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On asymptotic distribution of maxima of complete and incomplete samples from stationary sequences

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Abstract

Let (X_n) be a strictly stationary random sequence and $M_n = \max\{X_1, \dots, X_n\}$. Suppose that some of the random variables $X_1, X_2, \dots$ can be observed and denote by $\widetilde{M}_n$ the maximum of observed random variables from the set $\{X_1, \dots, X_n\}$. We determine the limiting distribution of random vector $(\widetilde{M}_n, M_n)$ under some condition of weak dependency which is more restrictive than the Leadbetter condition. An example concerning a storage process in discrete time with fractional Brownian motion as input is also given.

MSC: primary 60G70; secondary 60G10

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1. Introduction

Let (X_n) be a strictly stationary random sequence with the marginal distribution function $F(x) = P\{X_1 \leqslant x\}$. Suppose that some of the random variables $X_1, X_2, X_3, \dots$ can be observed. If ε_k is the indicator of the event that random variable X_k is observed, then $S_n = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n$ is the number of observed random variables from the set $\{X_1, X_2, \dots, X_n\}$.

Following [5], for a given stationary sequence (X_n) let us define the associated independent sequence (X_n^*) to be i.i.d. with the same distribution function $F(x) = P\{X_1^* \leqslant x\} = P\{X_1 \leqslant x\}$.

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Throughout this paper we shall use the following notation:

$$M_n = \max\{X_1, \ldots, X_n\},$$
$$M_n^* = \max\{X_1^*, \ldots, X_n^*\},$$
$$\widetilde{M}_n = \begin{cases} \max\{X_j, 1 \leqslant j \leqslant n, \varepsilon_j = 1\}, & \text{if } S_n \geqslant 1, \\ \inf\{t | F(t) > 0\}, & \text{if } S_n = 0. \end{cases}$$
$$\widetilde{M}_n^* = \begin{cases} \max\{X_j^*, 1 \leqslant j \leqslant n, \varepsilon_j = 1\}, & \text{if } S_n \geqslant 1, \\ \inf\{t | F(t) > 0\}, & \text{if } S_n = 0. \end{cases}$$

Under some conditions of weak dependence of random variables in the sequence (X_n) [6] proved that random variables M_n and M_n^* have the same limiting distribution with the same normalizing constants. In this paper we are interested in limiting distributions of random vectors $(\widetilde{M}_n^*, M_n^*)$ and $(\widetilde{M}_n, M_n)$. We show in Sections 2–7 that in natural Leadbetter-like weak dependence conditions the limit distributions indicate asymptotic independence of the components of the random vectors (given a first one is at most the second). As opposed to this general weak dependence approach, we give in Section 8 an example of the storage process with fractional Brownian motion (FBM) on input, in discrete time. In this case, when the Hurst parameter is greater than $1/2$, the components of the vector $(\widetilde{M}_n, M_n)$ are asymptotically perfectly dependent.

2. Some preliminaries and examples

A distribution function F belongs to the domain of attraction of a non-degenerate distribution function G (notation $F \in D(G)$) if there exist sequences $a_n > 0$ and $b_n \in \mathbf{R}$, $n \in \mathbf{N}$, such that the equality

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \tag{2.1}$$

holds for every continuity point of G. Every distribution function with non-empty domain of attraction is of one of the following three types:

$$\Lambda(x) = \exp(-\mathrm{e}^{-x}), \quad -\infty < x < +\infty; \tag{2.2}$$

$$\Phi_\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ \exp(-x^{-\alpha}), & \text{if } x \geqslant 0, \end{cases} \quad (\alpha > 0), \tag{2.3}$$

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha), & \text{if } x < 0, \\ 1, & \text{if } x \geqslant 0, \end{cases} \quad (\alpha > 0). \tag{2.4}$$

We shall refer to $\Lambda(x)$, $\Phi_\alpha(x)$ and $\Psi_\alpha(x)$ as extreme value distribution functions. The characterization of domains of attraction can be given in terms of the regular varying of tails of corresponding distribution functions. For example, $F \in D(\Phi_\alpha)$, for some $\alpha > 0$, if and only if

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad \text{for every } x > 0. \tag{2.5}$$

For more details about domains of attraction of extreme value distribution functions see [2,8,3,12].

Results concerning limiting distribution of random vector $(\widetilde{M}_n^*, M_n^*)$ will be formulated under some conditions on the sequence (ε_n) and random variable $S_n = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$.

Let (ε_n) be an i.i.d. sequence, independent of (X_n^*), and $P\{\varepsilon_k = 1\} = p$, $P\{\varepsilon_k = 0\} = 1 - p$, where $0 < p < 1$. Then $S_n \in \mathcal{B}(n, p)$, i.e. S_n is a binomial random variable with parameters n and p. If $u_n = a_n x + b_n$, $v_n = a_n y + b_n$, where $a_n > 0$, $b_n \in \mathbf{R}$, and $x < y$, then the following equalities hold:

$$\begin{aligned} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\} &= \sum_{k=0}^{n} P\{S_n = k\} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n \mid S_n = k\} \\ &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (F(u_n))^k (F(v_n))^{n-k} \\ &= (pF(u_n) + (1-p)F(v_n))^n. \end{aligned} \tag{2.6}$$

Example 2.1. Let (X_n^*) be an i.i.d. sequence with the common distribution function $F(t) = 1 - \mathrm{e}^{-t}$. Then, for every real x, $P\{M_n^* \leqslant x + \ln n\} \to \exp(-\mathrm{e}^{-x})$, as $n \to \infty$ and $F \in D(\Lambda)$. For $x < y$ let us define $u_n = x + \ln n$ and $v_n = y + \ln n$. If (ε_n) is the i.i.d. sequence, independent of (X_n^*), and $S_n \in \mathcal{B}(n, p)$, where $0 < p < 1$, then

$$\begin{aligned} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\} &= (pF(u_n) + (1-p)F(v_n))^n \\ &= \left\{ p\left(1 - \mathrm{e}^{-(x+\ln n)}\right) + (1-p)\left(1 - \mathrm{e}^{-(y+\ln n)}\right) \right\} \\ &= \left\{ 1 - \frac{p\mathrm{e}^{-x} + (1-p)\mathrm{e}^{-y}}{n} \right\}^n \\ &\to \mathrm{e}^{-p\mathrm{e}^{-x}} \mathrm{e}^{-(1-p)\mathrm{e}^{-y}}, \quad \text{as } n \to \infty. \end{aligned} \tag{2.7}$$

Example 2.2. Let $F \in D(\Phi_\alpha)$, where $\alpha > 0$, and $0 < x \leqslant y < +\infty$,

$$a_n = \left(\frac{1}{1 - F} \right)^{-1} (n) = \inf \left\{ t : F(t) \geqslant 1 - \frac{1}{n} \right\}. \tag{2.8}$$

If (ε_n) is the i.i.d. sequence, independent of (X_n^*), and $S_n \in \mathcal{B}(n, p)$, where $0 < p < 1$, then we shall prove that the following equality holds:

$$\lim_{n \to \infty} P\{\widetilde{M}_n^* \leqslant a_n x, M_n^* \leqslant a_n y\} = \mathrm{e}^{-px^{-\alpha}} \mathrm{e}^{-(1-p)y^{-\alpha}}. \tag{2.9}$$

If the constant a_n is given by (2.8), then $a_n \to \infty$ and $1 - F(a_n) \sim \frac{1}{n}$ as $n \to \infty$. Consequently, using (2.5) and (2.6), we obtain that

$$\begin{aligned} &P\{\widetilde{M}_n^* \leqslant a_n x, M_n^* \leqslant a_n y\} = (pF(a_n x) + (1-p)F(a_n y))^n \\ &= \left\{ 1 - p\frac{1 - F(a_n x)}{1 - F(a_n)}(1 - F(a_n)) - (1-p)\frac{1 - F(a_n y)}{1 - F(a_n)}(1 - F(a_n)) \right\}^n \\ &= \left\{ 1 - \frac{px^{-\alpha}}{n}(1 + o(1)) - \frac{(1-p)y^{-\alpha}}{n}(1 + o(1)) \right\}^n \\ &= \left\{ 1 - \frac{px^{-\alpha} + (1-p)y^{-\alpha}}{n} + o\left(\frac{1}{n}\right) \right\}^n \\ &\to \mathrm{e}^{-px^{-\alpha}} \mathrm{e}^{-(1-p)y^{-\alpha}}, \quad \text{as } n \to \infty. \end{aligned} \tag{2.10}$$

Results concerning limiting distribution of random vector $(\widetilde{M}_n, M_n)$ will be formulated under conditions of weak dependency of random variables from the sequence (X_n) and some conditions on the sequence (ε_n).

Definition 2.3. Let (X_n) be a strictly stationary random sequence, (u_n) and (v_n) two sequences of real numbers, and $\mathbf{N}_n = \{1, 2, \ldots, n\}$. *The condition* $D(u_n, v_n)$ is satisfied, if for all $A_1, A_2, B_1, B_2 \subset \mathbf{N}_n$, such that

$$
\begin{aligned}
& b - a \geqslant l, \quad \text{for all } a \in A_1 \cup A_2, b \in B_1 \cup B_2, \\
& A_1 \cap A_2 = \varnothing, \quad B_1 \cap B_2 = \varnothing,
\end{aligned}
$$

the following inequality holds:

$$
\begin{aligned}
& \left| P\left(\bigcap_{j \in A_1 \cup B_1} \{X_j \leqslant u_n\} \cap \bigcap_{j \in A_2 \cup B_2} \{X_j \leqslant v_n\} \right) \right. \\
& \quad \left. - P\left(\bigcap_{j \in A_1} \{X_j \leqslant u_n\} \cap \bigcap_{j \in A_2} \{X_j \leqslant v_n\} \right) \cdot P\left(\bigcap_{j \in B_1} \{X_j \leqslant u_n\} \cap \bigcap_{j \in B_2} \{X_j \leqslant v_n\} \right) \right| \\
& \quad \leqslant \alpha_{n,l},
\end{aligned}
$$

and $\alpha_{n,l_n} \to 0$ as $n \to \infty$ for some $l_n = o(n)$.

The condition $D(u_n, v_n)$ is a modification of the condition $D(u_n)$ that was introduced by [6]. Both of these two conditions are satisfied if, for example, the Rosenblatt strong mixing condition holds for the sequence (X_n).

Definition 2.4 (*[5]*). Let (X_n) be a strictly stationary random sequence and (u_n) a sequence of real numbers. *The condition* $D'(u_n)$ is satisfied if

$$
\lim_{k \to \infty} \limsup_{n \to \infty} \, n \cdot \sum_{j=2}^{[n/k]} P\{X_1 > u_n, X_j > u_n\} = 0.
$$

3. Main results

In this section two general results concerning limiting distributions of random vectors $(\widetilde{M}_n^*, M_n^*)$ and $(\widetilde{M}_n, M_n)$ will be formulated.

Theorem 3.1. *Let us suppose that the following conditions are satisfied:*

(a) $F \in D(G)$, *i.e. for some constants* $a_n > 0$ *and* $b_n \in \mathbf{R}$, $n \in \mathbf{N}$, *and every real* x *the equality* (2.1) *holds.*

(b) (ε_n) *is a sequence of indicators that is independent of* (X_n^*) *and such that*

$$
\frac{S_n}{n} \xrightarrow{P} p \in [0, 1] \quad \text{as } n \to \infty. \tag{3.1}
$$

Then, the following equality holds for all real $x < y$*:*

$$
\lim_{n \to \infty} P\{\widetilde{M}_n^* \leqslant a_n x + b_n, M_n^* \leqslant a_n y + b_n\} = G^p(x) G^{1-p}(y). \tag{3.2}
$$

Theorem 3.2. *Let us suppose that the following conditions are satisfied:*

(a) $F \in D(G)$, *i.e. for some constants* $a_n > 0$ *and* $b_n \in \mathbf{R}$, $n \in \mathbf{N}$, *and every real* x *the equality* (2.1) *holds.*

(b) (X_n) *is a strictly stationary random sequence, such that conditions* $D(u_n, v_n)$ *and* $D'(u_n)$ *are satisfied for* $u_n = a_n x + b_n$ *and* $v_n = a_n y + b_n$, *where* $x < y$.

(c) (ε_n) *is a sequence of indicators that is independent of* (X_n) *and such that*

$$\frac{S_n}{n} \xrightarrow{P} p \in [0, 1] \quad \text{as } n \to \infty, \tag{3.3}$$

Then, the following equality holds for all real $x < y$:

$$\lim_{n\to\infty} P\{\widetilde{M}_n \leqslant a_n x + b_n, M_n \leqslant a_n y + b_n\} = G^p(x)G^{1-p}(y). \tag{3.4}$$

Remark 3.3. The random variable S_n in Theorems 3.1 and 3.2 is not necessarily a binomial one.

Remark 3.4. The limit theorem for joint distribution of maxima of a Gaussian process in continuous and discrete time was proved by Piterbarg [11].

Remark 3.5. Theorem 8.3, Section 8, exhibits an opposite situation. For the storage process in discrete time with FBM on input, the limit distribution of $(\widetilde{M}_n, M_n)$ is $G(\min\{x, y\})$, that is we have perfect asymptotic dependence. This result is obtained for the Hurst parameter of FBM greater than $1/2$ and the condition $P\{S_n = 0\} \to 0$ as $n \to \infty$. By Lemma 8.2, the considered storage process satisfies $D(u_n, v_n)$, and therefore does not satisfy the condition $D'(u_n)$.

4. Some auxiliary results

In this section we shall formulate some lemmas needed for proving Theorems 3.1 and 3.2.

Lemma 4.1. *Let the condition* (a) *of Theorem* 3.2 *be satisfied,* $u_n = a_n x + b_n$, $v_n = a_n y + b_n$, *where* $x < y$ *and* $0 < G(x) \leqslant G(y) \leqslant 1$.

(a) *The following equality holds:*

$$\lim_{n\to\infty} n(1 - pF(u_n) - (1-p)F(v_n)) = -p \ln G(x) - (1-p) \ln G(y). \tag{4.1}$$

(b) *If* k *is a fixed positive integer and* $m = \left[\frac{n}{k}\right]$, *then the following equality holds:*

$$\lim_{n\to\infty} m(F(u_n) - F(v_n)) = \frac{\ln G(x) - \ln G(y)}{k}. \tag{4.2}$$

Lemma 4.2. *Let* (X_n) *be a strictly stationary random sequence such that the condition* $D(u_n, v_n)$ *is satisfied for* $u_n = a_n x + b_n$ *and* $v_n = a_n y + b_n$, *where* $x < y$. *Let* $I_1, I_2, \ldots, I_k$ *be subsets of* $\mathbf{N}_n = \{1, 2, \ldots, n\}$, *such that* $|b - a| \geqslant l$ *for all* $a \in I_s$, $b \in I_t$, *where* $s \neq t$, *and suppose that* (ε_n) *is a sequence of indicators independent of* (X_n). *If we define*

$$M(I_s) = \max\{X_j : j \in I_s\},$$
$$\widetilde{M}(I_s) = \max\{X_j : j \in I_s, \varepsilon_j = 1\},$$

then the following inequality holds:

$$\left| P\left(\bigcap_{s=1}^{k}\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\}\right) - \prod_{s=1}^{k} P\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\} \right| \leqslant (k-1)\alpha_{n,l}. \tag{4.3}$$

Lemma 4.3. *Let (X_n) be a strictly stationary random sequence such that condition $D(u_n, v_n)$ and condition* (a) *of Theorem* 3.2 *are satisfied. Let k be a fixed positive integer, $m = [n/k]$, and*

$$K_s = \{j : (s-1)m + 1 \leqslant j \leqslant sm\},$$
$$M(K_s) = \max\{X_j : j \in K_s\},$$
$$\widetilde{M}(K_s) = \max\{X_j : j \in K_s, \varepsilon_j = 1\},$$

for $s \in \{1, 2, \ldots, k\}$. Then the following equality holds:

$$\lim_{n\to\infty}\left(P\{\widetilde{M}_n \leqslant u_n, M_n \leqslant v_n\} - \prod_{s=1}^{k} P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\}\right) = 0. \tag{4.4}$$

5. Proof of Theorem 3.1

Let $0 < \varepsilon < p$ and let us define $u_n = a_n x + b_n$, $v_n = a_n y + b_n$. Then, we get

$$\begin{aligned} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\} &= \sum_{k=0}^{n} P\{S_n = k\} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n \mid S_n = k\} \\ &= \sum_{k=0}^{n} P\{S_n = k\}(F(u_n))^k (F(v_n))^{n-k}. \end{aligned} \tag{5.1}$$

Let us define

$$\Sigma_1 = \Sigma_1(n, p, \varepsilon) = \sum_{k:\left|\frac{k}{n}-p\right|>\varepsilon} P\{S_n = k\}(F(u_n))^k (F(v_n))^{n-k}, \tag{5.2}$$

$$\Sigma_2 = \Sigma_2(n, p, \varepsilon) = \sum_{k:\left|\frac{k}{n}-p\right|\leqslant\varepsilon} P\{S_n = k\}(F(u_n))^k (F(v_n))^{n-k}. \tag{5.3}$$

Using the condition (b), we obtain that

$$\Sigma_1 \leqslant \sum_{k:\left|\frac{k}{n}-p\right|>\varepsilon} P\{S_n = k\} \to 0, \quad \text{as } n \to \infty. \tag{5.4}$$

The following inequalities hold:

$$\Sigma_2 \leqslant (F(u_n))^{n(p-\varepsilon)} \cdot (F(v_n))^{n-n(p+\varepsilon)} \cdot \sum_{k:\left|\frac{k}{n}-p\right|\leqslant\varepsilon} P\{S_n = k\}, \tag{5.5}$$

$$\Sigma_2 \geqslant (F(u_n))^{n(p+\varepsilon)} \cdot (F(v_n))^{n-n(p-\varepsilon)} \cdot \sum_{k:\left|\frac{k}{n}-p\right|\leqslant\varepsilon} P\{S_n = k\}. \tag{5.6}$$

Using (5.5), (5.6) and (2.1) and the condition (b), we obtain that for every $\varepsilon \in (0, p)$ the following inequalities hold:

$$\limsup_{n\to\infty} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\} \leqslant G^{p-\varepsilon}(x) \cdot G^{1-p-\varepsilon}(y), \tag{5.7}$$

$$\liminf_{n\to\infty} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\} \geqslant G^{p+\varepsilon}(x) \cdot G^{1-p+\varepsilon}(y). \tag{5.8}$$

Finally, if $\varepsilon \downarrow 0$, then it follows from (5.7) and (5.8) that

$$\limsup_{n\to\infty} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\} \leqslant G^{p}(x)G^{1-p}(y), \tag{5.9}$$

$$\liminf_{n\to\infty} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\} \geqslant G^{p}(x)G^{1-p}(y), \tag{5.10}$$

and the statement of the theorem follows. □

6. Proof of auxiliary results

Proof of Lemma 4.1. Note that the following equalities hold:

$$n(1 - pF(u_n) - (1-p)F(v_n)) = p \cdot n(1 - F(u_n)) + (1-p) \cdot n(1 - F(v_n)), \tag{6.1}$$

$$m(F(u_n) - F(v_n)) = m(1 - F(v_n)) - m(1 - F(u_n)). \tag{6.2}$$

Equalities (4.1) and (4.2) are easy consequences of (2.1), equalities (6.1) and (6.2) and Theorem 1.5.1 from Leadbetter et al. [7]. □

Proof of Lemma 4.2. We shall use the method of mathematical induction. For $k = 2$, the inequality (4.3) is just the condition $D(u_n, v_n)$. Suppose that inequality (4.3) holds for arbitrary $k - 1$ sets, such that the distance between any two of them is not less then l.

Let us consider k sets $I_1, I_2, \ldots, I_k \subset \mathbf{N}_n$, for which conditions of Lemma 4.2 are satisfied. Define

$$B_s = \{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\}, \quad s \in \{1, 2, \ldots, k\}.$$

Using the condition $D(u_n, v_n)$ and the assumption that the statement of Lemma 4.2 holds for $k - 1$ sets, we obtain that

$$\begin{aligned}
&|P(B_1B_2\ldots B_k) - P(B_1)P(B_2)\ldots P(B_k)| \\
&\quad \leqslant |P(B_1B_2\ldots B_{k-1}B_k) - P(B_1B_2\ldots B_{k-1})P(B_k)| \\
&\qquad + |P(B_1B_2\ldots B_{k-1}) - P(B_1)P(B_2)\ldots P(B_{k-1})| \cdot P(B_k) \\
&\quad \leqslant \alpha_{n,l} + (k-2)\alpha_{n,l} = (k-1)\alpha_{n,l}. \quad \square
\end{aligned}$$

Proof of Lemma 4.3. For any positive integer n let us define $\mathbf{N}_n = \{1, 2, \ldots, n\}$. Let k be a fixed positive integer and $m = [n/k]$. For large values of n we can choose a positive integer l such that $k < l < m$. Let

$$\mathbf{N}_{mk} = (I_1 \cup J_1) \cup (I_2 \cup J_2) \cup \cdots \cup (I_k \cup J_k)$$

be the representation of the set $\mathbf{N}_{mk} = \{1, 2, \ldots, mk\}$ as the union of mutually disjoint sets, such that the following conditions are satisfied:

- Every one of the sets $I_1, J_1, I_2, J_2, \ldots, I_k, J_k$ consists of consecutive positive integers.
- Cardinal numbers of these sets are given by

$$|I_1| = |I_2| = \cdots = |I_k| = m - l,$$
$$|J_1| = |J_2| = \cdots = |J_k| = l.$$

- The set I_1 consists of the first $m - l$ positive integers; the set J_1 consists of the next l positive integers; I_2 consists of the next $m - l$ positive integers; J_2 consists of the next l positive integers; etc. Obviously, $K_s = I_s \cup J_s$ for all $s \in \{1, 2, \ldots, k\}$.

Since $mk \leqslant n < (m+1)k < mk + l$, we get $|\mathbf{N}_n \setminus \mathbf{N}_{mk}| < k < l$. Let us define sets I_{k+1} and J_{k+1} in the following way:

$$J_{k+1} = \{mk+1, mk+2, \ldots, mk+l\},$$
$$I_{k+1} = \{mk - m + l + 1, \ldots, mk - 1, mk\}.$$

Then, $|I_{k+1}| = m - l$, $|J_{k+1}| = l$. The set J_{k+1} contains the set $\mathbf{N}_n \setminus \mathbf{N}_{mk}$, and the set I_{k+1} is a subset of $\mathbf{N}_{mk}$. We shall use that maxima on the sets $I_1, I_2, \ldots, I_k$ are weakly dependent, and that small intervals $J_1, J_2, \ldots, J_k, J_{k+1}$ can be neglected.

Let us define

$$\Delta = P\{\widetilde{M}_n \leqslant u_n, M_n \leqslant v_n\} - \prod_{s=1}^{k} P\{\widetilde{M}(I_s \cup J_s) \leqslant u_n, M(I_s \cup J_s) \leqslant v_n\}, \tag{6.3}$$

$$\Delta_1 = P\left(\bigcap_{s=1}^{k}\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\}\right) - P\{\widetilde{M}_n \leqslant u_n, M_n \leqslant v_n\}, \tag{6.4}$$

$$\Delta_2 = P\left(\bigcap_{s=1}^{k}\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\}\right) - \prod_{s=1}^{k} P\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\}, \tag{6.5}$$

$$\Delta_3 = \prod_{s=1}^{k} P\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\} - \prod_{s=1}^{k} P\{\widetilde{M}(I_s \cup J_s) \leqslant u_n, M(I_s \cup J_s) \leqslant v_n, \}. \tag{6.6}$$

Then, the following equality holds:

$$\Delta = -\Delta_1 + \Delta_2 + \Delta_3. \tag{6.7}$$

Note that the following inclusion holds:

$$\left(\bigcap_{s=1}^{k}\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\}\right) \setminus \{\widetilde{M}_n \leqslant u_n, M_n \leqslant v_n\}$$
$$\subset \bigcup_{s=1}^{k+1} \left(\{\widetilde{M}(I_s) \leqslant u_n < \widetilde{M}(J_s)\} \cup \{M(I_s) \leqslant v_n < M(J_s)\}\right). \tag{6.8}$$

Using the condition that (X_n) is a strictly stationary random sequence, and relations (6.4) and (6.8), we obtain that

$$0 \leqslant \Delta_1 \leqslant \sum_{s=1}^{k+1} P\{\widetilde{M}(I_s) \leqslant u_n < \widetilde{M}(J_s)\} + (k+1)P\{M(I_1) \leqslant v_n < M(J_1)\}. \tag{6.9}$$

Using Lemma 4.2, we get

$$|\Delta_2| \leqslant (k-1)\alpha_{n,l}. \tag{6.10}$$

Note that

$$\left|\prod_{s=1}^{k} a_s - \prod_{s=1}^{k} b_s\right| \leqslant \sum_{s=1}^{k} |a_s - b_s|, \quad \text{for all } a_s, b_s \in [0,1]. \tag{6.11}$$

Indeed, for $a_1, a_2, b_1, b_2 \in [0, 1]$ we get

$$\begin{aligned} |a_1a_2 - b_1b_2| &= |a_1a_2 - b_1a_2 + b_1a_2 - b_1b_2| \\ &\leqslant |a_2| \cdot |a_1 - b_1| + |b_1| \cdot |a_2 - b_2| \leqslant |a_1 - b_1| + |a_2 - b_2| \end{aligned}$$

and for arbitrary k, inequality (6.11) follows by induction. The following inclusions also hold:

$$\{\widetilde{M}(I_s \cup J_s) \leqslant u_n, M(I_s \cup J_s) \leqslant v_n\} \subset \{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\}, \tag{6.12}$$

$$\begin{aligned} &\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\} \setminus \{\widetilde{M}(I_s \cup J_s) \leqslant u_n, M(I_s \cup J_s) \leqslant v_n\} \\ &\quad \subset \{\widetilde{M}(I_s) \leqslant u_n < \widetilde{M}(J_s)\} \cup \{M(I_s) \leqslant v_n < M(J_s)\}. \end{aligned} \tag{6.13}$$

Using relations (6.11)–(6.13) we obtain that

$$\begin{aligned} 0 &\leqslant \Delta_3 \leqslant \\ &\leqslant \sum_{s=1}^{k} \left(P\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\} - P\{\widetilde{M}(I_s \cup J_s) \leqslant u_n, M(I_s \cup J_s) \leqslant v_n\}\right) \\ &= \sum_{s=1}^{k} P\left(\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\} \setminus \{\widetilde{M}(I_s \cup J_s) \leqslant u_n, M(I_s \cup J_s) \leqslant v_n\}\right) \\ &\leqslant \sum_{s=1}^{k} P\{\widetilde{M}(I_s) \leqslant u_n < \widetilde{M}(J_s)\} + \sum_{s=1}^{k} P\{M(I_s) \leqslant v_n < M(J_s)\} \\ &= \sum_{s=1}^{k} P\{\widetilde{M}(I_s) \leqslant u_n < \widetilde{M}(J_s)\} + k \cdot P\{M(I_1) \leqslant v_n < M(J_1)\}. \end{aligned} \tag{6.14}$$

Using (6.7), (6.9), (6.10) and (6.14) we get

$$\begin{aligned} |\Delta| \leqslant (k-1)\alpha_{n,l} &+ 2\sum_{s=1}^{k+1} P\{\widetilde{M}(I_s) \leqslant u_n < \widetilde{M}(J_s)\} \\ &+ (2k+1)P\{M(I_1) \leqslant v_n < M(J_1)\}. \end{aligned} \tag{6.15}$$

Let us define

$$p_1 = P\{M(I_1) \leqslant v_n < M(J_1)\}, \tag{6.16}$$

$$\tilde{p}_s = P\{\widetilde{M}(I_s) \leqslant u_n < \widetilde{M}(J_s)\}, \quad s \in \{1, 2, \ldots, k+1\}. \tag{6.17}$$

Let us estimate p_1 and $\tilde{p}_s$. The following inequalities hold:

$$\begin{aligned} p_1 &\leqslant P\{v_n < M(J_1)\} = P\left(\bigcup_{j\in J_1}\{X_j > v_n\}\right) \\ &\leqslant \sum_{j\in J_1} P\{X_j > v_n\} = \frac{l}{n}\cdot n(1-F(u_n)), \end{aligned} \tag{6.18}$$

$$\begin{aligned} \tilde{p}_s &\leqslant P\{u_n < \widetilde{M}(J_s)\} \leqslant P\{u_n < M(J_s)\} \\ &\leqslant \sum_{j\in J_s} P\{X_j > u_n\} = \frac{l}{n}\cdot n(1-F(u_n)). \end{aligned} \tag{6.19}$$

Having in mind extreme value distributions $\Phi_\alpha(x)$ and $\Psi_\alpha(x)$, we shall suppose that real numbers $x < y$ are such that $0 < G(x) < G(y) < 1$. If $n \to \infty$ and $l = l_n = o(n)$, then $n(1-F_n(x)) \to -\ln G(x)$, $n(1-F_n(y)) \to -\ln G(y)$, and from (6.18) and (6.19) we get $p_1 \to 0$, $p_s \to 0$ for all $s \in \{1, 2, \ldots, k+1\}$. Now, (4.4) follows from (6.15). □

7. Proof of Theorem 3.2

Let k be a fixed positive integer, $m = [n/k]$, $u_n = a_n x + b_n$, $v_n = a_n y + b_n$, where $x < y$ and $0 < G(x) < G(y) < 1$. Let us define

$$K_s = \{j : (s-1)m + 1 \leqslant j \leqslant sm\}, \quad s \in \{1, 2, \ldots, k\}; \tag{7.1}$$

$$B_s = \{j : j \in K_s, \varepsilon_j = 1\}, \quad s \in \{1, 2, \ldots, k\}; \tag{7.2}$$

$$A_{sj} = \{X_{(s-1)m+j} > u_n\}, \quad j \in \{1, 2, \ldots, m\}. \tag{7.3}$$

Note that $S_{sm} - S_{(s-1)m}$ is the number of ε_j that equal 1 in $(s-1)m + 1 \leqslant j \leqslant sm$. Since $S_n/n \to_P p$, the following relation also holds:

$$\frac{S_{sm} - S_{(s-1)m}}{m} = s\frac{S_{sm}}{sm} - (s-1)\frac{S_{(s-1)m}}{(s-1)m} \xrightarrow{P} p, \quad \text{as } n \to \infty. \tag{7.4}$$

For a fixed $s \in \{1, 2, \ldots, k\}$ let us consider the event $\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\}$. The following equality holds:

$$\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\}^c = \bigcup_{j\in B_s}\{X_j > u_n\} \cup \bigcup_{j\in K_s\setminus B_s}\{X_j > v_n\}. \tag{7.5}$$

Using the equality (7.5) and the Bonferoni inequality, we obtain that

$$\begin{aligned} 1 - P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\} &= \sum_{t=0}^{m} P\{S_{sm} - S_{(s-1)m} = t\} \\ &\quad \times P\left(\bigcup_{j\in B_s}\{X_j > u_n\} \cup \bigcup_{j\in K_s\setminus B_s}\{X_j > v_n\} \,\middle|\, S_{sm} - S_{(s-1)m} = t\right) \\ &\leqslant \sum_{t=0}^{m} P\{S_{sm} - S_{(s-1)m} = t\}\left(t(1-F(u_n)) + (m-t)(1-F(v_n))\right). \end{aligned} \tag{7.6}$$

Since $\{X_i > u_n, X_j > v_n\} \subset \{X_i > u_n, X_j > u_n\}$ for $i \neq j$, we get in a similar way

$$1 - P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\} \geqslant \sum_{t=0}^{m} P\{S_{sm} - S_{(s-1)m} = t\}$$
$$\times \left(t(1 - F(u_n)) + (m - t)(1 - F(v_n)) - m \sum_{j=2}^{m} P(A_{s1} A_{sj}) \right). \tag{7.7}$$

Using inequalities (7.6) and (7.7), we obtain that

$$1 - \sum_{t=0}^{m} P\{S_{sm} - S_{(s-1)m} = t\} \, (s(1 - F(u_n)) + (m - t)(1 - F(v_n)))$$
$$\leqslant P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\} \leqslant 1 - \sum_{t=0}^{m} P\{S_{sm} - S_{(s-1)m} = t\}$$
$$\times \ (t(1 - F(u_n)) + (m - t)(1 - F(v_n))) + m \sum_{j=2}^{m} P(A_{s1} A_{sj}). \tag{7.8}$$

Let us define

$$T(n, k) = m \sum_{j=2}^{m} P(A_{s1} A_{sj}) = \left[\frac{n}{k}\right] \sum_{j=2}^{m} P(A_{s1} A_{sj}). \tag{7.9}$$

Since the sequence (X_n) is strictly stationary, the sum in (7.9) does not depend on s. Using the condition $D'(u_n)$, we obtain that

$$T_0(k) := \limsup_{n \to \infty} T(n, k) = o\left(\frac{1}{k}\right), \quad k \to \infty. \tag{7.10}$$

Using (7.8) and (7.9), we get

$$\begin{aligned}
P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\} &\leqslant 1 - (1 - F(u_n)) \cdot E(S_{sm} - S_{(s-1)m}) \\
&\quad - (1 - F(v_n))(m - E(S_{sm} - S_{(s-1)m})) + T(n, k) \\
&= 1 - m + F(u_n) \cdot E(S_{sm} - S_{(s-1)m}) \\
&\quad + F(v_n)(m - E(S_{sm} - S_{(s-1)m})) + T(n, k) \\
&= 1 - m \left\{ 1 - F(u_n) \frac{E(S_{sm} - S_{(s-1)m})}{m} \right. \\
&\quad \left. - \ F(v_n) \left(1 - \frac{E(S_{sm} - S_{(s-1)m})}{m} \right) \right\} + T(n, k).
\end{aligned} \tag{7.11}$$

Since the relation (7.4) holds and the sequence $(S_{sm} - S_{(s-1)m})/m$, $m = 1, 2, \ldots$, is uniformly integrable, we get for any fixed s the following equality:

$$\lim_{m \to \infty} E\left(\frac{S_{sm} - S_{(s-1)m}}{m}\right) = p.$$

Hence,

$$\frac{E(S_{sm} - S_{(s-1)m})}{m} = p + \eta_s, \quad \text{where } \eta_s \to 0 \quad \text{as } n \to \infty. \tag{7.12}$$

It follows from (7.11) and (7.12) that

$$\begin{aligned} &P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\} \\ &\quad \leqslant 1 - m\{1 - F(u_n)(p + \eta_s) - F(v_n)(1 - p - \eta_s)\} + T(n, k) \\ &\quad = 1 - m\{1 - pF(u_n) - (1 - p)F(v_n)\} + m\eta_s(F(u_n) - F(v_n)) + T(n, k). \end{aligned} \tag{7.13}$$

Using (4.1), (4.2) and (7.13), one obtains

$$\limsup_{n\to\infty} P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\} \leqslant 1 + \frac{p \ln G(x) + (1 - p) \ln G(y)}{k} + T_0(k). \tag{7.14}$$

Similarly we get

$$\liminf_{n\to\infty} P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\} \geqslant 1 + \frac{p \ln G(x) + (1 - p) \ln G(y)}{k}. \tag{7.15}$$

Since inequalities (7.14) and (7.15) hold for all $s \in \{1, 2, \dots, k\}$, it follows that

$$\begin{aligned} &\left(1 + \frac{p \ln G(x) + (1 - p) \ln G(y)}{k}\right)^k \\ &\quad \leqslant \liminf_{n\to\infty} \prod_{s=1}^{k} P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\} \\ &\quad \leqslant \limsup_{n\to\infty} \prod_{s=1}^{k} P\{\widetilde{M}(K_s) \leqslant u_n, M(K_s) \leqslant v_n\} \\ &\quad \leqslant \left(1 + \frac{p \ln G(x) + (1 - p) \ln G(y)}{k} + T_0(k)\right)^k. \end{aligned} \tag{7.16}$$

Using Lemma 4.3 and inequalities (7.16) we obtain

$$\begin{aligned} &\left(1 + \frac{p \ln G(x) + (1 - p) \ln G(y)}{k}\right)^k \\ &\quad \leqslant \liminf_{n\to\infty} P\{\widetilde{M}_n \leqslant u_n, M_n \leqslant v_n\} \leqslant \limsup_{n\to\infty} P\{\widetilde{M}_n \leqslant u_n, M_n \leqslant v_n\} \\ &\quad \leqslant \left(1 + \frac{p \ln G(x) + (1 - p) \ln G(y)}{k} + T_0(k)\right)^k. \end{aligned} \tag{7.17}$$

Finally, (3.4) follows from (7.17) if we let $k \to \infty$. □

8. A storage process in discrete time

Let $X(t)$, $t \geqslant 0$, $X(0) = 0$ a.s., be a fractional Brownian motion, that is a Gaussian zero mean process with stationary increments such that $E(X(t) - X(s))^2 = |t - s|^{2H}$, $0 < H \leqslant 1$. Consider a random sequence

$$Y_k = \sup_{j\geqslant k}(X(j) - X(k) - c(j - k)),$$

where $c > 0$ and $k, l \in Z$. Since X has stationary increments, Y_k is a stationary sequence; we call it a discrete time storage process with fractional Brownian motion as input, see for details [10], where the storage process with continuous time

$$Y(t) = \sup_{\sigma \geqslant t}(X(\sigma) - X(t) - c(\sigma - t))$$

was studied. We have

$$P\left\{\max_{k=1,\dots,K} Y_k \leqslant u\right\}$$
$$= P\{X(u(s+\tau)) - X(su) \leqslant u + cu\tau, \text{ for some } \tau \geqslant 0, s \in [0, u^{-1}K], su, \tau u \in Z\}$$
$$= P\left\{\sup_{\tau>0, s\in[0,u^{-1}K], su, \tau u \in Z} \frac{X(u(s+\tau)) - X(su)}{\tau^H u^H v(\tau)} \leqslant u^{1-H}\right\}, \tag{8.1}$$

where $v(\tau) = \tau^{-H} + c\tau^{1-H}$. The Gaussian random field

$$Z(s, \tau) := \frac{X(u(s+\tau)) - X(su)}{\tau^H u^H v(\tau)}, \quad \tau > 0,\ s \in R,$$

is studied in [10]. Its distribution does not depend on u, $Z(s, \tau)$ is stationary in s but not in τ, its variance $\sigma_Z^2(\tau)$ depends only on τ and has a single maximum point at $\tau_0 = H/(c(1-H))$; moreover,

$$\sigma_Z(\tau) = v^{-1}(\tau) = \frac{1}{A} - \frac{B}{2A^2}(\tau - \tau_0)^2 + O((\tau - \tau_0)^3) \tag{8.2}$$

as $\tau \to \tau_0$, where

$$A := \frac{1}{1-H}\left(\frac{H}{c(1-H)}\right)^{-H} = v(\tau_0), \tag{8.3}$$

$$B := H\left(\frac{H}{c(1-H)}\right)^{-H-2} = v''(\tau_0). \tag{8.4}$$

For the correlation function $r(s, \tau; s', \tau')$ of Z one has

$$r(s, \tau; s', \tau') = 1 - \frac{1 + o(1)}{2\tau_0^{2H}}(|s - s' + \tau - \tau'|^{2H} + |s - s'|^{2H}), \tag{8.5}$$

as $s - s' \to 0$, $\tau - \tau' \to 0$, and, for sufficiently large $|s - s'|$ and an absolute constant C,

$$|r(s, \tau; s', \tau')| \leqslant C|s - s'|^{2H-2}, \tag{8.6}$$

if $2H \neq 1$. For $2H = 1$, we have $r(s, \tau; s', \tau') = 0$ for large $|s - s'|$ since the increments of the Brownian motion on disjoint intervals are independent; see for details [10,4].

Here we consider the Gaussian field $Z(s, \tau)$ on the grid $\mathcal{R}_u = u^{-1}(Z_+ \times Z)$.

Lemma 8.1. *Assume that* $1/2 < H < 1$. *There exists* $\delta > 0$ *such that*

$$P\left\{\max_{k=1,\dots,K} Y_k > u\right\} \sim P\left\{\max_{t\in[0,K]} Y(t) > u\right\} \sim P\{Y(0) > u\} \sim P\{Y_1 > u\}$$

as $u \to \infty$, *where* K *may tend to infinity such that* $K = O(\mathrm{e}^{\delta u^2})$.

Proof. The second relation is proved in [10], Theorem 1. To prove the first and third relations, we use (8.1) and the fact that the grid $\mathcal{R}_u$ is *dense* for the Gaussian field Z and the level u^{1-H}, in the sense of [11]. Indeed, for the level u^{1-H}, in view of (8.2) and (8.5), we get that $(u^{1-H})^{-2/(2H)} = u^{1-1/H}$, and for $H > 1/2$, $u^{-1} = o(u^{1-1/H})$. □

Lemma 8.2. *Let $u_n, v_n \to +\infty$ as $n \to \infty$, such that u_n/v_n is bounded from zero and infinity. Then the condition $D(u_n, v_n)$ is fulfilled for the sequence (Y_k).*

Proof. By (8.1), Definition 2.3 reduces to the Gaussian field Z and levels u_n^{1-H} and v_n^{1-H}. Then the assertion of the Lemma follows from Theorem 1.1, [9], by an argument similar to that in the proofs of Theorems 2.1 and 2.2. Besides, in estimating α_{n,l_n} one uses relations (8.2) and (8.6). □

Theorem 8.3. *Assume that $1/2 < H < 1$. Let (ε_n) be a sequence of indicators that is independent of the sequence (Y_k) and let us define $S_n = \varepsilon_1 + \cdots + \varepsilon_n$,*

$$\widetilde{M}_n = \max\{Y_j \mid 1 \leqslant j \leqslant n, \varepsilon_j = 1\},$$
$$M_n = \max\{Y_1, \ldots, Y_n\},$$
$$a_n = \frac{(2A^{-2})^{1/(2(1-H))}}{2(1-H)}(\ln n)^{-(1-2H)/(2(1-H))},$$
$$b_n = (2A^{-2}\ln n)^{1/(2(1-H))} + \left[\frac{h(2A^{-2})^{1/(2(1-H))}\ln(2A^{-2}\ln n)}{4(1-H)^2} + \frac{(2A^{-2})^{1/(2(1-H))}\ln c}{2(1-H)}\right](\ln n)^{-(1-2H)/(2(1-H))},$$

where A is given by (8.3), $h = 2(1-H)^2/H - 1$, $c = a^{2/H}(2b)^{-1/2}\mathcal{H}_{2H}^2 A^{2/H-2}$, $a = 1/(2\tau_0^{2H})$, $b = B/(2A)$ and $\mathcal{H}_{2H}$ is the Pickands constant given by

$$H_{2H} = \lim_{T\to\infty}\frac{1}{T}E\exp\left\{\max_{0\leqslant t\leqslant T}(\sqrt{2}\,X(t) - |t|^{2H})\right\}.$$

If $P\{S_n = 0\} \to 0$ as $n \to \infty$, then the following equality holds:

$$\lim_{n\to\infty} P\{\widetilde{M}_n \leqslant a_n x + b_n, M_n \leqslant a_n y + b_n\} = \exp(-\exp(-\min\{x, y\})). \tag{8.7}$$

Proof. If $x \geqslant y$, the equality (8.7) follows immediately from Theorem 1 in [4]. Let $x < y$ and $u_n(x) = a_n x + b_n$, $u_n(y) = a_n y + b_n$. Then, the equality (8.7) also follows from above mentioned Theorem 1 and the following relations:

$$P\{\widetilde{M}_n \leqslant u_n(x), M_n \leqslant u_n(y)\} \sim P\{\widetilde{M}_n \leqslant u_n(x)\} \sim P\{M_n \leqslant u_n(x)\}. \tag{8.8}$$

The second asymptotic relation in (8.8) follows from Lemma 8.1 if $P\{S_n = 0\} \to 0$ as $n \to \infty$. The first relation in (8.8) is a consequence of the following relations:

$$\{\widetilde{M}_n \leqslant u_n(x)\} = \{\widetilde{M}_n \leqslant u_n(x), M_n \leqslant u_n(y)\} \cup \{\widetilde{M}_n \leqslant u_n(x), M_n > u_n(y)\},$$
$$\{\widetilde{M}_n \leqslant u_n(x), M_n > u_n(y)\} \subset \{\widetilde{M}_n \leqslant u_n(y)\} \setminus \{M_n \leqslant u_n(y)\},$$
$$\{\widetilde{M}_n \leqslant u_n(y)\} \sim \{M_n \leqslant u_n(y)\}. \quad \square$$

Remark 8.4. Let $X = \{X(t)\}_{t \geqslant 0}$ be an infinitely divisible stochastic process, with no Gaussian component, that is self-similar with index $H > 0$. For given constants $c > 0$ and $\gamma > H$, the storage process with a self-similar and infinitely divisible input is defined by

$$Y(t) = \sup_{s \geqslant t}(X(s) - X(t) - c(s-t)^{\gamma}), \quad t \geqslant 0. \tag{8.9}$$

For an input process X with stationary increments the storage process Y is stationary, if finite. Under some additional conditions on the process X and the function $t = t(u)$, [1] proved that

$$\lim_{u \to \infty} \frac{P\left\{\sup\limits_{s \in [0,t]} Y(s) > u\right\}}{P\left\{\inf\limits_{s \in [0,t]} Y(s) > u\right\}} = 1. \tag{8.10}$$

In some cases, the function $t(u)$ may tend to infinity (Corollary 2, [1]). Let (Y_k) be a sequence defined by $Y_k = Y(k)$, $k \in \{1, 2, \ldots\}$. We call it a discrete time storage process with a self-similar and infinitely divisible input. Apparently, using (8.10), it would be possible to get the result that the components of the vector $(\widetilde{M}_n, M_n)$ are asymptotically perfectly dependent, where $\widetilde{M}_n$ and M_n are defined as in our Theorem 8.3.

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