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On the ergodicity and mixing of max-stable processes

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Abstract

Max-stable processes arise in the limit of component-wise maxima of independent processes, under appropriate centering and normalization. In this paper, we establish necessary and sufficient conditions for the ergodicity and mixing of stationary max-stable processes. We do so in terms of their spectral representations by using extremal integrals.

The large classes of moving maxima and mixed moving maxima processes are shown to be mixing. Other examples of ergodic doubly stochastic processes and non-ergodic processes are also given. The ergodicity conditions involve a certain measure of dependence. We relate this measure of dependence to the one of Weintraub [K.S.Weintraub, Sample and ergodic properties of some min-stable processes, Ann. Probab. 19 (2) (1991) 706–723] and show that Weintraub's notion of '0-mixing' is equivalent to mixing. Consistent estimators for the dependence function of an ergodic max-stable process are introduced and illustrated over simulated data.

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Keywords: Max-stable processes; Ergodicity; Mixing; Dependence function; Spectral representation

1. Introduction

Extreme value distributions arise in the limit of the maxima of independent and identically distributed random variables or vectors, under appropriate normalization. They have thus important applications in many diverse areas such as insurance, finance, telecommunications,

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the environment, and flood and natural disaster modeling (see e.g. [12]). The class of extreme value distributions coincides with the class of max-stable laws. Recall that a random vector X in $\mathbb{R}^n$ has a max-stable distribution if, for all $a, b \in \mathbb{R}^n$, $a, b \geq 0$, there exist $c \geq 0,\ c, d \in \mathbb{R}^n$, such that

$$aX' \vee bX'' \stackrel{d}{=} cX + d,$$

where X' and X'' are independent copies of X and the inequalities, multiplications, the max '$\vee$', and the addition operations are taken coordinate-wise. When all finite-dimensional distributions of a process $X = \{X_t\}_{t\in\mathbb{R}}$ are max-stable, it is said to be max-stable. These max-stable processes are as important in extreme value theory as the Gaussian (or α-stable) processes are in the classical 'linear' theory.

Max-stable processes have been studied extensively in the past 30 years and many of their properties are well understood. For example, the structure of their finite-dimensional distributions is well known. Insightful Poisson point process or spectral representations are available. Path, large deviations, zero–one laws and many other properties have been established. This was done in the seminal works of Balkema and Resnick [1], de Haan [7,8], de Haan and Pickands [10], Giné, Hahn and Vatan [14], Resnick and Roy [20], and many others. More details and further references can be found in the monographs of Resnick [18,19], and de Haan and Ferreira [9].

Despite this large body of knowledge on max-stable processes, surprisingly little is known about their *ergodicity* and *mixing* properties. To the best of our knowledge, only Weintraub [26] has addressed these questions. He introduced a certain measure of dependence and studied several 'mixing conditions' based on this measure. Although natural, these conditions were not explicitly related either to the ergodic or to the mixing properties of the process. Our goal in this paper is to fill this gap. Here, by following a different approach, we obtain simple necessary and sufficient conditions for ergodicity and mixing. These conditions are in terms of the spectral representation of the process.

For convenience, we focus on max-stable processes with α-Fréchet distributions and use extremal integrals rather than de Haan's spectral representations to represent them. We start by introducing some notation. Recall that a random variable Z has an α-Fréchet distribution, $\alpha > 0$, if

$$\mathbb{P}\{Z \leq x\} = \exp\{-\sigma^\alpha x^{-\alpha}\}, \quad x \in (0, \infty),$$

where $\sigma > 0$ is the scale coefficient of Z. We will frequently use the notation

$$\|Z\|_\alpha := \sigma$$

for the scale coefficient of Z. Observe that $\|Z\|_\alpha \not\equiv (\mathbb{E}Z^\alpha)^{1/\alpha} = \infty$. The following definition extends the notion of a *Fréchet distribution* to the multivariate setting.

Definition 1.1. A process $X = \{X_t\}_{t\in\mathbb{R}}$ is said to be α-Fréchet, $\alpha > 0$, if for all $t_j \in \mathbb{R}$ and $a_j > 0,\ j = 1, \ldots, n$, the random variable,

$$Z := \max\{a_j X_{t_j},\ 1 \leq j \leq n\} \equiv \bigvee_{1\leq j\leq n} a_j X_{t_j}$$

has an α-Fréchet distribution. Namely, all *max-linear combinations* of the X_t's are α-Fréchet variables.

All α-Fréchet processes are max-stable. Conversely, as shown in [7], any max-stable process with α-Fréchet marginal distributions is an α-Fréchet process. Thus, for any (strictly) stationary max-stable process $X = \{X_t\}_{t\in\mathbb{R}}$ and $\alpha > 0$, the process $\widetilde{X}_t := h_\alpha(X_t),\ t \in \mathbb{R}$, is α-Fréchet, with a suitable strictly monotone function h_α. Since monotone transformations of the marginals do not affect the ergodicity and mixing properties of the process, it is enough to study the class of α-Fréchet processes, for some (any) $\alpha > 0$.

We focus on the class of stationary α-Fréchet processes $X = \{X_t\}_{t\in\mathbb{R}}$ with the *extremal integral* representation:

$$X_t = {\int^{\mathrm{e}}}_{E} f_t(u) M_\alpha(\mathrm{d}u), \quad t \in \mathbb{R}, \text{ with } f_t := U_t(f_0), \tag{1.1}$$

where M_α is an α-Fréchet random sup-measure with control measure μ and where $f_t \in L_+^\alpha(\mu) := \{g \geq 0 : \|g\|_{L^\alpha(\mu)}^\alpha = \int_E g^\alpha \mathrm{d}\mu < \infty\}$ are the deterministic 'spectral functions' of the process X. The extremal integral in (1.1) may be viewed as a maximum of the infinitesimal α-Fréchet noise $M_\alpha(\mathrm{d}u)$, weighted by the deterministic kernels $f_t(u)$. See Section 2.1 below, for more details.

The operators $U_t : L_+^\alpha(\mu) \to L_+^\alpha(\mu),\ t \in \mathbb{R}$ in (1.1) form a group of *max-linear isometries* on the space $L_+^\alpha(\mu)$; that is, they preserve the max-linear combinations of functions and their L^α-norms (Definition 2.3, below). The max-isometry and group properties of $\{U_t\}$ ensure the stationarity of the process X in (1.1). For example, when $E = \mathbb{R}$ and M_α has the Lebesgue control measure $\mu(\mathrm{d}u) = \mathrm{d}u$, with $U_t(f)(u) := f_0(t+u)$, the process X in (1.1) becomes the (stationary) moving maxima process:

$$X_t = {\int^{\mathrm{e}}}_{\mathbb{R}} f_0(t+u) M_\alpha(\mathrm{d}u), \quad t \in \mathbb{R}, \tag{1.2}$$

where $\int_{\mathbb{R}} f_0^\alpha(u)\mathrm{d}u < \infty$. More generally, when $E = \mathbb{R} \times F$, $\mu(\mathrm{d}u, \mathrm{d}v) = \mathrm{d}u\nu(\mathrm{d}v)$, for some measure $\mathrm{d}\nu$ on F, with $f_t(u, v) := U_t(f_0)(u, v) = f_0(t+u, v)$, one obtains the *mixed moving maxima*:

$$X_t = {\int^{\mathrm{e}}}_{\mathbb{R}\times F} f_0(t+u, v) M_\alpha(\mathrm{d}u, \mathrm{d}v), \quad t \in \mathbb{R}. \tag{1.3}$$

By choosing various spaces E and groups of max-linear isometries $\{U_t\}$, one can handle a wide variety of stationary Fréchet processes.

Max-linear isometries have been studied before in the seminal work of de Haan and Pickands [10]. These authors explicitly introduced a class of max-linear isometries called *pistons*. Their results imply that any strictly stationary and continuous in probability Fréchet process X has the representation (1.1) with $E = [0, 1]$, where $\{U_t\}$ is a group of *pistons*. In Section 2.2, we complete the picture of max-linear isometries by showing that all max-linear isometries have the structure of the *pistons* of de Haan and Pickands (Proposition 2.4). This follows from a surprising result that any max-linear isometry is also linear (Proposition 2.3). Thus the 'piston structure' of max-linear isometries is closely related, and in fact, follows from the Banach–Lamperti representation of the linear isometries in L^p-spaces, for $p \neq 2$.

In Section 3, we focus on the ergodicity and mixing properties of the process X in (1.1). We first establish the measurability of X. It turns out that X in (1.1) has a measurable modification if and only if it is continuous in probability. This parallels the result for α-stable processes of Theorem 0 in [5]. In Section 3.2, we establish necessary and sufficient conditions for the ergodicity and mixing of the process X. This is done in terms of a dependence measure involving explicitly the kernel functions $f_t = U_t(f_0)$. Our methods rely on the extremal integral representation in (1.1), which resembles closely the corresponding stochastic integral representations of α-stable (sum-stable) processes (see e.g. [22]). Cambanis, Hardin and Weron [5] provide a thorough treatment of ergodicity and mixing for α-stable (sum-stable) processes. Guided by their ideas, we obtain a complete characterization of ergodicity and mixing in the max-stable setting (see Theorems 3.2–3.4 below). Our results are similar in spirit to ones in the α-stable setting, but the proofs involve different tools.

Several examples of ergodic and non-ergodic processes are given in Section 4.1. The ergodicity and mixing conditions given in Section 3.2 are easy to verify for large classes of processes. The moving maxima or more generally the mixed moving maxima processes in (1.2) and (1.3), for example, are always mixing. We discuss an example of a 'doubly stochastic' stationary Fréchet process, introduced by Brown and Resnick [4], and show that it is mixing. In fact, we introduce a more general class of doubly stochastic Fréchet processes driven by Lévy processes and show that they are mixing. An interesting open question is whether or not these processes belong to the class of mixed moving maxima.

The *dependence function* of a stationary Fréchet process X (see (4.14) below) reflects the structure of its distribution and may be viewed as the counterpart of the auto-covariance function for Gaussian processes, for example. Thus, the estimation of the dependence function is an important statistical problem. In Section 4.2, we propose estimators for this function for an ergodic Fréchet process and show their strong consistency. We also illustrate their performance with simulated data.

We conclude by comparing our results to the work of Weintraub [26]. We show that '0-mixing' in the sense of Weintraub is equivalent to mixing. This is done by relating two measures of dependence.

The paper is organized as follows. In Section 2, we briefly review extremal integral representations and establish structural results on max-linear isometries, which are used in the rest of the paper. Section 3 contains the main results of the paper on necessary and sufficient conditions for ergodicity and mixing of max-stable processes. Simpler conditions for mixing are given in Section 3.3 in terms of a natural measure of dependence. Examples and applications are presented in Section 4. Weintraub's results are discussed in Section 5. Appendix A contains some proofs and auxiliary results.

2. Preliminaries

2.1. Extremal integral representations

We review here the representations of max-stable processes used in the rest of the paper. More details and proofs can be found in [8,10] and also [23].

Definition 2.1. Let $(E, \mathcal{E}, \mu)$ be a measure space with σ-finite, positive measure μ. A set-indexed random process $M_\alpha = \{M_\alpha(A)\}_{A\in\mathcal{E}}$ is said to be an independently scattered α-Fréchet sup-measure with control measure μ, if:

(i) *(independently scattered)* For any collection of disjoint sets $A_j \in \mathcal{E},\ 1 \le j \le n,\ n \in \mathbb{N}$, with $\mu(A_j) < \infty$, the random variables $M_\alpha(A_j),\ 1 \le j \le n$ are independent.

(ii) (α-*Fréchet*) For any $A \in \mathcal{E}$, $\mu(A) < \infty$, we have

$$\mathbb{P}\{M_\alpha(A) \le x\} = \exp\{-\mu(A)x^{-\alpha}\}, \quad x > 0, \tag{2.1}$$

that is, $M_\alpha(A)$ is α-Fréchet with scale coefficient $\|M_\alpha(A)\|_\alpha = \mu(A)^{1/\alpha}$.

(iii) (σ-*sup-additive*) For any collection of disjoint sets $A_j \in \mathcal{E}$, $j \in \mathbb{N}$, such that $\mu(\cup_j A_j) < \infty$, we have that

$$M_\alpha\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \bigvee_{j\in\mathbb{N}} M_\alpha(A_j) := \sup_{j\in\mathbb{N}} M_\alpha(A_j), \quad \text{almost surely.}$$

By convention, we set $M_\alpha(A) = \infty$, if $\mu(A) = \infty$.

Random α-Fréchet sup-measures with control measure μ can be constructed on a sufficiently rich probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the help of Kolmogorov's extension theorem (Proposition 2.1 in [23]). Vervaat [24] introduced and studied random sup-measures in a general setting (see also [17,20]). The additional structure of α-Féchet sup-measures allows one to study in more detail *extremal integrals*. Specifically, let $f(x) := \sum_{j=1}^n a_j 1_{A_j}(x), a_j \geq 0$ be a *non-negative* simple function, where $A_j \in \mathcal{E}, j = 1, \ldots, n$ are *disjoint*. The *extremal integral* of f with respect to the α-Fréchet sup-measure M_α is defined as:

$$\int_E^{e} f \mathrm{d}M_\alpha \equiv \int_E^{e} f(x) M_\alpha(\mathrm{d}x) := \bigvee_{1\leq j\leq n} a_j M_\alpha(A_j).$$

Since the independent $M_\alpha(A_j)$'s are α-Fréchet, so is the resulting extremal integral $Z = \int_E^{e} f \mathrm{d}M_\alpha$. One has, moreover, that

$$\|Z\|_\alpha = \left\|\int_E^{e} f \mathrm{d}M_\alpha\right\|_\alpha = \left(\int_E f(x)^\alpha \mu(\mathrm{d}x)\right)^{1/\alpha} = \|f\|_{L^\alpha(\mu)}, \tag{2.2}$$

that is, the scale coefficient of the extremal integral equals the L^α-norm of the deterministic integrand. By using this property, one can extend the definition of the extremal integral for an arbitrary $f \in L_+^\alpha(\mu) := \{f \geq 0, \int_E f(x)^\alpha \mu(\mathrm{d}x) < \infty\}$.

Unlike the usual integrals, the extremal integrals are not linear but *max-linear*. In other words, for any $f, g \in L_+^\alpha(\mu)$, and $a, b \geq 0$, we have

$$\int_E^{e} (af \vee bg) \mathrm{d}M_\alpha = a \int_E^{e} f \mathrm{d}M_\alpha \vee b \int_E^{e} g \mathrm{d}M_\alpha, \quad \text{almost surely.} \tag{2.3}$$

Let now $f_t(x) \in L_+^\alpha(\mu(\mathrm{d}x)), t \in \mathbb{R}$ be an arbitrary collection of deterministic functions. Consider the stochastic process $X = \{X_t\}_{t\in\mathbb{R}}$, where

$$X_t := \int_E^{e} f_t \mathrm{d}M_\alpha, \quad t \in \mathbb{R}. \tag{2.4}$$

The max-linearity of the extremal integral implies that X is an α-Fréchet process. Conversely, Proposition 3.2 in [23] implies that any α-Fréchet process X, which is *separable in probability* has the representation in (2.4). This is so, in particular, if X is continuous in probability (see also Theorem 3 in [8]). Since in the sequel we will only work with stochastically continuous, strictly stationary α-Fréchet processes, we can always assume that X is expressed in terms of extremal integrals as in (2.4).

Observe that by applying (2.2) and (2.3), we obtain

$$\mathbb{P}\{X_{t_j} \leq x_j,\ 1 \leq j \leq n\} = \exp\left\{-\int_E \left(\vee_{1\leq j\leq n} x_j^{-1} f_{t_j}(u)\right)^\alpha \mu(\mathrm{d}u)\right\}, \tag{2.5}$$

for any $t_j \in \mathbb{R}, x_j > 0, j = 1, \ldots, n$. That is, one can explicitly handle the finite-dimensional distributions of X in terms of its extremal integral representation. When $(E, \mathcal{E}, \mu) \equiv ([0, 1], \mathcal{B}, \mathrm{d}u)$, the representation (2.5) is commonly known as the *spectral representation* of X and the functions $f_t(x)$ are called *spectral functions* (see, e.g. Proposition 5.11′ in [18]).

Two extremal integrals $\xi := \int^{e}_E f \mathrm{d}M_\alpha$ and $\eta := \int^{e}_E g \mathrm{d}M_\alpha$ are "close" if so are the integrands f and g. In fact, the convergence in probability of max-stable integrals is equivalent to the convergence of their integrands in a suitable metric on $L^\alpha_+(\mu)$. To best describe this isometry relation, we need the notions of Fréchet spaces and max-linear isometries.

Definition 2.2. A non-empty set $\mathcal{M}$ of random variables is said to be an α-Fréchet space, $\alpha > 0$, if:

(i) it is closed with respect to taking max-linear combinations (with non-negative scalars)
(ii) its elements are α-Fréchet.
An α-Fréchet space is said to be *closed* if it contains its limits in probability.

Any α-Fréchet space $\mathcal{M}$ can be equipped with the metric

$$\rho_{\alpha,\mathcal{M}}(\xi, \eta) := 2\|\xi \vee \eta\|^\alpha_\alpha - \|\xi\|^\alpha_\alpha - \|\eta\|^\alpha_\alpha,$$

which *metrizes the convergence in probability*. In fact, any closed α-Fréchet space $\mathcal{M}$ is a complete metric space with respect to the metric $\rho_{\alpha,\mathcal{M}}$, and we have moreover the following

Proposition 2.1. (i) *The set $\mathcal{M}$ of all extremal integrals of functions $f \in L^\alpha_+(\mu)$ is a closed α-Fréchet space.* (ii) *For any $\xi := \int^{e}_E f \mathrm{d}M_\alpha$ and $\eta := \int^{e}_E g \mathrm{d}M_\alpha$, $f, g \in L^\alpha_+(\mu)$,*

$$\rho_{\alpha,\mathcal{M}}(\xi, \eta) = \int_E |f(u)^\alpha - g(u)^\alpha| \mu(\mathrm{d}u) =: \rho_{\alpha,\mu}(f, g). \tag{2.6}$$

(iii) *$(L^\alpha_+(\mu), \rho_{\alpha,\mu})$ is a complete metric space.*

The proof of this result follows from Proposition 2.6 and Theorem 2.1 in [23]. Observe that Proposition 2.1 implies that the extremal integral operator $I(f) := \int^{e}_E f \mathrm{d}M_\alpha$, $f \in L^\alpha_+(\mu)$ is an isometry between the complete metric spaces $(L^\alpha_+(\mu), \rho_{\alpha,\mu})$ and $(\mathcal{M}, \rho_{\alpha,\mathcal{M}})$. In particular, for a sequence $\xi_n = I(f_n),\ f_n \in L^\alpha_+(\mu),\ n \in \mathbb{N}$,

$$\xi_n \xrightarrow{\mathrm{P}} \xi, n \to \infty, \quad \text{if and only if} \quad \int_E |f_n(u)^\alpha - f(u)^\alpha| \mu(\mathrm{d}u), \quad n \to \infty,$$

where $\xi = I(f)$.

2.2. Max-linear isometries

An α-Fréchet process X can have many equivalent extremal integral representations (2.4), defined over the same or different measure spaces. The notion of max-linear isometry can be used to relate two such representations.

Definition 2.3. Let $\alpha > 0$ and consider the spaces $L^\alpha_+(\mu)$ and $L^\alpha_+(\nu)$, defined over two measure spaces $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \nu)$, respectively, where μ and ν are positive and σ-finite measures. The map $G : L^\alpha_+(\mu) \to L^\alpha_+(\nu)$, is said to be a *max-linear isometry*, if:

(i) For any $f, g \in L^\alpha_+(\mu)$ and $a, b \geq 0$, $G(af \vee bg) = aG(f) \vee bG(g)$, ν-a.e.
(ii) For any $f \in L^\alpha_+(\mu)$, $\|G(f)\|_{L^\alpha_+(\nu)} = \|f\|_{L^\alpha_+(\mu)}$.

The *max-linear isometry* G is called a max-linear *isomorphism* if it is onto. It is an *automorphism*, if in addition, the two measure spaces coincide.

Observe that the max-linear isometries preserve the metrics, that is, $\rho_{\alpha,\nu}(G(f), G(g)) = \rho_{\alpha,\mu}(f, g), \forall f, g \in L^{\alpha}_{+}(\mu)$. Indeed, this follows from Definition 2.3 and the fact that $\forall f, g \in L^{\alpha}_{+}(\mu)$,

$$\rho_{\alpha,\mu}(f, g) = \int_E |f^{\alpha} - g^{\alpha}| \mathrm{d}\mu = 2\int_E (f^{\alpha} \vee g^{\alpha}) \mathrm{d}\mu - \int_E f^{\alpha} \mathrm{d}\mu - \int_E g^{\alpha} \mathrm{d}\mu.$$

The next result illustrates the role of max-linear isometries in relating two representations.

Proposition 2.2. *Let $X = \{X_t\}_{t\in\mathbb{R}}$ be an α-Fréchet process with extremal integral representation (2.4). Consider another measure space $(F, \mathcal{F}, \nu)$ with positive σ-finite measure, and let $G : (L^{\alpha}_{+}(\mu), \rho_{\alpha,\mu}) \to (L^{\alpha}_{+}(\nu), \rho_{\alpha,\nu})$ be a max-linear isometry.*

Then, the process $\widetilde{X} = \{\widetilde{X}_t\}_{t\in\mathbb{R}}$,

$$\widetilde{X}_t := {\int^{\mathrm{e}}}_{F} G(f_t) \mathrm{d}\widetilde{M}_{\alpha}$$

has the same finite-dimensional distributions as X. Here $\widetilde{M}_{\alpha}$ is an independently scattered α-Fréchet random sup-measure with control measure ν.

The result follows by applying the definition of max-linear isometry to Relation (2.5).

The following surprising result shows that the class of *max-linear isometries* coincides with the class of *linear isometries* between two metric spaces $(L^{\alpha}_{+}(\mu), \rho_{\alpha,\mu})$ and $(L^{\alpha}_{+}(\nu), \rho_{\alpha,\nu})$, for any $\alpha > 0$.

Proposition 2.3. *The map $G : (L^{\alpha}_{+}(\mu), \rho_{\alpha,\mu}) \to (L^{\alpha}_{+}(\nu), \rho_{\alpha,\nu})$ is a max-linear isometry of metric spaces, if and only if, it is a linear isometry of metric spaces. Note that $L^{\alpha}_{+}(\mu)$ and $L^{\alpha}_{+}(\nu)$ contain only non-negative functions.*

Any such max-linear isometry G extends uniquely to a linear isometry between the spaces $(L^{\alpha}(\mu), \widetilde{\rho}_{\alpha,\mu})$ and $(L^{\alpha}(\nu), \widetilde{\rho}_{\alpha,\nu})$, equipped with the metric $\widetilde{\rho}_{\alpha,\mu}(f, g) := \rho_{\alpha,\mu}(f_+, g_+) + \rho_{\alpha,\mu}(f_-, g_-)$, for all $f, g \in L^{\alpha}(\mu) = \{f : E \to \mathbb{R}, \int_E |f(u)|^{\alpha} \mu(\mathrm{d}u) < \infty\}$, where $f_{\pm} := \max\{\pm f, 0\}$.

In addition, one has the representation:

$$\widetilde{\rho}_{\alpha,\mu}(f, g) = \int_E |f^{\langle\alpha\rangle}(u) - g^{\langle\alpha\rangle}(u)| \mu(\mathrm{d}u), \tag{2.7}$$

where $f^{\langle\alpha\rangle}(u) = \mathrm{sign}(f(u))|f(u)|^{\alpha}$.

The proof is given in Appendix A.

Examples of max-linear isometries

1. Let $T : (E, \mathcal{E}, \mu) \to (F, \mathcal{F}, \nu)$ be a measure preserving transformation; then the map $G_T(f) := f \circ T^{-1}$, $f \in L^{\alpha}_{+}(\mu)$ is a max-linear isometry.
2. De Haan and Pickands [10] introduced a rich class of max-linear automorphisms for the Lebesgue space $L^1_{+}([0, 1], \mathrm{d}u)$ called *pistons*. The map $G : L^1_{+}([0, 1], \mathrm{d}u) \to L^1_{+}([0, 1], \mathrm{d}u)$ is said to be a *piston*, if:
   (i) $\int_0^1 G(f)(u)\mathrm{d}u = \int_0^1 f(u)\mathrm{d}u$, for all $f(u) \in L^1_{+}([0, 1], \mathrm{d}u)$ *and*
   (ii) $G(f) = r(u) f(H(u))$, for some measurable functions r and H, such that $r(u) > 0$ and $H : [0, 1] \to [0, 1]$ is a bijection.

When the spaces $L_+^\alpha(\mu)$ and $L_+^\alpha(\nu)$ coincide with $L_+^1([0,1], \mathrm{d}u)$, the transformation G_T in the previous example yields the special case of *permutation pistons*.

As stated in [10] (after Definition 2.2 therein), by Lamperti's Theorem (Ch. 15.5, Theorem 16 in [21]), the *pistons* are precisely the *linear automorphisms* on $L_+^1([0,1], \mathrm{d}u)$. The following Proposition 2.4 proves this statement and also shows that the *piston structure* of the max-linear automorphisms is valid over a general space $L_+^\alpha(\mu)$, $\alpha > 0$.

Consider a measure space $(E, \mathcal{E}, \mu)$ with a σ-*finite* measure μ. Following Lamperti [16], we say that a set–mapping $T : \mathcal{E} \to \mathcal{E}$, defined modulo sets of μ-measure zero, is a *regular set isomorphism* if:

(i) $T(E \setminus A) = T(E) \setminus T(A)$, for all $A \in \mathcal{E}$;
(ii) $T(\cup_{n=1}^\infty A_n) = \cup_{n=1}^\infty T(A_n)$, for disjoint A_n's in $\mathcal{E}$;
(iii) $\mu(T(A)) = \mu(A)$, for all $A \in \mathcal{E}$.

The map T induces a canonical *linear* transformation $T : L^\alpha(\mu) \to L^\alpha(\mu)$, defined as $T(1_A) := 1_{T(A)}$, for all indicator functions. We shall denote the image of $f \in L^\alpha(\mu)$ with respect to this transformation as $T(f)$.

Proposition 2.4. *Let $G : (L_+^\alpha(\mu), \rho_\alpha) \to (L_+^\alpha(\mu), \rho_\alpha)$, be a max-linear isometry, for some $\alpha > 0$ over the measure space $(E, \mathcal{E}, \mu)$ with σ-finite measure μ. Then, there exist a regular set isomorphism $T : \mathcal{E} \to \mathcal{E}$ and a measurable, non-negative function $h : E \to \mathbb{R}_+$, such that*

$$G(f)(u) = h(u)T(f)(u), \quad \forall f \in L_+^\alpha(\mu). \tag{2.8}$$

The function h is unique (mod μ), the map T is unique on the set $\{u \in E : h(u) > 0\}$ and

$$h = \left(\frac{\mathrm{d}\mu \circ T^{-1}}{\mathrm{d}\mu}\right)^{1/\alpha}. \tag{2.9}$$

Conversely, any G as in (2.8) where (2.9) holds is a max-linear isometry.

Proof. Consider the bijection $I_\alpha : L_+^\alpha(\mu) \to L_+^1(\mu)$, where $I_\alpha(f) := f^\alpha$. Then, the map $\widetilde{G} := I_\alpha \circ G \circ I_\alpha^{-1}$ is a max-linear isometry of $L_+^1(\mu)$ into itself *if and only if* G is a max-linear isometry of $L_+^\alpha(\mu)$ into itself.

By Proposition 2.3, $\widetilde{G}$ extends uniquely to a linear isometry of $L^1(\mu)$ into itself, equipped with the metric

$$\widetilde{\rho}_\alpha(f, g) = \int_E |f^{\langle 1\rangle} - g^{\langle 1\rangle}|\mathrm{d}\mu = \int_E |f - g|\mathrm{d}\mu.$$

Thus, in this case the metric $\widetilde{\rho}_\alpha(f, g)$ coincides with the usual norm $\|f - g\|_{L^1(\mu)}$. Hence, by Theorem 3.1 of [16], we obtain

$$\widetilde{G}(\widetilde{f}) = \widetilde{h} \cdot T(\widetilde{f}), \quad \forall \widetilde{f} \in L^1(\mu),$$

where $\widetilde{h}$ is unique (mod μ) and T is a regular set isomorphism, which is (mod μ) unique on $\{u \in E : \widetilde{h}(u) \neq 0\}$. One also has that $\widetilde{h}$ is the Radon–Nikodym derivative $\mathrm{d}\nu/\mathrm{d}\mu$ of the measure $\nu := \mu \circ T^{-1}$, defined on the range of T, with respect to μ. Observe that $\widetilde{h} \geq 0$ (mod μ) since the linear extension of $\widetilde{G}$ maps non-negative elements of $L^1(\mu)$ into non-negative ones.

Let now $\widetilde{f} = I_\alpha(f) = f^\alpha$, for any $f \in L_+^\alpha(\mu)$, and note that

$$I_\alpha \circ G(f) = \widetilde{G}(\widetilde{f}) = \widetilde{h} \cdot T(f^\alpha).$$

Since $T(f^\alpha) = T(f)^\alpha$, this implies (2.8) and (2.9) with $h := \widetilde{h}^{1/\alpha}$, and hence completes the proof of the proposition. $\square$

Remarks.

1. The equivalence between the max-linear and linear isometries established in Proposition 2.3 relies on the fact that we consider only sets of non-negative functions. This is why images of functions with disjoint supports have disjoint supports. This last property is also essential in Lamperti's characterization of the isometries of $L^p(\mu)$ spaces, for $1 \le p < \infty,\ p \neq 2$.
2. De Haan and Pickands [10] developed an elegant theory of *proper spectral representations* for stationary min-stable processes. This theory applies to the max-stable α-Fréchet processes and carries a strong resemblance to the theory of *minimal representations* for α-(sum-)stable processes of Hardin [15]. The structural results in Propositions 2.3 and 2.4 indicate that the theory of proper spectral representations extends to max-stable processes defined as extremal integrals over general measure spaces.

3. Ergodic properties of max-stable processes

Here, following Cambanis, Hardin and Weron [5], we establish necessary and sufficient conditions for the ergodicity of strictly stationary α-Fréchet processes. We do so in Section 3.2 below. We address first the measurability of these processes.

3.1. Measurability of stationary α-Fréchet processes

Fix $\alpha > 0$ and let M_α be a random α-Fréchet sup-measure, defined on the measure space $(E, \mathcal{E}, \mu)$ (see Definition 2.1). Let $f \in L^\alpha_+(\mu)$ be arbitrary, and consider the process $X = \{X_t\}_{t\in\mathbb{R}}$

$$X_t := {\int_E}^{\!\!\mathrm{e}} U_t(f)\mathrm{d}M_\alpha, \quad t \in \mathbb{R}, \tag{3.1}$$

where $U_t : L^\alpha_+(\mu) \to L^\alpha_+(\mu)$, is a group of max-linear automorphisms, indexed by $t \in \mathbb{R}$: $U_{t+s} \equiv U_t \circ U_s$, $s, t \in \mathbb{R}$ and $U_0(f) := \mathrm{id}(f) \equiv f$, $f \in L^\alpha_+(\mu)$.

By using Relation (2.5) and the max-isometry property of the U_t's, one can show that X is a (strictly) stationary α-Fréchet process (see also Proposition 2.2). By choosing various measure spaces and groups of max-linear automorphisms, one can construct rich classes of stationary α-Fréchet processes (see e.g. Section 4.1 below).

To be able to discuss the ergodicity of X, the process should admit a measurable modification. The following result shows that any measurable X as in (3.1) is necessarily continuous in probability (see also Theorem 0 in [5]).

Theorem 3.1. *The stationary α-Fréchet process $X = \{X_t\}_{t\in\mathbb{R}}$ in (3.1) has a measurable modification if and only if it is continuous in probability.*

Proof. If X is continuous in probability, then it has a measurable modification (see e.g. Ch. III, Theorem 1 in [13]). Conversely, if X has a measurable modification, then so does the process $X^\alpha = \{X_t^\alpha\}_{t\in\mathbb{R}}$. It is thus enough to show that X^α is continuous in probability. Observe that $\{X_t^\alpha\}$ is now a 1-Fréchet process with the representation $\{{\int_E}^{\mathrm{e}} U_t(f)(u)^\alpha M_1(\mathrm{d}u)\}$, where $M_1(\mathrm{d}u)$ is a 1-Fréchet sup-measure with control measure μ. Thus, without loss of generality, we will suppose that $\alpha = 1$.

The max-linear isometries $\{U_t\}$ extend uniquely to a group of *linear* isometries on the metric space $(L^1, \widetilde{\rho}_\alpha)$ (Proposition 2.3). Since $\alpha = 1$, however, $\widetilde{\rho}_\alpha(g_1, g_2) = \int_E |g_1 - g_2| d\mu$ is the usual L^1-norm $\|g_1 - g_2\|_{L^1(\mu)}$. Equip the normed space $L^1(\mu)$, $\|\cdot\|_{L^1(\mu)}$ with the Borel σ-algebra. By Theorem 3 in [6], and in view of Proposition 2.1, the measurability of $t \mapsto X_t^\alpha$ implies that the map $t \mapsto U_t(g)$ is Borel measurable for each $g \in F_U$. Therefore, the maps $t \mapsto U_t(g)$ are Borel measurable for all $g \in \overline{F_U} := \overline{\mathrm{span}}\{F_U\}_{L^1(\mu)}$. This, however, implies the strong continuity of the group of linear operators $\{U_t\}$ on $\overline{F_U}$ (see, e.g. page 616 in [11]). That is, for all $t_0 \in \mathbb{R}$, and $g \in \overline{F_U}$, we have $\|U_t(g) - U_{t_0}(g)\|_{L^1(\mu)} \to 0$, as $t \to t_0$. Hence, in particular

$$\rho_{1,\mathcal{M}}(X_t, X_{t_0}) = \rho_1(U_t(f), U_{t_0}(f)) = \|U_t(f) - U_{t_0}(f)\|_{L^1(\mu)} \to 0, \quad \text{as } t \to t_0.$$

Since $\rho_{1,\mathcal{M}}$ metrizes the convergence in probability (Proposition 2.1), this implies the continuity in probability of X and completes the proof. $\square$

Theorem 6.1 first presented in the seminal work de Haan and Pickands [10] implies (for the case of maxima) that any *continuous in probability* stationary α-Fréchet process has the representation in (3.1). Moreover, the space $(E, \mathcal{E}, \mu)$ can be chosen to be the Lebesgue space $([0, 1], \mathcal{B}, \mathrm{d}u)$, where the max–automorphisms U_t's were called *pistons* (see also Proposition 3.1 in [23]). In summary, we have the following

Corollary 3.1. *Any continuous in probability stationary α-Fréchet process $X = \{X_t\}_{t\in\mathbb{R}}$ has the representation (3.1), for some measure space $(E, \mathcal{E}, \mu)$.*

3.2. Necessary and sufficient conditions for ergodicity and mixing

Recall that a stationary process X is ergodic *if and only if*, for any bounded $h : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$, measurable with respect to the product Borel σ-algebra,

$$\xi_T(h) := \frac{1}{T}\int_0^T h \circ S_\tau(X)\mathrm{d}\tau \xrightarrow{a.s.} \mathbb{E}h(X), \quad \text{as } T \to \infty. \tag{3.2}$$

Here $S_\tau : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ denotes the *shift operator*, $S_\tau(x) = (x_{t+\tau})_{t\in\mathbb{R}}$, where $x = (x_t)_{t\in\mathbb{R}} \in \mathbb{R}^{\mathbb{R}}$. The measurability of X implies that $h \circ S_\tau(X)$ is measurable and one can consider the integrals in (3.2).

Let X be as in (3.1) and let

$$F_U(f) := \overline{\vee\text{−span}}\{U_t(f),\ t \in \mathbb{R}\}$$

denote the minimal closed subset of $(L_+^\alpha(\mu), \rho_{\alpha,\mu})$, which contains all max-linear combinations $\vee_{1\le j\le n} a_j U_{t_j}(f), a_j \ge 0, t_j \in \mathbb{R}, j = 1, \ldots, n$.

To gain information about the process X, it is natural to focus on the set of functions F_U. This is because the Fréchet space generated by the random variables $X_t, t \in \mathbb{R}$ is precisely the set of extremal integrals $\int^e_E g\mathrm{d}M_\alpha$, for all $g \in F_U(f)$. The next result provides conditions for the ergodicity of X in terms of $F_U(f)$.

Theorem 3.2. *Let X be a measurable α-Fréchet process, defined by (3.1). The process X is ergodic, if and only if, for some (any) $p > 0$,*

$$\frac{1}{T}\int_0^T \|U_\tau g \wedge g\|^p_{L^\alpha(\mu)}\mathrm{d}\tau \longrightarrow 0, \tag{3.3}$$

as $T \to \infty$, for all $g \in F_U(f)$, where $a \wedge b = \min\{a, b\}$.

Proof. The convergence (3.2) holds for all bounded Borel measurable h if and only if it holds for all $h(x) := I(a_1 < x_{t_1} \le b_1, \dots, a_n < x_{t_n} \le b_n)$, $a_i, b_i, t_i \in \mathbb{R}$, $1 \le i \le n$. In our case, it is enough to show that (3.3) is equivalent to (3.2) for all functions of the type

$$h(x) := I(x_{t_1} \le a_1, \dots, x_{t_n} \le a_n), \quad a_i > 0, 1 \le i \le n. \tag{3.4}$$

This is because the X_t's are positive and since the indicators of cylinder sets can be represented as finite linear combinations of functions as in (3.4).

Consider an arbitrary h as in (3.4). The Birkhoff theorem implies that $\xi_T(h)$ converges almost surely and in $\mathcal{L}^1$, as $T \to \infty$, to an integrable random variable ξ_{inv}, invariant with respect to shifts. Observe that, since h is bounded, so is $\xi_T(h)$, *uniformly in* T, and therefore since $\mathbb{E}\xi_T(h) = \mathbb{E}h(X)$, Relation (3.2) holds if and only if,

$$\mathbb{E}\xi_T(h)^2/(\mathbb{E}\xi_T(h))^2 \longrightarrow 1, \quad \text{as } T \to \infty. \tag{3.5}$$

Thus, to complete the proof, it is enough to show that (3.3) and (3.5) are equivalent for all h as in (3.4). The Fubini's theorem implies that $\mathbb{E}\xi_T(h)^2$ equals

$$\begin{aligned}\mathbb{E}\left(\frac{1}{T}\int_0^T h \circ S_\tau(X)\mathrm{d}\tau\right)^2 &= \mathbb{E}\left(\frac{1}{T}\int_0^T I(X_{\tau+t_1} \le a_1, \dots, X_{\tau+t_n} \le a_n)\mathrm{d}\tau\right)^2 \\ &= \frac{1}{T^2}\int_0^T\int_0^T \mathbb{P}\{X_{\tau'+t_i} \le a_i, X_{\tau''+t_i} \le a_i, \forall i, 1 \le i \le n\}\mathrm{d}\tau'\mathrm{d}\tau''.\end{aligned} \tag{3.6}$$

In view of (2.5) and (3.1), the last probability equals

$$\exp\left\{-\int_E \left(U_{\tau'}g \vee U_{\tau''}g\right)^\alpha \mathrm{d}\mu\right\} = \exp\left\{-\int_E \left(U_{\tau'-\tau''}g \vee g\right)^\alpha \mathrm{d}\mu\right\}, \tag{3.7}$$

where $g(u) := \bigvee_{1 \le i \le n} a_i^{-1} U_{t_i} f(u)$ and where in the last relation, we used the fact that $\{U_\tau\}_{\tau\in\mathbb{R}}$ is a group of max-linear isometries.

Similarly, since $\int_E g^\alpha \mathrm{d}\mu = \int_E (U_\tau g)^\alpha \mathrm{d}\mu$, $\forall \tau \in \mathbb{R}$ (by the max-linear isometry property), we have

$$\mathbb{E}\xi_T(h) = \frac{1}{T}\int_0^T \exp\left\{-\int_E (U_\tau g)^\alpha \mathrm{d}\mu\right\}\mathrm{d}\tau = \exp\left\{-\int_E g^\alpha \mathrm{d}\mu\right\}. \tag{3.8}$$

By combining Relations (3.6)–(3.8), we obtain

$$\begin{aligned}\frac{\mathbb{E}\xi_T(h)^2}{(\mathbb{E}\xi_T(h))^2} = \frac{1}{T^2}\int_0^T\int_0^T \exp&\left\{-\int_E \left(U_{\tau'-\tau''}g \vee g\right)^\alpha \mathrm{d}\mu + \int_E \left(U_{\tau'-\tau''}g\right)^\alpha \mathrm{d}\mu \right. \\ &\left. + \int_E g^\alpha \mathrm{d}\mu\right\}\mathrm{d}\tau'\mathrm{d}\tau''.\end{aligned} \tag{3.9}$$

Note that $-a \vee b + a + b = a \wedge b$, $\forall a, b \ge 0$, and thus the argument of the last exponent equals

$$R(\tau' - \tau'') := \int_E (U_{\tau'-\tau''}g \wedge g)^\alpha \mathrm{d}\mu = \int_E (U_{\tau'}g \wedge U_{\tau''}g)^\alpha \mathrm{d}\mu. \tag{3.10}$$

Observe that the function $(\tau', \tau'') \mapsto R(\tau' - \tau'')$ in (3.10) is non-negative definite. Indeed, for μ-almost all $u \in E$ and all $\tau_j \in \mathbb{R}, \theta_j \in \mathbb{C}$, $j = 1, \dots, m$, we have

$$\sum_{1 \le i,j \le m} \theta_i\overline{\theta}_j (U_{\tau_i}g(u))^\alpha \wedge (U_{\tau_j}g(u))^\alpha = \mathbb{E}\left(\sum_{1 \le i,j \le m} \theta_i\overline{\theta}_j Z_i Z_j\right) = \mathbb{E}|\sum_{i=1}^m \theta_i Z_i|^2 \ge 0,$$

with $Z_i := B(U_{\tau_j} g(u)^\alpha)$, where $B = \{B(s)\}_{s\geq 0}$ is the standard Brownian motion. This implies that $(\tau', \tau'') \mapsto R(\tau' - \tau'') = \int_E U_{\tau'} g(u)^\alpha \wedge U_{\tau''} g(u)^\alpha \mu(\mathrm{d}u)$ is non-negative definite.

Since X is measurable, it is also continuous in probability and $\tau \mapsto U_\tau g$ is continuous in $(L^\alpha_+(\mu), \rho_{\alpha,\mu})$ (Theorem 3.1, Section 2.1). Hence, $R(\cdot)$ is a continuous function, bounded by $R(0) = \int_E g^\alpha \mathrm{d}\mu$, and as shown above, non-negative definite. Thus, by Bochner's representation theorem

$$R(\tau' - \tau'') \equiv \int_E (U_{\tau'-\tau''} g \wedge g)^\alpha \mathrm{d}\mu = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\tau'-\tau'')u} \nu(\mathrm{d}u), \quad \forall \tau', \tau'' \in \mathbb{R}, \tag{3.11}$$

for some symmetric, positive and finite measure ν.

Now, note that

$$\left(\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\tau'-\tau'')u} \nu(\mathrm{d}u) \right)^n = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\tau'-\tau'')u} \nu^{*n}(\mathrm{d}u), \quad n \in \mathbb{N},$$

where $\nu^{*n}(\mathrm{d}u) = \nu(\mathrm{d}u) * \cdots * \nu(\mathrm{d}u)$ denotes the n-th convolution power of the measure ν. Thus, for the exponent in (3.9), we obtain

$$\begin{aligned} \frac{\mathbb{E}\xi_T(h)^2}{(\mathbb{E}\xi_T(h))^2} &= \frac{1}{T^2} \int_0^T \int_0^T \exp\{R(\tau' - \tau'')\} \mathrm{d}\tau' \mathrm{d}\tau'' \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!\, T^2} \int_0^T \int_0^T \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\tau'-\tau'')u} \nu^{*n}(\mathrm{d}u) \mathrm{d}\tau' \mathrm{d}\tau''. \end{aligned} \tag{3.12}$$

Since the measures $\nu^{*n}(\mathrm{d}u)$ are finite and symmetric, and since $\mathrm{e}^{\mathrm{i}(\tau'-\tau'')u}$ is bounded, by using the Fubini's theorem, we get $\forall n \in \mathbb{N}$,

$$\begin{aligned} &\frac{1}{T^2} \int_0^T \int_0^T \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\tau'-\tau'')u} \nu^{*n}(\mathrm{d}u) \mathrm{d}\tau' \mathrm{d}\tau'' \\ &= \int_{-\infty}^{\infty} \frac{\sin^2(Tu/2)}{(Tu/2)^2} \nu^{*n}(\mathrm{d}u) \longrightarrow \nu^{*n}\{0\}, \quad \text{as } T \to \infty. \end{aligned}$$

This convergence and the fact that the convergent series $\sum_{n=1}^{\infty} \nu^{*n}(\mathbb{R})/n!$ dominates the series on the right-hand side of (3.12) (uniformly in T) imply that

$$\frac{\mathbb{E}\xi_T(h)^2}{(\mathbb{E}\xi_T(h))^2} \longrightarrow 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \nu^{*n}\{0\}, \quad \text{as } T \to \infty.$$

Therefore, Relation (3.5) holds, if and only if $\nu^{*n}\{0\} = 0, \forall n \in \mathbb{N}$. Since ν is finite and symmetric, by the Fubini's theorem, we also have that

$$\frac{1}{T} \int_0^T R(\tau)^n \mathrm{d}\tau = \int_{-\infty}^{\infty} \frac{\sin(Tu)}{Tu} \nu^{*n}(\mathrm{d}u) \longrightarrow \nu^{*n}\{0\}, \quad \text{as } T \to \infty. \tag{3.13}$$

Since $R(\tau) = \int_E (U_\tau g \wedge g)^\alpha \mathrm{d}\mu = \|U_\tau g \wedge g\|^\alpha_{L^\alpha(\mu)}$, is a bounded and non-negative function, in view of Lemma A.2 (i) below, and (3.13), Relation (3.3) is equivalent to $\nu^{*n}\{0\} = 0,\ \forall n \in \mathbb{N}$. This completes the proof of the theorem. □

The next result provides a necessary and sufficient condition for mixing. Recall that a strictly stationary and measurable process $X = \{X_t\}_{t\in\mathbb{R}}$ is mixing if and only if,

$$\mathbb{P}(A \cap B_\tau) \longrightarrow \mathbb{P}(A)\mathbb{P}(B), \quad \text{as } \tau \to \infty, \tag{3.14}$$

for all $A \in \mathcal{F}_- := \sigma\{X_t, t \le 0\}$, and $B \in \mathcal{F}_+ := \sigma\{X_t, t \ge 0\}$, where B_τ denotes the 'time-shifted' version of B.

Theorem 3.3. *Let X be a measurable α-Fréchet process, defined by (3.1). The process X is mixing if and only if*

$$\|U_\tau h \wedge g\|_{L^\alpha(\mu)} \longrightarrow 0, \quad \text{as } \tau \to 0, \tag{3.15}$$

for all $g \in F_U^- := \overline{\vee\text{−span}}\{U_t(f), t \le 0\}$ *and* $h \in F_U^+ := \overline{\vee\text{−span}}\{U_t(f), t \ge 0\}$.

Proof. Since $X_t > 0$, a.s. as in the proof of Theorem 3.2, it is enough to prove (3.14), for all cylinder sets:

$$A = \{X_{t_i} \le a_i, i = 1, \ldots, n\} \quad \text{and} \quad B = \{X_{t_i} \le b_i, i = 1, \ldots, n\},$$

with $a_i, b_i \in (0, \infty]$, $i = 1, \ldots, n$, where $a_i = \infty$ for $t_i > 0$ and $b_i = \infty$ for $t_i \le 0$, ensures that $A \in \mathcal{F}_-$ and $B \in \mathcal{F}_+$.

We have

$$\mathbb{P}(A \cap B_\tau) = \mathbb{P}\{\vee_{1 \le i \le n} a_i^{-1} X_{t_i} \le 1, \vee_{1 \le i \le n} b_i^{-1} X_{t_i + \tau} \le 1\}.$$

Thus, in view of (3.1), we obtain

$$\mathbb{P}(A \cap B_\tau) = \exp\left\{-\int_E (U_\tau(h) \vee g)^\alpha \mathrm{d}\mu\right\},$$

where

$$g := \vee_{1 \le i \le n} a_i^{-1} U_{t_i}(f) \quad \text{and} \quad h := \vee_{1 \le i \le n} b_i^{-1} U_{t_i}(f).$$

Since

$$\mathbb{P}(A) = \exp\left\{-\int_E g^\alpha \mathrm{d}\mu\right\} \quad \text{and} \quad \mathbb{P}(B) = \mathbb{P}(B_\tau) = \exp\left\{-\int_E U_\tau(h)^\alpha \mathrm{d}\mu\right\},$$

we get (as in (3.9) and (3.10))

$$\begin{aligned}\frac{\mathbb{P}(A \cap B_\tau)}{\mathbb{P}(A)\mathbb{P}(B)} &= \exp\left\{-\int_E U_\tau(h)^\alpha \vee g^\alpha \mathrm{d}\mu + \int_E U_\tau(h)^\alpha \mathrm{d}\mu + \int_E g^\alpha \mathrm{d}\mu\right\} \\ &= \exp\left\{\int_E U_\tau(h)^\alpha \wedge g^\alpha \mathrm{d}\mu\right\}.\end{aligned}$$

This implies the equivalence of (3.14) and (3.15), and completes the proof of the theorem. □

Remarks.

1. Observe that as expected (3.15) implies (3.3) since mixing implies ergodicity (see e.g. [25]). Here by mixing we understand *strong mixing*.
2. Theorems 3.2 and 3.3 are similar in spirit to the criteria for ergodicity and mixing in the sum-stable case (see, Theorems 1 and 2 in [5]).
3. As in the sum-stable setting, (3.3) (and (3.15)) can be shown to hold for many classes of processes such as *moving maxima* and *mixed moving maxima*, for example (Section 4.1).
4. Theorems 3.2 and 3.3 are stated for continuous-time processes. Corresponding discrete-time analogs of these results are also valid. We omit them for the sake of conciseness.

In the next section, we provide further equivalent conditions for mixing that can be easier to check.

3.3. A measure of dependence

The conditions for ergodicity and mixing in Theorems 3.2 and 3.3 above suggest the following measure of dependence. For two jointly α-Fréchet variables ξ and η, define

$$d_\alpha(\xi,\eta) := \|\xi\|_\alpha^\alpha + \|\eta\|_\alpha^\alpha - \|\xi \vee \eta\|_\alpha^\alpha.$$

Let $\xi = \int^{e}_E f \mathrm{d}M_\alpha$ and $\eta = \int^{e}_E g \mathrm{d}M_\alpha$, with $f, g \in L_+^\alpha(\mu)$, for some α-Fréchet sup-measure M_α with control measure μ, defined on the measurable space $(E, \mathcal{E})$. We then have

$$d_\alpha(\xi,\eta) = \int_E f^\alpha \mathrm{d}\mu + \int_E g^\alpha \mathrm{d}\mu - \int_E f^\alpha \vee g^\alpha \mathrm{d}\mu = \int_E f^\alpha \wedge g^\alpha \mathrm{d}\mu. \tag{3.16}$$

In this case, we will also write $d_\alpha(f, g)$ for $d_\alpha(\xi, \eta)$.

Observe that $d_\alpha(cf, cg) = c^\alpha d_\alpha(f, g), \forall c > 0$ and $d_\alpha(f, g) \geq 0$. Moreover, $d_\alpha(f, g) = 0$ if and only if the functions f and g have disjoint supports (mod μ), or equivalently, if $\xi = \int^{e}_E f \mathrm{d}M_\alpha$ and $\eta = \int^{e}_E g \mathrm{d}M_\alpha$ are independent. In this sense, the measure of dependence $d_\alpha(f, g)$ for α-Fréchet processes, is analogous to the covariance function for Gaussian processes.

Let now X be as in (3.1), where $f_t(u) := U_t(f)(u)$, for some group of max-linear isometries $\{U_t\}_{t\in\mathbb{R}}$, and observe that

$$d_\alpha(X_\tau, X_0) = \int_{\mathbb{R}} f_\tau^\alpha \wedge f_0^\alpha \, \mathrm{d}\mu = \|U_\tau(f) \wedge f\|_{L^\alpha(\mu)}. \tag{3.17}$$

Thus, by Theorem 3.3, if X is mixing, then $d_\alpha(X_\tau, X_0) \to 0$, as $\tau \to 0$. The converse is also true:

Theorem 3.4. *Let X be a stationary and continuous in probability α-Fréchet process. The process X is mixing if and only if $d_\alpha(X_\tau, X_0) \to 0$, as $\tau \to 0$.*

Proof. In view of Corollary 3.1, we can assume that X has the representation (3.1). As argued above, the 'only if' part follows from Theorem 3.3 and (3.17). We now prove the 'if' part. Observe that $X^\alpha = \{X_t^\alpha\}_{t\in\mathbb{R}}$ is a 1-Fréchet process with the representation

$$\{X_t^\alpha\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \int^{e}_E f_t^\alpha \mathrm{d}M_1 \right\}_{t\in\mathbb{R}},$$

where M_1 is a 1-Fréchet random sup-measure with control measure μ (Proposition 2.9 in [23]). We also have $d_\alpha(X_t, X_s) = d_1(X_t^\alpha, X_s^\alpha)$, $s, t \in \mathbb{R}$, and the process X is mixing if and only if X^α is mixing. Thus, without loss of generality, we will assume that $\alpha = 1$.

It suffices to show that (3.15) holds. Since $\alpha = 1$, the metric $\widetilde{\rho}_\alpha(g, h)$ on $L^1 := L^1(\mu)$ coincides with the usual L^1-norm $\|g - h\|_{L^1}$ (see Proposition 2.3, above). Let now $g \in \overline{\vee\text{−span}}\{U_t(f),\ t \leq 0\}$ and $h \in \overline{\vee\text{−span}}\{U_t(f),\ t \geq 0\}$ be arbitrary.

For any $\epsilon > 0$, there exist $g_\epsilon \in \vee\text{−span}\{U_t(f),\ t \leq 0\}$ and $h_\epsilon \in \vee\text{−span}\{U_t(f),\ t \geq 0\}$ such that $\|g - g_\epsilon\|_{L^1} \leq \epsilon$ and $\|h - h_\epsilon\|_{L^1} \leq \epsilon$. Then, since $|a \wedge b - c \wedge d| \leq |a - c| + |b - d|$, for all $a, b, c, d \geq 0$,

$$|U_\tau(h) \wedge g - U_\tau(h_\epsilon) \wedge g_\epsilon| \leq |U_\tau(h) - U_\tau(h_\epsilon)| + |g - g_\epsilon|.$$

The max-linear isometries U_τ are also linear (Proposition 2.3), and thus

$$\Big| \|U_\tau(h) \wedge g\|_{L^1} - \|U_\tau(h_\epsilon) \wedge g_\epsilon\|_{L^1} \Big| \leq \|U_\tau(h - h_\epsilon)\|_{L^1} + \|g - g_\epsilon\|_{L^1} \leq 2\epsilon.$$

Therefore, it is enough to prove (3.15) for all $h = \vee_{1\le i\le n} x_i f_{t_i}$, and $g = \vee_{1\le i\le n} y_i f_{t_i}$ with $t_i \in \mathbb{R}$, where $x_i,\ y_i \ge 0,\ i = 1, \ldots, n$.

Notice that, for all $a_i, b_i \ge 0,\ i = 1, \ldots, n$, we have

$$(\vee_{1\le i\le n} a_i) \wedge (\vee_{1\le i\le n} b_i) = \bigvee_{1\le i,j\le n} (a_i \wedge b_j).$$

By setting $a_i := x_i U_\tau(f_{t_i})$ and $b_i := y_i f_{t_i}$, we get that $\|U_\tau(h) \wedge g\|_{L^1}$ equals

$$\int_{\mathbb{R}} U_\tau(h) \wedge g \,\mathrm{d}\mu \le C \bigvee_{1\le i,j\le n} \int_{\mathbb{R}} U_\tau(f_{t_i}) \wedge f_{t_j} \,\mathrm{d}\mu = C \bigvee_{1\le i,j\le n} d_1(X_{\tau+t_i-t_j}, X_0), \quad (3.18)$$

where $C := \vee_{1\le i\le n}(x_i \vee y_i)$. Thus, since C, n and the t_i's are fixed, Relation (3.18) implies that $\|U_\tau(h) \wedge g\|_{L^1} \to 0$ as $\tau \to 0$, which completes the proof of the theorem. □

Remarks.

1. Theorem 3.4 above provides a convenient way of proving mixing by using only the *dependence function* $d_\alpha(t - s) = d_\alpha(X_t, X_s)$. This is analogous to the classical result for Gaussian processes where mixing is equivalent to the convergence of the auto-covariance function to zero.
2. The continuity in probability assumption in Theorem 3.4 is used only in the continuous-time setting to ensure that X has a measurable version (Theorem 3.1). The result of Theorem 3.4 is also valid in the discrete-time setting, where continuity is irrelevant.
3. Theorem 3.4 applies to *any* continuous in probability stationary α-Fréchet process. Since
$$d_\alpha(X_t, X_s) = \|X_t\|_\alpha^\alpha + \|X_s\|_\alpha^\alpha - \|X_t \vee X_s\|_\alpha^\alpha$$
one can calculate the dependence function $d_\alpha(t - s) = d_\alpha(X_t, X_s)$ without knowing the functions $U_t(f)$ or the isometries U_t explicitly. This feature is important in applications (see, e.g. the doubly stationary process of Brown and Resnick in Section 4 below).

4. Applications

Here, we first give several important examples of ergodic, mixing and non-ergodic Fréchet processes and then, we discuss the estimation of the dependence function d_α.

4.1. Examples: Mixed moving maxima, doubly stochastic processes and random fields

• *Moving maxima and mixed moving maxima*

Theorem 3.4 implies that the moving maxima and more generally, the mixed moving-maxima processes are mixing. Namely, let

$$X_t := {\int^{\mathrm{e}}}_{\mathbb{R}\times E} f(t-u, v) M_\alpha(\mathrm{d}u, \mathrm{d}v), \quad t \in \mathbb{R}, \tag{4.1}$$

where $M_\alpha(\mathrm{d}u, \mathrm{d}v)$ is an α-Fréchet sup-measure with control measure $\mathrm{d}u \times \nu(\mathrm{d}v)$. $X = \{X_t\}_{t\in\mathbb{R}}$ is a strictly stationary α-Fréchet process called mixed moving maxima (see p. 256 in [23]). By the Lebesgue's theorem,

$$\rho_{\alpha,\mathcal{M}}(X_t, X_0) = \int_{\mathbb{R}} |f(t+u, v)^\alpha - f(u, v)^\alpha| \mathrm{d}u\nu(\mathrm{d}v)$$

tends to zero as $t \to 0$. Hence X is continuous in probability (Proposition 2.1) and it has a measurable modification.

Proposition 4.1. *The mixed moving maxima process X in (4.1) is mixing (and hence ergodic).*

Proof. By Theorem 3.4, it is enough to prove that $d_\alpha(X_\tau, X_0) \to 0,\ \tau \to 0$. Observe that, by the Monotone Convergence Theorem,

$$\int_{\mathbb{R}\times E} f_n^\alpha(u,v)\mathrm{d}u\nu(\mathrm{d}v) := \int_{\mathbb{R}\times E} 1_{\{|u|\ge n\}} f^\alpha(u,v)\mathrm{d}u\nu(\mathrm{d}v) \to 0,$$

as $n \to \infty$. Thus, by writing $f^\alpha(u,v) = f_n^\alpha(u,v) + g_n^\alpha(u,v)$, where $g_n^\alpha(u,v) = 1_{\{|u|<n\}} f^\alpha(u,v)$ and using the inequality $(a+b)\wedge(c+d) \le (a\wedge c) + b + d$, valid for all $a,b,c,d \ge 0$, we obtain

$$\begin{aligned} d_\alpha(X_t, X_0) \le& \int_{\mathbb{R}\times E} g_n^\alpha(t+u,v) \wedge g_n^\alpha(u,v)\,\mathrm{d}u\nu(\mathrm{d}v) \\ &+ \int_{\mathbb{R}\times E} f_n^\alpha(t+u,v)\,\mathrm{d}u\nu(\mathrm{d}v) + \int_{\mathbb{R}\times E} f_n^\alpha(u,v)\,\mathrm{d}u\nu(\mathrm{d}v). \end{aligned} \tag{4.2}$$

The last two terms are equal, they do not depend on t, and vanish as $n \to \infty$. Now, fix n to be sufficiently large, and note that the first term in the right-hand side of (4.2) also vanishes, as $t \to \infty$ since the supports of the functions $g_n^\alpha(t+\cdot,\cdot)$ and $g_n^\alpha(\cdot,\cdot)$, are disjoint for $t > 2n$. This shows that $d_\alpha(X_t, X_0) \to 0$, as $t \to \infty$, which completes the proof of the proposition. $\square$

• *Doubly stochastic processes*

Let M_α be an α-Fréchet random sup-measure on the space $(E, \mathcal{E}, \mu)$ with control measure μ. We now suppose that the measure μ is a *probability measure*, and thus $(E, \mathcal{E}, \mu)$ becomes a probability space. Consider the extremal integrals

$$X_t := {\int^{\mathrm{e}}_E} f_t(u) M_\alpha(\mathrm{d}u), \quad t \in \mathbb{R}, \tag{4.3}$$

where $f_t = f_t(u) \ge 0$ and $\mathbb{E}_\mu f_t^\alpha := \int_E f_t^\alpha(u)\mu(\mathrm{d}u) < \infty$, $t \in \mathbb{R}$. One can view the kernels $\{f_t\}_{t\in\mathbb{R}}$ as a stochastic process on the space $(E, \mathcal{E}, \mu)$. Thus, the Fréchet process $X = \{X_t\}_{t\in\mathbb{R}}$ can be viewed as *doubly stochastic*, since it involves the stochastic measure M_α and also the 'random' kernels f_t's. Note, however, that from the perspective of the measure M_α, the kernels f_t are deterministic.

The value of this approach is in the fact that many non-negative stochastic processes can be plugged-in as the kernels f_t to the extremal integrals above. Therefore, insights about the stochastic process $\{f_t\}_{t\in\mathbb{R}}$ can yield results for the corresponding Fréchet process X.

We now discuss an interesting example of a doubly stochastic process introduced by Brown and Resnick [4]. Let $\alpha = 1$ and $w = \{w_t(u)\}_{t\ge 0}$ be the standard Brownian motion defined on the probability space $(E, \mathcal{E}, \mu)$. Set

$$X_t := {\int^{\mathrm{e}}_E} \mathrm{e}^{w_t(u) - t/2} M_1(\mathrm{d}u), \quad t \ge 0. \tag{4.4}$$

The process $X = \{X_t\}_{t\ge 0}$ is a well-defined, (one-sided) strictly stationary and continuous in probability 1-Fréchet process. Since $\mathrm{e}^{w(t)-t/2}$, $t \ge 0$ is an exponential martingale, it is easy to see that the marginal distributions of X are all the same. It is not trivial to show, however, that X is stationary (see e.g. Proposition 4.2 below, [4], or p. 323 of [9]).

All kernels $f_t(u) = \mathrm{e}^{w_t(u)-t/2}$ have common supports and hence the X_t's are dependent in t. Perhaps contrary to intuition however, the process X is ergodic and in fact mixing. This follows from Theorem 4.1 below, which applies to an even more general family of Fréchet processes.

The process in (4.4) of Brown and Resnick can be obtained as a special case of doubly stochastic Fréchet processes driven by a general infinitely divisible Lévy process with a finite Laplace transform. Indeed, let $\Lambda = \{\Lambda_t\}_{t\geq 0}$ be a continuous in probability process with stationary and independent increments, i.e. a Lévy process, defined on the probability space $(E, \mathcal{E}, \mu)$. Suppose that the Laplace transform $\mathbb{E}_\mu \mathrm{e}^{-\alpha\Lambda_t} < \infty$ is finite for some $\alpha > 0$. Thus, by the independence and the stationarity of the increments, we have

$$\mathbb{E}_\mu \mathrm{e}^{-\xi\Lambda_t} = \mathrm{e}^{-t\phi(\xi)} < \infty, \quad \text{for all } \xi \in [0, \alpha], \tag{4.5}$$

where $\phi(\xi)$ is the *Laplace exponent* of the random variable Λ_1. Notice that by the stationarity of increments, we have $\Lambda_0 = 0$, almost surely.

As in the the above example of Brown and Resnick, we let

$$f_t := \mathrm{e}^{-\Lambda_t + t\phi(\alpha)/\alpha}, \quad t \geq 0, \tag{4.6}$$

so that $\mathbb{E}_\mu f_t^\alpha < \infty$. The independence and the stationarity of the increments of Λ, readily imply that $f_t^\alpha,\ t \in \mathbb{R}$ is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma\{\Lambda_s,\ 0 \leq s \leq t\}$. Furthermore, for the doubly stochastic process X with kernels f_t, we have the following result:

Proposition 4.2. *Let $\Lambda = \{\Lambda_t\}_{t\geq 0}$ be the Lévy process in (4.5) and let $f_t, t \geq 0$ be as in (4.6). Then the α-Fréchet process $X = \{X_t\}_{t\geq 0}$ in (4.3) is one-sided strictly stationary and continuous in probability.*

Proof. Since $\mathbb{E}_\mu f_t^\alpha < \infty$, the process X is well-defined. To prove stationarity, let $t_i \geq 0$ and $x_i > 0,\ i = 1, \ldots, n$ be arbitrary. For any $h > 0$, we then have

$$\begin{aligned}\mathbb{P}\{X_{t_i+h} \leq x_i,\ i = 1, \ldots, n\} &= \mathbb{P}\left\{\bigvee_{i=1}^n x_i^{-1} X_{t_i+h} \leq 1\right\} \\ &= \exp\left\{-\int_E \left(\bigvee_{1\leq i\leq n} x_i^{-\alpha} f_{t_i+h}^\alpha\right) \mathrm{d}\mu\right\}.\end{aligned}$$

Thus to establish the stationarity of X, it is enough to show that the integral $\int_E (\vee_{1\leq i\leq n} x_i^{-\alpha} f_{t_i+h}^\alpha)\mathrm{d}\mu$ does not depend on $h > 0$. We have that $\vee_i x_i^{-\alpha} f_{t_i+h}^\alpha$ equals

$$\bigvee_i x_i^{-\alpha} \mathrm{e}^{-\alpha\Lambda_{t_i+h} + (t_i+h)\phi(\alpha)} = \mathrm{e}^{-\alpha\Lambda_h + h\phi(\alpha)} \cdot \bigvee_i x_i^{-\alpha} \mathrm{e}^{-\alpha(\Lambda_{t_i+h} - \Lambda_h) + t_i\phi(\alpha)}. \tag{4.7}$$

Since $\{\Lambda_t, t \geq 0\}$ is a Lévy process, the vector $\{\Lambda_{t_i+h} - \Lambda_h\}_{i=1}^n$ is independent from Λ_h and equals $\{\Lambda_{t_i}\}_{i=1}^n$ in distribution. Thus, by taking expectation $\mathbb{E}_\mu$ in Relation (4.7), we obtain:

$$\begin{aligned}\int_E \bigvee_{1\leq i\leq n} x_i^{-\alpha} f_{t_i+h}^\alpha \mathrm{d}\mu &= \mathbb{E}_\mu \left(\mathrm{e}^{-\alpha\Lambda_h + h\phi(\alpha)}\right) \mathbb{E}_\mu \left(\bigvee_i a_i \mathrm{e}^{-\alpha(\Lambda_{t_i+h} - \Lambda_h) + t_i\phi(\alpha)}\right) \\ &= \int_E \bigvee_{1\leq i\leq n} x_i^{-\alpha} f_{t_i}^\alpha \mathrm{d}\mu.\end{aligned}$$

This completes the proof of the stationarity of $\{X_t, t \geq 0\}$.

We now show that X is continuous in probability. By stationarity, it is enough to prove that $X_t \to^{\mathrm{P}} X_0$, as $t \downarrow 0$. Observe that, for all $t \geq 0$,

$$\rho_{\alpha,\mathcal{M}}(X_t, X_0) = 2\|X_t \vee X_0\|_\alpha^\alpha - \|X_t\|_\alpha^\alpha - \|X_0\|_\alpha^\alpha = \int_E |f_t^\alpha - f_0^\alpha| \mathrm{d}\mu.$$

The last integral equals

$$\mathbb{E}_\mu|\mathrm{e}^{-\alpha\Lambda_t+t\phi(\alpha)}-1| =: \mathbb{E}_\mu|p_t-p_0|,$$

where $p_t \geq 0$ and $p_0 := 1 \geq 0$ can be viewed as probability densities with respect to the measure μ, since $\mathbb{E}_\mu p_t = \mathbb{E}_\mu e^{-\alpha\Lambda_t+t\phi(\alpha)} = 1$.

Now let $t_n \downarrow 0, n \to \infty$. Since $\Lambda_{t_n} \to^{\mathrm{P}} 0, n \to \infty$, there exists a sub–sequence $n_k \to \infty, k \to \infty$, such that $\Lambda_{t_{n_k}} \to 0$, (μ-)almost everywhere, as $k \to \infty$. Thus, $p_{t_{n_k}} \to 1 = p_0$, (μ-) almost everywhere, as $k \to \infty$ and by Sheffé's lemma (see e.g. Appendix II in [3]), we have

$$\mathbb{E}_\mu|\mathrm{e}^{-\alpha\Lambda_{t_{n_k}}+t_{n_k}\phi(\alpha)}-1| = \mathbb{E}_\mu|p_{t_{n_k}}-p_0| \longrightarrow 0, \quad \text{as } k \to \infty.$$

We have thus shown that for any $t_n \downarrow 0,\ n \to \infty$, there exists a sub–sequence $n_k \to \infty$, such that $\rho_{\alpha,\mathcal{M}}(X_{t_{n_k}}, X_0) \to 0$, as $k \to \infty$. This, since $\rho_{\alpha,\mathcal{M}}$ metrizes the convergence in probability (Proposition 2.1), implies that $X_t \to^{\mathrm{P}} X_0$, as $t \downarrow 0$, which completes the proof of the proposition. □

Under some minor additional conditions on the Laplace transform of Λ, one can also show that the process X is *exponentially mixing*:

Theorem 4.1. *Let $\alpha > 0$ and $\mathbb{E}_\mu\mathrm{e}^{-\alpha\Lambda_t} = \mathrm{e}^{-t\phi(\alpha)} < \infty$. If for some $\xi > 0$, $\mathbb{E}_\mu\mathrm{e}^{-\xi\Lambda_t} = \mathrm{e}^{-t\phi(\xi)} < \infty$ and $\phi(\xi)/\xi > \phi(\alpha)/\alpha$, then the process X in (4.5) is mixing. In fact,*

$$d_\alpha(X_\tau, X_0) = \mathbb{E}_\mu(\mathrm{e}^{-\alpha\Lambda_\tau+\tau\phi(\alpha)} \wedge 1) \leq 2\mathrm{e}^{-c\tau}, \tag{4.8}$$

for some $c > 0$ and all $\tau > 0$.

Proof. The process X is continuous in probability (Proposition 4.2) and in view of Theorem 3.4, to prove that X is mixing, it is enough to show that (4.8) holds. Let $\delta > 0$ and observe that for any $p, 0 < p \leq \max\{1, \xi/\alpha\}$,

$$\begin{aligned}\mathbb{E}_\mu(\mathrm{e}^{-\alpha\Lambda_t+t\phi(\alpha)} \wedge 1) &\leq \mu\{\alpha\Lambda_t \leq t(\phi(\alpha)+\delta)\} + \mathbb{E}_\mu\mathrm{e}^{-\alpha\Lambda_t+t\phi(\alpha)}1_{\{\alpha\Lambda_t > t(\phi(\alpha)+\delta)\}}\\ &\leq \mathbb{E}_\mu\mathrm{e}^{(-\alpha\Lambda_t+t(\phi(\alpha)+\delta))p} + \mathrm{e}^{-\delta t} = \mathrm{e}^{-t\phi(\alpha p)+tp(\phi(\alpha)+\delta)} + \mathrm{e}^{-\delta t},\end{aligned} \tag{4.9}$$

where in the last relation we used that $\mathbb{E}_\mu\mathrm{e}^{-\alpha p\Lambda_t} = \mathrm{e}^{-t\phi(\alpha p)} < \infty$, for all $0 < p \leq \max\{1, \xi/\alpha\}$.

Let now $p := \xi/\alpha$, and observe that the first term on the right-hand side of (4.9) equals

$$\mathrm{e}^{-t(\phi(\alpha p)-p\phi(\alpha)-p\delta)} = \mathrm{e}^{-t\xi(\phi(\xi)/\xi-\phi(\alpha)/\alpha-\delta/\alpha)}.$$

Since $\phi(\xi)/\xi - \phi(\alpha)/\alpha > 0$ by choosing $\delta > 0$ sufficiently small, one can make the argument of the last exponent negative. This, in view of (4.9) implies (4.8) and completes the proof of the theorem. □

The conditions in the last result are easy to verify for the process of Brown and Resnick. Indeed, since $\mathbb{E}_\mu\mathrm{e}^{-\xi w_t} = \mathrm{e}^{t\xi^2/2} < \infty$, for all $t \geq 0$ and $\xi \in \mathbb{R}$. We thus have $\phi(\xi)/\xi = -\xi/2$, which is a monotone decreasing function and hence the condition $\phi(\xi)/\xi > \phi(\alpha)/\alpha$ of Theorem 4.1 holds, for any $\xi < \alpha$. Moreover, the conditions of Theorem 4.1 always hold when the process Λ_t is *non-negative*, that is, when $\Lambda = \{\Lambda_t\}_{t\geq 0}$ is a *Lévy subordinator*.

Corollary 4.1. *Let $\Lambda := \{\Lambda_t,\ t \geq 0\}$ be an arbitrary Lévy subordinator. Then, for any $\alpha > 0$ the one-sided stationary α-Fréchet process X in (4.5) is well-defined and exponentially mixing in the sense of (4.8).*

Proof. Observe that $\mathbb{E}e^{-\xi\Lambda_t} = e^{-t\phi(\xi)} < \infty$ for all $\xi \ge 0$, since $\Lambda_t \ge 0$, almost surely. Thus, it suffices to show that, for any $\alpha > 0$, there exists $\xi > 0$, such that

$$\phi(\xi)/\xi > \phi(\alpha)/\alpha. \tag{4.10}$$

The Laplace transform of the Lévy subordinator Λ_t, however, can be conveniently expressed through its corresponding Lévy measure Π. By Ch. III.1, page 72 in [2], we have

$$\phi(\xi)/\xi = d + \int_0^\infty e^{-\xi t}\Pi(t,\infty)dt, \tag{4.11}$$

where $\Pi(t,\infty)$ denotes the *tail* of the Lévy measure Π and where $d \in \mathbb{R}$ stands for the drift of the subordinator $\{\Lambda_t, t \ge 0\}$. Relation (4.11) readily implies that the function $\phi(\xi)/\xi$ is strictly monotone decreasing in ξ and hence any $\xi < \alpha$ satisfies (4.10). This completes the proof of the corollary. □

• *Random fields*

Theorem 3.2 can be readily applied to random fields. Namely, let $d \in \mathbb{N},\ d > 1$ and define

$$X_t = X_{t_1,\dots,t_d} = \int_E U_t(f)M_\alpha(du), \quad t \in \mathbb{R}^d, \tag{4.12}$$

where $M_\alpha, \alpha > 0$ is an α-Fréchet sup-measure with control measure μ defined on the measure space $(E,\mathcal{E},\mu)$ and where $f \in L_+^\alpha(\mu)$. Here the $U_t : L_+^\alpha(\mu) \to L_+^\alpha(\mu)$ are max-linear isometries, which form a group parameterized by $t \in \mathbb{R}^d$ with respect to the composition: $U_{t+s} \equiv U_t \circ U_s$, with $U_{\vec{0}} \equiv \text{id}$.

It can be shown that the α-Fréchet field $\{X_t\}_{t\in\mathbb{R}^d}$ is strictly stationary. Its ergodicity properties along any direction $\theta \in \mathbb{R}^d \setminus \{\vec{0}\}$ can be determined by using Theorem 3.2. It may turn out that the field X_t *is ergodic* along some directions and *non-ergodic* along others.

For example, let $d = 2$ and consider the random field

$$X_{t_1,t_2} := \int_{\mathbb{R}\times[0,2\pi]} f(t_1-u)\sin^2(t_2-v)M_\alpha(du,dv), \tag{4.13}$$

where $f(u) \in L_+^\alpha(du)$ and $M_\alpha(du,dv)$ is defined on $\mathbb{R}\times[0,2\pi]$, with the Lebesgue control measure $du\,dv$.

The field X_t is stationary and it has the representation (4.12). It is a mixed moving maxima process along the direction $\theta = (1\ 0)' \in \mathbb{R}^d$, and hence it is mixing and in particular ergodic (Proposition 4.1). Along the direction $\theta = (0\ 1)'$, however, X_t is non-ergodic. Indeed, let

$$Y_\tau := X_{0,\tau} = \int_{\mathbb{R}\times[0,2\pi]} f(-u)\sin^2(\tau-v)M_\alpha(du,dv), \quad \tau \in \mathbb{R},$$

and observe that

$$d_\alpha(Y_\tau, Y_0) = \int_{\mathbb{R}} f^\alpha(u)du \int_0^{2\pi} |\sin(\tau-v)|^{2\alpha} \wedge |\sin(-v)|^{2\alpha}dv.$$

The periodicity of the sine function implies that the function $d_\alpha(\tau) = d_\alpha(Y_\tau, Y_0)$ is 2π-periodic. It is positive and therefore its Cesaro limit $T^{-1}\int_0^T d_\alpha(\tau)d\tau$ cannot be zero, as $T \to \infty$. This, in view of Theorem 3.2, implies that $\{Y_\tau\}_{\tau\in\mathbb{R}}$ is non-ergodic.

It is interesting to note that the *non-ergodicity* of X_{t_1,t_2} in (4.13) is an *unstable* property. That is, the field X is ergodic along any direction other than $(0\ 1)'$. Indeed

Proposition 4.3. *The process $Y_\tau := X_{\tau\theta_1,\tau\theta_2}$, $\tau \in \mathbb{R}$ is mixing (and hence ergodic), for all $\theta_1 \neq 0$, $\theta_2 \in \mathbb{R}$.*

Proof. For the dependence function of $Y = \{Y_\tau\}_{\tau\in\mathbb{R}}$, we have

$$d_\alpha(\tau) = d_\alpha(Y_\tau, Y_0) = \int_{\mathbb{R}} \int_0^{2\pi} f^\alpha(\tau\theta_1 - u)|\sin(\tau\theta_1 - v)|^{2\alpha} \wedge f^\alpha(-u)|\sin(-v)|^{2\alpha} \mathrm{d}u\mathrm{d}v.$$

Note that $(a_1b_1) \wedge (a_2b_2) \le (a_1 \wedge a_2)(b_1 \vee b_2)$, for all $a_i, b_i \ge 0, i = 1, 2$. Thus, with $a_1 := f^\alpha(\tau\theta_1 - u)$, $a_2 := f^\alpha(-u)$ and $b_1 := |\sin(\tau\theta_1 - v)|^{2\alpha}$, $b_2 := |\sin(-v)|^{2\alpha}$, we obtain:

$$\begin{aligned} d_\alpha(\tau) &\le \int_{\mathbb{R}} f^\alpha(\tau\theta_1 - u) \wedge f^\alpha(-u)\mathrm{d}u \int_0^{2\pi} |\sin(\tau\theta_1 - v)|^{2\alpha} \vee |\sin(-v)|^{2\alpha}\mathrm{d}v \\ &\le 2\pi \int_{\mathbb{R}} f^\alpha(\tau\theta_1 + u) \wedge f^\alpha(u)\mathrm{d}u. \end{aligned}$$

As in the proof of Proposition 4.1, the last integral can be shown to vanish, as $\tau \to \infty$, for all $\theta_1 \neq 0$. This, in view of Theorem 3.4, implies that Y is mixing. □

By using the extremal stochastic integrals of suitably chosen kernels, one can construct many other interesting examples of random fields. These fields can be chosen to be ergodic or non-ergodic in various directions, not necessarily along the standard coordinates in $\mathbb{R}^d$.

4.2. On the estimation of the dependence function

Our goal here is to estimate the dependence function

$$d_\alpha(\tau) := d_\alpha(X_\tau, X_0) = \int_E f_\tau^\alpha \wedge f_0^\alpha \mathrm{d}\mu \tag{4.14}$$

of a stationary and ergodic α-Fréchet process X as in (3.1). The dependence function $d_\alpha(\tau)$ of the Fréchet process X can be viewed as the counterpart of the auto-covariance function for Gaussian processes. Therefore its estimation is of practical interest.

Observe first that for an α-Fréchet variable X, we have $\mathbb{E}X^p < \infty$, for all $p \in (0, \alpha)$. More precisely,

$$\mathbb{E}X^p = \|X\|_\alpha^p \int_0^\infty x^p \mathrm{d}e^{-x^{-\alpha}} = \Gamma(1 - p/\alpha)\|X\|_\alpha^p,$$

where $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\mathrm{d}x$, $a > 0$ is the Gamma function.

If X is ergodic, then the moments $\mathbb{E}X_0^p$ and $\mathbb{E}(X_\tau \vee X_0)^p$ can be estimated consistently by their sample counterparts. Thus, for a given sample X_k, $k = 1, \ldots, n$, we define

$$\widehat{d}_{\alpha,p,n}(\tau) := 2c_{p,\alpha}\left(\frac{1}{n}\sum_{k=1}^n X_k^p\right)^{\alpha/p} - c_{p,\alpha}\left(\frac{1}{n-\tau}\sum_{k=1}^{n-\tau} X_{k+\tau}^p \vee X_k^p\right)^{\alpha/p}, \quad \tau \in \mathbb{N}, \tag{4.15}$$

where $c_{p,\alpha} := \Gamma(1 - p/\alpha)^{-\alpha/p}$. This discussion and Birkhoff's ergodicity theorem imply the following result.

Proposition 4.4. *Let X_k, $k = 1, \ldots, n$ be a sample from a stationary ergodic α-Fréchet process. Then, for all $p \in (0, \alpha)$ and $\tau \in \mathbb{N}$, we have*

$$\widehat{d}_{\alpha,p,n}(\tau) \xrightarrow{a.s.} d_\alpha(\tau), \quad \text{as } n \to \infty, \tag{4.16}$$

where $\widehat{d}_{\alpha,p,n}(\tau)$ and $d_\alpha(\tau)$ are as in (4.14) and (4.15), respectively. In addition, for all $\gamma \in (0,1)$, we have

$$\mathbb{E}|\widehat{d}_{\alpha,p,n}(\tau) - d_\alpha(\tau)|^\gamma \longrightarrow 0, \quad \textit{as } n \to 0. \tag{4.17}$$

Proof. Birkhoff's ergodicity theorem implies the convergence in (4.16) (since X is ergodic). To show (4.17), observe that for all $\gamma \in (0,1)$,

$$|\widehat{d}_{\alpha,p,n}(\tau) - d_\alpha(\tau)|^\gamma \le |\widehat{d}_{\alpha,p,n}(\tau)|^\gamma + |d_\alpha(\tau)|^\gamma.$$

Thus, if the random variables $\{|\widehat{d}_{\alpha,p,n}(\tau)|^\gamma, n \in \mathbb{N}\}$ are uniformly integrable, then so are $\{|\widehat{d}_{\alpha,p,n}(\tau) - d_\alpha(\tau)|^\gamma,\ n \in \mathbb{N}\}$. Since uniform integrability and convergence in probability are equivalent to a convergence in mean, by Relation (4.16), it is enough to establish the uniform integrability of $\{|\widehat{d}_{\alpha,p,n}(\tau)|^\gamma, n \in \mathbb{N}\}$. Since $|a-b|^\delta \le |a|^\delta + |b|^\delta$, for all $a, b \in \mathbb{R}$, $\delta \in (0,1)$, by (4.15), we have

$$\begin{aligned}\mathbb{E}|\widehat{d}_{\alpha,p,n}(\tau)|^\delta &\le 2^\delta c_{p,\alpha}^\delta \mathbb{E}\left(\frac{1}{n}\sum_{k=1}^n X_k^p\right)^{\delta\alpha/p} + c_{p,\alpha}^\delta \mathbb{E}\left(\frac{1}{n-\tau}\sum_{k=1}^{n-\tau} X_{k+\tau}^p \vee X_k^p\right)^{\delta\alpha/p}\\ &=: 2^\delta c_{p,\alpha}^\delta \mathbb{E}A_n^{\delta\alpha/p} + c_{p,\alpha}^\delta \mathbb{E}B_n^{\delta\alpha/p}.\end{aligned} \tag{4.18}$$

Let now $\gamma < \delta < 1$, be such that also $\delta > p/\alpha$. By applying the Minkowski inequality with $q := \delta\alpha/p \ge 1$ to the term A_n in the last relation, we obtain:

$$(\mathbb{E}A_n^q)^{1/q} = \left(\mathbb{E}\left|\frac{1}{n}\sum_{k=1}^n X_k^p\right|^q\right)^{1/q} \le \frac{1}{n}\sum_{k=1}^n (\mathbb{E}X_k^{pq})^{1/q} = (\mathbb{E}X_1^{\delta\alpha})^{1/q} < \infty.$$

In the last relation, we used the stationarity of the X_k's and the fact that $\mathbb{E}X_1^{\delta\alpha} < \infty, \forall\delta \in (0,1)$. Similarly, we bound above the term $(\mathbb{E}B_n^q)^{1/q}$ in (4.18) and obtain:

$$\mathbb{E}|\widehat{d}_{\alpha,p,n}(\tau)|^\delta \le C < \infty, \quad \text{uniformly in } n \in \mathbb{N},$$

for some $\delta > \gamma$ and $C > 0$. This implies the uniform integrability of $|\widehat{d}_{\alpha,p,n}(\tau)|^\gamma$ in $n \in \mathbb{N}$ and completes the proof of the proposition. □

Remarks.

1. The convergence in (4.17) is not valid for $\gamma \ge 1$, since the $\widehat{d}_{\alpha,p,n}(\tau)$'s have infinite means. When $\gamma \in (0,1)$, Relation (4.17) provides additional insight to the convergence of the estimators $\widehat{d}_{\alpha,p,n}(\tau)$.
2. The moving maxima and mixed moving maxima processes are mixing (Proposition 4.1). Thus, by Proposition 4.4, the estimator $\widehat{d}_{\alpha,p,n}(\tau)$ of the dependence function $d_\alpha(\tau)$ is strongly consistent, for *all* mixed moving maxima, and for the moving maxima in particular.

Fig. 1 illustrates the performance of the estimator $\widehat{d}_{\alpha,p,n}(\tau)$ for one moving maxima time series. Observe that the estimate $\widehat{d}_{\alpha,p,n}(\tau)$ tracks relatively well the theoretical value of $d_\alpha(\tau)$. We chose a 'non-standard' randomly generated and irregular spectral function f. Our experiments (not shown here) indicate that the estimator $\widehat{d}_{\alpha,p,n}(\tau)$ continues to perform well for many other choices of the spectral function f, and it can be successfully used in practice.

Another important problem is to recover the kernel (i.e. the spectral function) f of the moving maxima process $X_t := {\int\!\!\!\!\!e}_{\mathbb{R}} f(t-u)M_\alpha(\mathrm{d}u)$ from data. This cannot be always done by using the dependence function d_α. When f is one-sided and monotone, however, the relationship between d_α and f is simple.

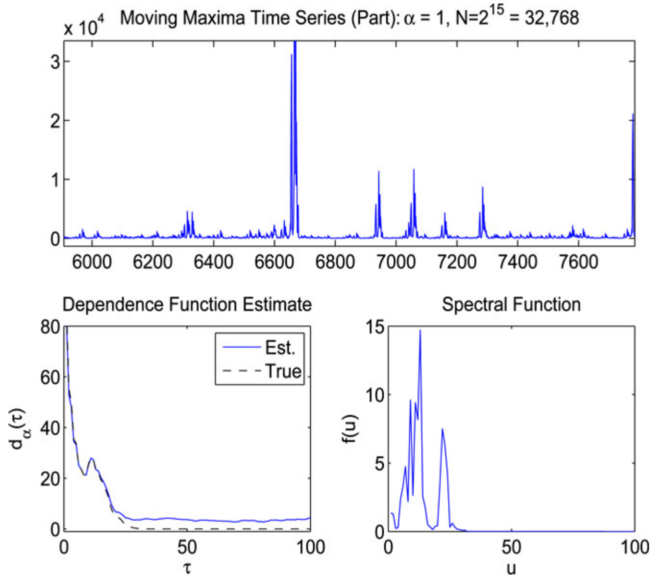

Fig. 1. *Top panel:* Part of a moving maxima α-Fréchet ($\alpha = 1$) time series of length $n = 2^{15}$ with spectral function f given in the *bottom right panel*. *Bottom left panel:* The estimate $\widehat{d}_{\alpha,p,n}(\tau)$ with $p = 1/4$ (solid line) and the true dependence function $d_\alpha(\tau)$ (broken line). Observe the close agreement between the estimate and the true function $d_\alpha(\tau)$ for a wide range of lags $0 \le \tau \le 20$.

Proposition 4.5. *Let* $f(u) = 0, u < 0$, $f(u_1) \ge f(u_2), \forall 0 \le u_1 \le u_2$, *and* $\int_{\mathbb{R}} f^\alpha(u)\mathrm{d}u < \infty$. *Then, for* $\tau > 0$,

$$d_\alpha(\tau) = \int_\tau^\infty f(u)^\alpha \mathrm{d}u \quad \textit{and hence} \quad f(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau} d_\alpha(\tau).$$

Proof. The result follows from the fact that for all $\tau > 0$, $d_\alpha(\tau)$ equals:

$$\int_{\mathbb{R}} f(\tau-u)^\alpha \wedge f(-u)^\alpha \mathrm{d}u = \int_{-\infty}^0 f(\tau-u)^\alpha \wedge f(-u)^\alpha \mathrm{d}u$$
$$= \int_{-\infty}^0 f(\tau-u)^\alpha \mathrm{d}u = \int_\tau^\infty f(v)^\alpha \mathrm{d}v.$$

This is because $f(\tau-u)^\alpha \wedge f(-u)^\alpha = f(\tau-u)^\alpha 1_{\{u\le 0\}}$, for all $\tau > 0$, by monotonicity. □

This result suggests an estimate of monotone, one-sided f's based on finite differences of $\widehat{d}_{\alpha,p,n}(\tau)$. In general, however, the function f is hard to identify from the dependence function d_α. One possible approach is to use M-estimation. That is, one can choose f from a parametric family of functions $f(x) = f(x;\theta)$ by minimizing the integrated error

$$\int_0^T |\widehat{d}_{\alpha,p,n}(\tau) - d_\alpha(\tau;\theta)|^p \mathrm{d}\tau,$$

for some $p > 0$, where $d_\alpha(\tau;\theta) = \int_{\mathbb{R}} |f(\tau-u;\theta) \wedge f(-u;\theta)|^\alpha \mathrm{d}u$. This, however, leads to non-regular and computationally demanding optimization problems which are beyond the scope of this paper.

5. On the mixing conditions of Weintraub

Weintraub [26] proposed several mixing conditions based on the de Haan's spectral representation of min-stable processes. They apply, by duality, to the max-stable setting. Although these conditions are quite natural, their connection to the ergodic and mixing properties of the processes was not shown. In this section, we fill this gap by proving in particular that the '0-mixing' stationary processes in the sense of Weintraub are necessarily mixing and hence ergodic.

Let $X = \{X_t\}_{t\in\mathbb{R}}$ be a strictly stationary 1-Fréchet process with the de Haan spectral representation:

$$X_t = \bigvee_{j\in\mathbb{N}} \frac{f_t(U_j)}{Y_j}, \quad t \in \mathbb{R},$$

where $\{(U_j, Y_j)\}_{j\in\mathbb{N}}$ is a homogeneous Poisson process on $\mathbb{R}\times\mathbb{R}_+$ with unit intensity, and where $f_t(u) \in L^1_+(\mathbb{R}, \mathrm{d}u)$, $t \in \mathbb{R}$. Then, by Proposition 3.1 in [23], one also has

$$\{X_t\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ {\int_{\mathbb{R}}}^{\mathrm{e}} f_t(u) M_1(\mathrm{d}u) \right\}, \tag{5.1}$$

where $M_1(\mathrm{d}u)$ is 1-Fréchet sup-measure on $\mathbb{R}$ with the Lebesgue control measure.

Weintraub [26] (Section 3 therein) deals with the min-stable process $\widetilde{X}_t := 1/X_t, t \in \mathbb{R}$, which has Exponential marginal distributions, and proposes mixing conditions based on the following measure of dependence:

$$q(\widetilde{X}_t, \widetilde{X}_s) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{f_t(v)}{f_t(u)} \vee \frac{f_s(v)}{f_s(u)} \mathrm{d}v \right)^{-1} \mathrm{d}u. \tag{5.2}$$

The process $\widetilde{X}$ is deemed to be 0-*mixing*, if

$$q(\widetilde{X}_t, \widetilde{X}_{t+s}) = q(\widetilde{X}_0, \widetilde{X}_s) \longrightarrow 0, \quad \text{as } s \to \infty. \tag{5.3}$$

We will also say that X is 0-mixing if its corresponding min-stable process $\widetilde{X}$ is 0-mixing. Notice that $\widetilde{X} = \{\widetilde{X}_t\}_{t\in\mathbb{R}}$ is ergodic/mixing if and only if X is ergodic/mixing.

The following result relates Weintraub's measure of dependence q in (5.2) to the measure of dependence d_α in Theorem 3.4.

Proposition 5.1. *Let X be as in (5.1). Then, for all $t, s \in \mathbb{R}$, we have*

$$\frac{1}{2} d_1(X_t/\sigma_t, X_s/\sigma_s) \le q(\widetilde{X}_t, \widetilde{X}_s) \le 2 d_1(X_t/\sigma_t, X_s/\sigma_s), \tag{5.4}$$

where $\sigma_\tau = \|X_\tau\|_1,\ \tau \in \mathbb{R}$.

Proof. Since X is stationary, without loss of generality we suppose that $\|X_t\|_1 = \|X_s\|_1 = \int_{\mathbb{R}} f_0(u)\mathrm{d}u = 1$. By using the inequality $(a \vee b)^{-1} \le 2/(a + b)$, we obtain

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \frac{f_t(v)}{f_t(u)} \vee \frac{f_s(v)}{f_s(u)} \mathrm{d}v \right)^{-1} &\le 2 \left(\frac{1}{f_t(u)} \int_{\mathbb{R}} f_t(v)\mathrm{d}v + \frac{1}{f_s(u)} \int_{\mathbb{R}} f_s(v)\mathrm{d}v \right)^{-1} \\ &= \frac{2 f_t(u) f_s(u)}{f_t(u) + f_s(u)}. \end{aligned}$$

Similarly, since $(a+b)^{-1} \le (a \vee b)^{-1}$, we get

$$\frac{f_t(u) f_s(u)}{f_t(u) + f_s(u)} \le \left(\int_{-\infty}^{\infty} \frac{f_t(v)}{f_t(u)} \vee \frac{f_s(v)}{f_s(u)} \mathrm{d}v \right)^{-1} \le \frac{2 f_t(u) f_s(u)}{f_t(u) + f_s(u)}. \tag{5.5}$$

Now, by applying the inequality

$$\frac{x \wedge y}{2} \le \frac{xy}{x+y} \le x \wedge y, \quad \text{with } x := f_t(u) \quad \text{and } y := f_s(u),$$

to (5.5) and integrating, we obtain

$$\frac{1}{2} \int_{\mathbb{R}} f_t(u) \wedge f_s(u) \mathrm{d}u \le q(\widetilde{X}_t, \widetilde{X}_s) \le 2 \int_{\mathbb{R}} f_t(u) \wedge f_s(u) \mathrm{d}u,$$

which is (5.4). □

Proposition 5.1 and Theorem 3.4 imply that the process X in (5.1) is 0-mixing (in the sense of Weintraub) if and only if it is mixing. It provides a more complete picture of the intuition behind the mixing conditions of Weintraub, which relate the degree of dependence of the process X to its Poisson point process representation.

Appendix A

In the proof of Proposition 2.3 below, we use the following elementary result.

Lemma A.1. *For any two functions f and $g : E \to \mathbb{R}$, we have*

$$|f(x) - g(x)| = |f_+(x) - g_+(x)| + |f_-(x) - g_-(x)|, \quad \textit{for all } x \in E,$$

where $f_\pm(x) = \max\{\pm f(x), 0\}$.

Proof. Consider the following cases: If $f(x) > 0$, then $f_+(x) = f(x)$, $f_-(x) = 0$ and

$$|f_+(x) - g_+(x)| + |f_-(x) - g_-(x)| = |f(x) - g_+(x)| + |g_-(x)|.$$

If $g(x) < 0$, then the last equals $|f(x)| + |g(x)| = |f(x) - g(x)|$. Otherwise, if $g(x) \ge 0$, we have $|f(x) - g_+(x)| + |g_-(x)| = |f(x) - g(x)|$ since $g_-(x) = 0$ and $g(x) = g_+(x)$. The other cases can be treated similarly. □

Proof of Proposition 2.3. Let first G be a *max-linear* isometry. Define $\widetilde{G}$, such that $\widetilde{G}(a 1_A) := G(a 1_A) = a G(1_A)$, for any $A \in \mathcal{E}$ and $a \ge 0$ and such that for any simple function $f(x) = \sum_{j=1}^n a_j 1_{A_j}(x) \in L_+^\alpha(\mu)$ with disjoint A_j's,

$$\widetilde{G}(f) := \sum_{j=1}^n a_j \widetilde{G}(1_{A_j}).$$

That is, $\widetilde{G}$ is defined first on the set of indicator functions, and then extended by linearity.

We will now show that $\widetilde{G}(f) = G(f)$, for all simple functions f. Indeed, for *disjoint* A and B in $\mathcal{E}$, by the isometry of G, we get

$$\begin{aligned}
&\int_F |G(1_A)^\alpha(u) - G(1_B)^\alpha(u)| \nu(\mathrm{d}u) \\
&\quad = 2 \int_F G(1_A)^\alpha(u) \vee G(1_B)^\alpha(u) \nu(\mathrm{d}u) - \int_F G(1_A)^\alpha(u) \nu(\mathrm{d}u) - \int_F G(1_B)^\alpha(u) \nu(\mathrm{d}u)
\end{aligned}$$

$$= 2\int_E 1_A(v) \vee 1_B(v)\mu(\mathrm{d}v) - \int_E 1_A(v)\mu(dv) - \int_E 1_B(v)\mu(\mathrm{d}v)$$
$$= \int_E |1_A(v) - 1_B(v)|\mu(\mathrm{d}v) = \mu(A) + \mu(B).$$

Thus, by setting $g(u) := G(1_A)^\alpha(u)$ and $h(u) = G(1_B)^\alpha(u)$, we obtain

$$\int_F |g(u) - h(u)|\nu(\mathrm{d}u) = \mu(A) + \mu(B) = \int_F g(u)\nu(\mathrm{d}u) + \int_F h(u)\nu(\mathrm{d}u), \tag{A.1}$$

since $\int_F G(1_A)^\alpha(u)\nu(\mathrm{d}u) = \mu(A)$, and $\int_F G(1_B)^\alpha(u)\nu(\mathrm{d}u) = \mu(B)$. Since $g(u), h(u) \geq 0$, ν-almost everywhere, Relation (A.1) is valid if and only if $g(u)h(u) = 0$ ν-almost everywhere.

We have thus shown that the functions $g(u) = G(1_A)^\alpha(u)$ and $h(u) = G(1_B)^\alpha(u)$ have disjoint supports, for any two disjoint A and B in $\mathcal{E}$. Since the max-linear combinations of non-negative functions with disjoint supports coincide with their linear combinations, we have

$$\widetilde{G}(f) = G(f), \quad \nu\text{-}a.e. \quad \text{for all simple functions } f \in L_+^\alpha(\mu).$$

Thus G is a linear operator on the set of simple functions in $(L_+^\alpha(\mu), \rho_{\alpha,\mu})$, which are dense in $(L_+^\alpha(\mu), \rho_{\alpha,\mu})$. Now, the fact that the operation addition is continuous in the metric $\rho_{\alpha,\mu}$ (see Lemma 2.3 in [23]) implies that the operator $G : L_+^\alpha(\mu) \to L_+^\alpha(\nu)$ is linear. It is an isometry since it preserves the metrics $\rho_{\alpha,\mu}$ and $\rho_{\alpha,\nu}$.

Conversely, if $G : L_+^\alpha(\mu) \to L_+^\alpha(\nu)$ is a *linear* isometry of metric spaces, by using a similar argument, one can show that $G(1_A)$ and $G(1_B)$ have disjoint supports ν-a.e. if $\mu(A \cap B) = 0$. One can therefore construct a max-linear operator $\widetilde{G}$ which coincides with G on the set of simple functions in $L_+^\alpha(\mu)$. Using the continuity of the max operation with respect to $\rho_{\alpha,\mu}$ and the density of the simple functions implies that G is also a max-linear isometry.

We now prove the second statement. For any $f \in L^\alpha(\mu)$, define $G(f) := G(f_+) - G(f_-)$. This implies that $G(af) = aG(f)$, for any $a \in \mathbb{R}$ and $f \in L^\alpha(\mu)$. Also, for any two simple functions f and g, it is easy to show that $G(f+g) = G(f) + G(g)$. The simple functions are dense in $(L^\alpha(\mu), \widetilde{\rho}_{\alpha,\mu})$, and thus the continuity of the addition implies the linearity of G. The isometry property of G with respect to $\widetilde{\rho}_{\alpha,\mu}$ follows from the isometry property of G with respect to $\rho_{\alpha,\mu}$ and $\rho_{\alpha,\nu}$.

To show (2.7), note that the metric $\widetilde{\rho}_{\alpha,\mu}$ on $L^\alpha(\mu)$ is defined as:

$$\widetilde{\rho}_{\alpha,\mu}(f, g) := \int_E |f_+^\alpha - g_+^\alpha|\mathrm{d}\mu + \int_E |f_-^\alpha - g_-^\alpha|\mathrm{d}\mu.$$

Thus, Lemma A.1, applied to the functions $f^{\langle\alpha\rangle}$ and $g^{\langle\alpha\rangle}$, implies (2.7). □

In the next result we collect some elementary facts about Cesaro convergence for bounded positive functions, as used in Section 3.2.

Lemma A.2. *Let $f : [0, \infty) \to [0, \infty)$ be a non-negative, measurable and bounded function:* $\sup_{\tau\geq 0} f(\tau) \leq M < \infty$.

(i) *If for some $p > 0$, $T^{-1}\int_0^T f(\tau)^p \mathrm{d}\tau \to 0$, as $T \to \infty$, then, for all $q > 0$, $T^{-1}\int_0^T f(\tau)^q \mathrm{d}\tau \to 0$, as $T \to \infty$.*

(ii) *If $f(\tau) \to 0$, as $\tau \to \infty$, then $T^{-1}\int_0^T f(\tau)^p \mathrm{d}\tau \to 0$, as $T \to \infty$, for some (any) $p > 0$. The converse is not always true.*

Proof. (i): If $q < p$, the result follows from the Hölder inequality. Indeed,

$$\int_0^T f(\tau)^q \mathrm{d}\tau \le \left(\int_0^T f(\tau)^p \mathrm{d}\tau\right)^{q/p} \left(\int_0^T 1\mathrm{d}\tau\right)^{1-q/p},$$

and hence $T^{-1}\int_0^T f(\tau)^q \mathrm{d}\tau \le (T^{-1}\int_0^T f(\tau)^p \mathrm{d}\tau)^{q/p} \to 0$, as $T \to \infty$.

If $q > p$, let $A := \{x \ge 0 : f(x) > 1\}$ and observe that $f(\tau)^q \le M^q f(\tau)^p, \tau \in A$ and $f(\tau)^q \le f(\tau)^p, \tau \notin A$. Thus,

$$\frac{1}{T}\int_0^T f(\tau)^q \mathrm{d}\tau \le \frac{M^q}{T}\int_{[0,T]\cap A} f(\tau)^p \mathrm{d}\tau + \frac{1}{T}\int_{[0,T]\setminus A} f(\tau)^p \mathrm{d}\tau,$$

which vanishes as $T \to \infty$.

(ii): The first statement is obvious. For the second, let $f(\tau) = \sum_{n=1}^{\infty} 1_{[n,n+1/n)}(\tau)$. Observe that

$$T^{-1}\int_0^T f(\tau)\mathrm{d}\tau \le T^{-1}\sum_{1\le n\le T+1} n^{-1} \le T^{-1}\log(T+1) \to 0, \quad T \to \infty,$$

whereas $f(\tau) \not\to 0, \tau \to \infty$. □

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