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Occupation time theorems for one-dimensional random walks and diffusion processes in random environments

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Abstract

The long time asymptotics of the time spent on the positive side are discussed for one-dimensional diffusion processes in random environments. The limiting distributions under the log–log scale are obtained for the diffusion processes in the stable medium as well as for the Brox model. Similar problems are discussed for random walks in random environments and it is proved that the limiting laws are the same as in the case of diffusions.

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1. Introduction

In this paper, we are concerned with the long time asymptotics of the occupation times on the positive side

$$A(n) = \sum_{i=1}^{n} 1_{\{Z(i-1)\geq 0, Z(i)\geq 0\}}, \quad n = 1, 2, \ldots \quad \text{and}$$

$$A(t) = \int_0^t 1_{[0,\infty)}(X(s))\mathrm{d}s, \quad t \geq 0 \tag{1}$$

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for a class of one-dimensional random walks $\mathbf{Z} = (Z(n))$ on $\mathbb{Z}$ and diffusion processes $\mathbf{X} = (X(t))$ on $\mathbb{R}$, respectively, in random environments. For a better understanding of the problem and results, we would review some recent studies in the case of usual generalized diffusions on the real line.

Let $X = (X(t), P_x)$ be a one-dimensional (generalized) diffusion process on $\mathbb{R}$ with the Feller generator $\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$. Here, $s : x \in \mathbb{R} \mapsto s(x) \in \mathbb{R}$ is a strictly increasing continuous function such that $s(0) = 0$ and $\lim_{x\to\pm\infty} s(x) = \pm\infty$ and $\mathrm{d}m(x)$ is a non-zero positive Radon measure on $\mathbb{R}$. The state space of X is the support of the measure $\mathrm{d}m$ so that X is a birth-and-death process when $\mathrm{d}m$ is supported on the one-dimensional lattice $\mathbb{Z}$. Note that X is always conservative (i.e., $P(X(t) \in \mathbb{R}) = 1$) and recurrent. For the occupation time (1), the class of possible limit random variables in the law of $A(t)/t$ as $t \to \infty$ coincides with the class of Lamperti random variables $\{Y_{p,\alpha}\}_{0\le p\le 1,\, 0\le\alpha\le 1}$: $Y_{p,\alpha}$ is a [0, 1]-valued random variable with the Stieltjes transform

$$E\left(\frac{1}{\lambda + Y_{p,\alpha}}\right) = \frac{p(\lambda+1)^{\alpha-1} + (1-p)\lambda^{\alpha-1}}{p(\lambda+1)^{\alpha} + (1-p)\lambda^{\alpha}}, \qquad \lambda > 0.$$

Also, a sufficient condition, which turns out to be necessary and sufficient when $0 < p < 1$, can be given in terms of $s(x)$ and $\mathrm{d}m(x)$ for the convergence

$$\frac{1}{t}A(t) \xrightarrow{d} Y_{p,\alpha} \quad \text{as } t \to \infty \tag{2}$$

to hold (cf. [15]). Note that $Y_{1/2,1/2}$ is arc-sine distributed so that these limit laws may be regarded as a generalization of the arc-sine law.

When $\alpha = 1$, $Y_{p,1}$ is a constant random variable; $P(Y_{p,1} = p) = 1$ so that (2) is a law of large numbers. We have studied in [10] some improvements such as behaviors of fluctuations $\frac{1}{t}A(t) - p$ for these laws of large numbers.

When $\alpha = 0$, the limit random variable $Y_{p,0}$ is a Bernoulli random variable; $P(Y_{p,0} = 1) = 1 - P(Y_{p,0} = 0) = p$. The convergence (2) in this case occurs when X is recurrent but is *nearly transient*, or in other words, is *apt to localize*. We have studied in [9] its improvement in the log–log scale: For example, for the diffusion $X = (X(t), P_x)$ with generator

$$\mathcal{L} = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x}{1+x^2}\frac{\mathrm{d}}{\mathrm{d}x}\right) \quad \text{on } \mathbb{R},$$

the asymptotic (2) holds with the limit random variable $Y_{\frac{1}{2},0}$ (under P_x for any but fixed x) and it can be improved as

$$\frac{1}{\log t}\log A(t) \xrightarrow{d} U, \quad \text{as } t \to \infty, \tag{3}$$

where U is a [0, 1]-valued random variable with $P(U < 1) = P(U = 1) = 1/2$; actually, in this case, the law of U is given by

$$\frac{1}{(1+x)^2}1_{\{0\le x<1\}}\mathrm{d}x + \frac{1}{2}\,\delta_{\{1\}}(\mathrm{d}x), \tag{4}$$

(cf. [9], Example 1).

It is in this case of $\alpha = 0$ that we can see some similarity with a diffusion process in a random environment because of its localization character. In this model, we cannot expect the convergence (2) to a Bernoulli random variable if the environment is frozen; i.e., in the *quenched* model. However, for the *annealed* model in which we average on the environments, the convergence (2) to a Bernoulli random variable actually takes place. We can further obtain its improvement in the log–log scale. For example, in the Brox model (cf. [5]) which is given as $\mathcal{L}^w = \frac{1}{2}\mathrm{e}^{w(x)}\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-w(x)}\frac{\mathrm{d}}{\mathrm{d}x})$-diffusion $X^w = (X^w(t), P_x^w)$ on $\mathbb{R}$ for a given Brownian path $w(x)$, we have that, under the annealed probability $\overline{P}_0(\cdot) = \int P_x^w(\cdot)\,P(\mathrm{d}w)$, $P(\mathrm{d}w)$ being the Wiener measure over the paths $w : x \in \mathbb{R} \mapsto w(x) \in \mathbb{R}$, the convergence (2) holds to the Bernoulli random variable $Y_{\frac{1}{2},0}$ and its improvement in the log–log scale is given in the same way as (3) where U is a $[0, 1]$-valued random variable with its law given by $\frac{1}{2}1_{\{0\le x\le 1\}}\mathrm{d}x + \frac{1}{2}\delta_{\{1\}}(\mathrm{d}x)$. This can be obtained as a particular and typical example of results in Theorems 20 and 23 given in Section 4.

Finally, we summarize the contents of this paper. In Section 2, we develop a general theory for the growth in time, in the log scale, of a class of one-dimensional Brownian additive functionals. In Section 3, the results in Section 2 will be applied to obtain general results on asymptotic growth in time, in the log scale, of occupation times on the positive side of a family of one-dimensional diffusion processes depending on a parameter $\lambda > 0$. In Section 4, we discuss the long time asymptotics of occupation times on the positive side for diffusions and random walks in random environments by appealing to the general results in Section 3. Here, the scaling property, i.e., a self-similarity, is essential in reducing the problem to the results obtained in Section 3, so that we fundamentally assume that the environment is self-similar in the case of diffusions and is asymptotically self-similar in the case of random walks. Our model in Section 4.1 is a particular case of models introduced and studied by Suzuki [14]; the environments on the positive and the negative sides are mutually independent symmetric stable processes which may have different exponents. The case of random walks will be studied in Section 4.2 by imbedding them in birth-and-death processes with asymptotically self-similar environments (cf. Kawazu–Tamura–Tanaka [11], Hu–Shi [6]).

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2. Asymptotics of a family of Brownian additive functionals

In this section, we prepare some results on long time asymptotics of a family of increasing Brownian additive functionals parametrized by $\lambda > 0$. These results will be a basic tool to obtain results on occupation times for random walks and diffusions in random environments in Section 4. Before proceeding, we set up a general framework.

Let Φ be the set of all càdlàg non-decreasing functions $\varphi : [0,\infty) \ni x \mapsto \varphi(x) \in [0,\infty)$. We set always $\varphi(0-) = 0$ for $\varphi \in \Phi$. We identify φ with its associated Lebesgue–Stieltjes measure $\mathrm{d}\varphi$, which is the positive Radon measure on $[0,\infty)$ such that $(\mathrm{d}\varphi)([0,a]) = \varphi(a)$, $a \in [0,\infty)$. For $\varphi_\lambda, \varphi \in \Phi$, we define $\varphi_\lambda \to \varphi$ in Φ as $\lambda \to \infty$, and denote it as $\lim_{\lambda\to\infty}\varphi_\lambda = \varphi$, if $\lim_{\lambda\to\infty}\varphi_\lambda(x) = \varphi(x)$ for every $x \in [0,\infty)$ such that $\varphi(x) = \varphi(x-)$, equivalently, for every x in a dense subset of $[0,\infty)$. This definition can also be stated as follows: $\lim_{\lambda\to\infty}\varphi_\lambda = \varphi$ in Φ if and only if; for every $x \in [0,\infty)$,

$$\varphi(x-) \le \liminf_{\lambda\to\infty}\varphi_\lambda(x-) \le \limsup_{\lambda\to\infty}\varphi_\lambda(x) \le \varphi(x). \tag{5}$$

As is well-known, $\varphi_\lambda \to \varphi$ in Φ as $\lambda \to \infty$ if and only if

$$\lim_{\lambda\to\infty}\int_{[0,\infty)} f(x)\mathrm{d}\varphi_\lambda(x) = \int_{[0,\infty)} f(x)\mathrm{d}\varphi(x)$$

for every continuous function f on $[0,\infty)$ with a compact support. The composite $\varphi\circ\psi$, for $\varphi,\psi\in\Phi$, is defined as usual by $\varphi\circ\psi(x)=\varphi(\psi(x)),\ x\in[0,\infty)$. It does *not* hold, in general, that $\varphi_\lambda\to\varphi$ and $\psi_\lambda\to\psi$ in Φ imply $\varphi_\lambda\circ\psi_\lambda\to\varphi\circ\psi$ in Φ. We can, however, deduce from (5) that $\varphi_\lambda\to\varphi$ and $\psi_\lambda\to\psi$ in Φ imply the following:

$$\lim_{\lambda\to\infty}\varphi_\lambda(\psi_\lambda(x))=\varphi(\psi(x)) \quad \text{for every } x\in[0,\infty) \text{ such that } \varphi(\psi(x-)-)=\varphi(\psi(x)). \tag{6}$$

Thus we have

Lemma 1. *Let $\varphi_\lambda,\varphi,\psi_\lambda,\psi\in\Phi$. If $\varphi_\lambda\to\varphi$ and $\psi_\lambda\to\psi$ in Φ and if the set $\{x>0\mid\varphi(\psi(x-)-)=\varphi(\psi(x))\}$ is dense in $[0,\infty)$, then $\varphi_\lambda\circ\psi_\lambda\to\varphi\circ\psi$ in Φ.*

It is well-known that there exists a metric ρ on Φ such that Φ is a Lusin space (a standard measurable space, i.e., a Borel subset of a Polish space with the relative topology, cf. [3] or [13]) and, for $\varphi_\lambda,\varphi\in\Phi$, $\varphi_\lambda\to\varphi$ in Φ is equivalent to $\lim_{\lambda\to\infty}\rho(\varphi_\lambda,\varphi)=0$.

Let

$$\Phi_\infty=\{\varphi\in\Phi\mid\lim_{x\to\infty}\varphi(x)=\infty\}.$$

Φ_∞ is a Borel subset of Φ. For $\varphi\in\Phi_\infty$, we define the right-continuous inverse $\varphi^{-1}\in\Phi_\infty$ by

$$\varphi^{-1}(x)=\inf\{y>0\mid\varphi(y)>x\},\quad x\in[0,\infty).$$

It is easy to see that $(\varphi^{-1})^{-1}=\varphi$ for any $\varphi\in\Phi_\infty$ and, $(\psi\circ\varphi)^{-1}=\varphi^{-1}\circ\psi^{-1}$ if $D(\varphi^{-1})\cap D(\psi)=\emptyset$, where

$$D(\varphi)=\{x\in[0,\infty)\mid\varphi(x)\neq\varphi(x-)\}.$$

Also, it holds that, for $\varphi_\lambda,\varphi\in\Phi_\infty$,

$$\varphi_\lambda\to\varphi\quad\text{in }\Phi\quad\text{if and only if}\quad\varphi_\lambda^{-1}\to\varphi^{-1}\text{ in }\Phi. \tag{7}$$

We introduce the following notation: For $\lambda>0$, $e_\lambda\in\Phi_\infty$ is defined by

$$e_\lambda(x)=\mathrm{e}^{\lambda x}-1,\quad x\in[0,\infty). \tag{8}$$

Thus,

$$e_\lambda^{-1}(x)=\frac{1}{\lambda}\log(x+1),\quad x\in[0,\infty). \tag{9}$$

Now we consider Φ-valued random variables. Since Φ is a Lusin space so that Φ is a Borel subset of a Polish space $\widetilde{\Phi}$, a Φ-valued random variable can be identified with a $\widetilde{\Phi}$-valued random variable. Thus Φ-valued random variables may be regarded as random variables with values in a Polish space.

For Φ-valued random variables $X_\lambda,\lambda>0$ and X, $X_\lambda\xrightarrow{d}X$ as $\lambda\to\infty$ denotes the *convergence in law*, that is, the law on Φ of X_λ converges in the weak (or *narrow* in the terminology of [4]) topology to the law on Φ of X. Also, $X_\lambda\xrightarrow{p}X$ denotes the *convergence in probability*, that is, $P(\rho(X_\lambda,X)>\epsilon)\to0$ as $\lambda\to\infty$ for every $\epsilon>0$. Similarly, $X_\lambda\longrightarrow X,$ a.s. denotes the *almost-sure convergence*, that is, $P(\rho(X_\lambda,X)\to0\text{ as }\lambda\to\infty)=1$.

The following implication is well-known:

$$X_\lambda \longrightarrow X, \quad \text{a.s.} \Rightarrow X_\lambda \xrightarrow{p} X \Rightarrow X_\lambda \xrightarrow{d} X.$$

Conversely, the following Skorohod realization theorem (cf. [7], p.9) holds: If $X_\lambda \xrightarrow{d} X$ as $\lambda \to \infty$, then we can realize $\widetilde{X}_\lambda$ and $\widetilde{X}$ on a suitable probability space in such a way that $\widetilde{X}_\lambda \stackrel{d}{=} X_\lambda$ for each $\lambda > 0$, $\widetilde{X} \stackrel{d}{=} X$ and $\widetilde{X}_\lambda \longrightarrow \widetilde{X}$, a.s., as $\lambda \to \infty$. Therefore, problems on the convergence in law may be reduced to the discussions on almost-sure convergence. Note also that $X_\lambda \xrightarrow{d} X$ implies $X_\lambda(x) \xrightarrow{d} X(x)$ for each x such that $P(X(x) = X(x-)) = 1$.

The proof of the following proposition is easy and omitted.

Proposition 2. (i) *If X is a deterministic Φ-valued random variable so that $X \equiv \varphi$ for some $\varphi \in \Phi$, then $X_\lambda \xrightarrow{d} X$ as $\lambda \to \infty$ if and only if $X_\lambda \xrightarrow{p} X$ as $\lambda \to \infty$.*

(ii) *For a fixed $\varphi \in \Phi$, $X_\lambda \xrightarrow{p} \varphi$ as $\lambda \to \infty$ if and only if, for every $x \in [0, \infty)$ such that $\varphi(x) = \varphi(x-)$ (equivalently, for every x in a dense subset of $[0, \infty)$), $X_\lambda(x) \xrightarrow{p} \varphi(x)$ as $\lambda \to \infty$ as real random variables.*

Let $B = (B(t))$ be $BM^0(\mathbb{R})$; a one-dimensional standard Brownian motion starting at 0 and $\{\ell(t, x);\ t \geq 0, x \in \mathbb{R}\}$ be the local time of B with respect to the measure $2\mathrm{d}x$:

$$\int_0^t 1_E(B(s))\mathrm{d}s = 2 \int_E \ell(t, x)\mathrm{d}x, \quad E \in \mathcal{B}(\mathbb{R}).$$

Suppose we are given $\widetilde{m}_\lambda = (\widetilde{m}_\lambda(x)) \in \Phi$ for each $\lambda > 0$. Define a Φ-valued random variables S_λ by

$$S_\lambda(t) = \int_{[0,\infty)} \ell(t, x)\mathrm{d}\widetilde{m}_\lambda(x), \quad t \geq 0. \tag{10}$$

Lemma 3. *Assume that, for some $c = (c(x)) \in \Phi$,*

$$e_\lambda^{-1} \circ \widetilde{m}_\lambda \circ e_\lambda \longrightarrow c \quad \text{in } \Phi \text{ as } \lambda \to \infty. \tag{11}$$

Then

$$e_\lambda^{-1} \circ S_\lambda \circ e_{2\lambda} \xrightarrow{p} \xi \quad \text{in } \Phi \text{ as } \lambda \to \infty, \tag{12}$$

where $\xi \in \Phi$ (deterministic) is defined by

$$\xi(t) = t + c(t), \quad t \geq 0. \tag{13}$$

Proof. By Proposition 2(ii), it is sufficient to show that, for any $t \in [0, \infty)$ such that $\xi(t) = \xi(t-)$,

$$\frac{1}{\lambda} \log\{S_\lambda(\mathrm{e}^{2\lambda t} - 1) + 1\} \xrightarrow{p} \xi(t) \quad \text{as } \lambda \to \infty. \tag{14}$$

We have the following by the scaling property of the Brownian local time; for each $c > 0$,

$$\{\ell(t, x);\ t \geq 0,\ x \geq 0\} \stackrel{d}{=} \left\{ \frac{1}{\sqrt{c}} \ell(ct, \sqrt{c}\,x);\ t \geq 0,\ x \geq 0 \right\}.$$

Therefore,

$$\begin{aligned}\frac{1}{\lambda}\log\{S_\lambda(\mathrm{e}^{2\lambda t})\} &= \frac{1}{\lambda}\log\left\{\int_{[0,\infty)}\ell(\mathrm{e}^{2\lambda t},x)\mathrm{d}\widetilde{m}_\lambda(x)\right\}\\ &\overset{d}{=} \frac{1}{\lambda}\log\left\{\int_{[0,\infty)}\mathrm{e}^{\lambda t}\,\ell(1,\mathrm{e}^{-\lambda t}x)\mathrm{d}\widetilde{m}_\lambda(x)\right\}\\ &= t+\frac{1}{\lambda}\log\left\{\int_{[0,\infty)}\ell(1,x)\mathrm{d}\widetilde{m}_\lambda(\mathrm{e}^{\lambda t}x)\right\}. \end{aligned} \tag{15}$$

Notice that the right-hand side (RHS) of (14) is asymptotically equal to the RHS of (15). First, note that

$$\text{the RHS of (15)} \le t+\frac{1}{\lambda}\log\{a\cdot\widetilde{m}_\lambda(\mathrm{e}^{\lambda t}M)\},\quad \text{a.s.,}$$

where $M=\sup_{0\le u\le 1}B(u)$ and $a=\sup_{x\in[0,\infty)}\ell(1,x)$. Since t is chosen from the continuity points of ξ, it is a continuity point of c and hence, we have by (11) that

$$\frac{1}{\lambda}\log\widetilde{m}_\lambda(\mathrm{e}^{\lambda t})\to c(t),\quad \text{as } \lambda\to\infty.$$

Then we can deduce that

$$\limsup_{\lambda\to\infty}\ \text{RHS of (15)} \le t+c(t)=\xi(t).$$

Secondly, note that, almost surely, $\ell(1,0)>0$ and $x\mapsto\ell(1,x)$ is continuous. Hence, almost surely, there exists $\alpha>0$ such that

$$b:=\inf_{0\le x\le\alpha}\ell(1,x)>\frac{1}{2}\ell(1,0)>0.$$

From this, we have

$$\text{RHS of (15)} \ge t+\frac{1}{\lambda}\log\{b\cdot\widetilde{m}_\lambda(\mathrm{e}^{\lambda t}\cdot\alpha)\},\quad \text{a.s.,}$$

and hence, we can deduce that

$$\liminf_{\lambda\to\infty}\ \text{RHS of (15)} \ge \xi(t),\quad \text{a.s.}$$

This completes the proof because the LHS of (14) converges to a positive value as $\lambda\to\infty$ if and only if the LHS of (15) does so with the same limiting value. □

From now on, we are particularly interested in the case when $s_\lambda\in\Phi$ and $m_\lambda\in\Phi$ are given such that $s_\lambda\in\Phi_\infty$ and $x\mapsto s_\lambda(x)$ is strictly increasing and continuous. And $\widetilde{m}_\lambda$ is defined by

$$\widetilde{m}_\lambda=m_\lambda\circ s_\lambda^{-1},\quad \text{i.e., } \widetilde{m}_\lambda(x)=m_\lambda(s_\lambda^{-1}(x)). \tag{16}$$

In such a case, we have the Feller generator $\mathcal{L}_\lambda=\frac{\mathrm{d}}{\mathrm{d}m_\lambda}\frac{\mathrm{d}}{\mathrm{d}s_\lambda}$ on $[0,\infty)$ and the reflecting diffusion process $X_\lambda=(X_\lambda(t),P_x)$ on $[0,\infty)$ is uniquely associated with $\mathcal{L}_\lambda$. Note that X_λ under P_0 is given from $BM^0(\mathbb{R})B=(B(t))$ by

$$X_\lambda(t)=s_\lambda^{-1}(B(S_\lambda^{-1}(t))),\quad t\ge 0.$$

We assume that, for some $a=(a(x))\in\Phi_\infty$ and $b=(b(x))\in\Phi$,

(A.1) $e_\lambda^{-1} \circ s_\lambda \longrightarrow a$ in Φ as $\lambda \to \infty$,
(A.2) $e_\lambda^{-1} \circ m_\lambda \longrightarrow b$ in Φ as $\lambda \to \infty$.

Noting (6) and the relation $e_\lambda^{-1} \circ \widetilde{m}_\lambda \circ e_\lambda = (e_\lambda^{-1} \circ m_\lambda) \circ (e_\lambda^{-1} \circ s_\lambda)^{-1}$, we can deduce the following:

Lemma 4. *Under the assumptions* (A.1) *and* (A.2), (11) *holds with $c = b \circ a^{-1}$ if the following condition*

$$b(a^{-1}(x-)-) = b(a^{-1}(x)) \tag{17}$$

holds at every continuity point x of $b \circ a^{-1}$. In particular, (11) *holds if the following assumption* (A.3) *is satisfied:*

(A.3) *The set $\{x \in [0, \infty) \mid b(a^{-1}(x-)-) = b(a^{-1}(x))\}$ is dense in* $[0, \infty)$.

The following theorem immediately follows from Lemmas 3 and 4.

Theorem 5. *Let $s_\lambda \in \Phi$ and $m_\lambda \in \Phi$ be given as above. Under the assumptions* (A.1)–(A.3), *it holds that*

$$e_\lambda^{-1} \circ S_\lambda \circ e_{2\lambda} \xrightarrow{p} \xi \quad \text{in } \Phi \text{ as } \lambda \to \infty, \tag{18}$$

where

$$\xi(t) = t + b(a^{-1}(t)), \quad t \geq 0. \tag{19}$$

Example 6. Let $s(x) = x$ and $m(x) = cx^{\frac{1}{\alpha}-1}$ where $c > 0$ and $0 < \alpha < 1$. Set $s_\lambda(x) = s \circ e_\lambda$ and $m_\lambda = m \circ e_\lambda$, $\lambda > 0$. Then (A.1) and (A.2) hold with $a(x) = x$ and $b(x) = (\frac{1}{\alpha} - 1)\,x$. Then (A.3) obviously holds and hence, by Theorem 5, we have $e_\lambda^{-1} \circ S_\lambda \circ e_{2\lambda} \xrightarrow{p} \xi$ in Φ as $\lambda \to \infty$. Note that $\widetilde{m}_\lambda = m \circ s^{-1} = m$ and $\xi(x) = x + b(a^{-1}(x)) = \frac{x}{\alpha}$ so that this implies that

$$\frac{1}{\lambda} \log \left(\int_0^{e^{2\lambda t}} (B(s) \vee 0)^{\frac{1}{\alpha}-2} \mathrm{d}s \right) \xrightarrow{p} \frac{t}{\alpha} \quad \text{as } \lambda \to \infty.$$

Example 7. Let $w : [0, \infty) \to \mathbb{R}$ be a càdlàg function such that $w(0) = 0$, $\limsup_{u \to \infty} w(u) = \infty$ and $\int_0^\infty e^{\lambda w(u)} \mathrm{d}u = \infty$ for every $\lambda > 0$. Set

$$s_\lambda(x) = \int_0^x e^{\lambda w(u)} \mathrm{d}u \quad \text{and} \quad m_\lambda(x) = 2 \int_0^x e^{-\lambda w(u)} \mathrm{d}u. \tag{20}$$

Then (A.1) and (A.2) hold with

$$a(x) = \overline{w}(x) := \sup_{0 \leq u \leq x} w(u) \quad \text{and} \quad b(x) = \underline{w}(x) := \sup_{0 \leq u \leq x} (-w(u)). \tag{21}$$

This fact will be proved under a more general situation in Example 8. Then $c(x) = b(a^{-1}(x)) = \underline{w}(\overline{w}^{-1}(x))$ and hence (A.3) in this case means that the set

$$\{x \in [0, \infty) | \underline{w}(\overline{w}^{-1}(x-)-) = \underline{w}(\overline{w}^{-1}(x))\} \tag{22}$$

is dense in $[0, \infty)$. Hence, by Theorem 5, the convergence of Φ-valued random variables

$$\frac{1}{\lambda} \log S_\lambda(e^{2\lambda t}) \xrightarrow{p} \xi(t) = t + \underline{w}(\overline{w}^{-1}(t)) \quad \text{as } \lambda \to \infty \tag{23}$$

can be concluded if the set defined by (22) is dense in $[0, \infty)$.

Example 8. This is a slight generalization of Example 7: This result will be used in Section 4.2 through Example 14 of Section 3. Suppose we are given two families of càdlàg functions $w_\lambda : [0, \infty) \to \mathbb{R}$ and $v_\lambda : [0, \infty) \to \mathbb{R}$ for $\lambda > 0$, each satisfying the following conditions; denoting by f_λ either w_λ or v_λ, $f_\lambda(0) = 0$, $\limsup_{x\to\infty} f_\lambda(x) = \infty$ and $\int_0^\infty e^{\lambda f_\lambda(u)} du = \infty$. Suppose also $\phi_\lambda \in \Phi$, $\lambda > 0$, is given and assume $\phi_\lambda(x) \to x$ for every $x \in [0, \infty)$ as $\lambda \to \infty$. Assume further that a càdlàg function $w : [0, \infty) \to \mathbb{R}$ is given such that $w(0) = 0$, $\limsup_{x\to\infty} w(x) = \infty$ and the following holds:

$$w_\lambda \to w \quad \text{in } J_1 \quad \text{and} \quad v_\lambda \to w \quad \text{in } J_1 \text{ as } \lambda \to \infty, \tag{24}$$

where J_1 is the Skorohod topology (cf. [12]) on the space of càdlàg functions.

Under these assumptions, s_λ and m_λ are defined by

$$s_\lambda(x) = \int_0^x e^{\lambda w_\lambda(u)} du \quad \text{and} \quad m_\lambda(x) = \int_0^{\phi_\lambda(x)} \{e^{-\lambda w_\lambda(u)} + e^{-\lambda v_\lambda(u)}\} du. \tag{25}$$

Then, if $a(x)$ and $b(x)$ are defined by (21) using this limit function w, the same conclusions as Example 7 hold: For example, the proof of "$e_\lambda^{-1} \circ s_\lambda \to \overline{w}$ in Φ as $\lambda \to \infty$" is as follows: Let $x > 0$ be such that $\overline{w}(x) = \overline{w}(x-)$. Since $w_\lambda \to w$ in J_1-metric, it is easy to see that, for any $\epsilon > 0$, there exist $0 < x_1 < x_2 < x$ and $\lambda_0 > 0$ such that $w_\lambda(u) \geq \overline{w}(x) - \epsilon$ for all $u \in [x_1, x_2]$ and $\lambda > \lambda_0$. Then

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda} \log \left(\int_0^x e^{\lambda w_\lambda(u)} du \right) \geq \liminf_{\lambda\to\infty} \frac{1}{\lambda} \log \left((x_2 - x_1) e^{\lambda(\overline{w}(x) - \epsilon)} \right) = \overline{w}(x) - \epsilon.$$

On the other hand, $w_\lambda \to w$ in J_1 implies $\overline{w}_\lambda \to \overline{w}$ in J_1 and hence,

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \left(\int_0^x e^{\lambda w_\lambda(u)} du \right) \leq \limsup_{\lambda\to\infty} \frac{1}{\lambda} \log(x e^{\lambda \overline{w}_\lambda(x)}) = \overline{w}(x).$$

The proof of "$e_\lambda^{-1} \circ m_\lambda \to \underline{w}$ in Φ as $\lambda \to \infty$" is similar.

Remark 9. The same conclusion obviously holds if s_λ and m_λ are modified as

$$s_\lambda(x) = \int_0^x e^{\lambda w(u)} du \cdot \rho_1(\lambda) \quad \text{and}$$

$$m_\lambda(x) = \int_0^{\phi_\lambda(x)} \{e^{-\lambda w_\lambda(u)} + e^{-\lambda v_\lambda(u)}\} du \cdot \rho_2(\lambda),$$

where $\rho_i(\lambda) > 0$ and $\frac{1}{\lambda} \log \rho_i(\lambda) \to 0$ as $\lambda \to \infty$, $i = 1, 2$. A typical example is the case when $\rho_i(\lambda) = \alpha_i \lambda^{\beta_i}$ with $\alpha_i > 0$ and $\beta_i \geq 0$, $i = 1, 2$.

3. Asymptotics of occupation times on the positive side of a class of diffusions on $\mathbb{R}$

Suppose we are given, for each $\lambda > 0$, a strictly increasing and continuous function $s_\lambda : \mathbb{R} \to \mathbb{R}$ and a positive Radon measure $m_\lambda(dx)$ on $\mathbb{R}$. We assume that $\lim_{x\to-\infty} s_\lambda(x) = -\infty$

and $\lim_{x\to\infty} s_\lambda(x) = \infty$ for each $\lambda > 0$. Then the unique recurrent (generalized) diffusion $\mathbf{X}_\lambda = (X_\lambda(t), P_x)$ on $\mathbb{R}$ is associated with the Feller generator $\mathcal{L}_\lambda = \frac{\mathrm{d}}{\mathrm{d}m_\lambda}\frac{\mathrm{d}}{\mathrm{d}s_\lambda}$: Strictly speaking, the state space of $\mathbf{X}_\lambda$ is the support E_λ of the measure m_λ. We are interested in the long time asymptotic behavior of the occupation time

$$A_\lambda(t) = \int_0^t 1_{[0,\infty)}(X_\lambda(s))\mathrm{d}s, \quad t \ge 0. \tag{26}$$

In order to apply the results in Section 2, we define $s^\lambda_\pm$ and $m^\lambda_\pm$ as elements in Φ as follows:

$$s^\lambda_+(x) = s_\lambda(x) - s_\lambda(0), \qquad s^\lambda_-(x) = s_\lambda(0) - s_\lambda(-x), \quad x \ge 0 \tag{27}$$

and

$$m^\lambda_+(x) = m_\lambda([0,x]), \qquad m^\lambda_-(x) = m_\lambda([-x,0)), \quad x \ge 0. \tag{28}$$

Define

$$\widetilde{m}^\lambda_+ = m^\lambda_+ \circ (s^\lambda_+)^{-1} \quad \text{and} \quad \widetilde{m}^\lambda_- = m^\lambda_- \circ (s^\lambda_-)^{-1}. \tag{29}$$

For a $BM^0(\mathbb{R})$ $B = (B(t))$ with the local time $\ell(t,x)$, set

$$S^+_\lambda(t) = \int_{[0,\infty)} \ell(t,x)\mathrm{d}\widetilde{m}^\lambda_+(x), \qquad S^-_\lambda(t) = \int_{[0,\infty)} \ell(t,-x)\mathrm{d}\widetilde{m}^\lambda_-(x), \tag{30}$$

and

$$S_\lambda(t) = S^+_\lambda(t) + S^-_\lambda(t). \tag{31}$$

We assume that $(s^\lambda_+, m^\lambda_+)$ and $(s^\lambda_-, m^\lambda_-)$ satisfy the assumptions (A.1)–(A.3) of Section 2; namely, for each of $+$ and $-$,

(A.1)′ $e_\lambda^{-1} \circ s^\lambda_\pm \longrightarrow a_\pm$ in Φ as $\lambda \to \infty$, with $a_\pm \in \Phi_\infty$
(A.2)′ $e_\lambda^{-1} \circ m^\lambda_\pm \longrightarrow b_\pm$ in Φ as $\lambda \to \infty$
and
(A.3)′ the sets $\{x \in [0,\infty) \mid b_+(a_+^{-1}(x-)-) = b_+(a_+^{-1}(x))\}$

and $\{x \in (0,\infty) \mid b_-(a_-^{-1}(x-)-) = b_-(a_-^{-1}(x))\}$ are dense in $[0,\infty)$.

Then, by Theorem 5, we have the following joint convergence:

$$\left(e_\lambda^{-1} \circ S^+_\lambda \circ e_{2\lambda}, e_\lambda^{-1} \circ S^-_\lambda \circ e_{2\lambda}\right) \xrightarrow{p} (\xi_+, \xi_-) \quad \text{in } \Phi \times \Phi \text{ as } \lambda \to \infty, \tag{32}$$

where

$$\xi_+(t) = t + b_+(a_+^{-1}(t)) \quad \text{and} \quad \xi_-(t) = t + b_-(a_-^{-1}(t)). \tag{33}$$

By a standard argument using the inequality

$$S^+_\lambda(t) \vee S^-_\lambda(t) \le S^+_\lambda(t) + S^-_\lambda(t) \le 2\left(S^+_\lambda(t) \vee S^-_\lambda(t)\right),$$

we deduce from (32) that

$$e_\lambda^{-1} \circ S_\lambda \circ e_{2\lambda} \xrightarrow{p} \xi_+ \vee \xi_- \quad \text{in } \Phi \text{ as } \lambda \to \infty, \tag{34}$$

where $\xi_+ \vee \xi_- \in \Phi$ is defined by

$$(\xi_+ \vee \xi_-)(t) = \max\{\xi_+(t), \xi_-(t)\}. \tag{35}$$

Summing up these results, we have obtained the following

Theorem 10. *Assume that s_λ and m_λ satisfy the assumptions* (A.1)′–(A.3)′. *Then we have the following joint convergence:*

$$\begin{aligned}&\left(e_\lambda^{-1} \circ S_\lambda^+ \circ e_{2\lambda},\ e_\lambda^{-1} \circ S_\lambda^- \circ e_{2\lambda},\ e_\lambda^{-1} \circ S_\lambda \circ e_{2\lambda}\right)\\ &\quad \xrightarrow{p} (\xi_+,\ \xi_-,\ \xi_+ \vee \xi_-) \quad \text{in } \Phi \times \Phi \times \Phi \text{ as } \lambda \to \infty,\end{aligned} \tag{36}$$

where ξ_+ and ξ_- are defined by (33).

As in [9], we have

$$A_\lambda = S_\lambda^+ \circ S_\lambda^{-1}, \quad \text{i.e., } A_\lambda(t) = S_\lambda^+(S_\lambda^{-1}(t)),\ t \geq 0. \tag{37}$$

By the Skorohod realization theorem, we can realize, on a suitable probability space, a family of Φ-valued random variables $\widetilde{S}_\lambda^+, \widetilde{S}_\lambda^-$ such that, for each λ, $(S_\lambda^+, S_\lambda^-) \overset{d}{=} (\widetilde{S}_\lambda^+, \widetilde{S}_\lambda^-)$ and, setting $\widetilde{S}_\lambda = \widetilde{S}_\lambda^+ + \widetilde{S}_\lambda^-$,

$$\begin{aligned}&\left(e_\lambda^{-1} \circ \widetilde{S}_\lambda^+ \circ e_{2\lambda},\ e_\lambda^{-1} \circ \widetilde{S}_\lambda^- \circ e_{2\lambda},\ e_\lambda^{-1} \circ \widetilde{S}_\lambda \circ e_{2\lambda}\right)\\ &\quad \longrightarrow (\xi_+,\ \xi_-,\ \xi_+ \vee \xi_-) \quad \text{a.s. in } \Phi \times \Phi \times \Phi \text{ as } \lambda \to \infty.\end{aligned} \tag{38}$$

Hence, if we set $\widetilde{A}_\lambda(t) = \widetilde{S}_\lambda^+(\widetilde{S}_\lambda^{-1}(t))$, then we have, as Φ-valued random variables,

$$A_\lambda \overset{d}{=} \widetilde{A}_\lambda \quad \text{for each } \lambda > 0. \tag{39}$$

Note that

$$e_\lambda^{-1} \circ \widetilde{S}_\lambda^+ \circ e_{2\lambda} \circ (e_\lambda^{-1} \circ \widetilde{S}_\lambda \circ e_{2\lambda})^{-1} = e_\lambda^{-1} \circ \widetilde{S}_\lambda^+ \circ (\widetilde{S}_\lambda)^{-1} \circ e_\lambda = e_\lambda^{-1} \circ \widetilde{A}_\lambda \circ e_\lambda.$$

Also, note that $(\xi_+ \vee \xi_-)^{-1}(t)$ is continuous in t because ξ_+ and ξ_-, hence $\xi_+ \vee \xi_-$, are strictly increasing. Then, by noting the general fact concerning (6) and (7), we can now deduce from (38) the following: If $t \in [0, \infty)$ is such that

$$\xi_+((\xi_+ \vee \xi_-)^{-1}(t)) = \xi_+((\xi_+ \vee \xi_-)^{-1}(t)-), \tag{40}$$

then,

$$\begin{aligned}&(e_\lambda^{-1} \circ \widetilde{A}_\lambda \circ e_\lambda)(t) \left(= \frac{1}{\lambda} \log\left\{\widetilde{A}_\lambda(\mathrm{e}^{\lambda t} - 1) + 1\right\}\right)\\ &\quad \longrightarrow \xi_+((\xi_+ \vee \xi_-)^{-1}(t)) \quad \text{as } \lambda \to \infty, \text{ a.s.}\end{aligned} \tag{41}$$

From this fact, we obtain the following:

Theorem 11. *We assume that s_λ and m_λ satisfy* (A.1)′–(A.3)′. *If $t \in [0, \infty)$ is such that*

$$\xi_+((\xi_-)^{-1}(t)-) = \xi_+((\xi_-)^{-1}(t)), \tag{42}$$

then it holds that

$$\frac{1}{\lambda}\log A_\lambda(\mathrm{e}^{\lambda t}) \xrightarrow{p} \xi_+((\xi_-)^{-1}(t)) \wedge t, \quad \textit{as } \lambda \to \infty. \tag{43}$$

Proof. It is sufficient to prove in the above realization that, if (42) holds, then,

$$\frac{1}{\lambda}\log \widetilde{A}_\lambda(\mathrm{e}^{\lambda t}) \longrightarrow \xi_+((\xi_-)^{-1}(t)) \wedge t, \quad \text{as } \lambda \to \infty, \text{ a.s.} \tag{44}$$

If $(\xi_-)^{-1}(t) \le (\xi_+)^{-1}(t)$, then (40) follows from (42) and hence, (44) follows from (41).

Next, consider the case when

$$(\xi_-)^{-1}(t) > (\xi_+)^{-1}(t) \tag{45}$$

holds. Setting, for simplicity,

$$\varphi_\lambda^\pm(s) = \frac{1}{\lambda}\log \widetilde{S}_\lambda^\pm(\mathrm{e}^{2\lambda s}) \quad \text{and} \quad \psi_\lambda(s) = \frac{1}{\lambda}\log \widetilde{S}_\lambda(\mathrm{e}^{2\lambda s}),$$

we have

$$(\varphi_\lambda^\pm)^{-1}(s) \to (\xi_\pm)^{-1}(s) \quad \text{and} \quad \psi_\lambda(s)^{-1} \to (\xi_+)^{-1} \wedge (\xi_-)^{-1}(s)$$

as $\lambda \to \infty$ uniformly in s on any bounded interval, $a.s$. (Note that $(\xi_\pm)^{-1}$ and $(\xi_+)^{-1} \wedge (\xi_-)^{-1}$ are continuous because ξ_+ and ξ_- are strictly increasing.) Then, for fixed t satisfying (42) and (45), we have, from the inequality

$$\widetilde{S}_\lambda(s) = \widetilde{S}_\lambda^+(s) + \widetilde{S}_\lambda^-(s) \le 2(\widetilde{S}_\lambda^+(s) \vee \widetilde{S}_\lambda^-(s)),$$

that, for some λ_0 large enough,

$$\psi_\lambda^{-1}(t) \ge (\varphi_\lambda^+)^{-1}\left(t - \frac{\log 2}{\lambda}\right) \wedge (\varphi_\lambda^-)^{-1}\left(t - \frac{\log 2}{\lambda}\right) = (\varphi_\lambda^+)^{-1}\left(t - \frac{\log 2}{\lambda}\right),$$

for all $\lambda > \lambda_0$. Hence

$$\varphi_\lambda^+(\psi_\lambda^{-1}(t)) \ge \varphi_\lambda^+\left((\varphi_\lambda^+)^{-1}\left(t - \frac{\log 2}{\lambda}\right)\right) \longrightarrow t, \quad \text{as } \lambda \to \infty$$

and therefore,

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda}\log \widetilde{A}_\lambda(\mathrm{e}^{\lambda t}) = \liminf_{\lambda\to\infty} \varphi_\lambda^+(\psi_\lambda^{-1}(t)) \ge t, \quad \text{a.s.}$$

On the other hand, we have obviously,

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda}\log \widetilde{A}_\lambda(\mathrm{e}^{\lambda t}) \le t, \quad \text{a.s.},$$

so that

$$\lim_{\lambda\to\infty} \frac{1}{\lambda}\log \widetilde{A}_\lambda(\mathrm{e}^{\lambda t}) = t = \xi_+((\xi_-)^{-1}(t)) \wedge t, \quad \text{a.s.}$$

in this case as well. □

Example 12. This example corresponds to Example 6 of Section 2. We consider the $\mathcal{L} = \frac{d}{dm}\frac{d}{ds}$-diffusion process $X = (X(t), P_x)$ on $\mathbb{R}$ where $s(x) = x,\ x \in \mathbb{R}$, and

$$m(x) = \begin{cases} c_+ x^{\frac{1}{\alpha}-1}, & x \geq 0, \\ -c_-(-x)^{\frac{1}{\beta}-1}, & x < 0, \end{cases}$$

for some $c_+ > 0, c_- > 0$ and $0 < \alpha < 1, 0 < \beta < 1$. Let $s^+(x) = s^-(x) = x$ for $x \geq 0$ and $m^+(x) = c_+ x^{\frac{1}{\alpha}-1},\ m^-(x) = c_- x^{\frac{1}{\beta}-1},\ x \geq 0$. Set $s_\lambda^\pm = s^\pm \circ e_\lambda$ and $m_\lambda^\pm = m^\pm \circ e_\lambda$. From Example 6, we have $\xi_+(x) = \frac{x}{\alpha}$ and $\xi_-(x) = \frac{x}{\beta}$ so that

$$\xi_+((\xi_-)^{-1}(t)) = \frac{\beta}{\alpha} t, \quad t \geq 0.$$

Hence

$$\xi_+((\xi_-)^{-1}(t)) \wedge t = \begin{cases} \dfrac{\beta}{\alpha} t, & \beta < \alpha, \\ t, & \alpha \leq \beta. \end{cases}$$

In this case, the scale s_λ on $\mathbb{R}$ is determined by $s_\lambda(x) = s_\lambda^+(x),\ x \geq 0$ and $s_\lambda(x) = -s_\lambda^-(-x),\ x < 0$. Similarly the speed measure dm_λ on $\mathbb{R}$ is determined by $m_\lambda([0, x]) = m_\lambda^+(x),\ x \geq 0$ and $m_\lambda([x, 0]) = m_\lambda^-(-x),\ x < 0$. Then, if $X_\lambda = (X_\lambda(t), P_x^\lambda)$ is the $\mathcal{L}_\lambda = \frac{d}{dm_\lambda}\frac{d}{ds_\lambda}$-diffusion, we have

$$\{X_\lambda(t)\} \overset{d}{=} \{s_\lambda^{-1}(X(t))\}$$

and hence $\{A_\lambda(t)\} \overset{d}{=} \{A(t)\}$. Thus, from Theorem 11, we can deduce that, as $\lambda \to \infty$,

$$\frac{1}{\lambda} \log A(e^{\lambda t}) \longrightarrow \begin{cases} \dfrac{\beta}{\alpha} t, & \beta < \alpha, \\ t, & \alpha \leq \beta, \end{cases} \tag{46}$$

in the sense of Φ-valued random variables and also, in the sense of finite-dimensional distributions. In particular, we have

$$\frac{\log A(t)}{\log t} \longrightarrow \begin{cases} \dfrac{\beta}{\alpha}, & \beta < \alpha, \\ 1, & \alpha \leq \beta. \end{cases} \tag{47}$$

This is an improvement of the law convergence

$$\frac{A(t)}{t} \xrightarrow{p} \begin{cases} 0, & \beta < \alpha, \\ 1, & \alpha \leq \beta, \end{cases} \quad \text{as } t \to \infty.$$

Example 13. This example corresponds to Example 7 of Section 2. Let $w : \mathbb{R} \ni t \mapsto w(t) \in \mathbb{R}$ be a càdlàg function such that

$$w(0) = w(0-) = 0, \qquad \limsup_{u\to\infty} w(u) = \limsup_{u\to-\infty} w(u) = \infty$$

and

$$\int_0^\infty e^{\lambda w(u)} du = \int_{-\infty}^0 e^{\lambda w(u)} du = \infty \quad \text{for every } \lambda > 0.$$

Set

$$s_\lambda(x) = \int_0^x e^{\lambda w(u)} du, \quad -\infty < x < \infty \tag{48}$$

and

$$m_\lambda(x) = 2\int_0^x e^{-\lambda w(u)} du, \quad -\infty < x < \infty, \tag{49}$$

so that $m_\lambda(dx) = dm_\lambda(x) = 2e^{-\lambda w(x)}dx$. $X_\lambda = (X_\lambda(t), P_x)$ is the recurrent diffusion process on $\mathbb{R}$ associated with the Feller generator $\mathcal{L}_\lambda = \frac{d}{dm_\lambda}\frac{d}{ds_\lambda}$. Let $A_\lambda(t) = \int_0^t 1_{[0,\infty)}(X_\lambda(s))ds$, $\lambda > 0$. In order to apply the results in Example 7, we define $s_\pm^\lambda$,$m_\pm^\lambda$,$\overline{w_\pm}$, and $\underline{w_\pm}$ as elements in Φ;

$$s_+^\lambda(x) = \int_0^x e^{\lambda w(u)} du, \qquad s_-^\lambda(x) = \int_{-x}^0 e^{\lambda w(u)} du, \quad x \ge 0, \tag{50}$$

$$m_+^\lambda(x) = 2\int_0^x e^{-\lambda w(u)} du, \qquad m_-^\lambda(x) = 2\int_{-x}^0 e^{-\lambda w(u)} du, \quad x \ge 0. \tag{51}$$

We define $w_\pm : [0,\infty) \to \mathbb{R}$ by

$$w_+(x) = w(x) \quad \text{and} \quad w_-(x) = w(-x-0). \tag{52}$$

Then $\overline{w_\pm}$ and $\underline{w_\pm}$ are defined by (21); namely, for each of $+$ and $-$,

$$\overline{w_\pm}(x) = \sup_{0\le u\le x} w_\pm(u) \quad \text{and} \quad \underline{w_\pm}(x) = \sup_{0\le u\le x}(-w_\pm(u)). \tag{53}$$

Then (A.1)′ and (A.2)′ hold with $a_\pm$ and $b_\pm$ defined by (21) through $w_\pm$, respectively. (A.3)′ holds if the following holds:

$$\begin{aligned}&\text{The sets } \left\{x \in [0,\infty) \mid \underline{w_+}((\overline{w_+})^{-1}(x-)-) = \underline{w_+}((\overline{w_+})^{-1}(x))\right\}\\ &\quad \text{and } \left\{x \in [0,\infty) \mid \underline{w_-}((\overline{w_-})^{-1}(x-)-) = \underline{w_-}((\overline{w_-})^{-1}(x))\right\}\\ &\qquad \text{are both dense in } [0,\infty).\end{aligned} \tag{54}$$

Hence, by Theorem 10, we have the following conclusion: Assume that (54) holds. Let $\xi_\pm \in \Phi_\infty$ be defined by

$$\xi_+(t) = t + \underline{w_+} \circ \overline{w_+}^{-1}(t) \quad \text{and} \quad \xi_-(t) = t + \underline{w_-} \circ \overline{w_-}^{-1}(t), \quad t \ge 0. \tag{55}$$

If t is such that

$$\xi_+((\xi_-)^{-1}(t)-) = \xi_+((\xi_-)^{-1}(t)), \tag{56}$$

then it holds that

$$\frac{1}{\lambda}\log A_\lambda(e^{\lambda t}) \xrightarrow{p} \xi_+((\xi_-)^{-1}(t)) \wedge t, \quad \lambda \to \infty. \tag{57}$$

Example 14. This is a modification of Example 13, which corresponds to that in Example 8 of Section 2. Suppose we are given, for each of $+$ and $-$, and for $\lambda > 0$, families $w_\lambda^\pm$, $v_\lambda^\pm$, $w^\pm$ and $\phi_\pm^\lambda$ satisfying the same conditions as w_λ, v_λ, w and ϕ_λ in Example 8. Then, we have $s_\lambda^\pm$ and $m_\lambda^\pm$, respectively, in the same way as (25) in Example 8. Now, these can be extended to s_λ and m_λ on $\mathbb{R}$ by setting $s_\lambda(x) = s_\lambda^+(x),\ x \ge 0$ and $s_\lambda(x) = -s_\lambda^-(-x),\ x < 0$, and $m_\lambda(x) = m_\lambda^+(x),\ x \ge 0$ and $m_\lambda(x) = -m_\lambda^-(-x-0),\ x < 0$. Hence we have the Feller generator $\mathcal{L}_\lambda = \frac{\mathrm{d}}{\mathrm{d}m_\lambda}\frac{\mathrm{d}}{\mathrm{d}s_\lambda}$ on $\mathbb{R}$ and the associated recurrent (generalized) diffusion $X_\lambda = (X_\lambda(t), P_x)$ on the support of $\mathrm{d}m_\lambda$. Then, for the occupation time $A_\lambda(t) = \int_0^t 1_{[0,\infty)}(X_\lambda(u))\mathrm{d}u$, we have the same conclusion as in Example 13; in particular, the limit process can be described in terms of the limit function $w^\pm$ on $[0, \infty)$.

Remark 15. In Examples 13 and 14, we have obviously the same conclusion if $s_\lambda^\pm$ and $m_\lambda^\pm$ are replaced, respectively, by those multiplied by some $\rho(\lambda) > 0$ such that $\lim_{\lambda\to\infty} \frac{1}{\lambda}\log\rho(\lambda) = 0$. A typical example is the case $\rho(\lambda) = c_1\lambda^{c_2}$ with positive constants c_1 and c_2.

4. Long time asymptotics of occupation times for diffusions and random walks in random environments

We would apply the results of Section 3 to the study of the long time asymptotic behaviors of occupation times on the positive side for diffusions on $\mathbb{R}$ and random walks on $\mathbb{Z}$ in random environments.

4.1. *The case of Brox model and its generalization to stable environments*

Let $w : [0,\infty) \to \mathbb{R}$ be a càdlàg function such that $w(0) = 0$, $\limsup_{u\to\infty} w(u) = \infty$ and $\int_0^\infty \mathrm{e}^{\lambda w(u)}\mathrm{d}u = \infty$ for every $\lambda > 0$. We denote the totality of such functions by W_+, which is a Lusin space under the Skorohod topology. For a pair $w = (w_+, w_-) \in W_+ \times W_+ =: W$, define $s_\pm^w \in \Phi_\infty$ and $m_\pm^w \in \Phi$, for each of $+$ and $-$, by

$$s_\pm^w(x) = \int_0^x \mathrm{e}^{w_\pm(u)}\mathrm{d}u \quad \text{and} \quad m_\pm^w(x) = 2\int_0^x \mathrm{e}^{-w_\pm(u)}\mathrm{d}u. \tag{58}$$

These $s_\pm^w$ and $m_\pm^w$ determine the scale s^w and the speed measure $\mathrm{d}m^w$ on $\mathbb{R}$;

$$s^w(x) = 1_{\{x\ge 0\}}s_+^w(x) - 1_{\{x<0\}}s_-^w(-x) \tag{59}$$

and

$$\mathrm{d}m^w(x) = 2\left\{1_{\{x\ge 0\}}\mathrm{e}^{-w_+(x)}\mathrm{d}x + 1_{\{x<0\}}\mathrm{e}^{-w_-(-x)}\mathrm{d}x\right\}. \tag{60}$$

Then we have the Feller generator $\mathcal{L}^w = \frac{\mathrm{d}}{\mathrm{d}m^w}\frac{\mathrm{d}}{\mathrm{d}s^w}$ and the unique recurrent diffusion $X^w = (X^w(t), P_x^w)$ on $\mathbb{R}$ associated with $\mathcal{L}^w$. We suppose that X^w is realized on the space Ω of all continuous functions $\omega : [0,\infty) \to \mathbb{R}$ as $X^w(t) = \omega(t)$ so that P_x^w is the Borel probability on Ω concentrated on $\{\omega \mid \omega(0) = x\}$.

Let $0 < \alpha_+ \le 2$ and $0 < \alpha_- \le 2$. Let $(X_+(t), X_-(t))$ be a pair of mutually independent one-dimensional symmetric stable processes $X_+(t)$ and $X_-(t)$ of indexes α_+ and α_-, respectively, with $X_+(0) = X_-(0) = 0$, so that

$$E[\mathrm{e}^{\mathrm{i}(\xi_+X_+(t)+\xi_-X_-(t))}] = \mathrm{e}^{-(c_+|\xi_+|^{\alpha_+}+c_-|\xi_-|^{\alpha_-})t}, \quad \xi_+, \xi_- \in \mathbb{R}$$

for some positive constants c_+ and c_-. The values c_+ and c_- are irrelevant in future discussions. It is well-known that $\limsup_{t\to\infty} X_\pm(t) = \infty$ and $\int_0^\infty \mathrm{e}^{\lambda X_\pm(t)}\mathrm{d}t = \infty$ for every $\lambda > 0$ almost surely, so that $(X_+(t), X_-(t))$ is a $W = W_+ \times W_+$-valued random variable. We denote its law on W by $P_{\alpha_+,\alpha_-}(\mathrm{d}w)$. Then we have a stochastic process $\overline{X} = (\overline{X}(t), \overline{P}_x)$ realized on $\Omega \times W$ where $\overline{X}(t) = \omega(t), \omega \in \Omega$, and $\overline{P}_x(\mathrm{d}\omega, \mathrm{d}w) = P_x^w(\mathrm{d}\omega) \cdot P_{\alpha_+,\alpha_-}(\mathrm{d}w)$. This model has been introduced by Suzuki [14]: Indeed, she introduced a more general model in which P_{α_+,α_-} is the law of the pair of mutually independent self-similar processes on $\mathbb{R}$ with indexes $\alpha_+ > 0$ and $\alpha_- > 0$, respectively. We restrict ourselves to the case of the pair of independent symmetric stable processes: As we shall see, this restriction is only needed below in verifying Lemmas 18 and 19, and in obtaining an explicit formula for the limit law in Theorem 23. The case of mutually independent Wiener processes is well-known as the Brox model (cf. [5]).

For $w = (w_+, w_-) \in W(= W_+ \times W_+)$ and $\lambda > 0$, we introduce, for each of $+$ and $-$,

$$s_\pm^{\lambda,w}(x) = \lambda^{\alpha_\pm} \int_0^x \mathrm{e}^{\lambda w_\pm(u)} \mathrm{d}u, \quad x \geq 0$$

and

$$m_\pm^{\lambda,w}(x) = 2\lambda^{\alpha_\pm} \int_0^x \mathrm{e}^{-\lambda w_\pm(u)} \mathrm{d}u, \quad x \geq 0.$$

These $s_\pm^{\lambda,w}$ and $m_\pm^{\lambda,w}$ determine the scale s_λ^w and the speed measure $\mathrm{d}m_\lambda^w$ on $\mathbb{R}$ in the same way as (59) and (60). Then, we have the unique $\mathcal{L}^{\lambda,w} = \frac{\mathrm{d}}{\mathrm{d}m_\lambda^w} \frac{\mathrm{d}}{\mathrm{d}s_\lambda^w}$-diffusion process $X^{\lambda,w} = \left(X(t), P_x^{\lambda,w}\right)$ on $\mathbb{R}$ which we realize on Ω as $X(t) = \omega(t), \omega \in \Omega$. Also we have the stochastic process $\overline{X}^\lambda = \left(\overline{X}^\lambda(t), \overline{P}_x^\lambda\right)$ realized on $\Omega \times W$ where $\overline{X}^\lambda(t) = \omega(t), \omega \in \Omega$, and $\overline{P}_x^\lambda(\mathrm{d}\omega, \mathrm{d}w) = P_x^{\lambda,w}(\mathrm{d}\omega)\, P_{\alpha_+,\alpha_-}(\mathrm{d}w)$.

Let, for $a > 0$ and $b > 0$, $T_{a,b} : \mathbb{R} \to \mathbb{R}$ be a homeomorphism on $\mathbb{R}$ defined by

$$T_{a,b}(x) = a \cdot 1_{\{x \geq 0\}} \cdot x + b \cdot 1_{\{x<0\}} \cdot x.$$

Lemma 16. *For each $\lambda > 0$,*

$$\left(T_{\lambda^{-\alpha_+},\lambda^{-\alpha_-}}(\overline{X}), \overline{P}_0\right) \stackrel{d}{=} \left(\overline{X}^\lambda, \overline{P}_0^\lambda\right). \tag{61}$$

Proof. Generally, if $Y = (Y(t), P_x)$ is $\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}m} \frac{\mathrm{d}}{\mathrm{d}s}$-Feller diffusion on $\mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ is a homeomorphism of $\mathbb{R}$, then $T(Y) = (T(Y(t)), P_x)$ is $\widetilde{\mathcal{L}} = \frac{\mathrm{d}}{\mathrm{d}\widetilde{m}} \frac{\mathrm{d}}{\mathrm{d}\widetilde{s}}$-Feller diffusion on $\mathbb{R}$ with $\mathrm{d}\widetilde{m} = \mathrm{d}(m \circ T^{-1})$ and $\widetilde{s} = s \circ T^{-1}$. Using this fact and the self-similarity of stable processes:

$$\left\{\left(w_+(\lambda^{\alpha_+} t), w_-(\lambda^{\alpha_-} t), P_{\alpha_+,\alpha_-}\right)\right\} \stackrel{d}{=} \{(\lambda w_+(t), \lambda w_-(t), P_{\alpha_+,\alpha_-})\},$$

(61) is easily concluded. $\square$

Corollary 17. *If $A(t) = \int_0^t 1_{[0,\infty)}(\overline{X}(u))\mathrm{d}u$ and $A_\lambda(t) = \int_0^t 1_{[0,\infty)}(\overline{X}_\lambda(u))\mathrm{d}u$, then, for each $\lambda > 0$, we have*

$$\left(\{A(t)\}_{t \geq 0}, \overline{P}_0\right) \stackrel{d}{=} \left(\{A_\lambda(t)\}_{t \geq 0}, \overline{P}_0^\lambda\right).$$

The limiting property of $A_\lambda(\mathrm{e}^{\lambda t})$ under $\overline{P}_0^\lambda$ as $\lambda \to \infty$ can be studied by results in Example 13 and Remark 15. For $w = (w_+, w_-) \in W_+ \times W_+(= W)$, we define $\xi_+(t)$ and $\xi_-(t)$ by (55). Note that these may be regarded as stochastic processes under $\overline{P}_0$ as well as under $\overline{P}_0^\lambda$ on $\Omega \times W$.

Lemma 18. *Under P_{α_+,α_-} (and hence, under $\overline{P}_0^\lambda$), (54) holds almost surely.*

Proof. Note first that $w_+ : [0,\infty) \ni x \mapsto w_+(x) \in \mathbb{R}$ and $w_- : [0,\infty) \ni x \mapsto w_-(x) \in \mathbb{R}$ are mutually independent stable processes under P_{α_+,α_-}. Then, by known properties of stable processes, we can easily see that, for each fixed $x \geq 0$, it holds a.s.(P_{α_+,α_-}) that

$$\underline{w_+}((\overline{w_+})^{-1}(x-)-) = \underline{w_+}((\overline{w_+})^{-1}(x))$$

and

$$\underline{w_-}((\overline{w_-})^{-1}(x-)-) = \underline{w_-}((\overline{w_-})^{-1}(x)).$$

By a standard *Fubini argument,* we deduce that these identities hold for almost all $x \in [0,\infty)$, P_{α_+,α_-}-almost surely. Thus (54) holds for $a.a.w\ (P_{\alpha_+,\alpha_-})$. □

Lemma 19. *For each $t \geq 0$, (56) holds almost surely under P_{α_+,α_-} (and hence, under $\overline{P}_0^\lambda$).*

Proof. This is easily provided by the independence of ξ_+ and ξ_- combined with the fact that ξ_+ and ξ_- have no time of fixed discontinuity. □

Thus by Example 13 and Remark 15, combined with Lemmas 18 and 19, we can conclude that, under $\overline{P}_0^\lambda$,

$$\left\{\frac{1}{\lambda}\log A_\lambda(\mathrm{e}^{\lambda t})\right\}_{t\geq 0} \longrightarrow \left\{\xi_+((\xi_-)^{-1}(t)) \wedge t\right\}_{t\geq 0}$$

as $\lambda \to \infty$ in the sense of convergence of all finite-dimensional distributions as well as the convergence in law as Φ-valued random variables. Then, by Corollary 17, we immediately obtain the following:

Theorem 20. *Under $\overline{P}_0$, we have*

$$\left\{\frac{1}{\lambda}\log A(\mathrm{e}^{\lambda t})\right\}_{t\geq 0} \longrightarrow \left\{\xi_+((\xi_-)^{-1}(t)) \wedge t\right\}_{t\geq 0}, \quad \lambda \to \infty$$

in the sense of convergence of all finite-dimensional distributions as well as the convergence in law as Φ-valued random variables. In particular,

$$\frac{1}{\log t}\log A(t) \xrightarrow{d} \xi_+((\xi_-)^{-1}(1)) \wedge 1, \quad \text{as } t \to \infty. \tag{62}$$

From (62), we have, for every $\beta < 1$,

$$\liminf_{t\to\infty} \overline{P}_0\left(\frac{1}{\log t}\log A(t) < \beta\right) \geq \overline{P}_0\left(\xi_+((\xi_-)^{-1}(1)) < \beta\right).$$

From this, we can easily deduce that, for every $\epsilon > 0$,

$$\liminf_{t\to\infty} \overline{P}_0\left(\frac{A(t)}{t} < \epsilon\right) \geq \overline{P}_0\left((\xi_-)^{-1}(1) < (\xi_+)^{-1}(1)\right).$$

By the same argument applied to $t - A(t)$ in place of $A(t)$, we obtain

$$\liminf_{t\to\infty} \overline{P}_0\left(\frac{A(t)}{t} > 1-\epsilon\right) \geq \overline{P}_0\left((\xi_+)^{-1}(1) < (\xi_-)^{-1}(1)\right).$$

Since $\overline{P}_0((\xi_+)^{-1}(1) = (\xi_-)^{-1}(1)) = 0$, we conclude the following

Corollary 21. *Under* $\overline{P}_0$,

$$\frac{A(t)}{t} \xrightarrow{d} 1_{\{(\xi_+)^{-1}(1)<(\xi_-)^{-1}(1)\}} \quad \text{as } t \to \infty. \tag{63}$$

Thus, the limit random variable in the law of $A(t)/t$ as $t \to \infty$ is the Bernoulli random variable $Y_{p,0}$ where

$$p = \overline{P}_0\left((\xi_+)^{-1}(1) < (\xi_-)^{-1}(1)\right)\left(= P_{\alpha_+,\alpha_-}\left((\xi_+)^{-1}(1) < (\xi_-)^{-1}(1)\right)\right). \tag{64}$$

As we remark below, this value p is always $1/2$ for every α_+ and α_-.

Remark 22. As for the almost sure result for the ratio $A(t)/t$ as $t \to \infty$, we have the following: $\overline{P}_0$-almost surely, (equivalently, P_0^w-almost surely for P_{α_+,α_-}-almost all environments $w = (w_+, w_-)$), it holds that

$$\limsup_{t\to\infty} \frac{A(t)}{t} = 1 \quad \text{and} \quad \liminf_{t\to\infty} \frac{A(t)}{t} = 0.$$

This fact can be proved by applying a result of Bertoin [1]: Bertoin obtained an integral test for $s_\pm^w(x)$ and $m_\pm^w(x)$ in order that the above asymptotics for the ratio $A(t)/t$ should hold. We can show that the conditions of the integral test are fulfilled by P_{α_+,α_-}-almost all $w = (w_+, w_-)$: For this, we use the above fact (63) and combine this with the Hewitt–Savage 0–1 law for stable processes w_+ and w_-. We omit the details.

The distribution of $\xi_+((\xi_-)^{-1}(1)) \wedge 1$ can be obtained explicitly as follows.

Theorem 23. *It holds that*

$$\xi_+((\xi_-)^{-1}(1)) \stackrel{d}{=} \frac{U_2}{U_1}, \tag{65}$$

where U_1 and U_2 are mutually independent $(0,1)$-valued random variables such that $U_1 \stackrel{d}{=} V(\alpha_+)$ and $U_2 \stackrel{d}{=} V(\alpha_-)$, where $V(\alpha)$, $0 < \alpha \le 2$, are $(0,1)$-valued random variables with the distribution function $P(V(\alpha) \le x) = u_\alpha(x)$, $0 < x < 1$, given by

$$u_\alpha(x) = \begin{cases} \dfrac{\sin\frac{\alpha\pi}{2}}{\pi} x^{\frac{\alpha}{2}} \displaystyle\int_0^\infty \frac{d\xi}{\xi^{\frac{\alpha}{2}}(\xi+1)(1+(1-x)\xi)^{\frac{\alpha}{2}}}, & 0 < \alpha < 2, \\ x, \quad \alpha = 2. \end{cases} \tag{66}$$

The random variable $V(\alpha)$ has the following symmetry:

$$1 - V(\alpha) \stackrel{d}{=} V(\alpha), \tag{67}$$

or equivalently, the function $u_\alpha(x)$ in (66) satisfies

$$1 - u_\alpha(x) = u_\alpha(1-x), \quad 0 \le x \le 1. \tag{68}$$

So the limit random variable in the law of (62) is given by $(U_2/U_1) \wedge 1$.

Proof. Recall that $\{\xi_+(t)\}_{t\geq 0}$ is defined from a symmetric stable process $\{w_+(t)\}_{t\geq 0}$ of exponent α_+ by $\xi_+(t) = t + \underline{w_+}((\overline{w_+})^{-1}(t))$ where $\overline{w_+}(t) = \sup_{0\leq u\leq t} w_+(u)$ and $\underline{w_+}(t) = \sup_{0\leq u\leq t}(-w_+(u))$. Also, $\{\xi_-(t)\}$ is defined from $\{w_-(t)\}_{t\geq 0}$ in the same way. Denoting by $\{x(t), P_a\}$ the stable process of index α_+ starting at $a \in \mathbb{R}$ and by σ_b the first leaving time from the interval $(-b, b)$; $\sigma_b = \inf\{t > 0 \mid |x(t)| \geq b\}$ for $b > 0$, we have for $x > t > 0$,

$$\begin{aligned} P(\xi_+(t) \geq x) &= P\left(\underline{w_+}((\overline{w_+})^{-1}(t)) \geq x - t\right) \\ &= P_{\frac{x}{2}-t}\left(x(\sigma_{\frac{x}{2}}) \in (-\infty, -x/2]\right). \end{aligned} \tag{69}$$

We have a general formula (cf. [2]): If $|a| < b$,

$$P_a(x(\sigma_b) \in \mathrm{d}y) = \begin{cases} 1_{\{|y|>b\}} \cdot \dfrac{\sin\frac{\alpha_+\pi}{2}}{\pi} \dfrac{(b^2-a^2)^{\frac{\alpha_+}{2}}}{(y^2-b^2)^{\frac{\alpha_+}{2}}} \cdot \dfrac{\mathrm{d}y}{|y-a|}, & 0 < \alpha_+ < 2, \\ \dfrac{a+b}{2b}\delta_b(\mathrm{d}y) + \dfrac{b-a}{2b}\delta_{-b}(\mathrm{d}y), & \alpha_+ = 2. \end{cases}$$

Using this formula, we can easily calculate the RHS of (69) to obtain

$$P(\xi_+(t) \geq x) = u_{\alpha_+}\left(\frac{t}{x}\right), \quad 0 < t < x,$$

where $u_\alpha(x)$, $0 < x < 1$, is given by (66). Hence

$$P\left(\frac{1}{\xi_+(t)} \leq x\right) = u_{\alpha_+}(tx), \quad 0 < x < \frac{1}{t}. \tag{70}$$

By the self-similarity of stable processes, it is easy to deduce under P_{α_+,α_-} that $\{\xi_\pm(ct)\}_{t\geq 0} \stackrel{d}{=} \{c\,\xi_\pm(t)\}_{t\geq 0}$, respectively, for each $c > 0$. Then, noting the independence of ξ_+ and ξ_-, we have under P_{α_+,α_-},

$$\xi_+((\xi_-)^{-1}(1)) \stackrel{d}{=} (\xi_-)^{-1}(1) \cdot \xi_+(1).$$

Also,

$$P\left((\xi_-)^{-1}(1) \leq x\right) = P(1 \leq \xi_-(x)) = P\left(\frac{1}{\xi_-(x)} \leq 1\right) = u_{\alpha_-}(x).$$

Therefore, setting $U_1 = 1/\xi_+(1)$ and $U_2 = (\xi_-)^{-1}(1)$, we can now conclude the assertion of the theorem. A proof of (68) is as follows: (69) and (70) imply that

$$u_\alpha(x) = P_{\frac{1}{2}-x}(x(\sigma_{1/2}) \in (-\infty, -1/2]), \quad 0 \leq x \leq 1.$$

Then we have

$$\begin{aligned} 1 &= P_{\frac{1}{2}-x}(x(\sigma_{1/2}) \in (-\infty, -1/2]) + P_{\frac{1}{2}-x}(x(\sigma_{1/2}) \in [1/2, \infty)) \\ &= u_\alpha(x) + P_{\frac{1}{2}-x}(x(\sigma_{1/2}) \in [1/2, \infty)) \end{aligned}$$

and, using the symmetry of the symmetric stable process as stated in the form $\{x(t), P_a\} \stackrel{d}{=} \{-x(t), P_{-a}\}$,

$$P_{\frac{1}{2}-x}(x(\sigma_{1/2}) \in [1/2, \infty)) = P_{-\frac{1}{2}+x}(x(\sigma_{1/2}) \in (-\infty, -1/2]) = u_\alpha(1-x).$$

Hence, $1 = u_\alpha(x) + u_\alpha(1-x)$ and (68) is proved. $\square$

In the case of $\alpha_+ = \alpha_- = 2$, in particular, the case of the Brox model, we have the following convergence:

$$P\left(\frac{1}{\log t}\log A(t) \leq x\right) \longrightarrow \begin{cases} \frac{x}{2}, & 0 < x < 1, \\ 1, & x \geq 1, \end{cases} \quad \text{as } t \to \infty. \tag{71}$$

Note that p in (64) is given by

$$p = \int_0^1 u_{\alpha_+}(x)\mathrm{d}u_{\alpha_-}(x),$$

and by the symmetry (68), we can easily deduce that *this value is always* $1/2$ *for every* α_+ *and* α_-.

4.2. The case of random walks on $\mathbb{Z}$

Given a sequence $\mathbf{p} = (p_i)$, $i \in \mathbb{Z}$, such that $0 < p_i < 1$, let $Z = (Z(n), P_i^Z)$ be a time-homogeneous Markov chain on $\mathbb{Z}$ with the discrete time $n = 0, 1, 2, \ldots$ and with the one-step transition probability $p_{i,j}$, $i, j \in \mathbb{Z}$ given by

$$p_{i,j} = p_i \cdot \delta_{j,i+1} + (1 - p_i) \cdot \delta_{j,i-1}. \tag{72}$$

P_i^Z is the probability law governing the paths starting at i so that $P_i^Z(Z(0) = i) = 1$ and $P_i(Z(n+1) = j \mid Z(1), \ldots, Z(n)) = p_{Z(n),j}$, $j \in \mathbb{Z}$. We denote it as $Z^{\mathbf{p}} = (Z(n), P_i^{Z,\mathbf{p}})$ when we emphasize its dependence on $\mathbf{p} = (p_i)$.

As is well-known, Z can be imbedded in a birth-and-death process on $\mathbb{Z}$ which is a generalized diffusion process associated with the Feller generator $\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$ with the speed measure $\mathrm{d}m$ supported on $\mathbb{Z}$. Here, the scale $s(x)$ and the speed measure $\mathrm{d}m(x)$ are given as follows:

$$\begin{cases} s(0) = 0, \qquad s(l) = \displaystyle\sum_{k=0}^{l-1}\prod_{i=0}^{k}\frac{1-p_i}{p_i}, \quad l = 1, 2, \ldots, \\ s(-1) = -1, \qquad s(l) = -1 - \displaystyle\sum_{k=1}^{-l-1}\prod_{i=1}^{k}\frac{p_{-i}}{1-p_{-i}}, \quad l = -2, -3, \ldots \end{cases} \tag{73}$$

and $s(x)$, $-\infty < x < \infty$, is given by the piecewise linear continuous extension;

$$s(x) = s(l) + (s(l+1) - s(l))(x - l), \quad l < x < l + 1.$$

The speed measure $\mathrm{d}m(x)$ is defined by setting

$$m(x) = \begin{cases} \displaystyle\sum_{k=0}^{l}\frac{1}{p_k}\prod_{i=0}^{k}\frac{p_i}{1-p_i}, & l \leq x < l+1,\ l = 0, 1, 2, \ldots \\ 0, \quad -1 \leq x < 0, \\ -\displaystyle\sum_{k=1}^{-l}\frac{1}{1-p_{-k}}\prod_{i=1}^{k}\frac{1-p_{-i}}{p_{-i}}, & l - 1 \leq x < l,\ l = -1, -2, \ldots, \end{cases} \tag{74}$$

so that

$$\mathrm{d}m(x) = \sum_{k=0}^{\infty}\frac{1}{p_k}\prod_{i=0}^{k}\frac{p_i}{1-p_i}\,\delta_k(\mathrm{d}x) + \sum_{k=1}^{\infty}\frac{1}{1-p_{-k}}\prod_{i=1}^{k}\frac{1-p_{-i}}{p_{-i}}\,\delta_{-k}(\mathrm{d}x).$$

We assume that $\lim_{x\to\infty} s(x) = -\lim_{x\to-\infty} s(x) = \infty$ so that the random walk is recurrent. Then, with the Feller generator $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$, a unique recurrent generalized diffusion $X = (X(t), P_i^X)$ is associated on the support of dm, which is $\mathbb{Z}$ in this case, so that X is a birth-and-death process on $\mathbb{Z}$. We denote it as $X^{\mathbf{p}} = (X(t), P_i^{X,\mathbf{p}})$ when we emphasize its dependence on $\mathbf{p}$. If σ_n is the nth jumping time of $X(t)$, then

$$(X(\sigma_n), P_i^X)_{n=0,1,2,\ldots} \stackrel{d}{=} (Z(n), P_i^Z)_{n=0,1,2,\ldots}.$$

Also, it is easy to see that the family $\{\sigma_n - \sigma_{n-1},\ n = 1, 2, \ldots\}$ is i.i.d. with mean 1 exponential distribution, so that, by the strong law of large numbers, we have

$$P_i^X\left(\text{for every } t_0 > 0,\ \sup_{t_0\le t<\infty}\left|\frac{\sigma_{[\mathrm{e}^{\lambda t}]}}{\mathrm{e}^{\lambda t}} - 1\right| \to 0 \quad \text{as } \lambda\to\infty\right) = 1. \tag{75}$$

Here $[x]$ denotes the largest integer not exceeding x: $[x] = n$ if $n \le x < n+1$, $n \in \mathbb{Z}$. It is now obvious that the random walk $(Z(n), P_i^Z)$ is recurrent.

We define ξ_i, $i \in \mathbb{Z}$, by

$$\xi_i = \begin{cases} \log \dfrac{1-p_i}{p_i}, & i = 0, 1, 2, \ldots \\ \log \dfrac{p_i}{1-p_i}, & i = -1, -2, \ldots. \end{cases} \tag{76}$$

Also, we define two càdlàg functions $\theta_+(x)$ and $\theta_-(x)$ on $[0, \infty)$ by setting

$$\theta_+(x) = \sum_{i=0}^{[x]} \xi_i, \quad 0 \le x < \infty \tag{77}$$

and

$$\theta_-(x) = \begin{cases} 0, & 0 \le x < 1 \\ \displaystyle\sum_{i=1}^{[x]} \xi_{-i}, & 1 \le x < \infty. \end{cases} \tag{78}$$

We extend $\theta_\pm$ to be defined on $\mathbb{R}$ by setting $\theta_\pm(x) = 0,\ x < 0$. We put the following assumption (A.4) on the sequence $\mathbf{p} = (p_i)$:

(A.4) There exist two càdlàg functions $w_+(x) \in W_+$ and $w_-(x) \in W_+$ on $[0, \infty)$, and two positive increasing functions $\psi_+(\lambda)$ and $\psi_-(\lambda)$ of $\lambda > 0$ with $\lim_{\lambda\to\infty} \psi_\pm(\lambda) = \infty$, such that the following convergence holds:

$$\left(\frac{1}{\lambda}\theta_+(\psi_+(\lambda)t), \frac{1}{\lambda}\theta_-(\psi_-(\lambda)t)\right) \longrightarrow (w_+(t), w_-(t)) \quad \text{as } \lambda \to \infty \tag{79}$$

in the sense of Skorohod's J_1-topology on $W_+ \times W_+$. (For the definition of W_+, see the beginning of Section 4.1).

We define continuous increasing functions $s_+(x)$ and $s_-(x)$ of $x \in [0, \infty)$ by

$$s_\pm(x) = \int_0^x \mathrm{e}^{\theta_\pm(u)}\mathrm{d}u, \quad x \ge 0. \tag{80}$$

We also define càdlàg increasing functions $m_+(x)$ and $m_-(x)$ of $x \in [0, \infty)$ by

$$m_+(x) = \int_0^{[x]+1} \mathrm{e}^{-\theta_+(u)} \mathrm{d}u + \int_0^{[x]+1} \mathrm{e}^{-\theta_+(u-1)} \mathrm{d}u, \quad x \geq 0 \tag{81}$$

and

$$m_-(x) = \int_0^{[x]} \mathrm{e}^{-\theta_-(u)} \mathrm{d}u + \int_0^{[x]} \mathrm{e}^{-\theta_-(u+1)} \mathrm{d}u, \quad x \geq 0. \tag{82}$$

We can easily verify from (73) and (74) that

$$s(x) = s_+(x), \qquad m(x) = m_+(x) \quad \text{for } x \geq 0, \tag{83}$$

and

$$s(x) = -s_-(-x), \qquad m(x) = -m_-(-x-0) \quad \text{for } x < 0. \tag{84}$$

In verifying the identities $m(x) = m_+(x)$ for $x \geq 0$ and $m(x) = -m_-(-x-0)$ for $x < 0$, note the simple relations $\frac{1}{p_k} = \frac{1-p_k}{p_k} + 1$, $k = 0, 1, \ldots$ and $\frac{1}{1-p_{-k}} = \frac{p_{-k}}{1-p_{-k}} + 1$, $k = 1, 2, \ldots$.

We now set, for $\lambda > 0$ and each of $+$ and $-$,

$$s_\lambda^\pm(x) = s_\pm(\psi_\pm(\lambda)x), \quad x \geq 0, \tag{85}$$

$$m_\lambda^\pm(x) = m_\pm(\psi_\pm(\lambda)x), \quad x \geq 0, \tag{86}$$

$$w_\lambda^\pm(x) = \frac{1}{\lambda}\theta_\pm(\psi_\pm(\lambda)x), \quad x \geq 0, \tag{87}$$

$$v_\lambda^+(x) = \frac{1}{\lambda}\theta_+(\psi_+(\lambda)x - 1), \quad x \geq 0, \tag{88}$$

$$v_\lambda^-(x) = \frac{1}{\lambda}\theta_-(\psi_-(\lambda)x + 1), \quad x \geq 0, \tag{89}$$

$$\varphi_\lambda^+(x) = \frac{[\psi_+(\lambda)x] + 1}{\psi_+(\lambda)}, \quad x \geq 0 \tag{90}$$

and

$$\varphi_\lambda^-(x) = \frac{[\psi_-(\lambda)x]}{\psi_-(\lambda)}, \quad x \geq 0. \tag{91}$$

Then, it easy to deduce from (80)–(82) that, for each of $+$ and $-$,

$$s_\lambda^\pm(x) = \psi_\pm(\lambda) \cdot \int_0^x \mathrm{e}^{\lambda w_\lambda^\pm(u)} \mathrm{d}u$$

and

$$m_\lambda^\pm(x) = \psi_\pm(\lambda) \cdot \int_0^{\varphi_\lambda^\pm(x)} \left\{ \mathrm{e}^{-\lambda w_\lambda^\pm(u)} + \mathrm{e}^{-\lambda v_\lambda^\pm(u)} \right\} \mathrm{d}u.$$

(79) implies that

$$(w_\lambda^+(t), w_\lambda^-(t)) \longrightarrow (w_+(t), w_-(t))$$

and

$$(v_\lambda^+(t), v_\lambda^-(t)) \longrightarrow (w_+(t), w_-(t))$$

as $\lambda \to \infty$ in the J_1-topology on $W_+ \times W_+$.

Thus, our family defined above satisfies the conditions in Example 14 so that we have the same conclusion for the occupation time $A_\lambda(t) = \int_0^t 1_{[0,\infty)}(X_\lambda(u))\mathrm{d}u$ where $X_\lambda = (X_\lambda(t))$ is $\mathcal{L}_\lambda = \frac{\mathrm{d}}{\mathrm{d}m_\lambda}\frac{\mathrm{d}}{\mathrm{d}s_\lambda}$-generalized diffusion. Here, noting (80)–(82) combined with (85) and (86), we have $s_\lambda = s \circ T_{\psi_+(\lambda),\psi_-(\lambda)}$ and $m_\lambda = m \circ T_{\psi_+(\lambda),\psi_-(\lambda)}$, where the map T is defined, as above, by $T_{a,b}(x) = ax\, 1_{\{x\ge 0\}} + bx\, 1_{\{x<0\}}$. Hence,

$$X_\lambda(t) \overset{d}{=} T^{-1}_{\psi_+(\lambda),\psi_-(\lambda)} \circ X(t) \left(= T_{1/\psi_+(\lambda),\,1/\psi_-(\lambda)}(X(t))\right)$$

from the birth-and-death process $X = (X(t), P_i^X)$ introduced at the beginning of this subsection.

This conclusion can be applied, in the same way as we did in the previous Section 4.1, to the study of the occupation time for the birth-and-death processes in random environment and thereby for random walk in random environment. For the relationship between these two processes, see Zeitouni [16].

First of all, we set up our model. Let Π be the totality of sequences $\mathbf{p} = (p_i), i \in \mathbb{Z}$, such that $0 < p_i < 1$ and $\lim_{x\to\infty} s(x) = -\lim_{x\to-\infty} s(x) = \infty$, where $s(x)$ is defined as above by (73). For $\mathbf{p} \in \Pi$, we have the random walk $Z^{\mathbf{p}} = (Z(n), P_i^{Z,\mathbf{p}})$ on $\mathbb{Z}$ as above which we realize canonically on $\Omega_{\mathbb{Z}}^{(1)} = \{\omega : \mathbb{Z}_+ \ni n \mapsto \omega(n) \in \mathbb{Z}\}$ so that $Z(n) = \omega(n), \omega \in \Omega_{\mathbb{Z}}^{(1)}$ and $P_i^{Z,\mathbf{p}}$ is the probability on $\Omega_{\mathbb{Z}}^{(1)}$ supported on $\{\omega : \omega(0) = i\}$. Here, we use the notation $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}_- = \{-1, -2, \ldots\}$ so that $\mathbb{Z} = \mathbb{Z}_+ \cup \mathbb{Z}_-$. A random walk in random environment is determined by giving a probability P on Π; we realize this on $\Omega_{\mathbb{Z}}^{(1)} \times \Pi$ with the annealed probability $\overline{P}_i^Z(\mathrm{d}\omega\, d\mathbf{p}) = P(d\mathbf{p})\, P_i^{Z,\mathbf{p}}(\mathrm{d}\omega)$, so that $P_i^{Z,\mathbf{p}}$ can be regarded as the conditional probability given the environment $\mathbf{p}$.

In the same way, for given $\mathbf{p} \in \Pi$, we have the birth-and-death process $X^{\mathbf{p}} = (X(t), P_i^{X,\mathbf{p}})$ on $\mathbb{Z}$, which we realize on $\Omega_{\mathbb{Z}}^{(2)} = \{\omega : [0,\infty) \ni t \mapsto \omega(t),\ \text{càdlàg}\ \}$ as $X(t) = \omega(t),\ \omega \in \Omega_{\mathbb{Z}}^{(2)}$ and $P_i^{X,\mathbf{p}}$ is the law of $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$- generalized diffusion starting at i with s and m defined by (73) and (74), which is a probability on $\Omega_{\mathbb{Z}}^{(2)}$ supported on $\{\omega : \omega(0) = i\}$. The birth-and-death process in random environment is defined by the probability $\overline{P}_i^X(\mathrm{d}\omega\,\mathrm{d}\mathbf{p}) = P(\mathrm{d}\mathbf{p})\, P_i^{X,\mathbf{p}}(\mathrm{d}\omega)$.

We put, in accordance with the assumption (A.4), the following fundamental assumption (A.5) on the probability P on Π:

(A.5) Under P on Π, $\mathbf{p} = (p_i)$ satisfies the following; the families $\mathbf{p}_+ := \{p_0, p_1, \ldots\}$ and $\mathbf{p}_- := \{p_{-1}, p_{-2}, \ldots\}$ are mutually independent i.i.d. families. Furthermore, there exist two constants $0 < \alpha_+ \le 2$ and $0 < \alpha_- \le 2$, two positive increasing functions $\psi_+(\lambda)$ and $\psi_-(\lambda)$ of $\lambda > 0$ with $\lim_{\lambda\to\infty} \psi_\pm(\lambda) = \infty$ such that the following convergence holds:

$$\left(\frac{1}{\lambda}\theta_+(\psi_+(\lambda)t), \frac{1}{\lambda}\theta_-(\psi_-(\lambda)t)\right) \xrightarrow{d} (w_+(t), w_-(t)) \quad \text{as } \lambda \to \infty \tag{92}$$

in the sense of J_1-topology on $W_+ \times W_+$, where $\{w_+(t)\}$ and $\{w_-(t)\}$ are mutually independent symmetric stable processes of exponents α_+ and α_-, respectively.

Here, càdlàg increasing processes $\theta_+(t)$ and $\theta_-(t)$ are defined from $\mathbf{p} = (p_i)$ by (76)–(78).

Now, for almost all $\mathbf{p} \in \Pi$, we can define, for $\lambda > 0$ and each of $+$ and $-$, $s_\lambda^\pm$, $m_\lambda^\pm$, $w_\lambda^\pm$, $v_\lambda^\pm$, $\varphi_\lambda^\pm$ as above through (85)–(91). From $s_\lambda^\pm$ and $m_\lambda^\pm$, càdlàg increasing functions $s_\lambda(x)$ and $m_\lambda(x)$ on $\mathbb{R}$ are defined as in Example 14 and the unique recurrent generalized diffusion $X^\lambda = (X^\lambda(t), P_x^\lambda)$ is associated with the Feller generator $\mathcal{L}_\lambda = \frac{\mathrm{d}}{\mathrm{d}m_\lambda}\frac{\mathrm{d}}{\mathrm{d}s_\lambda}$. As we remarked above, X^λ is obtained from $X^{\mathbf{p}} = (X(t), P_i^{X,\mathbf{p}})$ by $\{X^\lambda(t)\} \stackrel{d}{=} \left\{T_{1/\psi_+(\lambda),\,1/\psi_-(\lambda)}(X(t))\right\}$ so that the state space of X^λ is $\frac{1}{\psi_+(\lambda)}\mathbb{Z}_+ \cup \frac{1}{\psi_-(\lambda)}\mathbb{Z}_-$. Define

$$A_\lambda(t) = \int_0^t 1_{[0,\infty)}(X^\lambda(s))\mathrm{d}s \quad \text{and} \quad A^X(t) = \int_0^t 1_{[0,\infty)}(X(s))\mathrm{d}s.$$

Then obviously

$$\{A_\lambda(t), \overline{P}_0^\lambda\} \stackrel{d}{=} \{A^X(t), \overline{P}_0^X\},$$

where $\overline{P}_0^\lambda(\mathrm{d}\omega\,\mathrm{d}\mathbf{p}) = P(\mathrm{d}\mathbf{p})\,P_0^\lambda(\mathrm{d}\omega)$. For $A_\lambda(t)$, we can apply the result of Example 14 with Remark 15 and, using this result, we can give the same arguments as in Section 4.1 to obtain the following:

Theorem 24. *Under $\overline{P}_0^X$,*

$$\left\{\frac{1}{\lambda}\log A^X(\mathrm{e}^{\lambda t})\right\}_{t\ge 0} \longrightarrow \left\{\xi_+((\xi_-)^{-1}(t)) \wedge t\right\}_{t\ge 0}, \quad \text{as } \lambda \to \infty \tag{93}$$

in the sense of convergence of all finite-dimensional distributions as well as the convergence in law as Φ-valued random variables. In particular

$$\frac{1}{\log t}\log A^X(t) \xrightarrow{d} \xi_+((\xi_-)^{-1}(1)) \wedge 1 \quad \text{as } t \to \infty. \tag{94}$$

Here, the processes ξ_+ and ξ_- are defined from independent symmetric stable processes w_+ and w_- in the same way as in Theorem 20, namely by (53) and (55), and the limit law in (94) is given in the same way as Theorem 23.

We can now deduce from this our final result for the random walk in the random environment $Z = (Z(n), \overline{P}_i^Z)$.

Theorem 25. *Let $A^Z(n) = \sum_{k=1}^n 1_{\{Z(k-1)\ge 0, Z(k)\ge 0\}}$. Then, under $\overline{P}_0^Z$,*

$$\left\{\frac{1}{\lambda}\log A^Z([\mathrm{e}^{\lambda t}])\right\}_{t\ge 0} \longrightarrow \left\{\xi_+((\xi_-)^{-1}(t)) \wedge t\right\}_{t\ge 0}, \quad \text{as } \lambda \to \infty \tag{95}$$

in the sense of convergence of all finite-dimensional distributions as well as the convergence in law as Φ-valued random variables. In particular,

$$\frac{1}{\log n}\log A^Z(n) \xrightarrow{d} \xi_+((\xi_-)^{-1}(1)) \wedge 1 \quad \text{as } n \to \infty. \tag{96}$$

Here, the processes ξ_+ and ξ_- are defined from independent symmetric stable processes w_+ and w_- by (53) and (55) so that the law of the limit random variable is given by Theorem 23.

Proof. For $\mathbf{p} \in \Pi$, let $X^{\mathbf{p}} = (X(t), P_i^{X,\mathbf{p}})$ be the birth-and-death process on $\mathbb{Z}$ as given above realized on the space $\Omega_{\mathbb{Z}}^{(2)}$. It is a $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$-generalized diffusion process where $s(x)$ and $m(x)$ are defined by (84) through $\mathbf{p}$. We assume $\mathbf{p}$ possesses the property that both $\lim_{x\to\infty} s(x) = \lim_{x\to-\infty}(-s(x)) = \infty$ and $\lim_{x\to\infty} m(x) = \lim_{x\to-\infty}(-m(x)) = \infty$ hold. We note that almost every $\mathbf{p} \in \Pi$ under $P(d\mathbf{p})$ possesses this property.

Let σ_n, $n = 0, 1, 2, \ldots$, be the nth jumping time of $X(t)$:

$$\sigma_0 = 0, \qquad \sigma_n = \inf\{t > \sigma_{n-1} | X(t) \neq X(\sigma_{n-1})\}, \quad n = 1, 2, \ldots. \tag{97}$$

Set

$$Z(n) = X(\sigma_n), \quad n = 0, 1, \ldots \tag{98}$$

and

$$\tau_n = \sigma_n - \sigma_{n-1}, \quad n = 1, 2, \ldots. \tag{99}$$

Then $\{Z(n)\}$ under $P_i^{X,\mathbf{p}}$ is a realization of the random walk (in a fixed environment $\mathbf{p}$) such that $Z(0) = i$. Note that $\{\tau_n\}$ is i.i.d. with mean 1 exponential distribution. Also, $\{X(\sigma_n)\}$ and $\{\sigma_n\}$ are independent.

In the following, we consider under the probability $P_0^{X,\mathbf{p}}$: Set, as above,

$$A^Z(n) = \sum_{k=1}^{n} 1_{\{Z(k-1)\geq 0, Z(k)\geq 0\}}, \quad n = 1, 2, \ldots, \tag{100}$$

and also

$$A(n) = \sum_{k=1}^{n} 1_{\{Z(k-1)\geq 0\}}, \quad n = 1, 2, \ldots, \tag{101}$$

$$A^X(t) = \int_0^t 1_{[0,\infty)}(X(s))\mathrm{d}s. \tag{102}$$

Then, we have

$$A^X(\sigma_n) = \sum_{k=1}^{n} 1_{\{Z(k-1)\geq 0\}} \cdot \tau_k \tag{103}$$

and hence

$$\frac{A^X(\sigma_n)}{A(n)} = \frac{\sum_{k=1}^{n} \tau_k \, 1_{\{Z(k-1)\geq 0\}}}{\sum_{k=1}^{n} 1_{\{Z(k-1)\geq 0\}}}.$$

Let a random sequence $\{k_i\}$ of positive integers be defined, successively, by

$$k_1 = 1, \qquad k_i = \min\{k > k_{i-1} | Z(k-1) \geq 0\}, \quad i = 2, 3, \ldots.$$

Since $X^{\mathbf{p}}$ is recurrent, $\{k_i\}$ is well-defined a.s. Set $\theta_i = \tau_{k_i}$, $i = 1, 2, \ldots$. By the strong Markov property of $X^{\mathbf{p}}$, we can easily deduce that $\{\theta_i\}$ is also i.i.d. with mean 1 exponential distribution. Obviously, we have

$$A^X(\sigma_n) = \theta_1 + \theta_2 + \cdots + \theta_i \quad \text{and} \quad A(n) = i \quad \text{if } k_i \leq n < k_{i+1}.$$

Hence, by the strong law of large numbers, the following convergence occurs almost surely:

$$\frac{A^X(\sigma_n)}{A(n)} = \frac{\theta_1 + \theta_2 + \cdots + \theta_i}{i} \longrightarrow 1, \quad \text{as } n \to \infty.$$

Thus we have

$$A(n) = A^X(\sigma_n)(1 + o(1)) \quad \text{as } n \to \infty \text{ a.s.} \tag{104}$$

If we set $\widetilde{a}(n) = \int_0^{\sigma_n} 1_{\{0\}}(X(s))\mathrm{d}s$, then $\widetilde{a}(n) = \sum_{k=1}^n \tau_k 1_{\{Z(k-1)=0\}}$ and, by the same argument as for the proof of (104), we have

$$a(n) = \widetilde{a}(n)(1 + o(1)) \quad \text{as } n \to \infty \text{ a.s.,}$$

where $a(n) = \sum_{k=1}^n 1_{\{Z(k-1)=0\}}$. Since

$$\begin{aligned} A(n) - A^Z(n) &= \sum_{k=1}^n 1_{\{Z(k-1)\geq 0, Z(k)<0\}} \\ &= \sum_{k=1}^n 1_{\{Z(k-1)=0, Z(k)=-1\}} \leq \sum_{k=1}^n 1_{\{Z(k-1)=0\}} = a(n), \end{aligned}$$

we have

$$0 \leq A(n) - A^Z(n) \leq \widetilde{a}(n)(1 + o(1)) \quad \text{as } n \to \infty \text{ a.s.} \tag{105}$$

Finally, we have

$$\widetilde{a}(n) = o(A^X(\sigma_n)) \quad \text{as } n \to \infty \text{ a.s.} \tag{106}$$

This follows from the well-known ratio ergodic theorem for a recurrent $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$-generalized diffusion $X(t)$ on $\mathbb{R}$, which states that, if B_1 and B_2 are Borel subsets of $\mathbb{R}$ such that $0 < \widetilde{m}(B_1) < \infty$ and $0 < \widetilde{m}(B_2) \leq \infty$, then

$$\frac{\int_0^t 1_{B_1}(X(s))\mathrm{d}s}{\int_0^t 1_{B_2}(X(s))\mathrm{d}s} \longrightarrow \frac{\widetilde{m}(B_1)}{\widetilde{m}(B_2)} \quad \text{as } t \to \infty, \text{ a.s.,}$$

where $\widetilde{m}(\mathrm{d}x) = d(m \circ s^{-1}(x))$, (cf. [8]).

Note that, in our case,

$$\frac{\widetilde{a}(n)}{A^X(\sigma_n)} = \frac{\int_0^{\sigma_n} 1_{\{0\}}(X(s))\mathrm{d}s}{\int_0^{\sigma_n} 1_{[0,\infty)}(X(s))\mathrm{d}s}, \quad \text{and} \quad \widetilde{m}(\{0\}) < \infty \quad \text{but } \widetilde{m}([0,\infty)) = \infty.$$

From (104)–(106), we have

$$A^Z(n) = A^X(\sigma_n)(1 + o(1)) \quad \text{as } n \to \infty, \text{ a.s.}$$

for $P_0^{X,\mathbf{p}}$ and hence, a.s. for $\overline{P}_0^X$. We can now deduce the convergence (95) from the convergence (93) combined with (75). □

In the same way as Corollary 21, we have the following

Corollary 26. *Under* $\overline{P}_0^Z$,

$$\frac{A^Z(n)}{n} \xrightarrow{d} 1_{\{(\xi_+)^{-1}(1)<(\xi_-)^{-1}(1)\}} \quad \text{as } n \to \infty. \tag{107}$$

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