one of the proof of Theorem 3.1, we only sketch it. First replace all the m_n by k except in "$\mathbb{F}_p(n)$", "$\mathbb{F}_p^+(n)$", "$\mathbb{F}_p^-(n)$" and "W_n" that do not change. This will change of course the definitions of x_1, x_2 and λ.

Then the only modifications needed are based on the following fact, let $k \in \mathbb{F}_p(n)$ and $l \in W_n, l \neq k$, with Q probability 1 we have

$$\begin{aligned}\exp(S_l - S_k) &\equiv \exp(S_k - S_{m_n}) \exp(S_l - S_{m_n}) \\ &\leq \left(\log \frac{1-\eta_0}{\eta_0}\right)^{|k-m_n|} \exp(S_l - S_{m_n}) \leq (\log_2 n) \exp(S_l - S_{m_n}). \end{aligned} \tag{4.66}$$

As we will see in Section 5.5, it is from (4.66) and (4.13) (respectively (4.12)) that we get (3.21) (respectively (3.22)).

The main steps of the proof of Theorem 3.8 have to be modified as follows:

First remark that, using (3.21), (4.38) remains true. In Proposition 4.11, (4.40) becomes:

$$\mathbb{P}_k^{\alpha}\left[\mathcal{L}(k,n) < \eta_1 n\right] \leq \frac{4}{\eta_0} \frac{1}{(\log_p n)^{\rho}}.$$

Notice that, comparing with (4.40) the $\log_2 n$ disappears, this comes from the fact that we use (3.21) (instead of (3.10)) to prove the equivalent of (4.36) (replacing $\mathbb{P}_{m_n}^{\alpha}$ by $\mathbb{P}_k^{\alpha}$). Moreover it is important to notice that Lemma 4.13 remains true and λ still verifies (4.64).

Moreover (4.41) becomes

$$\mathbb{P}_k^{\alpha}[\mathcal{L}(k,n) > \eta_2 n] \leq \frac{32(\log_p n)^3 \log_2 n}{\eta_0^2 \sigma^6 (\log n)^{\frac{1}{2}}}$$

since we use (4.66) for the proof of the equivalent of Lemma 4.12 (replacing, as always, m_n by k). □

5. Proof of the good properties for the environment

Here we give the main ideas for the proof of Propositions 3.4, 3.6, 3.11, 3.12 and 4.2. This section is organized as follows. In Section 5.1 we recall elementary results on sums of independent random variables, in Section 5.2 we prove Lemma 2.7 and give standard results on the basic valley, in Section 5.3 (respectively Section 5.4) we give some details of the proof of Proposition 3.12 (respectively Proposition 3.4). Collecting the results of Section 5.2, Lemma 2.7, Propositions 3.4 and 3.12 we get Proposition 4.2. In Section 5.5 we sketch the proof of (3.21) and (3.22). Throughout this section we assume that the parameter γ (see Definition 2.6) is strictly positive.

5.1. Elementary results

We will always work on the right hand side of the origin, that means with $(S_m, m \in \mathbb{N})$, by symmetry we obtain the same result for $m \in \mathbb{Z}_-$.

We introduce the following stopping times, for $a > 0$,

$$V_a^+ \equiv V_a^+(S_j, j \in \mathbb{N}) = \begin{cases} \inf\{m \in \mathbb{N}^*, \ S_m \geq a\}, \\ +\infty, \quad \text{if such a } m \text{ does not exist.} \end{cases} \tag{5.1}$$

$$V_a^- \equiv V_a^-(S_j, j \in \mathbb{N}) = \begin{cases} \inf\{m \in \mathbb{N}^*, S_m \le -a\}, \\ +\infty, \quad \text{if such a } m \text{ does not exist.} \end{cases} \tag{5.2}$$

The following lemma is an immediate consequence of the Wald equality (see [41]).

Lemma 5.1. *Assume (2.2)–(2.4), let $a > 0$, $d > 0$; we have*

$$Q[V_a^- < V_d^+] \le \frac{d + I_{\eta_0}}{d + a + I_{\eta_0}}, \tag{5.3}$$

$$Q[V_a^- > V_d^+] \le \frac{a + I_{\eta_0}}{d + a + I_{\eta_0}}, \tag{5.4}$$

recalling $I_{\eta_0} = \log((1 - \eta_0)(\eta_0)^{-1})$.

The following lemma is easy to prove when the $\epsilon_i = \pm 1$ with a probability $1/2$ and is a simple extension in our more general case.

Lemma 5.2. *Assume (2.2)–(2.4) hold, there exists $c_0 \equiv C_0(Q) > 0$ and $n_0 \equiv n_0(Q)$ such that for all $n > n_0$*

$$Q[V_0^- > r(n)] \le \frac{c_0}{\sqrt{r(n)}}, \tag{5.5}$$

$(r(n), n)$ is a strictly positive increasing sequence.

5.2. *Standard results on the basic valley $\{M_n, m_n, M_n{}'\}$*

First let us give the main ideas of the proof of Lemma 2.7.

Proof of Lemma 2.7. To prove that $\{M_n', m_n, M_n\} \neq \emptyset$ in probability, it is enough to find a valley $\{M', m, M\}$ that satisfies the three properties of Definition 2.6. It is easy to show that $\{M' = \bar{V}_{\Gamma_n}^+, m = \bar{m}, M = V_{\Gamma_n}^+\}$ with $\bar{V}_{\Gamma_n}^+ = \sup\{k < 0, S_k > \Gamma_n\}$ and $\bar{m} = \inf\{|k| > 0, S_k = \min_{\{\bar{V}_{\Gamma_n}^+ \le m \le V_{\Gamma_n}^+\}} S_m\}$ satisfies these properties in probability. Indeed by definition $\{\bar{V}_{\Gamma_n}^+, \bar{m}, V_{\Gamma_n}^+\}$ satisfies the first two properties of Definition 2.6. For the third one, assume for simplicity that $\bar{m} > 0$, by definition of $\bar{V}_{\Gamma_n}^+$ and hypothesis (2.4) we have $\Gamma_n \le S_{\bar{V}_{\Gamma_n}^+} \le \Gamma_n + I_{\eta_0}$ with probability 1. So we are left to prove that there exists $c_0 > 0$ and $n_0 \equiv n_0(Q)$ such that for all $n \ge n_0$

$$Q\left[S_{\bar{V}_{\Gamma_n}^+} - \max_{0 \le t \le \bar{m}}(S_t) \le \gamma \log_2 n\right] \equiv Q\left[\Gamma_n - \gamma \log_2 n \le \max_{0 \le t \le \bar{m}}(S_t) \le \Gamma_n + I_{\eta_0}\right]$$
$$\le \frac{c_0 \gamma \log_2 n}{\log n}. \tag{5.6}$$

To get this upper bound we make the following remark, the event $\{\Gamma_n - \gamma \log_2 n \le \max_{0 \le t \le \bar{m}}(S_t) \le \Gamma_n + I_{\eta_0}\}$ asks for the walk to reach a point larger than $\Gamma_n - \gamma \log_2 n$ and then to touch a point $S_{\bar{m}} \le 0$ before a point larger than or equal to $\Gamma_n + I_{\eta_0}$. So the probability on the right hand side of (5.6) can be bounded from above by a constant times the probability $Q_{\Gamma_n - \gamma \log_2 n}[V_0^- < V_{\Gamma_n + I_{\eta_0}}]$ (where $Q_y[\cdots] \equiv Q[\cdots | S_0 = y]$) which gives (5.6) by Lemma 5.1. For more details of this computation see for example [42] pages 56–58. □

Lemma 5.3. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $p \geq 2$ there exists $n_0 \equiv n_0(I_{\eta_0}, \sigma, \mathbb{E}_Q[|\epsilon_0|^3])$ such that for all $n > n_0$*

$$Q[M_n \leq (\sigma^{-1} \log n)^2 \log_p n, {M_n}' \geq -(\sigma^{-1} \log n)^2 \log_p n] \geq 1 - cR_p(n), \tag{5.7}$$

$$Q[M_n > m_n + f_p(n)] \geq 1 - cR_p(n), \tag{5.8}$$

$$Q\left[S_{M_1} - S_{m_1} \leq \log n - \gamma \log_2 n, S_{M_1'} - S_{m_1'} \leq \log n - \gamma \log_2 n\right] \geq 1 - cR_p(n). \tag{5.9}$$

See just before (4.8) for the definitions of M_1, m_1, M_1' and m_1', $f_p(n)$ is given by (3.3).

Proof. The proof of this lemma is easy and is omitted. □

Lemma 5.4. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $p \geq 2$ there exists $n_0 \equiv n_0(Q)$ such that for all $n > n_0$*

$$Q\left[\min_{k \in \mathbb{F}_p^+(n)} \left(\beta_k \mathbb{P}_{k-1}^{\alpha}\left[T_k^{k-1} > T_{m_n}^{k-1}\right]\right) \leq \frac{1}{g_1(n)}, \ m_n > 0\right] \leq cR_p(n),$$

$$Q\left[\min_{k \in \mathbb{F}_p^-(n)} \left(\alpha_k \mathbb{P}_{k+1}^{\alpha}\left[T_k^{k+1} > T_{m_n}^{k+1}\right]\right) \leq \frac{1}{g_1(n)}, \ m_n > 0\right] \leq cR_p(n), \tag{5.10}$$

with $g_1(n) = \exp[((2\sqrt{3}\sigma f_p(n))^2 \log_3(n))^{1/2}]$, recall that $\mathbb{F}_p^-(n)$ and $\mathbb{F}_p^+(n)$ have been defined just before (4.10).

Proof. The proof is omitted it makes use of Lemma 5.8 and elementary facts on sums of i.i.d. random variables. □

5.3. *Proof of Propositions 3.6, 3.9, 3.11 and 3.12*

5.3.1. *Preliminaries*

By linearity of the expectation we have:

$$\mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(W_n, T_{m_n})] \equiv \sum_{j=m_n+1}^{M_n} \mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(j, T_{m_n})] + \sum_{j=M_n'}^{m_n-1} \mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(j, T_{m_n})] + 1, \tag{5.11}$$

so using Lemma 4.3 we get Proposition 3.9. Now using Lemma 4.3 and hypothesis (2.4) we easily get the following lemma.

Lemma 5.5. *Assume (2.4), for all ${M_n}' \leq k \leq M_n$, $k \neq m_n$*

$$\frac{\eta_0}{1 - \eta_0} \frac{1}{\mathrm{e}^{S_k - S_{m_n}}} \leq \mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(k, T_{m_n})] \leq \frac{1}{\eta_0} \frac{1}{\mathrm{e}^{S_k - S_{m_n}}}, \tag{5.12}$$

with a Q probability equal to one.

Proposition 3.11 is a trivial consequence of Lemma 5.5 and (5.11). The following lemma is easy to prove:

Lemma 5.6. *For all $\alpha \in \Omega_1$ and $n > 3$, with a Q probability equal to one we have*

$$\sum_{j=m_n+1}^{M_n} \frac{1}{e^{S_j - S_{m_n}}} \le \sum_{i=1}^{N_n+1} \frac{1}{e^{a(i-1)}} \sum_{j=m_n+1}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}, \tag{5.13}$$

$$\sum_{j=M_n'}^{m_n-1} \frac{1}{e^{S_j - S_{m_n}}} \le \sum_{i=1}^{N_n+1} \frac{1}{e^{a(i-1)}} \sum_{j=M_n'}^{m_n-1} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}, \tag{5.14}$$

where $a = \frac{I_{\eta_0}}{4}$, $N_n = [(\Gamma_n + I_{\eta_0})/a]$, we recall that $I_{\eta_0} = \log((1-\eta_0)(\eta_0)^{-1})$ and $\mathbb{I}$ is the indicator function.

Using Proposition 3.11 and Lemma 5.6, we have for all $n > 3$

$$\mathbb{E}_Q[\mathbb{E}^{\alpha}_{m_n}[\mathcal{L}(W_n, T_{m_n})]] \le 1 + \sum_{i=1}^{N_n+1} \frac{1}{e^{a(i-1)}} \mathbb{E}_Q\left[\sum_{j=m_n+1}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}\right] + \sum_{i=1}^{N_n+1} \frac{1}{e^{a(i-1)}} \mathbb{E}_Q\left[\sum_{j=M_n'}^{m_n-1} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}\right]. \tag{5.15}$$

The next step for the proof of Proposition 3.12 is to show that the two expectations $\mathbb{E}_Q[\ldots]$ on the right hand side of (5.15) are bounded by a constant depending only on the distribution Q times a polynomial in i. This result is given by Lemma 5.9.

5.3.2. Proof of Proposition 3.12

Remark 5.7. We give some details of the proof of Proposition 5.11 mainly because it is based on a very nice cancellation that occurs between two $\Gamma_n \equiv \log n + \gamma \log_2 n$, see formulas (5.23) and (5.24). Similar cancellation is already present in [6].

Let us define the following stopping times, let $i > 1$:

$$u_0 = 0,$$
$$u_1 \equiv V_0^- = \inf\{m > 0,\ S_m < 0\},$$
$$u_i = \inf\{m > u_{i-1},\ S_m < S_{u_{i-1}}\}.$$

The following lemma gives a way to characterize the point m_n, it is inspired by the work of [6] and is just from inspection.

Lemma 5.8. *Let $n > 3$ and $\gamma > 0$, recall $\Gamma_n = \log n + \gamma \log_2 n$, assume $m_n > 0$, for all $l \in \mathbb{N}^*$ we have*

$$m_n = u_l \Rightarrow \begin{cases} \bigcap_{i=0}^{l-1} \{\max_{u_i \le j \le u_{i+1}} (S_i) - S_{u_i} < \Gamma_n\} \quad \text{and} \\ \max_{u_l \le j \le u_{l+1}} (S_i) - S_{u_i} \ge \Gamma_n \quad \text{and} \\ M_n = V^+_{\Gamma_n, l} \end{cases} \tag{5.16}$$

where

$$V^+_{z,l} \equiv V^+_{z,l}(S_j, j \ge 1) = \inf(m > u_l, S_m - S_{u_l} \ge z). \tag{5.17}$$

A similar characterization of m_n if $m_n \leq 0$ can be done (the case $m_n = 0$ is trivial).

Lemma 5.9. *There exists $c_0 \equiv c_0(Q)$ such that for all $i \geq 1$ and all n:*

$$\mathbb{E}_Q\left[\sum_{j=m_n+1}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}\right] \leq c_0 \times i^2, \tag{5.18}$$

$$\mathbb{E}_Q\left[\sum_{j=M_n'}^{m_n-1} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}\right] \leq c_0 \times i^2. \tag{5.19}$$

Proof. We will only prove (5.18), we get (5.19) symmetrically; moreover we assume that $m_n > 0$, computations are similar for the case $m_n \leq 0$. Thinking of the basic definition of the expectation, we need an upper bound for the probability:

$$Q\left[\sum_{j=m_n+1}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k\right].$$

First we make a partition over the values of m_n and then we use Lemma 5.8, we get:

$$\begin{aligned} Q\left[\sum_{j=m_n+1}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k\right] &\equiv \sum_{l \geq 0} Q\left[\sum_{j=m_n+1}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k,\ m_n = u_l\right] \\ &\leq \sum_{l \geq 0} Q\left[\mathcal{A}_l^+, \max_{u_l \leq j \leq u_{l+1}} (S_j) - S_{u_l} \geq \Gamma_n, \mathcal{A}_l^-\right] \end{aligned} \tag{5.20}$$

where

$$\mathcal{A}_l^+ = \sum_{s=u_l+1}^{V^+_{\Gamma_n,l}} \mathbb{I}_{\{S_j - S_{u_l} \in [a(i-1), ai[\}} = k,$$

$$\mathcal{A}_l^- = \bigcap_{r=0}^{l-1} \left\{\max_{u_r \leq j \leq u_{r+1}} (S_r) - S_{u_r} < \Gamma_n\right\}, \qquad \mathcal{A}_0^- = \Omega_1$$

for all $l \geq 0$. By the strong Markov property we have:

$$Q\left[\mathcal{A}_l^+, \max_{u_l \leq j \leq u_{l+1}} (S_j) - S_{u_l} \geq \Gamma_n, \mathcal{A}_l^-\right] \leq Q[\mathcal{A}_0^+, V_0^- > V_{\Gamma_n}^+] Q[\mathcal{A}_l^-]. \tag{5.21}$$

The strong Markov property gives also that the sequence $(\max_{u_r \leq j \leq u_{r+1}} (S_r) - S_{u_r} < \Gamma_n, r \geq 1)$ is independent and identically distributed, therefore:

$$Q[\mathcal{A}_l^-] \leq (Q[V_0^- < V_{\Gamma_n}^+])^{l-1}. \tag{5.22}$$

We notice that $Q[\mathcal{A}_0^+,\ V_0^- > V_{\Gamma_n}^+]$ does not depend on l, therefore, using (5.20)–(5.22) we get:

$$\begin{aligned} Q\left[\sum_{j=m_n+1}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k.\right] &\leq (1 + (Q[V_0^- \geq V_{\Gamma_n}^+])^{-1}) \\ &\quad \times Q[\mathcal{A}_0^+, V_0^- > V_{\Gamma_n}^+]. \end{aligned} \tag{5.23}$$

To get an upper bound for $Q[\mathcal{A}_0^+,\ V_0^- > V_{\Gamma_n}^+]$, first we introduce the following sequence of stopping times, let $s > 0$:

$$H_{ia,0} = 0,$$
$$H_{ia,s} = \inf\{m > H_{ia,s},\ S_m \in [(i-1)a, ia[\}.$$

Making a partition over the values of $H_{ia,k}$, by the Markov property we get:

$$\begin{aligned}
&Q[\mathcal{A}_0^+,\ V_0^- > V_{\Gamma_n}^+] \\
&\le \sum_{w\ge 0}\int_{(i-1)a}^{ia} Q\left[H_{ia,k} = w,\, S_w \in \mathrm{d}x,\, \bigcap_{s=0}^{w}\{S_s > 0\},\ \bigcap_{s=w+1}^{\inf\{l>w, S_l\ge \Gamma_n\}}\{S_s > 0\}\right] \\
&\le Q[H_{ia,k} < V_0^-] \max_{(i-1)a\le x\le ia}\{Q_x[V_{\Gamma_n-x}^+ < V_x^-]\} \\
&\equiv Q[H_{ia,k} < V_0^-] Q_{ia}[V_{\Gamma_n-ia}^+ < V_{ia}^-].
\end{aligned} \tag{5.24}$$

To finish we need an upper bound for $Q[H_{ia,k} < V_0^-]$, we do not want to give details of the computations for this because it is not difficult, however the reader can find these details in [42] pages 142–145. We have for all $i > 1$:

$$\begin{aligned}
Q[H_{ia,k} < V_0^-] \le\ & Q\left[V_0^- > V_{(i-1)a}^+\right]\left(1 - Q\left[\epsilon_0 < -\frac{I_{\eta_0}}{2}\right] Q_{(i-1)a-\frac{I_{\eta_0}}{4}}\right. \\
&\times \left.\left[V_{(i-1)a}^+ \ge V_0^-\right]\right)^{k-1},
\end{aligned} \tag{5.25}$$

$$Q[H_{a,k} < V_0^-] \le Q[\epsilon_0 \ge 0]\left(1 - Q\left[\epsilon_0 < -\frac{I_{\eta_0}}{4}\right]\right)^{k-1}. \tag{5.26}$$

So using (5.23)–(5.26) and Lemma 5.1 one can find a constant c_0 that depends only on the distribution Q such that for all $i \ge 0$:

$$\mathbb{E}_Q\left[\sum_{j=m_n+1}^{M_n} \mathbb{I}_{S_j - S_{m_n}\in[a(i-1),ai[}\right] \equiv \sum_{k=1}^{+\inf} kQ\left[\sum_{j=m_n+1}^{M_n} \mathbb{I}_{S_j - S_{m_n}\in[a(i-1),ai[} = k\right] \le c_0 \times i^2,$$

which provides (5.18) □

5.3.3. Proof of Proposition 3.6

To show Proposition 3.6 we use the same method as was previously used to prove Proposition 3.12, the key point is to show the following lemma:

Lemma 5.10. *There exists a constant $c_0 \equiv c_0(Q)$ such that for all strictly positive increasing sequences $(\theta(n), n)$, $i \ge 1$ and all n:*

$$\mathbb{E}_Q\left[\sum_{j=m_n+\theta(n)}^{M_n} \mathbb{I}_{S_j - S_{m_n}\in[a(i-1),ai[}\right] \le \frac{c_0 \times i^2}{\sqrt{\theta(n)}}, \tag{5.27}$$

$$\mathbb{E}_Q\left[\sum_{j=M_n'}^{m_n-\theta(n)} \mathbb{I}_{S_j - S_{m_n}\in[a(i-1),ai[}\right] \le \frac{c_0 \times i^2}{\sqrt{\theta(n)}}. \tag{5.28}$$

We will not give the details of this proof because it is very similar to the proof of Proposition 3.4. Just notice that $\sqrt{\theta(n)}$ comes from the fact that the probability $Q[\bigcap_{s=1}^{\theta_n}\{S_s > 0\}] \equiv Q[V_0 > \theta(n)]$ appears when we give an upper bound of $Q[\sum_{j=m_n+\theta(n)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1),ai[} = k]$. This comes from the fact that the event $\bigcap_{l=1}^{\theta(n)}\{S_{m_n+l} - S_{m_n} > 0\}$ is hidden in the definition of m_n. A last remark, the upper bound we get here is good for sequences $(\theta(n), n)$ that grow very slowly. In the next section we use another more powerful method for sequences that grow more rapidly.

5.4. *Proof of Proposition* 3.4

5.4.1. *Preliminaries*

It is here that the explicit form of $f_p(n)$ given by (3.3) will become clear. Using (5.11) and Lemma 5.5 we only need to find an upper bound for

$$\sum_{l=m_n+f_p(n)+1}^{M_n} \frac{1}{\mathrm{e}^{S_l - S_{m_n}}} + \sum_{l=M_n'}^{m_n - f_p(n)-1} \frac{1}{\mathrm{e}^{S_l - S_{m_n}}}. \tag{5.29}$$

Assume for the moment that

$$S_k - S_{m_n} \geq 2\log(|k-m_n|), \quad \forall k \in \{M_n{}', \ldots, m_n - f_p(n), m_n + f_p(n), \ldots, M_n{}'\}, \tag{5.30}$$

with a Q probability larger than $1 - cR_p(n)$ with $c > 0$. Then for all $k \in \{M_n{}', \ldots, m_n - f_p(n), m_n + f_p(n), \ldots, M_n{}'\}$, we have

$$\frac{1}{\mathrm{e}^{S_k - S_{m_n}}} \leq \frac{1}{(k-m_n)^2}.$$

This implies the convergence of the two partial sums in (5.29) and therefore with a Q probability close to 1

$$\begin{aligned}\mathbb{E}_{m_n}^{\alpha}[\mathcal{L}(V_n^{c,r}, T_{m_n})] &\leq \frac{1}{\eta_0}\left(\sum_{l=m_n+f_p(n)+1}^{M_n} \frac{1}{\mathrm{e}^{S_l - S_{m_n}}} + \sum_{l=M_n'}^{m_n - f_p(n)-1} \frac{1}{\mathrm{e}^{S_l - S_{m_n}}}\right) \\ &\leq \frac{1}{\eta_0}\frac{2}{f_p(n)+1},\end{aligned}$$

this entails (3.10). (5.30) is a consequence of Proposition 5.11 proved in the following section.

5.4.2. *Study of the potential* $(S_m, m \in \mathbb{Z})$ *in* $\bar{\mathbb{F}}_p(n)$

We will assume $m_n > 0$, the case $m_n \leq 0$ is similar.

Proposition 5.11. *There exists $c > 0$ such that if (2.2)–(2.4) hold, for all $p \geq 2$ there exists $c_0 \equiv c_0(Q)$ and $n_0 \equiv n_0(Q)$ such that for all $n > n_0$*

$$Q\left[\bigcup_{k=m_n+f_p(n)}^{M_n} \{S_k - S_{m_n} \leq 2\log(k-m_n)\}, m_n > 0\right] \leq cR_p(n) + \frac{c_0}{\log_p n}, \tag{5.31}$$

$$Q\left[\bigcup_{k=M_{n'}}^{m_n+f_p(n)} \{S_k - S_{m_n} \leq 2\log(m_n-k)\}, m_n > 0\right] \leq cR_p(n) + \frac{c_0}{\log_p n}. \tag{5.32}$$

Proof of Proposition 5.11. We will only prove (5.31); we get (5.32) with the same arguments. Let $n \geq 3$, we define $L(n) = (\sigma^{-1} \log n)^2 \log_p n$ and $[L(n)]$ is the integer part of $L(n)$. By (5.7) and (5.8), there exists $n_1 \equiv n_1(I_{\eta_0}, \sigma, \mathbb{E}_Q[|\epsilon_0|^3])$ such that for all $n > n_1$

$$Q\left[\bigcup_{k=m_n+f_p(n)}^{M_n} \{S_k - S_{m_n} \leq 2\log(k - m_n)\}, m_n > 0\right] \leq cR_p(n) + Q[\mathcal{A}], \tag{5.33}$$

where

$$\mathcal{A} = \left\{\bigcup_{k=m_n+f_p(n)}^{M_n} \left\{S_k - S_{m_n} \leq 2\log(k - m_n)\right\}, m_n + f_p(n) < M_n \leq L(n)\right\}.$$

Using the same method details as for the proof of Lemma 5.9 (from lines (5.20)–(5.23)), we get:

$$Q[\mathcal{A}] \leq (1 + (Q[V_0^- \geq V_{\Gamma_n}^+])^{-1})Q[\mathcal{A}_0^+] \tag{5.34}$$

where

$$\mathcal{A}_0^+ = \left\{\bigcup_{j=f_p(n)}^{V_{\Gamma_n}^+} \left\{S_j \leq 2\log(j)\right\}, f_p(n) < V_{\Gamma_n}^+ \leq L(n)\right\}.$$

Let us define $\mathcal{C}_n = \bigcup_{j=f_p(n)}^{V_{\Gamma_n}^+} \{S_j \leq 2\log j\}$, to estimate $Q[\mathcal{A}_0^+]$ we make a partition over the values of $S_{f_p(n)}$,

$$Q[\mathcal{A}_0^+] = Q[\mathcal{C}_n, V_0^- > V_{\Gamma_n}^+, f_p(n) < V_{\Gamma_n}^+ \leq L(n), S_{f_p(n)} \leq 2\log f_p(n)] \tag{5.35}$$

$$+ Q[\mathcal{C}_n, V_0^- > V_{\Gamma_n}^+, f_p(n) < V_{\Gamma_n}^+ \leq L(n), S_{f_p(n)} > 2\log f_p(n)]. \tag{5.36}$$

First let us estimate (5.35), we remark that $\{V_0^- > V_{\Gamma_n}^+, f_p(n) < V_{\Gamma_n}^+ \leq L(n)\} \Rightarrow S_{f_p(n)} \geq 0$, so by the Markov property

$$\begin{aligned} &Q[\mathcal{C}_n, V_0^- > V_{\Gamma_n}^+, f_p(n) < V_{\Gamma_n}^+ \leq L(n), S_{f_p(n)} \leq 2\log f_p(n)] \\ &\leq \int_0^{2\log f_p(n)} Q[V_0^- > V_{\Gamma_n}^+,\ f_p(n) < V_{\Gamma_n}^+ \leq L(n), S_{f_p(n)} \in \mathrm{d}y] \\ &= \int_0^{2\log f_p(n)} Q\left[\bigcap_{k=1}^{f_p(n)} \{S_k \geq 0\}, S_{f_p(n)} \in \mathrm{d}y\right] Q_y[V_0^- > V_{\Gamma_n - y}^+] \end{aligned} \tag{5.37}$$

where $Q_y[.] = Q[.|S_0 = y]$. Since $Q_y[V_0^- > V_{\Gamma_n-y}^+]$ is increasing on y, the term in the right hand side of (5.37) is bounded by

$$\begin{aligned} &Q_{2\log f_p(n)}\left[V_0^- > V_{\Gamma_n - 2\log(f_p(n))}^+\right] \int_0^{2\log f_p(n)} Q\left[\bigcap_{k=1}^{f_p(n)} \{S_k \geq 0\},\ S_{f_p(n)} \in \mathrm{d}y\right] \\ &= Q[V_0^- > f_p(n)]Q_{2\log f_p(n)}\left[V_0^- > V_{\Gamma_n - 2\log f_p(n)}^+\right]. \end{aligned} \tag{5.38}$$

To estimate (5.36), let us define the stopping time $U_{f_p(n)} = \inf\{m > f_p(n),\ S_m \le \log m\}$. We remark that

$$\{\mathcal{C}_n, S_{f_p(n)} \ge 2\log f_p(n)\} \Rightarrow \{f_p(n) \le U_{f_p(n)} \le V^+_{\Gamma_n}\}.$$

Defining

$$\mathcal{A}'(l) = \{\mathcal{C}_n,\ V_0^- > V^+_{\Gamma_n}, f_p(n) < V^+_{\Gamma_n}, S_{f_p(n)} \ge 2\log f_p(n), U_{f_p(n)} = l\},$$

we have

$$Q[\mathcal{C}_n, V_0^- > V^-_{\Gamma_n}, f_p(n) < V^+_{\Gamma_n}, S_{f_p(n)} \ge 2\log f_p(n)] = \sum_{l=f_p(n)}^{[L(n)]} Q[\mathcal{A}'(l)]. \tag{5.39}$$

Since by hypothesis $Q[-I_{\eta_0} \le \epsilon_0 \le I_{\eta_0}] = 1$, we have $U_{f_p(n)} = l \Rightarrow \log l - I_{\eta_0} \le S_l \le \log l$ Q. a.s., so

$$\begin{aligned}
&\sum_{l=f_p(n)}^{[L(n)]} Q[A'(l)] \\
&\quad = \sum_{l=f_p(n)}^{[L(n)]} \int_{\log l - I_{\eta_0}}^{\log l} Q[\mathcal{C}_n, V_0^- > V^+_{\Gamma_n}, f_p(n) < V^+_{\Gamma_n}, S_{f_p(n)} \ge 2\log f_p(n), \mathcal{D}_{l,y}]
\end{aligned}$$

with $\mathcal{D}_{l,y} = \{\bigcap_{j=f_p(n)}^{l-1}\{S_j < \log j\}, S_l \in \mathrm{d}y\}$. By the Markov property, we get

$$\begin{aligned}
\sum_{l=f_p(n)}^{[L(n)]} Qt[A'(l)] \le{}& \sum_{l=f_p(n)}^{[L(n)]} \int_{\log l - I_{\eta_0}}^{\log l} Q_y\left[V_0^- > V^+_{\Gamma_n - y}\right] \\
&\times Q\left[S_{f_p(n)} \ge 2\log f_p(n),\ \bigcap_{j=0}^{l}\{S_j \ge 0\}, \mathcal{D}_{l,y}\right].
\end{aligned}$$

Using that $Q_y[V_0^- > V^+_{\Gamma_n - y}]$ is increasing in y, we obtain

$$\begin{aligned}
\sum_{l=f_p(n)}^{[L(n)]} Q[A'(l)] &\le \sum_{l=f_p(n)}^{[L(n)]} Q_{\log l}\left[V_0^- > V^+_{\Gamma_n - \log l}\right] \int_{\log l - I_{\eta_0}}^{\log l} Q\left[\bigcap_{i=1}^{l}\{S_i > 0\}, \mathcal{D}_{l,y}\right] \\
&\le Q_{\log([L(n)])}\left[V_0^- > V^+_{\Gamma_n - \log([L(n)])}\right] \\
&\quad \times \sum_{l=f_p(n)}^{[L(n)]} Q\left[\bigcap_{i=1}^{f_p(n)}\{S_i > 0\},\ U_{f_p(n)} = l\right] \\
&= Q_{\log([L(n)])}\left[V_0^- > V^+_{\Gamma_n - \log([L(n)])}\right] Q[V_0^- > f_p(n)].
\end{aligned} \tag{5.40}$$

Using (5.40), (5.39), (5.38) and (5.37) we get

$$\begin{aligned}
Q[\mathcal{A}_0^+] \le Q[V_0^- > f_p(n)]\Big(&Q_{2\log f_p(n)}\left[V_0^- > V^+_{\Gamma_n - 2\log f_p(n)}\right] \\
&+ Q_{\log([L(n)])}\left[V_0^- > V^+_{\Gamma_n - \log([L(n)])}\right]\Big).
\end{aligned} \tag{5.41}$$

Now using Lemmata 5.1 and 5.2, there exists $n_2 \equiv n_2(Q)$ and $c_0 \equiv c_0(Q) > 0$ such that for all $n > n_2$:

$$Q[\mathcal{A}_0^+] \leq \frac{c_0 \log_2(n)}{\Gamma_n \sqrt{f_p(n)}}. \tag{5.42}$$

Using (5.42), (5.34), once again Lemma 5.1, (5.33) and finally taking $n_0' = n_1 \vee n_2$ ends the proof of (5.31). □

5.5. *Proof of* (3.21) *and* (3.22)

For (3.21), we use the fact that Lemma 5.5 remains true if we replace m_n by some $l \in \mathbb{F}_p(n)$. Then we use (4.66) and finally we follow the method used in Section 5.4. (3.22) is obtained in a similar way.

Acknowledgments

This article is a part of my Ph.D. thesis written under the supervision of P. Picco. I would like to thank him for helpful discussions throughout the last three years. I would like to thank R. Correa and all the members of the C.M.M. (Santiago, Chili) for their hospitality throughout the year 2002. I would like to thank the anonymous referee for nice comments.

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