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On mean curvature functions of Brownian paths

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Abstract

We consider the path Z^t described by a standard Brownian motion in $\mathbb{R}^d$ on some time interval $[0, t]$. This is a random compact subset of $\mathbb{R}^d$. Using the support (curvature) measures of [D. Hug, G. Last, W. Weil, A local Steiner-type formula for general closed sets and applications, Math. Z. 246 (2004) 237–272] we introduce and study two mean curvature functions of Brownian motion. The geometric interpretation of these functions can be based on the Wiener sausage $Z^t_{\oplus r}$ of radius $r > 0$ which is the set of all points $x \in \mathbb{R}^d$ whose Euclidean distance $d(Z^t, x)$ from Z^t is at most r. The mean curvature functions can be easily expressed in terms of the Gauss and mean curvature of $Z^t_{\oplus r}$ as integrated over the positive boundary of $Z^t_{\oplus r}$. We will show that these are continuous functions of locally bounded variation. A consequence is that the volume of $Z^t_{\oplus r}$ is almost surely differentiable at any fixed $r > 0$ with the derivative given as the content of the positive boundary of $Z^t_{\oplus r}$. This will imply that also the expected volume of $Z^t_{\oplus r}$ is differentiable with the derivative given as the expected content of the positive boundary of $Z^t_{\oplus r}$. In fact it has been recently shown in [J. Rataj, V. Schmidt, E. Spodarev, On the expected surface area of the Wiener sausage (2005) (submitted for publication) http://www.mathematik.uni-ulm.de/stochastik/] that for $d \leq 3$ the derivative is just the expected surface content of $Z^t_{\oplus r}$ and that for $d \geq 4$ this is true at least for almost all $r > 0$. The paper [J. Rataj, V. Schmidt, E. Spodarev, On the expected surface area of the Wiener sausage (2005) (submitted for publication) http://www.mathematik.uni-ulm.de/stochastik/] then proceeds to use a result from [A.M. Berezhkovskii, Yu.A. Makhnovskii, R.A. Suris, Wiener sausage volume moments, J. Stat. Phys. 57 (1989) 333–346] to get explicit formulae for this expected surface content. We will use here this result to derive a linear constraint on the mean curvature functions. For $d = 3$ we will provide a more detailed analysis of the mean curvature functions based on a classical formula in [F. Spitzer, Electrostatic capacity, heat flow, and Brownian motion, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 3 (1964) 110–121].

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1. Introduction

We consider a standard Brownian motion $(W_t)_{t\geq 0}$ in $\mathbb{R}^d$ $(d \geq 2)$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let

$$Z^t := \{W_s : 0 \leq s \leq t\}$$

denote the path of the motion between 0 and $t \geq 0$. This is a random closed set. Therefore we may consider the (random) support (curvature) measures $\mu_0(Z^t;\cdot), \ldots, \mu_{d-1}(Z^t;\cdot)$ as introduced and studied (for arbitrary closed sets) in [7]. These are signed measures on $\mathbb{R}^d \times S^{d-1}$ that are supported by the *normal bundle* $N(Z^t) \subset Z^t \times S^{d-1}$ of Z^t, where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. In contrast to Z^t the normal bundle $N(Z^t)$ is a $(d-1)$-dimensional set which is sufficiently smooth to allow for the definition of (generalized) local curvatures. The *local reach* $\delta(Z^t, x, u)$ at a point $(x, u) \in Z^t \times S^{d-1}$ is the maximal value $s > 0$ such that an open ball with centre $x + su$ and radius s does not intersect Z^t. In a sense $\delta(Z^t, x, \cdot)$ measures the degree of fractality at $x \in Z^t$. If $\delta(Z^t, x, u) > 0$ for some $u \in S^{d-1}$ then $(x, u) \in N(Z^t)$ and $x \in Z^t$ is the *metric projection* of some point in $\mathbb{R}^d \setminus Z^t$ onto Z^t. All those points x make up the *positive boundary* of Z^t. All other points of Z^t cannot occur as metric projections. For any $i \in \{0, \ldots, d-1\}$ and any $s > 0$ the restriction of $\mu_i(Z^t;\cdot)$ to $\{\delta(Z^t,\cdot) > s\}$ has a finite total variation with an even finite expectation (see Proposition 3.5). The subject of the present paper is the *mean curvature functions* of Brownian motion, defined by

$$F_i(t, s) := \mathbb{E}\left[\int \mathbf{1}\{\delta(Z^t, x, u) > s\}\, \mu_i(Z^t; \mathrm{d}(x, u))\right], \quad i \in \{0, \ldots, d-1\}.$$

Self-similarity will imply that these functions can be represented either as a function of time $t > 0$ or of minimal local reach $s > 0$. Because the two-dimensional Hausdorff measure of Z^t vanishes (see [14]) we have in fact that $F_i(t, s) = 0$ for $i \geq 2$.

A more direct geometric interpretation of the mean curvature functions can be based on the *Wiener sausage*

$$Z^t_{\oplus r} := \{x \in \mathbb{R}^d : d(Z^t, x) \leq r\}$$

of radius $r \geq 0$. This is the set of all points $x \in \mathbb{R}^d$ whose Euclidean distance $d(Z^t, x)$ from Z^t is at most r. We will see below (see (2.5) and Section 4) that

$$\mathbb{E}[\mu_i(Z^t_{\oplus s}; \mathbb{R}^d \times S^{d-1})] = s^i \binom{d}{d-i} \frac{\kappa_d}{\kappa_{d-i}} F_0(t, s) + \mathbf{1}\{i \geq 1\} s^{i-1} \binom{d-1}{d-i} \frac{\kappa_{d-1}}{\kappa_{d-i}} F_1(t, s) \tag{1.1}$$

holds for all $i \in \{0, \ldots, d-1\}$, where κ_j is the (j-dimensional) volume of the Euclidean unit ball in $\mathbb{R}^j$. In particular

$$F_0(t, s) = \mathbb{E}[\mu_0(Z^t_{\oplus s}; \mathbb{R}^d \times S^{d-1})],$$

$$F_1(t, s) = \mathbb{E}[\mu_1(Z^t_{\oplus s}; \mathbb{R}^d \times S^{d-1})] - sd\frac{\kappa_d}{\kappa_{d-1}} \mathbb{E}[\mu_0(Z^t_{\oplus s}; \mathbb{R}^d \times S^{d-1})].$$

The boundary of $Z^t_{\oplus s}$ is in a sense smooth for almost all $r > 0$ (see e.g. [4]). According to Corollary 2.5 in [7] we may interpret $\mu_0(Z^t_{\oplus s}; \mathbb{R}^d \times S^{d-1})$ as the integral of the *Gauss curvature* over the positive boundary of $Z^t_{\oplus s}$. The number $\mu_1(Z^t_{\oplus s}; \mathbb{R}^d \times S^{d-1})$ equals the length of the positive boundary of $Z^t_{\oplus s}$ for $d = 2$. For $d = 3$ it can be interpreted as the integral of the *mean curvature* over the positive boundary of $Z^t_{\oplus s}$.

The Lebesgue measure $\mathcal{H}^d(Z^t_{\oplus r})$ of the Wiener sausage has been extensively studied in the literature (see e.g. [12,5,13,1]). By means of the generalized Steiner formula in [7] this volume can be expressed in terms of support measures and the reach function of Z^t. Therefore the expected volume $\mathbb{E}[\mathcal{H}^d(Z^t_{\oplus r})]$ can be expressed in terms of the mean curvature functions of Brownian motion (see Theorem 4.5). We will show that these are continuous functions of locally bounded variation. A consequence is that the volume of $Z^t_{\oplus r}$ is almost surely differentiable at any fixed $r > 0$ with the derivative given as the content of the positive boundary of $Z^t_{\oplus r}$. This will imply that also the expected volume of $Z^t_{\oplus r}$ is differentiable with the derivative given as the expected content of the positive boundary of $Z^t_{\oplus r}$. In fact it has been recently shown in [10] that for $d \leq 3$ the derivative is just the expected surface content of $Z^t_{\oplus r}$ while for $d \geq 4$ this is true at least for almost all distances $r > 0$. Moreover, the authors of [10] have used a result from [1] to get explicit formulae for the expected surface content of $Z^t_{\oplus r}$. It is somewhat surprising that earlier papers on the Wiener sausage paid little attention to the (expected) surface content. Using the result from [1] we will derive in Theorem 4.8 a linear constraint on the mean curvature functions. For the three-dimensional case we will use a classical formula from [12] to show in Theorem 5.1 (see also Remark 5.2) that $F_0(t, s) = 1 + c\sqrt{t}/s + o(1/s)$ as $s \to \infty$, where the constant $c \geq 0$ satisfies the inequality $c \leq 4/\sqrt{2\pi}$.

As a technical tool, curvatures have been used in [10]. However, to the best of our knowledge we make here the first systematic attempt to define and study mean curvatures of Brownian motion. But even if these functions were known explicitly, the probabilistic behaviour of the random support measures would still remain to be studied. Obtaining more detailed information is a challenging but very interesting and promising task.

2. Support measures

The aim of this section is to define the support measures of [7] and to summarize some of their basic properties. We are working in $\mathbb{R}^d$ with Euclidean norm $|\cdot|$ and unit sphere $S^{d-1} := \{z \in \mathbb{R}^d : |z| = 1\}$. The i-dimensional Hausdorff measure in Euclidean space is denoted by $\mathcal{H}^i$. The closed Euclidean ball with centre $a \in \mathbb{R}^d$ and radius $r \geq 0$ is denoted by $B(a, r)$, while $B^0(a, r)$ denotes the corresponding open ball.

We now consider an arbitrary non-empty closed subset A of $\mathbb{R}^d$. The distance $d(A, z)$ from a point $z \in \mathbb{R}^d$ to A is defined as $\inf\{|y - z| : y \in A\}$, where $\inf \emptyset := \infty$. We put $p(A, z) := y$ whenever y is a uniquely determined point in A with $\mathrm{d}(A, z) = |y - z|$, and we call this point the *metric projection* of z on A. If $0 < \mathrm{d}(A, z) < \infty$ and $p(A, z)$ is defined, then $p(A, z)$ lies on the boundary ∂A of A and we put $u(A, z) := (z - p(A, z))/\mathrm{d}(A, z)$. The *exoskeleton* $\mathrm{exo}(A)$ of A consists of all points of $\mathbb{R}^d \backslash A$ which do not admit a metric projection on A. This is a measurable set (see, e.g., Lemma 6.1 in [7]) satisfying $\mathcal{H}^d(\mathrm{exo}(A)) = 0$. If A is convex, then $\mathrm{exo}(A) = \emptyset$. We extend the definition of $p(A, z) \in \mathbb{R}^d$ and $u(A, z) \in S^{d-1}$ in a suitable and measurable way to all $z \in \mathbb{R}^d$.

The *normal bundle* of A is defined by

$$N(A) := \{(p(A, z), u(A, z)) : z \notin A \cup \mathrm{exo}(A)\}.$$

It is a measurable subset of $\partial A \times S^{d-1}$. The *positive boundary* $\partial^+ A$ of A is defined as the set of all $x \in \partial A$ for which there is some $u \in S^{d-1}$ such that $(x, u) \in N(A)$. This is the set of all boundary points occurring as metric projections on A. The *reach function* $\delta(A, \cdot) : \mathbb{R}^d \times S^{d-1} \to [0, \infty]$ of A is defined by

$$\delta(A, x, u) := \inf\{s \geq 0 : x + su \in \mathrm{exo}(A)\}, \quad (x, u) \in N(A),$$

and $\delta(A, x, u) := 0$ for $(x, u) \notin N(A)$. Note that $\delta(A, \cdot) > 0$ on $N(A)$. By Lemma 6.2 in [7], $\delta(A, \cdot)$ is a measurable function. If $x \in A$, then we have for any $u \in S^{d-1}$ that

$$\begin{aligned}\delta(A, x, u) &= \sup\{s \geq 0 : B(x + su, s) \cap A = \{x\}\} \\ &= \sup\{s \geq 0 : B^0(x + su, s) \cap A = \emptyset\}.\end{aligned}$$

This implies for any $r > 0$ that $\delta(A, x, u) = r$ if and only if $(B(x + ru, r) \backslash \{x\}) \cap A \neq \emptyset$ and $B^0(x + ru, r) \cap A = \emptyset$. These facts will be used in Section 3.

A *reach measure* μ of A is a real-valued function defined on all Borel subsets of $N(A)$ which are contained in

$$\{(x, u) : x \in B, \delta(A, x, u) \geq s\},$$

for some $s > 0$ and some compact $B \subset \mathbb{R}^d$. Moreover, it is a *bounded signed measure* on $\{(x, u) : x \in B, \delta(A, x, u) \geq s\}$, for each $s > 0$ and each compact $B \subset \mathbb{R}^d$. If μ is a reach measure, then it is convenient to extend the domain of μ by setting $\mu(C) := 0$ for any Borel set $C \subset \mathbb{R}^d \times S^{d-1} \backslash N(A)$. The Hahn decomposition theorem then implies a unique representation $\mu = \mu^+ - \mu^-$ with mutually singular σ-finite measures μ^+ and μ^- on $\mathbb{R}^d \times S^{d-1}$ vanishing outside $N(A)$. Although μ^+ and μ^- are defined on all measurable sets, it is in general not possible to extend μ to all measurable sets using $\mu = \mu^+ - \mu^-$. The measure $|\mu| := \mu^+ + \mu^-$ is the *total variation measure* of μ. For any measurable function $f : \mathbb{R}^d \times S^{d-1} \to [-\infty, \infty]$, we define the integral $\int f \mathrm{d}\mu$ as $\int f \mathrm{d}\mu^+ - \int f \mathrm{d}\mu^-$ whenever this difference is well defined, i.e. whenever the integrals $\int f \mathrm{d}\mu^+$ and $\int f \mathrm{d}\mu^-$ are both defined and the above difference is not of the form $-\infty + \infty$ or $\infty - \infty$. Subsequently, we write $a \wedge b := \min\{a, b\}$ for $a, b \in \mathbb{R}$.

By Theorem 2.1 in [7], there exist reach measures $\mu_0(A; \cdot), \ldots, \mu_{d-1}(A; \cdot)$ of A satisfying

$$\int \mathbf{1}\{x \in B\}(\delta(A, x, u) \wedge r)^{d-j} \, |\mu_j|(A; \mathrm{d}(x, u)) < \infty, \tag{2.1}$$

$j = 0, \ldots, d - 1$, for all compact sets $B \subset \mathbb{R}^d$ and all $r > 0$, and, for any measurable bounded function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support,

$$\begin{aligned}\int_{\mathbb{R}^d \backslash A} f(z) \, \mathrm{d}z = \sum_{i=0}^{d-1} (d - i)\kappa_{d-i} \int_0^\infty \int & s^{d-1-i} \mathbf{1}\{\delta(A, x, u) > s\} \\ & \times f(x + su) \, \mu_i(A; \mathrm{d}(x, u)) \, \mathrm{d}s.\end{aligned} \tag{2.2}$$

These measures are called *support measures* of A. They are uniquely defined by the *local Steiner formula* (2.2).

Eq. (2.2) yields in particular a formula for the volume of the *parallel set* $A_{\oplus r}$ of a compact set A at distance $r \geq 0$, i.e. of the set of all points $x \in \mathbb{R}^d$ whose distance from A is at most r:

$$\mathcal{H}^d(A_{\oplus r}) = \mathcal{H}^d(A) + \sum_{i=0}^{d-1} (d - i)\kappa_{d-i} \int_0^r s^{d-i-1} f_i(A, s) \, \mathrm{d}s$$

$$= \mathcal{H}^d(A) + \sum_{i=0}^{d-1} \kappa_{d-i} \int \delta(A,x,u)^{d-i} \wedge r^{d-i}\, \mu_i(A; \mathrm{d}(x,u)), \tag{2.3}$$

where

$$f_i(A,s) := \int \mathbf{1}\{\delta(A,x,u) > s\}\, \mu_i(A; \mathrm{d}(x,u)), \quad s > 0, i \in \{0,\dots,d-1\}.$$

If A is convex, then this simplifies to the classical Steiner formula

$$\mathcal{H}^d(A_{\oplus r} \backslash A) = \sum_{i=0}^{d-1} r^{d-i} \kappa_{d-i} V_i(A), \tag{2.4}$$

where the coefficients $V_0(A), \dots, V_{d-1}(A)$ are the *intrinsic volumes* of the convex body A (see, e.g., [11, (4.2.27)]). Corollary 4.4 in [7] provides a geometric interpretation of the functions $f_i(A,s)$ in the general case:

$$\mu_i(A_{\oplus s}; \mathbb{R}^d \times S^{d-1}) = \sum_{j=0}^{i} s^{i-j} \binom{d-j}{d-i} \frac{\kappa_{d-j}}{\kappa_{d-i}} f_j(A,s), \quad i \in \{0,\dots,d-1\}. \tag{2.5}$$

In particular,

$$f_0(A,s) = \mu_0(A_{\oplus s}; \mathbb{R}^d \times S^{d-1}),$$
$$f_1(A,s) = \mu_1(A_{\oplus s}; \mathbb{R}^d \times S^{d-1}) - sd\frac{\kappa_d}{\kappa_{d-1}} \mu_0(A_{\oplus s}; \mathbb{R}^d \times S^{d-1}).$$

Let $A \subset \mathbb{R}^d$ be compact. Proposition 4.9 in [7] implies for any $c > 0$, any $s > 0$, and any $j \in \{0,\dots,d-1\}$ that

$$\int \mathbf{1}\{\delta(cA,x,u) > s\}\, \mu_j(cA; \mathrm{d}(x,u)) = c^j \int \mathbf{1}\{\delta(A,x,u) > c^{-1}s\}\, \mu_j(A; \mathrm{d}(x,u)). \tag{2.6}$$

The same equation holds if μ_j is replaced with $|\mu_j|$.

In the remainder of this section we will state a few more basic properties of support measures that are not explicitly mentioned in [7] but needed later. The first property follows from Corollary 2.5 in [7] and the fact that the principal curvatures occurring there are locally defined (see the first paragraph of the proof of Theorem 2.1 in [7]).

Proposition 2.1. *Let $A, B \subset \mathbb{R}^d$ be non-empty closed sets. Then we have for all $i \in \{0,\dots,d-1\}$ that*

$$\int \mathbf{1}\{(x,u) \in \cdot\}\mathbf{1}\{(x,u) \in N(A) \cap N(B)\}\, \mu_i(A; \mathrm{d}(x,u))$$
$$= \int \mathbf{1}\{(x,u) \in \cdot\}\mathbf{1}\{(x,u) \in N(A) \cap N(B)\}\, \mu_i(B; \mathrm{d}(x,u)).$$

The next result is a consequence of inequality (2.11) and equation (2.23) in [7].

Proposition 2.2. *For any $i \in \{0,\dots,d-1\}$*

$$\int \mathbf{1}\{(x,u) \in \cdot, \delta(A,x,u) = \infty\}\, \mu_i(A; \mathrm{d}(x,u)) \geq 0.$$

Next we prove:

Proposition 2.3. *If* $A \subset \mathbb{R}^d$ *is non-empty and compact then*

$$\int \mathbf{1}\{\delta(A, x, u) = \infty\}\, \mu_0(A; \mathrm{d}(x, u)) = 1.$$

Proof. From the argument leading to Proposition 2.2 and Proposition 4.10 in [7] we get

$$\mathrm{d}\kappa_d \int \mathbf{1}\{\delta(A, x, u) = \infty\}\, \mu_0(A; \mathrm{d}(x, u)) = \int_{S^{d-1}} \sum_{x \in N(A,u)} \mathbf{1}\{\delta(A, x, u) = \infty\}\, \mathcal{H}^{d-1}(\mathrm{d}u), \tag{2.7}$$

where $N(A, u) := \{x \in \mathbb{R}^d : (x, u) \in N(A)\}$. Fixing $u \in S^{d-1}$ we first show that $N(A, u) \neq \emptyset$. Indeed, since A is compact there is an $x \in A$ such that

$$\langle y, u\rangle \le \langle x, u\rangle, \quad y \in A,$$

where $\langle y, u\rangle$ denotes the inner product of y and u. Hence $B(x + su, s) \cap A = \{x\}$ for all $s > 0$ proving that $\delta(A, x, u) = \infty$. Hence (2.7) implies that

$$b_0 := \int \mathbf{1}\{\delta(A, x, u) = \infty\}\, \mu_0(A; \mathrm{d}(x, u)) \ge 1.$$

To show the converse inequality we take $r > 0$ and apply the Steiner formula (2.2) to get

$$\int \mathbf{1}\{0 < \mathrm{d}(A, z) \le r, \delta(A, p(A, z), u(A, z)) = \infty\}\, \mathrm{d}z = \sum_{i=0}^{d-1} \kappa_{d-i} r^{d-i} b_i, \tag{2.8}$$

where

$$b_i := \int \mathbf{1}\{\delta(A, x, u) = \infty\}\, \mu_i(A; \mathrm{d}(x, u)).$$

Choosing $R > 0$ such that A is contained in a ball with radius R, we obtain from (2.8) that

$$\kappa_d (R + r)^d \ge \kappa_d r^d b_0 + \sum_{i=1}^{d-1} \kappa_{d-i} r^{d-i} b_i.$$

Dividing by r^d and letting $r \to \infty$ gives the desired inequality $1 \ge b_0$. □

Finally we will need the following result.

Proposition 2.4. *Let* $A \subset \mathbb{R}^d$ *be non-empty and closed and* $k \in \{1, \ldots, d-1\}$ *such that* $\mathcal{H}^k(\partial A) = 0$. *Then* $\mu_k(A; \cdot) \equiv 0$.

Proof. In view of Lemma 2.3 in [7] it is sufficient to show that

$$\mu_k(A; N(B) \cap \cdot) = 0$$

holds for any compact $B \subset \mathbb{R}^d$ with positive reach. By Proposition 2.1 we have

$$\mu_k(A; N(B) \cap \cdot) = \mu_k(B; N(A) \cap \cdot).$$

Since $N(A) \subset \partial A \times S^{d-1}$, the desired equality is a consequence of Theorem 5.5 in [2]. □

3. Preliminaries on Brownian paths

The first lemma permits to consider Z^t as a random element in the space $\mathcal{F}^d$ equipped with the usual Fell–Matheron "hit-or-miss" topology (see [9]). We skip the proof which is quite standard. For further measurability properties of support measures and the reach function (tacitly used in the sequel) we refer the reader to Section 6 in [7].

Lemma 3.1. *The mapping $(\omega, t) \mapsto Z^t(\omega)$ from $\Omega \times [0, \infty)$ into $\mathcal{F}^d$ is measurable.*

From (2.3) we obtain that

$$\mathcal{H}^d(Z^t_{\oplus r}) = \mathcal{H}^d(Z^t) + \sum_{i=0}^{d-1} (d-i)\kappa_{d-i} \int_0^r s^{d-i-1} f_i(t, s)\, \mathrm{d}s, \tag{3.1}$$

where

$$f_i(t, s) := \int \mathbf{1}\{\delta(Z^t, x, u) > s\}\, \mu_i(Z^t; \mathrm{d}(x, u)). \tag{3.2}$$

A geometric interpretation of these functions is provided by Eq. (2.5). From (3.1) we see that $r \mapsto \mathcal{H}^d(Z^t_{\oplus r})$ is differentiable everywhere with the exception of at most countably many points (see also Corollary 4.5 in [7]). On the other hand we obtain from Lemma 1 in [6] (a version of Lemma 3.2.34 in [3]) that

$$\mathcal{H}^d(Z^t_{\oplus r}) = \mathcal{H}^d(Z^t) + \int_0^r \mathcal{H}^{d-1}(\partial Z^t_{\oplus s})\, \mathrm{d}s, \quad r \geq 0,$$

everywhere on Ω. Together with (3.1) this implies

$$\mathcal{H}^{d-1}(\partial Z^t_{\oplus s}) = \sum_{i=0}^{d-1} (d-i)\kappa_{d-i} s^{d-i-1} f_i(t, s) \quad \mathcal{H}^1\text{-a.e. } s \geq 0 \tag{3.3}$$

everywhere on Ω. Hence the mapping $s \mapsto \mathcal{H}^d(Z^t_{\oplus s})$ is everywhere on Ω and for $\mathcal{H}^1$-a.e. $r > 0$ differentiable at r with the derivative given by the surface content $\mathcal{H}^{d-1}(\partial Z^t_{\oplus r})$. However, this does not imply that $s \mapsto \mathcal{H}^d(Z^t_{\oplus s})$ is almost surely differentiable at a given *fixed* $r > 0$. For $d \leq 3$ this property has recently been established in [10] using the theory of Lipschitz manifolds. In the next section we will prove this in any dimension, using a different method.

It is known (see [14]) that Brownian paths have almost surely Hausdorff dimension 2 and that $\mathcal{H}^2(Z^t) = 0$ $\mathbb{P}$-almost surely. Therefore Proposition 2.4 implies that

$$\mu_i(Z^t, \cdot) \equiv 0, \quad t \geq 0,\ i \in \{2, \ldots, d-1\} \quad \mathbb{P}\text{-a.s.} \tag{3.4}$$

Hence

$$\mathcal{H}^d(Z^t_{\oplus r}) = d\kappa_d \int_0^r s^{d-1} f_0(t, s)\, \mathrm{d}s + (d-1)\kappa_{d-1} \int_0^r s^{d-2} f_1(t, s)\, \mathrm{d}s, \quad r \geq 0, \tag{3.5}$$

holds outside a measurable set of $\mathbb{P}$-measure zero.

The next lemma provides an important continuity property of the reach function. In contrast to all further arguments it is of a purely geometric nature. Only continuity of Brownian paths is being used.

Lemma 3.2. *Let $t, r > 0$ and assume that $W_t \notin \partial^+ Z^t$. Then*

$$\lim_{s \to t-} \mathbf{1}\{(x,u) \in N(Z^t) : \delta(Z^s, x, u) = r\} = \mathbf{1}\{(x,u) \in N(Z^t) : \delta(Z^t, x, u) = r\}.$$

Proof. Let $Z^{t-} := \{W_s : 0 \le s < t\}$. Then

$$\begin{aligned}
&\mathbf{1}\{(x,u) \in N(Z^t) : \delta(Z^s, x, u) = r\} \\
&\quad = \mathbf{1}\{(x,u) \in N(Z^t) : \delta(Z^s, x, u) = r, B^0(x + ru, r) \cap Z^t = \emptyset\} \\
&\qquad + \mathbf{1}\{(x,u) \in N(Z^t) : \delta(Z^s, x, u) = r, B^0(x + ru, r) \cap Z^{t-} \neq \emptyset\},
\end{aligned} \tag{3.6}$$

where we have used the fact that the continuity of Brownian paths implies the equivalence of $B^0(x+ru, r) \cap Z^t = \emptyset$ with $B^0(x+ru, r) \cap Z^{t-} = \emptyset$. The second term on the above right-hand side is less than

$$\mathbf{1}\{(x,u) \in N(Z^t) : B^0(x + ru, r) \cap Z^s = \emptyset, B^0(x + ru, r) \cap Z^{t-} \neq \emptyset\}$$

and hence converges to 0 as $s \to t-$. The first term on the right-hand side of (3.6) equals

$$h(s, x, u) := \mathbf{1}\{(x,u) \in N(Z^t) : x \in Z^s, \delta(Z^t, x, u) = r, (B(x + ru, r) \setminus \{x\}) \cap Z^s \neq \emptyset\}.$$

Obviously

$$\begin{aligned}
&\lim_{s \to t-} h(s, x, u) \\
&\quad = \mathbf{1}\{(x,u) \in N(Z^t) : x \in Z^{t-}, \delta(Z^t, x, u) = r, (B(x + ru, r) \setminus \{x\}) \cap Z^{t-} \neq \emptyset\}.
\end{aligned}$$

From now on we assume that $W_t \notin \partial^+ Z^t$ and prove next that

$$\begin{aligned}
&\{(x,u) \in N(Z^t) : x \in Z^{t-}, \delta(Z^t, x, u) = r, (B(x + ru, r) \setminus \{x\}) \cap Z^{t-} \neq \emptyset\} \\
&\qquad = \{(x,u) \in N(Z^t) : \delta(Z^t, x, u) = r\}.
\end{aligned} \tag{3.7}$$

Let (x, u) be a member of the right-hand side of (3.7). In particular $x \in \partial^+ Z^t$ so that $x \neq W_t$. Hence $x \in Z^{t-}$ and it remains to show that

$$(B(x + ru, r) \setminus \{x\}) \cap Z^{t-} \neq \emptyset.$$

Assuming on the contrary that this is not the case we get from $\delta(Z^t, x, u) = r$ that

$$W_t \in B(x + ru, r) \setminus \{x\}.$$

Since on the other hand $(B^0(x+ru, r) \setminus \{x\}) \cap Z^t = \emptyset$ we obtain that $W_t \in \partial^+ Z^t$, a contradiction. This concludes the proof of the lemma. □

The next result is due to the fact that Brownian motion is an isotropic diffusion.

Lemma 3.3. *For any $t > 0$ we have that $\mathbb{P}(0 \in \partial^+ Z^t) = \mathbb{P}(W_t \in \partial^+ Z^t) = 0$.*

Proof. For any $v \in S^{d-1}$ we consider the open spherical cone

$$C_v := \{x \in \mathbb{R}^d : \langle x, v \rangle > |x|/2\}.$$

We then find a finite set $V \subset S^{d-1}$ such that

$$\bigcup_{v \in V} C_v = \mathbb{R}^d \setminus \{0\}$$

and

$$\min\{|u-v| : v \in V\} \le 1/4, \quad u \in S^{d-1}. \tag{3.8}$$

Due to the spherical symmetry of Brownian motion and Blumenthal's 0–1-law (see Corollary 19.18 in [8]) we have

$$\inf\{s > 0 : W_s \in C_v \cap B(0, k^{-1})\} = 0, \quad v \in V, k \in \mathbb{N}, \tag{3.9}$$

on a set $A \in \mathcal{A}$ with $\mathbb{P}(A) = 1$. Now we argue on A and assume that $0 \in \partial^+ Z^t$. Then there are $u \in S^{d-1}$ and $\varepsilon > 0$ (both depending on $\omega \in A$) such that

$$B^0(\varepsilon u, \varepsilon) \cap Z^t = \emptyset. \tag{3.10}$$

By (3.9) we find some $v \in V$ such that $|u-v| \le 1/4$. Choosing $k \in \mathbb{N}$ such $1/k < \varepsilon/2$, we will prove that

$$C_v \cap B^0(0, k^{-1}) \subset B^0(\varepsilon u, \varepsilon). \tag{3.11}$$

Because this contradicts (3.9) and (3.10) we obtain $A \cap \{0 \in \partial^+ Z^t\} = \emptyset$ and hence $\mathbb{P}(0 \in \partial^+ Z^t) = 0$.

To show (3.11) we take $x \in C_v \cap B^0(0, k^{-1})$ and obtain

$$\begin{aligned}
|x-\varepsilon u|^2 - \varepsilon^2 &= |x|^2 - 2\varepsilon\langle x, u\rangle \\
&= |x|^2 - 2\varepsilon\langle x, v\rangle + 2\varepsilon\langle x, v-u\rangle \\
&\le |x|^2 - \varepsilon|x| + 2\varepsilon|x||v-u| \\
&= |x|(|x| - \varepsilon + 2\varepsilon|v-u|) \\
&\le |x|(1/k - \varepsilon/2) < 0.
\end{aligned}$$

To prove that also $\mathbb{P}(W_t \in \partial^+ Z^t) = 0$ we consider the process

$$\tilde{W}_s := W_{t-s} - W_t, \quad 0 \le s \le t,$$

which is a Brownian motion on $[0, t]$. Hence $\mathbb{P}(0 \in \partial^+ \tilde{Z}) = 0$, where

$$\tilde{Z} := \{\tilde{W}_s : 0 \le s \le t\} = Z^t - W_t.$$

Since $\partial^+ \tilde{Z} = \partial^+ Z^t - W_t$, the desired result follows. □

Lemma 3.4. *For any $a, t > 0$, we have that $Z^{at} \overset{d}{=} \sqrt{a} Z^t$.*

Proof. The result follows easily from Lemma 3.1 and Brownian scaling:

$$(W_s) \overset{d}{=} (\sqrt{a} W_{s/a}). \quad \square$$

The lemma implies that

$$\mathcal{H}^d(Z^t_{\oplus r}) \overset{d}{=} t^{d/2} \mathcal{H}^d(Z^1_{\oplus r/\sqrt{t}}), \quad t, r > 0. \tag{3.12}$$

We will need the following integrability properties.

Proposition 3.5. *For any $t > 0$*

$$\mathbb{E}\left[\int (\delta(Z^t, x, u)^{d-i} \wedge r^{d-i}) \, |\mu_i|(Z^t; \mathrm{d}(x, u))\right] < \infty, \quad i = 0, \ldots, d-1. \tag{3.13}$$

In particular, we have for any $t, \varepsilon > 0$,

$$\mathbb{E}\left[\int \mathbf{1}\{\delta(Z^t, x, u) \geq \varepsilon\}\, |\mu_i|(Z^t; \mathrm{d}(x, u))\right] < \infty, \quad i = 0, \ldots, d-1. \tag{3.14}$$

Proof. We will prove that

$$\mathbb{E}[\mathcal{H}^d(Z^t_{\oplus r})] < \infty, \tag{3.15}$$

so that Theorem 6.5 in [7] implies the assertion (3.13). We proceed as in [10] and define $\xi := \max\{|W_s| : 0 \leq s \leq t\}$. Since $Z^t_{\oplus r} \subset B(0, \xi + r)$ it is enough to prove that ξ has a finite dth moment. But it is well known that ξ has finite moments of all orders. Indeed, ξ is bounded by the sum of d independent random variables each of which has distribution of $B^*_t := \max\{B_s : 0 \leq s \leq t\}$, where (B_s) is a Brownian motion in $\mathbb{R}$. It is a basic property of Brownian motion (see e.g. Proposition 13.13 in [8]) that $B^*_t \overset{d}{=} |B_t|$. □

4. Mean curvature functions

Proposition 3.5 implies that the functions

$$\begin{aligned} F_i(t, s) &:= \mathbb{E}\left[\int \mathbf{1}\{\delta(Z^t, x, u) > s\}\, \mu_i(Z^t; \mathrm{d}(x, u))\right], \\ G_i(t, s) &:= \mathbb{E}\left[\int \mathbf{1}\{\delta(Z^t, x, u) > s\}\, |\mu_i|(Z^t; \mathrm{d}(x, u))\right] \end{aligned} \tag{4.1}$$

are well defined and real valued for $i \in \{0, 1\}$ and all $t, s > 0$. We call F_0 and F_1 the *mean curvature functions* of Brownian motion. Because of (2.5) these functions satisfy (1.1).

We will now show that these functions are continuous in both variables. Moreover, they are actually functions of just one variable, where we may either choose time or minimal local reach.

Proposition 4.1. *The functions* F_0, F_1, G_0 *and* G_1 *are continuous on* $(0, \infty) \times (0, \infty)$ *and satisfy*

$$F_i(s^2 t, s) = s^i F_i(t, 1), \qquad G_i(s^2 t, s) = s^i G_i(t, 1), \quad s, t > 0, i \in \{0, 1\}. \tag{4.2}$$

Proof. We fix $i \in \{0, 1\}$. From (2.6) and Lemma 3.4 we obtain that

$$G_i(at, bs) = a^{i/2} G_i(t, a^{-1/2} bs), \quad a, b, s, t > 0, \tag{4.3}$$

and a similar equation for F_i. Choosing $a = s^2$ and $b = 1$ gives (4.2). Choosing $a = t^{-1}$ and $b = t^{-1/2}$ gives the first equality in

$$G_i(t, s) = t^{i/2} G_i(1, t^{-1/2} s) = s^i G_i(ts^{-2}, 1), \quad s, t > 0, \tag{4.4}$$

while the second follows from (4.3) upon taking $a = s^{-2}$ and $b = s^{-1}$. Clearly $G_i(t, \cdot)$ is decreasing and right-continuous for any $t > 0$ and we have from (4.4) that

$$G_i(t, s-) = t^{i/2} G_i(1, (t^{-1/2} s)-), \quad s, t > 0. \tag{4.5}$$

We now prove that G_i is continuous. By (4.4) we have to show that $G_i(1, \cdot)$ is left-continuous. Let us fix some $t > 0$. Since the set of discontinuities of the function $G_i(1, \cdot)$ is at most countable, we find a sequence $t_n \uparrow t$ such that

$$0 = t_n^{i/2} G_i(1, t_n^{-1/2}-) - t_n^{i/2} G_i(1, t_n^{-1/2}) = G_i(t_n, 1-) - G_i(t_n, 1), \quad n \in \mathbb{N},$$

where we have used (4.5) (with $s = 1$) to get the second equation. Hence

$$\limsup_{n\to\infty}(G_i(t_n, 1-) - G_i(t_n, 1)) = 0.$$

On the other hand we will show below that

$$\limsup_{n\to\infty}(G_i(t_n, 1-) - G_i(t_n, 1)) \geq G_i(t, 1-) - G_i(t, 1). \tag{4.6}$$

It follows that the function $t \mapsto G_i(t, 1-) - G_i(t, 1)$ vanishes everywhere on $(0, \infty)$. By (4.5) (for $s = 1$) this means that

$$G_i(1, t^{-1/2}-) - G_i(1, t^{-1/2}) = t^{-i/2}(G_i(t, 1-) - G_i(t, 1)) = 0, \quad t > 0.$$

Hence $G_i(1, \cdot)$ is left-continuous.

From (4.2) we have

$$F_i(t, 1) = t^{i/2} F_i(1, t^{-1/2}), \qquad F_i(1, t) = t^i F_i(t^{-2}, 1), \quad t > 0. \tag{4.7}$$

As above we conclude that continuity of F_i is a consequence of left-continuity of $F_i(1, \cdot)$. The latter follows from

$$0 \leq |F_i(1, t-) - F_i(1, t)| \leq G_i(1, t-) - G_i(1, t) = 0, \quad t > 0.$$

It remains to prove (4.6). We will use Lemma 3.2 with $r = 1$. Taking $s \in (0, t)$ and recalling that $\{(x, u) \in \mathbb{R}^d \times S^{d-1} : \delta(Z^s, x, u) = 1\} \subset N(Z^s)$ we first note that for all $\varepsilon \in (0, 1)$:

$$\begin{aligned}
&\int \mathbf{1}\{\delta(Z^s, x, u) = 1\}\, |\mu_i|(Z^s; \mathrm{d}(x, u)) \\
&\quad \geq \int \mathbf{1}\{\delta(Z^t, x, u) \geq \varepsilon, \delta(Z^s, x, u) = 1\}\, |\mu_i|(Z^s; \mathrm{d}(x, u)) \\
&\quad = \int \mathbf{1}\{\delta(Z^t, x, u) \geq \varepsilon, \delta(Z^s, x, u) = 1\}\, |\mu_i|(Z^t; \mathrm{d}(x, u)),
\end{aligned} \tag{4.8}$$

where the equality is due to Proposition 2.1. Together with bounded convergence, Lemmas 3.2 and 3.3 imply that the expression (4.8) converges, as $s \to t-$, almost surely towards

$$\begin{aligned}
&\int \mathbf{1}\{\delta(Z^t, x, u) \geq \varepsilon, \delta(Z^t, x, u) = 1\}\, |\mu_i|(Z^t; \mathrm{d}(x, u)) \\
&\quad = \int \mathbf{1}\{\delta(Z^t, x, u) = 1\}\, |\mu_i|(Z^t; \mathrm{d}(x, u)).
\end{aligned}$$

In view of (3.14) we can use bounded convergence (now with respect to $\mathbb{P}$) to obtain that

$$\begin{aligned}
\limsup_{n\to\infty}(G_i(t^n, 1-) - G_i(t^n, 1)) &= \limsup_{n\to\infty} \mathbb{E}\left[\int \mathbf{1}\{\delta(Z^{t_n}, x, u) = 1\}\, |\mu_i|(Z^{t_n}; \mathrm{d}(x, u))\right] \\
&\geq \mathbb{E}\left[\int \mathbf{1}\{\delta(Z^t, x, u) = 1\}\, |\mu_i|(Z^t; \mathrm{d}(x, u))\right].
\end{aligned}$$

Assertion (4.6) follows. □

Remark 4.2. Let $i \in \{0, 1\}$ and $t > 0$. The function $F_i(t, \cdot)$ is not only continuous but also of locally bounded variation. Indeed, using the Hahn decomposition of $\mu_i(Z^t; \cdot)$ we have

$$F_i(t, \cdot) = F_i^+(t, \cdot) - F_i^-(t, \cdot),$$

where

$$F_i^+(t,s) := \mathbb{E}\left[\int \mathbf{1}\{\delta(Z^t,x,u) > s\}\,\mu_i^+(Z^t;\mathrm{d}(x,u))\right], \quad s>0,$$

and $F_i^-(t,\cdot)$ is defined similarly. The functions $F_i^+(t,\cdot)$ and $F_i^-(t,\cdot)$ are continuous and decreasing.

Theorem 4.3. *Let $t > 0$ and $r > 0$. Then the function $s \mapsto \mathcal{H}^d(Z^t_{\oplus s})$ is almost surely differentiable at r. Outside a $\mathbb{P}$-null set the derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}^d(Z^t_{\oplus s})\bigg|_{s=r} = \mathrm{d}\kappa_d r^{d-1} f_0(t,r) + (d-1)\kappa_{d-1} r^{d-2} f_1(t,r), \tag{4.9}$$

(with $f_i(t,r)$ given by (3.2)) and equals $\mathcal{H}^{d-1}(\partial^+ Z^t_{\oplus r})$.

Proof. It follows from (3.1) (see also Corollary 4.5 in [7]) that $s \mapsto \mathcal{H}^d(Z^t_{\oplus s})$ is differentiable at r if

$$\int \mathbf{1}\{\delta(Z^t,x,u) = r\}\,|\mu_i|(Z^t;\mathrm{d}(x,u)) = 0, \quad i=0,\ldots,d-1. \tag{4.10}$$

In this case the derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}^d(Z^t_{\oplus s})\bigg|_{s=r} = \sum_{i=0}^{d-1}(d-i)\kappa_{d-i} r^{d-i-1} f_i(t,r).$$

By Corollary 4.6 in [7] the right-hand side of this equation is just $\mathcal{H}^{d-1}(\partial^+ Z^t_{\oplus r})$. Since Proposition 4.1 and (3.4) imply that Eq. (4.10) holds almost surely, we obtain the assertion from (3.4). □

Remark 4.4. For $d \le 3$ it has been shown in [10] that outside a $\mathbb{P}$-null set the derivative of $s \mapsto \mathcal{H}^d(Z^t_{\oplus s})$ at some (arbitrary) fixed $r > 0$ is given by $\mathcal{H}^{d-1}(\partial Z^t_{\oplus r})$. This means that

$$\mathcal{H}^{d-1}(\partial^+ Z^t_{\oplus r}) = \mathcal{H}^{d-1}(\partial Z^t_{\oplus r})$$

holds $\mathbb{P}$-almost surely.

For brevity we now define

$$V(t,r) := \mathbb{E}[\mathcal{H}^d(Z^t_{\oplus r})], \quad t,r>0. \tag{4.11}$$

The partial derivatives of V are denoted by $V_{(1)}(t,r)$ and $V_{(2)}(t,r)$, whenever they exist.

Theorem 4.5. *Let $t > 0$. Then the function $V(t,\cdot)$ is continuously differentiable on $(0,\infty)$. The derivative is given by*

$$V_{(2)}(t,r) = d\kappa_d r^{d-1} F_0(t,r) + (d-1)\kappa_{d-1} r^{d-2} F_1(t,r), \tag{4.12}$$

where the mean curvature functions F_0 and F_1 are defined by (4.1). Moreover,

$$V_{(2)}(t,r) = \mathbb{E}[\mathcal{H}^{d-1}(\partial^+ Z^t_{\oplus r})], \quad r>0, \tag{4.13}$$

$$V_{(2)}(t,r) = \mathbb{E}[\mathcal{H}^{d-1}(Z^t_{\oplus r})] \quad \mathcal{H}^1\text{-a.e. } r>0. \tag{4.14}$$

Proof. Taking into account (3.4) we take the expectations on both sides of (3.1). By Proposition 3.5 we are allowed to use Fubini's theorem to get

$$\mathbb{E}[\mathcal{H}^d(Z^t_{\oplus r})] = \sum_{i=0}^{1}(d-i)\kappa_{d-i}\int_0^r s^{d-i-1}F_i(t,s)\,\mathrm{d}s. \tag{4.15}$$

Eq. (4.12) follows from Proposition 4.1 and the fundamental theorem of calculus. Integrating (3.3) and using (3.4) gives

$$\int_0^r \mathbb{E}[\mathcal{H}^{d-1}(Z^t_{\oplus s})]\,ds = \sum_{i=0}^{1}(d-i)\kappa_{d-i}\int_0^r s^{d-i-1}F_i(t,s)\,\mathrm{d}s, \quad r>0,$$

and hence assertion (4.14). Eq. (4.13) follows from the last part of Theorem 4.3. □

Remark 4.6. On the basis of different methods, relationship (4.14) has been proved in [10]. For $d \le 3$ equation Eq. (4.14) even holds for all $r > 0$ (see Remark 4.4).

Corollary 4.7. *The function V is continuously differentiable on $(0,\infty)\times(0,\infty)$ and the partial derivatives satisfy the equation*

$$V_{(1)}(t,r) + \frac{1}{2t}V_{(2)}(t,r) = t^{d/2}V(r,t). \tag{4.16}$$

Proof. Eq. (3.12) implies that

$$V(t,r) = t^{d/2}V(1, rt^{-1/2}), \quad t,r>0.$$

The assertion can now easily be derived from Theorem 4.5. □

It has been pointed out in [10] that Corollary 4.7 is actually an old result. Indeed, a formula in [1] says that

$$V(t,r) = \kappa_d r^d + \frac{d(d-2)}{2}\kappa_d t r^{d-2} + \frac{4d\kappa_d r^d}{\pi^2}\int_0^\infty \frac{1-\exp(-y^2t/2r^2)}{y^3(J_\nu^2(y)+Y_\nu^2(y))}\,\mathrm{d}y, \tag{4.17}$$

where J_ν and Y_ν are Bessel functions of the first and second kind, respectively, of order $\nu := (d-2)/d$ (see [15]). As noticed in [10] this formula can be differentiated with respect to r to give

$$\begin{aligned} V_{(2)}(t,r) &= d\kappa_d r^{d-1} + \frac{\kappa_d d(d-2)^2}{2}tr^{d-3} + \frac{4d^2\kappa_d}{\pi^2}r^{d-1}\int_0^\infty \frac{1-\exp(-y^2t/2r^2)}{y^3(J_\nu^2(y)+Y_\nu^2(y))}\,\mathrm{d}y \\ &\quad - \frac{4d\kappa_d}{\pi^2}tr^{d-3}\int_0^\infty \frac{\exp(-y^2t/2r^2)}{y(J_\nu^2(y)+Y_\nu^2(y))}\,\mathrm{d}y. \end{aligned} \tag{4.18}$$

By (4.13) the right-hand side of (4.18) gives the expected content of $\partial^+ Z^t_{\oplus r}$ (cf. also Remark 4.4 and (4.14)).

Our next result yields an explicit formula for a linear combination of the mean curvature functions. It does not seem to be possible to extract from (4.18) further information on the mean curvature functions.

Theorem 4.8. *We have*

$$\mathrm{d}\kappa_d F_0(t,1) + (d-1)\kappa_{d-1}F_1(t,1) = d\kappa_d + \frac{\kappa_d d(d-2)^2}{2}t + \frac{4d^2\kappa_d}{\pi^2}\int_0^\infty \frac{1-\exp(-y^2t/2)}{y^3(J_\nu^2(y)+Y_\nu^2(y))}\,\mathrm{d}y - \frac{4d\kappa_d}{\pi^2}t\int_0^\infty \frac{\exp(-y^2t/2)}{y(J_\nu^2(y)+Y_\nu^2(y))}\,\mathrm{d}y. \tag{4.19}$$

Proof. From (4.12) and the scaling relation (4.2) we obtain that

$$V_{(2)}(tr^2, r) = r^{d-1}(d\kappa_d F_0(t,1) + (d-1)\kappa_{d-1}F_1(t,1)). \tag{4.20}$$

Using (4.18) to express the above left-hand side gives the result. □

Remark 4.9. From (3.12) we have

$$V(tr^2, t) = r^d V(t,1), \quad t, r > 0, \tag{4.21}$$

while (4.20) implies that

$$V_{(2)}(tr^2, t) = r^{d-1}V(t,1), \quad t, r > 0, \tag{4.22}$$

in accordance with (4.13) and Brownian scaling.

5. The three-dimensional case

In this section we assume that $d = 3$. Then the last integral in (4.17) can be simplified to yield

$$V(t,r) = 2\pi r t + 4\sqrt{2\pi}r^2\sqrt{t} + \frac{4}{3}\pi r^3. \tag{5.1}$$

This is a classical result of Spitzer [12]. To state the main result of this section we introduce the constant

$$c_1 := \mathbb{E}\left[\int \mathbf{1}\{\delta(Z^1, x, u) = \infty\}\,\mu_1(Z^1; \mathrm{d}(x,u))\right]. \tag{5.2}$$

Proposition 2.2 implies that $c_1 \geq 0$.

Theorem 5.1. *The limit*

$$c_0 := \lim_{r\to\infty} r\mathbb{E}\left[\int \mathbf{1}\{r < \delta(Z^1, x, u) < \infty\}\,\mu_0(Z^1; \mathrm{d}(x,u))\right] \tag{5.3}$$

exists, is non-negative, and satisfies the linear equation

$$2c_0 + c_1 = \frac{8}{\sqrt{2\pi}}, \tag{5.4}$$

where c_1 is given by (5.2). Furthermore

$$c_0 = \lim_{s\to 0} s^{-1/2}\mathbb{E}\left[\int \mathbf{1}\{1 < \delta(Z^s, x, u) < \infty\}\,\mu_0(Z^s; \mathrm{d}(x,u))\right]. \tag{5.5}$$

Proof. By (4.12),

$$V_{(2)}(t,r) = 4\pi r^2 F_0(t,r) + 2\pi r F_1(t,r). \tag{5.6}$$

For $i \in \{0, 1\}$ we write

$$F_i^{<\infty}(t,r) := \mathbb{E}\left[\int \mathbf{1}\{r < \delta(Z^t, x, u) < \infty\}\, \mu_i(Z^t; \mathrm{d}(x,u))\right],$$

$$F_i^{\infty}(t) := \mathbb{E}\left[\int \mathbf{1}\{\delta(Z^t, x, u) = \infty\}\, \mu_i(Z^t; \mathrm{d}(x,u))\right].$$

By Proposition 2.3, $F_0^{\infty}(t) = 1$. Hence we obtain that

$$V_{(2)}(t,r) = 4\pi r^2 + 4\pi r^2 F_0^{\infty}(t) + 2\pi r F_1^{<\infty}(t,r) + 2\pi r F_1^{\infty}(t). \tag{5.7}$$

Lemma 3.4 and Eq. (2.6) imply that

$$F_1^{\infty}(t) = \sqrt{t}\,\mathbb{E}\left[\int \mathbf{1}\{\delta(Z^1, x, u) = \infty\}\, \mu_1(Z^1; \mathrm{d}(x,u))\right] = c_1\sqrt{t}.$$

Inserting this and (5.1) into (5.7) we obtain that

$$2\pi t + 8\sqrt{2\pi}\, r\sqrt{t} = 4\pi r^2 F_0^{<\infty}(t,r) + 2\pi r F_1^{<\infty}(t,r) + 2\pi c_1\sqrt{t} r. \tag{5.8}$$

From (3.14) and monotone convergence we have

$$\lim_{r\to\infty} \mathbb{E}\left[\int \mathbf{1}\{r < \delta(Z^t, x, u) < \infty\}\, |\mu_1|(Z^t; \mathrm{d}(x,u))\right] = 0.$$

Dividing both sides of (5.8) by r and letting $r \to \infty$ yields the existence of the limit

$$c_0(t) := \lim_{r\to\infty} r F_0^{<\infty}(t,r)$$

as well as the equation

$$8\sqrt{2\pi}\sqrt{t} = 4\pi c_0(t) + 2\pi c_1\sqrt{t}. \tag{5.9}$$

The function $F_0^{<\infty}$ has the same scaling properties as F_0 and G_0, i.e. (see (4.2) and (4.4))

$$F_0^{<\infty}(s^2 t, s) = F_0^{<\infty}(t, 1) = F_0^{<\infty}(1, t^{-1/2}), \quad t, s > 0. \tag{5.10}$$

We obtain that

$$c_0(t) = \lim_{r\to\infty} r F_0^{<\infty}(1, r t^{-1/2}) = \sqrt{t} \lim_{r\to\infty} F_0^{<\infty}(1, r) = c_0(1)\sqrt{t} = c_0\sqrt{t}.$$

Substituting in (5.9) gives (5.4). The inequality $c_1 \geq 0$ comes from Fatou's Lemma. Eq. (5.5) is a consequence of (5.10). □

Remark 5.2. Let $t > 0$. Recalling $\int \mathbf{1}\{\delta(Z^t, x, u) = \infty\}\, \mu_0(Z^t; \mathrm{d}(x,u)) = 1$ and using the scaling properties of F_0, we can write Theorem 5.1 as

$$\lim_{r\to\infty} r(F_0(t,r) - 1) = \sqrt{t} \lim_{r\to\infty} r(F_0(t,r) - 1) = \sqrt{t}\left(\frac{4}{\sqrt{2\pi}} - \frac{c_1}{2}\right)$$

or

$$F_0(t,r) = 1 + \frac{\sqrt{t}}{r}\left(\frac{4}{\sqrt{2\pi}} - \frac{c_1}{2}\right) + o\left(\frac{1}{r}\right) \quad \text{as } r \to \infty.$$

Remark 5.3. In view of Corollary 2.5 in [7] and the intrinsic symmetry properties of Brownian motion it is tempting to guess that

$$\mathbb{E}\left[\int \mathbf{1}\{r < \delta(Z^t, x, u) < \infty\}\, \mu_1(Z^t; \mathrm{d}(x, u))\right] = 0, \quad r > 0. \tag{5.11}$$

Then (5.8) would imply

$$2\pi t + 8\sqrt{2\pi} r \sqrt{t} = 4\pi r^2 F_0^{<\infty}(t, r) + 2\pi c_1 \sqrt{t} r, \tag{5.12}$$

i.e.

$$F_0(t, r) = 1 + \frac{\sqrt{t}}{r}\left(\frac{4}{\sqrt{2\pi}} - \frac{c_1}{2}\right) + \frac{t}{2r^2}. \tag{5.13}$$

This is a polynomial in $\sqrt{t}/r$.

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