

# The small world effect on the coalescing time of random walks

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## Abstract

A small world is obtained from the  $d$ -dimensional torus of size  $2L$  adding randomly chosen connections between sites, in a way such that each site has exactly one random neighbour in addition to its deterministic neighbours. We study the asymptotic behaviour of the meeting time  $T_L$  of two random walks moving on this small world and compare it with the result on the torus. On the torus, in order to have convergence, we have to rescale  $T_L$  by a factor  $C_1 L^2$  if  $d = 1$ , by  $C_2 L^2 \log L$  if  $d = 2$  and  $C_d L^d$  if  $d \geq 3$ . We prove that on the small world the rescaling factor is  $C'_d L^d$  and identify the constant  $C'_d$ , proving that the walks always meet faster on the small world than on the torus if  $d \leq 2$ , while if  $d \geq 3$  this depends on the probability of moving along the random connection. As an application, we obtain results on the hitting time to the origin of a single walk and on the convergence of coalescing random walk systems on the small world.

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## 1. Introduction

Graphs provide a mathematical model in many scientific areas, from physics (magnetization properties of metals and evolution of gases) to biology (neural networks and disease spreading)

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and sociology (social networks and opinion spreading). Individuals (atoms, molecules, neurons and animals) are identified with vertices and an edge drawn between two vertices identifies a relation as proximity or existence of some sort of contact. When a part or the totality of the edges are subject to some randomness, it is natural to deal with random graphs (see [5] for a survey). One can construct a random graph starting from a deterministic graph either by adding random connections, or by removing some connections randomly, as in percolation. A particular class of random graphs of the first type are *small world graphs*, constructed starting from a  $d$ -dimensional (discrete) torus, whose edges are called *short range connections*, adding some random connections, called *long range connections*.

Bollobas and Chung [4] first noted that by the addition of a random matching in a cycle (i.e.  $d = 1$ ), the average distance between sites is considerably smaller than that on the deterministic graph. Watts and Strogatz [15] introduced, as a model for biological applications, the random graph obtained in  $d = 1$  with each site connected to the ones at Euclidean distance smaller than  $m$  and long range connections constructed by taking the deterministic ones and by moving with probability  $p$  one of the end sites to a new one chosen at random. Another possible construction was introduced by Newmann and Watts [12]: they took the same deterministic short range connections of Watts and Strogatz, but they added a density  $p$  of long range connections between randomly chosen sites. Average distance between sites and clustering coefficient of small world graphs have been well investigated [1,2,15]. See [9] for a historical introduction of small world graphs and main results.

Recently some authors have been focusing on processes taking place on random graphs. Durrett and Jung [10] have studied the contact process on the small world. Their version of the small world (which is also the one we study in the present work) is a generalization of the Bollobas–Chung model: they take the  $d$ -dimensional torus  $\Lambda^d(L) = \mathbb{Z}^d \bmod 2L$  with short range connections between each pair of vertices at Euclidean distance smaller than  $m$ . The long range connections are drawn choosing at random a partition of the  $(2L)^d$  vertices in pairs and connecting each pair of the partition (see Section 2.1 for more details about the construction). Note that all sites have exactly one long range neighbour, which may coincide with a short range one. Nevertheless with large probability most of the sites have a true long range neighbour, and we choose a random walk which makes the small world “stochastically homogeneous” (see the definition of the transition matrix  $P_S$  in Section 2.1). The main advantage of such a construction is that we can associate to the random graph a non-random translation invariant graph  $\mathcal{B}$ , called *big world*. To get an idea of how the big world looks like, see Fig. 1. One starts with a copy of  $\mathbb{Z}^d$  and then attaches to each site an edge joining this site to another copy of  $\mathbb{Z}^d$  and proceeds by (infinite) iteration. This edge represents the long range connection (thus for instance in dimension 1, if the long range neighbour of 0 is 3, then 3 is represented in the big world by the site three steps away from 0 in the first copy of  $\mathbb{Z}$ , but also by the site at the endpoint of the “long range” edge attached to 0, and indeed by many other sites). For more details on this deterministic graph and on its relationship with the small world, see [10] where the big world was first introduced and Section 2.2.

One expects that if the distance between sites plays an important role (as for random walks, coalescing random walk or the contact process), a process taking place on a small world will behave differently from the same one on the torus. We consider random walks on the small world and, under some assumptions on the starting sites, we study the asymptotic behaviour of three sequences of random times: the time  $W_L$  after which a single random walk first hits the origin, the time  $T_L$  after which two random walks first meet and the coalescing time  $\tau_L$  of a coalescing random walk starting from a fixed number of particles. Recall that the coalescing random walk

on a graph is a Markov process in which  $n$  particles perform independent random walks subject to the rule that when one particle jumps onto an already occupied site, the two particles coalesce to one. The time when we first have only one particle left is called *coalescing time*.

It is natural to compare our results with the corresponding results on the torus: for the simple symmetric continuous time random walk on the  $d$ -dimensional torus, Cox [6, Theorem 4] proved (under some assumptions on the initial position) that for  $d = 2W_L/C_2(2L)^2 \log(2L)$ , with  $C_2 = 2/\pi$ , and for  $d \geq 3$ ,  $W_L/C_d(2L)^d$ , with  $C_d$  equal to the expected number of visits to the origin of a discrete time simple symmetric random walk, converge to an exponential of mean 1. One can also get the same result for the random walk starting from the stationary distribution (this was proved in [11, Theorem 6.1] in discrete time) as a corollary.

Cox and Durrett (see [7, Theorem 2]) proved a result in the two-dimensional case under more general conditions on the starting point and on the transition matrix for a discrete time random walk. The case  $d = 1$  is slightly different: Flatto et al. [11, Theorem 6.1] proved that for the discrete time random walk starting from the uniform distribution  $W_L/L^2$  converges to a certain law. It is possible to show that these results also hold in continuous time.

Note that, by the symmetry of the walks on the torus, it is easy to show that the meeting time  $T_L$  of two independent random walks  $X_t$  and  $Y_t$  on the torus, conditioned to  $X_0 = x$  and  $Y_0 = y$ , coincides with the law of  $2W_L$  conditioned to the starting point  $x - y$ . Therefore from Theorems [6, Theorem 4], [7, Theorem 2], and [11, Theorem 6.1] one easily deduces the asymptotic behaviors of the meeting time of two particles.

Since a small world is a random graph, studying random walks on it we have two sources of randomness: the graph and the walk. We denote by  $S^L$  the random variable whose possible values are the small worlds of size  $L$  and by  $S$  (or  $S^L$  if we need to stress the dependence on  $L$ ) one of the possible realizations of  $S^L$ . If  $\Delta$  is the transition matrix of an adapted (i.e. transition from  $x$  to  $y$  may occur only if they are connected by an edge) and translation invariant symmetric random walk on  $\Lambda^d(L)$  and  $\beta \in (0, 1)$ , once a small world  $S$  is fixed, the random walk we consider on it moves according to  $\Delta$  with probability  $1 - \beta$  and moves along the long range connection with probability  $\beta$ . We denote by  $\mathbf{P}$  the uniform probability on all the small worlds; given a small world  $S$ , we denote by  $\mathbb{P}_S^{\mu, \nu}$  the joint law of two independent continuous time random walks starting from the probability distributions  $\mu$  and  $\nu$ . By  $\mathbb{P}$  (with no pedex) we denote the average over all small worlds (see Section 2.1 for the formal definitions).

We look for results for each graph  $S$  in a set of large  $\mathbf{P}$ -probability (“quenched” point of view – note that it is not possible to have almost sure results) and average results (i.e. with respect to  $\mathbb{P}$  – “annealed” point of view).

Durrett [9, Chapter 6] proved, for a large class of random graphs with  $N$  vertices, that for each sequence  $\{S^N\}_{N \geq 0}$  of small worlds chosen in sets of large  $\mathbf{P}$ -probability, if  $T_N$  is the meeting time of two particles starting from the stationary distribution, then  $T_N/CN$  (for some  $C > 0$ ) converges to the exponential distribution. In particular such a result holds for the small worlds we consider in dimension one (clearly with  $N = 2L$ ).

With different techniques we prove more accurate results in dimension  $d$ . We suppose that the two random walks start respectively from the origin 0 and from a site  $x_L$  and we prove convergence to an exponential law of  $T_L/C_d(2L)^d$ , under the assumption that either  $\{x_L\}_L$  is constant or moves towards infinity at a sufficiently large speed (bear in mind that on  $S$  there are two distances between two sites  $x$  and  $y$ : the Euclidean one  $|x - y|$  and the (random) graph distance  $d_S(x, y)$ ). Moreover, we identify the constant  $C_d$ , which is a fundamental tool to compare our results with the corresponding ones on the torus in  $d \geq 3$ .

Recall that there exists a deterministic graph called big world  $\mathcal{B}$  which can be mapped onto the small world. Through the inverse of this map (see Section 2.2 for details) we associate to each site  $x \in \Lambda^d(L)$  a unique site  $+(x)$  in  $\mathcal{B}$ , and to a random walk on the small world we associate a random walk on  $\mathcal{B}$ , whose law is denoted by  $\mathbb{P}_{\mathcal{B}}$ . We denote by  $G_{\mathcal{B}}^{ev}(x)$  the expected time spent at site 0 by the random walk at speed 2 starting from  $+(x)$  on  $\mathcal{B}$ , namely

$$G_{\mathcal{B}}^{ev}(x) := \int_0^\infty \mathbb{P}_{\mathcal{B}}^{+(x)}(X_{2t} = 0) dt. \quad (1.1)$$

Note that such constants depend on the probability  $\beta$ . We omit this dependence to avoid cumbersome notation. We also write 0 for  $+(0)$  for the sake of simplicity.

**Theorem 1.1.** *Let*

$$f(x, t) = \exp(-t/G_{\mathcal{B}}^{ev}(0)) \left\{ \frac{1}{G_{\mathcal{B}}^{ev}(0)} \delta_0(x) + \left(1 - \frac{G_{\mathcal{B}}^{ev}(x)}{G_{\mathcal{B}}^{ev}(0)}\right) (1 - \delta_0(x)) \right\},$$

and let  $g(t) = \exp(-t/G_{\mathcal{B}}^{ev}(0))$ .

1. Let  $x_L \in \Lambda(L)$  for all  $L$  such that  $x_L = x$  for all  $L$  sufficiently large. Then uniformly in  $t \geq 0$

$$\mathbb{P}^{x_L, 0} \left( \frac{T_L}{(2L)^d} > t \right) \xrightarrow{L \rightarrow \infty} f(x, t) \quad (1.2)$$

2. Let  $\alpha_L \geq (\log \log L)^2$ , then uniformly in  $t \geq 0$  and  $x_L$  such that  $|x_L| \geq \alpha_L$ ,

$$\mathbb{P}^{x_L, 0} \left( \frac{T_L}{(2L)^d} > t \right) \rightarrow g(t). \quad (1.3)$$

3. Let  $x_L \in \Lambda(L)$  for all  $L$  such that  $x_L = x$  for all  $L$  sufficiently large. For all  $\varepsilon > 0$

$$\mathbf{P} \left( S : \left| \mathbb{P}_S^{x_L, 0} \left( \frac{T_L}{(2L)^d} > t \right) - f(x, t) \right| < \varepsilon, \forall t \geq 0 \right) \xrightarrow{L \rightarrow \infty} 1. \quad (1.4)$$

4. Let  $\alpha_L \geq (\log \log L)^2$ . For all  $\varepsilon > 0$

$$\mathbf{P} \left( S : \sup_{\{x_L : d_S(0, x_L) \geq \alpha_L\}} \left| \mathbb{P}_S^{x_L, 0} \left( \frac{T_L}{(2L)^d} > t \right) - g(t) \right| < \varepsilon, \forall t \geq 0 \right) \xrightarrow{L \rightarrow \infty} 1. \quad (1.5)$$

The main tools in the proof of this result are: the use of the Laplace transform (much in the footsteps of [6]); the fact that with large  $\mathbf{P}$ -probability a large (but not too large) neighbourhood of a fixed vertex looks like the big world and the fact that for very large times the random walker is approximately uniformly distributed on the graph. We prove Theorem 1.1 for continuous time random walks, but for discrete time random walks (which we denote by  $\tilde{X}_n$ ) the same arguments lead to the corresponding result. The only difference is that instead of  $G_{\mathcal{B}}^{ev}(x)$  one has the expected number of visits at even times

$$\tilde{G}_{\mathcal{B}}^{ev}(x) := \sum_{n=0}^{\infty} \tilde{\mathbb{P}}_{\mathcal{B}}^{+(x)}(\tilde{X}_{2n} = 0). \quad (1.6)$$

Moreover, by a similar argument one proves the result for the hitting time to the origin (see Theorem 4.2). As a corollary one can get the law of the meeting time of two random walks and the law of the hitting time to the origin of a single walker starting from the uniform distribution. Note that while on a translation invariant graph one immediately deduces the results on the

meeting time of two walkers from the results on the hitting time (as on the torus), on random graphs like the small world this is not possible.

We are now able to compare the growth speed of  $T_L$  on the torus and on the small world (when the distance between the two starting points goes to infinity): depending on the dimension  $d$ , there are a function  $f_d(L)$  and a constant  $C_d$  such that  $T_L/C_d f_d(L)$  converges in law. The comparison is summarized in Table 1 (where  $G_{\mathbb{Z}^d}^{ev}(0)$  is the expected time spent at 0 by the process  $\{X_{2t}\}_{t \geq 0}$  on  $\mathbb{Z}^d$ ). If  $d \leq 2$  the small world effect is clear (adding random connection speeds up the meeting time); if  $d \geq 3$  we need to compare the two constants  $G_{\mathbb{Z}^d}^{ev}(0)$  and  $G_B^{ev}(0)$ . Recall that these

Table 1

$C_d$  and  $f_d$  such that  $T_L/(C_d f_d(L))$  converges in law.

$d$	Torus		Small world	
	$f_d(L)$	$C_d$	$f_d(L)$	$C_d$
1	$L^2$	$1/12$	$L$	$G_B^{ev}(0)$
2	$L^2 \log L$	$1/\pi$	$L^2$	$G_B^{ev}(0)$
$\geq 3$	$L^d$	$G_{\mathbb{Z}^d}^{ev}(0)$	$L^d$	$G_B^{ev}(0)$

quantities depend on  $\beta$  (the probability with which the random walk moves along the long range connection). We prove in the Appendix that if  $\beta$  is small then  $G_{\mathbb{Z}^d}^{ev}(0) > G_B^{ev}(0)$ , while if  $\beta$  is close to 1 then  $G_{\mathbb{Z}^d}^{ev}(0) < G_B^{ev}(0)$ . Thus two particles meet faster on the small world than on the torus if  $\beta$  is small, but meet faster on the torus if  $\beta$  is large. This means that where the limiting space  $\mathbb{Z}^d$  is transient ( $d \geq 3$ ), a small probability of taking a connection towards a distant site (the long range neighbour) makes it easier to meet, but if this probability is too large then it is easier for the two walkers to get lost instead of meeting. Unfortunately identifying the value of  $\beta$  at which the inequality between  $G_{\mathbb{Z}^d}^{ev}(0)$  and  $G_B^{ev}(0)$  reverses seems a difficult task, which goes beyond the aim of this paper. One strategy could be try to find numerical approximations of the two constants by evaluating with a combinatorial procedure the  $n$ -step return probabilities on  $\mathbb{Z}^d$  up to time  $n_0$  and substituting in (1.1) this evaluation up to step  $n_0$  and the asymptotic value of the return probabilities (see for instance [16, Theorem 13.10]) for  $n > n_0$ . Of course this has to be repeated for a large set of values of  $\beta$  and one also needs to tackle the question of how large  $n_0$  needs to be in order to make the error small.

The third random time we are interested in is the coalescing time. In [9, Chapter 6], the author sketched a proof that the number of particles of a normalized  $n$ -coalescing random walk (that is with  $n$  particles at time 0) starting from the stationary distribution, moving according to the simple symmetric random walk, in  $d = 1$ , on the small world, converges to Kingman's coalescent. Recall that Kingman's coalescent is a Markov process starting from  $n$  individuals without spatial structure: each couple has an exponential clock with mean 1 after which the two particles coalesce (see [6,8,14]). We use Theorem 1.1 to get new information about the number of particles  $(|\xi_t(A)|)_{t \geq 0}$  of the coalescing random walk  $(\xi_t(A))_{t \geq 0}$  starting from  $A = \{x_1, \dots, x_n\}$ ,  $x_i \in \Lambda^d(L)$  for  $1 \leq i \leq n$  in continuous time, extending the previous result to  $d$ -dimensional small worlds with general transition probabilities and more general initial distance between particles. We prove the following, where  $M$  is the number of deterministic neighbours of each site (depending on the model,  $M = 2d + 1$  or  $M = (2m + 1)^d$ ).

**Theorem 1.2.** Let  $h_L \geq (\log \log L)^2$  such that  $\lim_{L \rightarrow \infty} M^{4h_L}/(2L)^d = 0$ , then for each  $A = \{x_1, \dots, x_n\} \subset \Lambda^d(L)$  with  $|x_i - x_j| \geq h_L$  for  $i \neq j$ ,  $T > 0$  there exists a sequence

of sets  $\{H^L\}_L$  of small world graphs such that  $\mathbf{P}(H^L) \xrightarrow{L \rightarrow \infty} 1$  and for each sequence  $\{S^L\}_L$ ,  $S^L \in H^L$ , uniformly in  $0 \leq t \leq T$

$$\left| \mathbb{P}_{S^L}^A(|\xi_{s_L t}(A)| < k) - P_n(D_t < k) \right| \xrightarrow{L \rightarrow \infty} 0, \quad k = 2, \dots, n \quad (1.7)$$

where  $\mathbb{P}_{S^L}^A$  is the law of  $(\xi_t(A))_{t \geq 0}$  on  $S^L$ ,  $s_L = (2L)^d G_{\mathcal{B}}^{ev}(0)$  and  $P_n$  is the law of the number of particles  $D_t$  at time  $t \geq 0$  in a Kingman's coalescent starting from  $n$  particles.

We remark that the small world we consider is a random graph where each site has a single long range connection. One can show analogous results for random graphs with a fixed number  $K > 1$  (not depending on  $L$ ) of long range connections per site, added to the  $d$ -dimensional torus or to a translation invariant finite graph. The exponential limit will have a different parameter which we guess would be the expected time spent at the origin on a different big world structure.

We give here a brief outline of the paper. In Section 2.1 we give the formal definitions needed in what follows and give some technical results. In Section 2.1 we formally define the small world (actually two versions of it, depending on the notion of deterministic neighbourhood one chooses) and the random walk on it. In Section 2.2 we describe the big world and its relationship with the small world. Moreover, we prove that with large probability a ball of radius  $t(L)$  (with  $t(L)$  which does not grow too fast) in the small world looks exactly like the corresponding ball in the big world (that is there are no long range connections reaching inside the ball; see Proposition 2.6). Proposition 2.7 gives useful lower bounds on the probability that the graph distance and the Euclidean distance between two points are equal, and on the probability, if the latter is large, that also the former is large. As we already mentioned, one of the keys in our proofs is that when time is relatively small, thanks to Proposition 2.6 the random walker moves with large  $\mathbf{P}$ -probability as if she were on the big world. On the other hand, for large times we use the fact that she is close to the stationary distribution. In Section 2.4 we state Proposition 2.10 which is an estimate on the speed of convergence to equilibrium. Its proof uses known estimates, involving the isoperimetric constant. This is the reason why we need the results in Section 2.3, which roughly speaking say that with large  $\mathbf{P}$ -probability the isoperimetric constant is large. Section 3 is devoted to the estimates of the asymptotic behaviour of the Laplace transforms of the meeting time of two random walks, one starting at  $x$  and the other starting at 0:  $F^L(x, \lambda)$  is the “annealed” transform (i.e. with respect to  $\mathbb{P}$ ) and  $F_S^L(x, \lambda)$  is the “quenched” transform (i.e. with respect to  $\mathbb{P}_S$ ). To obtain these estimates, we need to evaluate the Laplace transforms of the time spent together of two random walks, namely  $G^L(x, \lambda)$  and  $G_S^L(x, \lambda)$ . These results are used in Section 4 where we prove Theorem 1.1 and the result on the hitting time of the origin. In Section 5 we introduce the coalescing random walk and we prove the convergence theorem to Kingman's coalescent. Finally in the Appendix we compare  $G_{\mathbb{Z}^d}^{ev}(0)$  and  $G_{\mathcal{B}}^{ev}(0)$ , which allows us to compare our results with the ones on the meeting time on the  $d$ -dimensional torus when  $d \geq 3$ .

## 2. Preliminaries

### 2.1. The small world

The vertices of the random graph are the ones of the  $d$ -dimensional torus, which we denote by

$$\Lambda(L) = \Lambda^d(L) = (\mathbb{Z} \bmod 2L)^d,$$

when there is no ambiguity, we will omit the dependence on  $d$ .

The set of edges  $\mathcal{E}^L$  of the graph is partly deterministic (short range connections) and partly random (long range connections). Note that we consider nonoriented edges, that is, if  $(x, y) \in \mathcal{E}^L$  then also  $(y, x) \in \mathcal{E}^L$  (thus we identify edges with subsets of order two).

We will consider two kinds of short range connections, one between neighbours (i.e. vertices  $x, y$  such that  $\|x - y\|_1 = 1$ ), and the other between vertices  $x, y$  such that  $\|x - y\|_\infty \leq m$ : the corresponding neighbourhoods are

$$\begin{aligned}\mathcal{N}(x) &= \{y \in \Lambda(L) : \|x - y\|_1 = 1\}, \quad x \in \Lambda(L), \\ \mathcal{N}_m^\infty(x) &= \{y \in \Lambda(L) : \|x - y\|_\infty \leq m\}, \quad x \in \Lambda(L), m \in \mathbb{N}.\end{aligned}$$

For all  $x, y \in \Lambda(L)$  we denote by  $d_S(x, y)$  the graph distance between  $x$  and  $y$ . Let  $\Omega$  be the set of all partitions of the set of  $\Lambda(L)$  into  $(2L)^d/2$  subsets of cardinality two. Let  $\mathbf{P}$  be the uniform probability on  $\Omega$ : the random choice of  $\omega \in \Omega$  represents the choice of the set of long range connections (some of which may coincide with short range ones). Note that both  $\Omega$  and  $\mathbf{P}$  depend on  $L$ .

**Definition 2.1.** Let  $\mathcal{G}^L$  be the family of all graphs with set of vertices  $\Lambda(L)$ . The small world  $\mathcal{S}^L$  is a random variable  $\mathcal{S}^L(\omega) : \Omega \rightarrow \mathcal{G}^L$  such that  $\mathcal{S}^L(\omega) = (\Lambda(L), \mathcal{E}^L(\omega))$ , where

$$\mathcal{E}^L(\omega) = \omega \cup \{\{x, y\} : x \in \Lambda(L), y \in \mathcal{N}(x)\}.$$

The set of edges of the small world  $\mathcal{S}_m^L(\Omega)$  is defined as

$$\mathcal{E}_m^L(\omega) = \omega \cup \{\{x, y\} : x \in \Lambda(L), y \in \mathcal{N}_m^\infty(x)\}.$$

We denote by  $\mathcal{S}^L(\Omega) = \{\mathcal{S}^L(\omega) : \omega \in \Omega\}$  and by  $\mathcal{S}_m^L(\Omega) = \{\mathcal{S}_m^L(\omega) : \omega \in \Omega\}$ .

For any fixed  $\omega$ , given two short range neighbours  $x$  and  $y$ , we write  $x \sim^{SR} y$ ; if they are long range neighbours we write  $x \sim^{LR} y$  (it may happen that  $x \sim^{SR} y$  and  $x \sim^{LR} y$  at the same time).

Note that  $\mathbf{P}$  clearly defines a probability measure on  $\mathcal{G}^L$ : with a slight abuse of notation we denote this measure with  $\mathbf{P}$  as well. Given  $\omega$ , we will also call “small world” the graph  $\mathcal{S}^L(\omega)$ . For the sake of simplicity we will focus here on the case  $\mathcal{S}^L$ , but our proofs can be adapted to  $\mathcal{S}_m^L$ . Moreover, when there is no ambiguity, we will write  $\mathcal{S}$  and  $\mathcal{S}_m$  instead of  $\mathcal{S}^L$  and  $\mathcal{S}_m^L$ .

**Remark 2.2.** We note that the small world could be defined imposing that we consider as probability space  $\Theta \subset \Omega$ , the family of partitions where no couple is a short range connection (thus the random graph has fixed degree), instead of  $\Omega$ . The results of the paper would not be different.

We consider discrete and continuous time random walks on small worlds; here is the definition regarding the discrete ones.

**Definition 2.3.** Let  $\Delta$  be an adapted, symmetric and translation invariant transition matrix on the torus,  $A_S$  be the (random) matrix where the  $x, y$  entry is 1 if and only if  $x$  and  $y$  are long range neighbours and 0 otherwise, and  $\mu$  be a probability measure on  $\Lambda(L)$ .

1. Given a small world  $\mathcal{S}$ , the transition matrix of the walk is  $P_S = \beta \Delta + (1 - \beta) A_S$  and we denote by  $\tilde{\mathbb{P}}_S^\mu$  the law of the discrete time random walk on  $\mathcal{S}$  with initial probability  $\mu$  and transitions ruled by  $P_S$ . If  $\mu = \delta_{x_0}$  we write  $\tilde{\mathbb{P}}_S^{x_0}$ .
2. We denote by  $\tilde{\mathbb{P}}^\mu$  the average of  $\tilde{\mathbb{P}}_S^\mu$  with respect to  $\mathbf{P}$ , that is

$$\tilde{\mathbb{P}}^\mu(\mathcal{C}(x_0, \dots, x_n)) = \sum_{S \in \Omega} \mathbf{P}(S) \mu(x_0) p_S(x_0, x_1) \cdots p_S(x_{n-1}, x_n),$$

where  $\mathcal{C}(x_0, \dots, x_n)$  is the set of all infinite sequences of vertices where the first  $n$  coordinates coincide with  $(x_0, \dots, x_n)$ .

We construct the continuous time version  $X_t$  of the random walk  $\tilde{X}_t$  by continuation. In other words we define  $X_t := \tilde{X}_{N_t}$  where  $N_t$  is a Poisson process with rate 1 independent of  $\tilde{X}_t$ : the law of  $X_t$  on  $S$  starting from a probability measure  $\mu$  on  $\Lambda(L)$  is given by

$$\mathbb{P}_S^\mu(X_t = y) = \sum_{k=0}^{\infty} \frac{e^{-t} t^k}{k!} \tilde{\mathbb{P}}_S^\mu(\tilde{X}_k = y). \quad (2.1)$$

From now on  $\Delta$ , hence also the family of transition matrices  $\{P_S\}_{S \in \mathcal{S}^L(\Omega)}$ , is considered fixed.

## 2.2. The big world

The small worlds  $\mathcal{S}^L$  and  $\mathcal{S}_m^L$  (which are random graphs) can be mapped into deterministic graphs, the *big worlds*  $\mathcal{B}$  and  $\mathcal{B}_m$  respectively, as in [10]. We recall here the construction. The sites are all vectors  $\pm(z_1, \dots, z_n)$ , with  $n \geq 1$  components, for all  $n \in \mathbb{N}$   $z_j \in \mathbb{Z}^d$  and  $z_j \neq 0$  for  $j < n$ . The edges in  $\mathcal{B}$  are drawn between  $+(z_1, \dots, z_n)$  and  $+(z_1, \dots, z_n + y)$  if and only if  $y \in \mathcal{N}(0)$ ; for  $\mathcal{B}_m$  we consider  $y \in \mathcal{N}_m^\infty(0)$  (we call these edges short range connections). The same is done between  $-(z_1, \dots, z_n)$  and  $-(z_1, \dots, z_n + y)$ . Moreover,  $+(z_1, \dots, z_n)$  has a long range neighbour, namely

$$\begin{aligned} &+(z_1, \dots, z_n, 0) \quad \text{if } z_n \neq 0, \\ &+(z_1, \dots, z_{n-1}) \quad \text{if } z_n = 0, n \geq 1, \\ &-(0) \quad \text{if } z_n = 0, n = 1. \end{aligned}$$

Analogously one defines the long range neighbour of  $-(z_1, \dots, z_n)$ . Note that the big world is a vertex transitive graph (i.e. the automorphism group acts transitively). We denote by  $|x|$  the graph distance on the big world from  $x$  to  $+(0)$  and we also write 0 instead of  $+(0)$ . See Fig. 1 (which is taken from [10]) for the case  $d = 1$ .

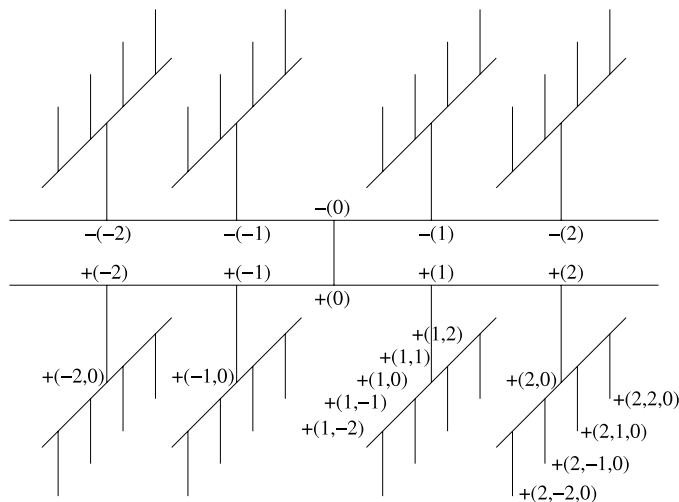


Fig. 1. A portion of the big world in  $d = 1$ .



We construct a random map  $\phi(\omega)$  which maps the big world onto the small world  $\mathcal{S}(\omega)$  in a way such that long range connections in the big world correspond to long range connections in the small world.

**Definition 2.4.** Given a small world  $\mathcal{S}$  and  $x \in \Lambda(L)$ , let  $LR_{\mathcal{S}}(x)$  be the long range neighbour of  $x$ . The map  $\phi : \Omega \rightarrow \Lambda(L)^{\mathcal{B}}$  is recursively defined as follows:

$$\begin{aligned}\phi(\omega)(+(z)) &= z \bmod (2L), \\ \phi(\omega)(-(z)) &= LR_{\mathcal{S}(\omega)}(0) + z \bmod (2L), \\ \phi(\omega)(\pm(z_1, \dots, z_n)) &= LR_{\mathcal{S}(\omega)}(\phi(\omega)(\pm(z_1, \dots, z_{n-1}))) + z_n \bmod (2L).\end{aligned}$$

Note that the transition matrix  $P_{\mathcal{S}}$  on the small world naturally induces a symmetric and translation invariant discrete time random walk  $\{\tilde{X}_n\}_{n \geq 0}$  on the big world (moving with probability  $\beta$  along the long range connection and with probability  $1 - \beta$  according to the transition matrix  $\Delta$ ). We denote by  $\mathbb{P}^x$  the law of  $\{\tilde{X}_n\}_{n \geq 0}$  with initial position  $x$ . One can prove, by using Cauchy–Schwarz’s inequality, the symmetry and the translational invariance of the walk, that for all  $x \in \mathcal{B}$  and  $n \in \mathbb{N}$ ,

$$\mathbb{P}_B^x(\tilde{X}_{2n} = 0) \leq \mathbb{P}_B^0(\tilde{X}_{2n} = 0); \quad \mathbb{P}_B^x(\tilde{X}_{2n+1} = 0) \leq \mathbb{P}_B^0(\tilde{X}_{2n} = 0). \quad (2.2)$$

Using (2.1) we get the continuous time version  $\{X_t\}_{t \geq 0}$  and we have that for each  $t \geq 0$

$$\mathbb{P}_B^x(X_{2t} = 0) \leq \mathbb{P}_B^0(X_{2t} = 0). \quad (2.3)$$

Let  $\tilde{G}_{\mathcal{B}}(x) := \sum_{n=0}^{\infty} \mathbb{P}_B^x(\tilde{X}_n = 0)$  be the expected number of visits to 0 of  $\{\tilde{X}_n\}_{n \geq 0}$  (recall that in (1.6) we introduced  $\tilde{G}_{\mathcal{B}}^{ev}(x)$ ) and let  $G_{\mathcal{B}}(x) = \int_0^{\infty} \mathbb{P}_B^x(X_t = 0) dt$  be the expected time spent at 0 by the continuous time process (recall (1.1) to compare with  $G_{\mathcal{B}}^{ev}(x)$ ). We can prove, starting from (2.1), that  $\tilde{G}_{\mathcal{B}}(x) = G_{\mathcal{B}}(x)$  and by a change of variable, that  $G_{\mathcal{B}}(x) = 2G_{\mathcal{B}}^{ev}(x)$ .

Clearly  $\tilde{G}_{\mathcal{B}}^{ev}(x) \leq \tilde{G}_{\mathcal{B}}(x)$  and they coincide if the random walk has period 2 (in which case they are nonzero only if  $|x|$  is even). Note that if  $m = 1$  the big world is the Cayley graph of  $\mathbb{Z}^d * \mathbb{Z}_2$  and the random walk on it is transient and  $G_{\mathcal{B}}(x)$  is finite. If  $m \geq 2$  the big world is the Cayley graph of  $\hat{\mathbb{Z}}^d * \mathbb{Z}_2$ , where  $\hat{\mathbb{Z}}^d$  has the  $m$ -neighbourhood relation, and the random walk is still transient (this can be proven via the flow criterion; see [16]). Moreover, by (2.3),  $G_{\mathcal{B}}^{ev}(x) \leq G_{\mathcal{B}}^{ev}(0)$ , and the analogous inequality holds in discrete time.

We are interested in the event where locally the small world does not differ from the big world.

**Definition 2.5.** If  $x \in \Lambda(L)$  and  $t > 0$ , we denote by  $I(x, t)$  the event in  $\Omega$

$$I(x, t) := \{\phi|_{B_{\mathcal{B}}(x, t)} \text{ is injective}\},$$

where  $B_{\mathcal{B}}(x, t)$  is the ball of radius  $t$  centered at  $x$  in the big world.

Clearly  $\mathbf{P}(I(x, t))$  does not depend on  $x$ .

**Proposition 2.6.** Let  $M = 2d + 1$  for  $\mathcal{B}$  and  $M = (2m + 1)^d$  for  $\mathcal{B}_m$  and let  $t = t(L)$  be a function of  $L$  such that  $M^{4t(L)} = o(L^d)$ . Then for sufficiently large  $L$

$$\mathbf{P}(I(x, t)) \geq 1 - \frac{CM^{4t}}{L^d} \xrightarrow{L \rightarrow \infty} 1,$$

where  $C$  is a positive constant.

**Proof.** Denote by  $K_t$  the number of long range connections in  $B_{\mathcal{B}}(0, t)$ , and by  $J_t$  the total number of vertices in the ball centered at 0 in  $\Lambda(L)$  and of radius  $t$ , which we denote by  $B_{\mathcal{B}}(0, t)$ . Note that the number of vertices in a graph with constant degree is always at most the number of vertices in the homogeneous tree of the same degree. Recall that the ball of radius  $t$  in the homogeneous tree of degree  $M \geq 3$  has exactly  $1 + M \sum_{j=0}^{\lfloor t \rfloor - 1} (M-1)^j \leq 3M^t$  vertices. Thus we get  $K_t \leq 3M^t$  and  $J_t \leq 3M^t$ .

Enumerate the long range connections in  $B_{\mathcal{B}}(0, t)$  from 1 to  $K_t$  and construct the mapping  $\phi$ . Note that  $I(0, t)$  contains the set  $A$  of  $\omega$  such that the long range connections in the image of  $B_{\mathcal{B}}(0, t)$  in the small world  $\mathcal{S}$  are all sites at distance at least  $2t$  on  $\Lambda(L)$ . Thus  $\mathbf{P}(I(0, t)) \geq \mathbf{P}(A)$  and

$$\begin{aligned} \mathbf{P}(A) &\geq \frac{(2L)^d - J_{2t}}{(2L)^d} \frac{(2L)^d - 2J_{2t}}{(2L)^d} \cdots \frac{(2L)^d - K_t J_{2t}}{(2L)^d} \\ &= \prod_{i=1}^{K_t} \left( 1 - \frac{i J_{2t}}{(2L)^d} \right) = \exp \left( \sum_{i=1}^{K_t} \log \left( 1 - \frac{i J_{2t}}{(2L)^d} \right) \right). \end{aligned}$$

Note that  $\log(1-x) \geq -2x$  if  $x \in [0, \bar{x}]$  for some  $\bar{x}$ . By our choice of  $t(L)$ , for  $L$  sufficiently large we have that  $K_t J_{2t}/L^d \leq \bar{x}$  and we get, for some positive  $C$  and  $C'$ ,

$$\begin{aligned} \mathbf{P}(A) &\geq \exp \left( -\frac{2J_{2t}}{(2L)^d} \sum_{i=1}^{K_t} i \right) \geq \exp \left( -C' \frac{J_{2t} K_t^2}{L^d} \right) \\ &\geq \exp \left( -\frac{CM^{4t}}{L^d} \right) \geq 1 - \frac{CM^{4t}}{L^d}. \quad \square \end{aligned}$$

By  $d_{\mathcal{S}}(x, y)$  we denote the (random) graph distance between  $x$  and  $y$ . Depending on  $\omega$ ,  $x$  and  $y$ , it may happen that  $d_{\mathcal{S}}(x, y) = d(x, y)$  or  $d_{\mathcal{S}}(x, y) < d(x, y)$ . The following proposition provides probability estimates of these events.

**Proposition 2.7.** Choose  $t$  as in Proposition 2.6. Then for sufficiently large  $L$

(a) if  $d(0, x) \leq t$ , then

$$\mathbf{P}(d_{\mathcal{S}}(0, x) = d(0, x)) \geq 1 - \frac{CM^{4t}}{L^d}; \quad (2.4)$$

(b) if  $d(0, x) > t$ , then

$$\mathbf{P}(d_{\mathcal{S}}(0, x) > t) \geq 1 - \frac{CM^{4t}}{L^d}. \quad (2.5)$$

**Proof.** (a) It suffices to note that the event  $(d_{\mathcal{S}}(0, x) = d(0, x))$  contains the event  $A$  of the previous proposition.

(b) We note that the event  $(d_{\mathcal{S}}(0, x) > t)$  contains  $C_x$  which is the event that all the  $K_{t/2}$  long range connections in  $B_{\mathcal{B}}(0, t/2)$  and  $B_{\mathcal{B}}(x, t/2)$  are mapped by  $\phi$  into vertices of  $\Lambda(L)$  at distance at least  $t$  from each other and from the balls of radius  $t$  centered at 0 and at  $x$  in  $\Lambda(L)$ . We work as in Proposition 2.6 to estimate

$$\mathbf{P}(C_x) \geq \frac{(2L)^d - 2J_t}{(2L)^d} \frac{(2L)^d - 3J_t}{(2L)^d} \cdots \frac{(2L)^d - K_{t/2} J_t}{(2L)^d}$$

and we proceed in a similar way to get the thesis.  $\square$

### 2.3. Isoperimetric constant

Estimates of the distance between the random walk and the equilibrium measure involve the isoperimetric constant. Thus we will get bounds for the *edge isoperimetric constant*

$$\iota = \min_{|V| \leq n/2} \frac{e(V, V^c)}{|V|},$$

where  $n$  is the total number of vertices in the graph and  $e(V, V^c)$  is the total number of edges between vertices in  $V$  and  $V^c$ , where  $V$  is a subset of the vertices of the graph.

Given  $\alpha > 0$ , we define

$$Q_\alpha^L := (S \in \mathcal{S}^L(\Omega) : \iota(S) > \alpha). \quad (2.6)$$

We want to prove that there exists  $\alpha$  (independent of  $L$ ) such that  $\mathbf{P}(Q_\alpha^L)$  is large when  $L$  is large. In order to prove this we need to recall some facts about random graphs.

Take  $n$  and  $r$  positive integers such that  $nr$  is even and consider the random multigraph with  $n$  vertices obtained in the following way: attach to each vertex  $r$  half edges, pick at random (with uniform probability  $\mathbb{P}$ ) a pairing of the  $nr$  half edges and join the half edges which are paired. Note that parallel edges and loops are possible and that the degree is at most  $r$ . We call this multigraph a random  $(n, r)$ -configuration. This procedure is usually proposed as a way to construct, with uniform probability, the random  $r$ -regular graph (one has to condition to the event that the multigraph has neither parallel edges nor loops, i.e. it is a graph); see [3] or [5].

Let us now recall [9, Theorem 6.3.2] (which is inspired by [3, Theorem 1]): it claims that, given  $r$ , there exists  $\alpha' > 0$  such that  $\mathbb{P}(\text{the } (n, r)\text{-configuration has } \iota \leq \alpha') = o(1)$  as  $n$  goes to infinity. It is not difficult to modify the proof of Durrett to get that for any fixed positive integer  $l$  one can refine the estimate and obtain  $o(n^{-l})$ .

**Proposition 2.8.** *Let  $n$ ,  $r$  and  $l$  be positive integers with  $nr$  even and let  $\mathbb{P}$  be the uniform probability on  $(n, r)$ -configurations. Then there exists  $\alpha' > 0$  independent of  $n$  and  $r$  (one may choose  $\alpha' = 1/10l$ ), such that  $\mathbb{P}(\iota \leq \alpha') = o(n^{-l})$ .*

Actually one can prove this proposition for more general random graphs. Indeed let  $h$  be a positive integer in  $[1, r - 1]$  and call  $(n, r, h)$ -configuration the multigraph obtained by a procedure similar to the one we used for  $(n, r)$ -configurations. The only difference is that here  $n$  vertices have  $r$  half edges each attached, and one vertex has  $h$  half edges attached ( $nr + h$  has to be even). It is not difficult to prove that Proposition 2.8 holds also for  $(n, r, h)$ -configurations.

Now we use this result to prove the analog for the small world. The ideas are taken from [9, Theorem 6.3.4].

**Proposition 2.9.** *Consider the small world  $\mathcal{S}^L$  and fix a positive integer  $l$ . Then there exists  $\alpha > 0$  such that  $\mathbf{P}(Q_\alpha^L) = o(L^{-dl})$ .*

**Proof.** We represent the vertices in  $\Lambda(L)$  by vertices in  $[-L, L]^d \cap \mathbb{Z}^d$  and partition this set into triplets plus possibly a singleton or a couple of vertices (if  $2L \bmod 3 = 1$  or  $2$  respectively). We enumerate the triplets from 1 to  $n = \lfloor (2L)^d/3 \rfloor$  and denote them by  $I_1, \dots, I_n$ . Note that it is possible to choose the triplets in a way such that each triplet has a vertex which is a short range neighbour of the other two vertices (see Fig. 2 for the case  $d = 2$  and  $L = 4$ ).

Now choose  $A \subset [-L, L]^d \cap \mathbb{Z}^d$  with  $|A| \leq (2L)^d/2$ : we need to prove that outside a set of small worlds of  $\mathbf{P}$ -probability which is  $o(L^{-dl})$  we have that  $e(A, A^c)/|A| > \alpha$ . Let  $J_A = \{j \in \{1, \dots, n\} : I_j \subset A\}$ ,  $K_A = \{j \notin J_A : I_j \cap A \neq \emptyset\}$  and  $B_A = \bigcup_{j \in J_A} I_j$ .

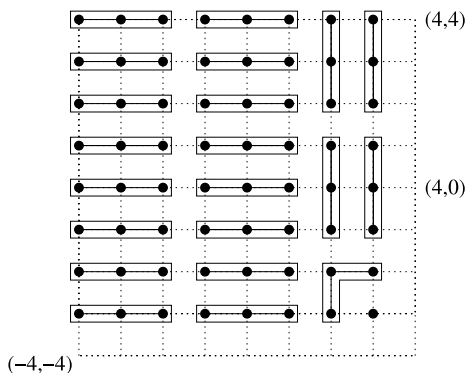


Fig. 2. The partition in triplets if  $d = 2$  and  $L = 4$ .

*Case 1.* If  $|J_A| \leq |A|/6$  then  $|B_A| \leq |A|/2$  and  $|K_A| \geq |A|/4$ . Thus there are at least  $|A|/4$  edges connecting  $A$  to  $A^c$  ( $|K_A|$  is a lower bound for  $e(A, A^c)$ ).

*Case 2.* If  $|J_A| > |A|/6$  we construct a  $(n, 3)$ -configuration associated to the small world: there is an edge between  $j$  and  $k$  for any long range edge between  $x \in I_j$  and  $y \in I_k$ . By Proposition 2.8 outside a set of  $\mathbf{P}$ -probability  $o(L^{-dl})$  we have  $e(J_A, J_A^c) \geq \alpha' |J_A|$  for some  $\alpha' > 0$ .

It is enough to show that there is a map  $\varphi$  from the set of edges between  $J_A$  and  $J_A^c$  to the set of edges between  $A$  and  $A^c$ , such that each edge between  $A$  and  $A^c$  has at most two preimages. Let  $j \in J_A, k \in J_A^c$  and let there be an edge between them. Then there exists  $x \in I_j \subset A, y \in I_k$  such that  $x \sim^{LR} y$ . If  $y \notin A$  then  $\varphi((j, k)) = (x, y)$ . If  $y \in A$  and it has a short range neighbour  $z \in I_k \cap A^c$  then we choose  $\varphi((j, k)) = (y, z)$  and  $(y, z)$  has no other preimages.

If  $y$  does not have a short range neighbour in  $I_k \cap A^c$  we know that it has a short range neighbour  $z \in I_k \cap A$  which is a short range neighbour of  $z' \in I_k \cap A^c$ . In this case,  $\varphi((j, k)) = (z, z')$  and  $(z, z')$  might have at most another preimage (the edge between some  $i$  and  $k$  originated by the long range edge between  $z$  and some  $x' \in I_i$ ).  $\square$

## 2.4. Convergence to equilibrium

Note by symmetry that the reversible distribution of the walk on  $\mathcal{S}^L$  is the uniform probability  $\pi$  on  $\mathcal{S}^L$ .

**Proposition 2.10.** Let  $\{X_t\}_{t \geq 0}$  be the continuous time random walk on the small world (recall Definition 2.3 and Eq. (2.1)). Fix  $l \in \mathbb{N}$  and pick  $\alpha$  as in Proposition 2.9. There exists  $\gamma > 0$  (depending only on  $\alpha, \Delta$  and  $\beta$ ) such that

$$\max_{x, y} |\mathbb{P}_S^x(X_t = y) - \pi(y)| \leq e^{-\gamma t}, \quad \text{for all } S \in \mathcal{Q}_\alpha^L; \quad (2.7)$$

$$\max_{x, y} |\mathbb{P}^x(X_t = y) - \pi(y)| \leq e^{-\gamma t} + o(L^{-dl}). \quad (2.8)$$

**Proof.** Recall that given a discrete time random walk on a finite set, with transition matrix  $P$  and reversible measure the uniform measure  $\pi$ , a result of Sinclair and Jerrum [13] gives an estimate of the speed of convergence to equilibrium. Indeed in this case  $P$  has all real eigenvalues, namely  $1 = \lambda_0 > \lambda_1 \geq \dots \geq \lambda_{n-1}$ . Let  $\lambda = \max\{|\lambda_i| : i = 1, \dots, n-1\}$ . It is well known that  $\lambda < 1$ .

Then for all  $t \in \mathbb{N}_0 := \{n \in \mathbb{Z} : n \geq 0\}$

$$\max_{x,y} |p^{(t)}(x, y) - \pi(y)| \leq \lambda^t \leq \exp(-(1 - \lambda)t),$$

where  $p^{(t)}(x, y)$  is a  $t$ -step probability of the walk. If we are able to estimate  $\lambda$  we are done. If  $\lambda = \lambda_1$  then the following result (which is known as Cheeger's inequality (see [9, Theorem 6.2.1])) is useful

$$\frac{1}{2}t^2 \left( \min_{x,y:p(x,y)>0} p(x, y) \right)^2 \leq 1 - \lambda_1.$$

A sufficient condition for  $\lambda = \lambda_1$  is that all the eigenvalues are positive, which for instance holds when we consider a lazy random walk, that is one which stays put with probability at least  $1/2$ . It is thus clear that for any small world  $S$  in  $\mathcal{Q}_\alpha^L$ , if the random walk  $\tilde{X}_t$  is such that  $\lambda = \lambda_1$ , then

$$\max_{x,y} |\tilde{\mathbb{P}}_S^x(\tilde{X}_t = y) - \pi(y)| \leq \exp(-c\alpha^2 t), \quad (2.9)$$

where  $c = \frac{1}{2} (\min_{x,y:p(x,y)>0} p(x, y))^2$  depends only on  $\Delta$  and is strictly positive (recall that  $\Delta$  is adapted and translation invariant on  $\Lambda(L)$ ). Moreover, by Proposition 2.8

$$\begin{aligned} \max_{x,y} |\tilde{\mathbb{P}}^x(\tilde{X}_t = y) - \pi(y)| &\leq \sum_S \mathbf{P}(S) \max_{x,y} |\tilde{\mathbb{P}}_S^x(\tilde{X}_t = y) - \pi(y)| \\ &\leq \exp(-c\alpha^2 t) \mathbf{P}(\mathcal{Q}_\alpha^L) + 2\mathbf{P}(\mathcal{Q}_\alpha^L) \\ &\leq \exp(-c\alpha^2 t) + o(L^{-dl}). \end{aligned} \quad (2.10)$$

It is easy starting from (2.1) to prove that (2.9) and (2.10) still hold in continuous time with a different constant in the exponential. Namely, one has to replace  $c\alpha^2$  with  $1 - \exp(-c\alpha^2)$ .

We are left with the proof that (2.9) and (2.10) hold for any random walk (not just for the lazy one) with different constants. This can be proven by coupling  $\tilde{X}$  with a random walk  $\tilde{Y} = \{\tilde{Y}_t\}_{t \geq 0}$  with transition matrix  $P'$  such that  $p'(x, x) = (1 + p(x, x))/2$ ,  $p'(x, y) = p(x, y)/2$ : the process  $\tilde{Y}$  is “lazy” and moves with  $\tilde{X}$  when a Bernoulli random variable with parameter  $1/2$  equals 1, otherwise it stays put. We leave the computation to the reader.  $\square$

### 3. Laplace transform estimates

Let  $T_L = \inf\{s > 0 : X_s = Y_s\}$  (respectively  $\tilde{T}_L$ ) be the first time, after time 0, that two independent continuous (respectively discrete) time random walks  $X_t$  and  $Y_t$  on the random graph  $S$  meet. Clearly the law of  $T_L$  (with respect to either  $\mathbb{P}_S$  or  $\mathbb{P}$ ) depends on the starting sites of the walkers. Without loss of generality, we assume that  $Y_0 = 0$  and  $X_0 = x$  (if we need to stress the dependence on  $L$ , we write  $X_0 = x_L$ ).

We introduce the following (annealed) Laplace transforms in continuous time,

$$\begin{aligned} G^L(x, \lambda) &:= \int_0^\infty e^{-\lambda t} \mathbb{P}^{x,0}(X_t = Y_t) dt = \int_0^\infty e^{-\lambda t} \mathbb{P}^x(X_{2t} = 0) dt, \\ F^L(x, \lambda) &:= \int_0^\infty e^{-\lambda t} \mathbb{P}^{x,0}(T_L \in dt), \end{aligned}$$

where  $\mathbb{P}^{x,0}$  denotes the product law of the two walkers. The corresponding quenched transforms are, given  $S \in \mathcal{S}^L(\Omega)$ ,

$$G_S^L(x, \lambda) := \int_0^\infty e^{-\lambda t} \mathbb{P}_S^{x,0}(X_t = Y_t) dt, \quad F_S^L(x, \lambda) := \int_0^\infty e^{-\lambda t} \mathbb{P}_S^{x,0}(T_L \in dt).$$

We are interested in the asymptotic behaviour, as  $L \rightarrow \infty$ , of  $T_L/(2L)^d$ , thus we study the previous Laplace transforms with parameter  $\lambda/(2L)^d$ .

The discrete time version of such Laplace transforms are defined in a similar way, but the integrals are replaced by sums. With a slight abuse of notation we omit the superscript  $\sim$  on the discrete time random walk when not necessary and we use  $X_t$ ,  $Y_t$ ,  $T_L$ ,  $\mathbb{P}_S^\mu$ ,  $G_S^L(x, \lambda)$  and  $F_S^L(x, \lambda)$  both in the discrete and the continuous time version of the process: since the proofs are similar, we detail the latter one and we only point out the differences.

### 3.1. Estimates for $G$

We first note that the evaluation of the limit of the annealed transforms can be done considering only small worlds with large isoperimetric constants, that is on  $Q_\alpha^L$  (which was defined by (2.6)). Let  $\mathcal{K} := \{K \subset \mathbb{R} : \inf K > 0\}$ .

**Lemma 3.1.** *Let*

$$g_L := \sum_{S \in (Q_\alpha^L)^c} \mathbf{P}(S) \int_0^\infty e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^x(X_{2t} = 0) dt,$$

$$f_L := \sum_{S \in (Q_\alpha^L)^c} \mathbf{P}(S) \int_0^\infty e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x,0}(T_L \in dt).$$

*There exists  $\alpha > 0$  such that  $\mathbf{P}(Q_\alpha^L) \xrightarrow{L \rightarrow \infty} 1$ ,  $g_L \xrightarrow{L \rightarrow \infty} 0$  and  $f_L \xrightarrow{L \rightarrow \infty} 0$  (for each  $K \in \mathcal{K}$ , uniformly for  $\lambda \in K$ ).*

**Proof.** By Proposition 2.9 we may choose  $\alpha$  such that  $\mathbf{P}((Q_\alpha^L)^c) = o(L^{-2d})$ . Then

$$0 \leq f_L \leq g_L \leq \mathbf{P}((Q_\alpha^L)^c) \int_0^\infty e^{-\frac{\lambda t}{(2L)^d}} dt = \mathbf{P}((Q_\alpha^L)^c) \frac{(2L)^d}{\lambda} \xrightarrow{L \rightarrow \infty} 0. \quad \square$$

The limit of the sum defining  $G$ , from  $\log \log L$  to infinity does not depend on the sequence of small worlds, provided that they are chosen with large isoperimetric constant. From now on, if not otherwise stated, we write  $t_L = \log \log L$ , fix  $\alpha$  such that  $\mathbf{P}((Q_\alpha^L)^c) = o(L^{-2d})$  and write  $Q^L$  instead of  $Q_\alpha^L$ .

**Lemma 3.2.** *If for all  $L$  we choose  $S \in Q^L$  and  $x_L \in \Lambda(L)$ , then for all  $\lambda > 0$*

$$\lim_{L \rightarrow \infty} \int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) dt = \frac{1}{\lambda}.$$

*Moreover, the convergence is uniform with respect to the choice of the sequences  $S \in Q^L$ ,  $x_L \in \Lambda(L)$  (and of  $\lambda$ ).*

**Proof.** Note that

$$\begin{aligned} & \int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) dt \\ &= \int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} \frac{1}{(2L)^d} dt + \int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} \left( \mathbb{P}_S^{x_L}(X_{2t} = 0) - \frac{1}{(2L)^d} \right) dt. \end{aligned} \quad (3.1)$$

The limit of the first term is uniform in  $\lambda$  and it is  $1/\lambda$ . Since  $S$  is chosen in  $Q^L$ , by (2.7) there exists a positive constant  $\gamma$  (recall that  $\gamma$  depends on  $\alpha$  which is now fixed) such that the second

sum on the right-hand side of (3.1) is smaller than or equal to

$$\int_{t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} e^{-\gamma 2t} dt = \frac{e^{-(\lambda/(2L)^d + 2\gamma)t_L}}{\lambda/(2L)^d + 2\gamma},$$

which tends to 0 as  $L$  goes to infinity (uniformly with respect to all the choices of the statement).  $\square$

Recall that, given a vertex  $x \in \Lambda(L)$ , there is a unique vertex  $+(x)$  in the big world (if  $x = 0$  we write 0 instead of  $+(0)$ ) and that by  $I(0, t)$  we denote the set of small worlds which look like the big world in a ball of radius  $t$  around 0 (see Definition 2.5). We now prove that, if  $L$  is sufficiently large, for a wide choice of  $S$  (i.e.  $S$  in a set with  $\mathbf{P}$ -probability which tends to 1 as  $L$  increases to infinity), we have that  $G_S^L(x_L, \lambda/(2L)^d)$  is close to  $1/\lambda + G_B^{ev}(x_L)$ .

**Theorem 3.3.** *Let*

$$h_S^L(\lambda) = \left| G_S^L(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - G_B^{ev}(x_L) \right|.$$

*For all  $\varepsilon > 0$  there exists  $\tilde{L}$  such that for all  $\lambda$ ,  $x_L$  and  $L \geq \tilde{L}$  we have that  $Q^L \cap I(0, t_L^2) \subset (S : h_S^L(\lambda) \leq \varepsilon)$ . If  $d_S(0, x_L) > t_L^2$  then  $Q^L \subset (S : h_S^L(\lambda) \leq \varepsilon)$ .*

**Proof.** Note that

$$\begin{aligned} h_S^L(\lambda) &\leq \left| \int_{t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) dt - \frac{1}{\lambda} \right| \\ &\quad + \left| \int_0^{t_L} e^{-\frac{\lambda t}{(2L)^d}} \left( \mathbb{P}_S^{x_L}(X_{2t} = 0) dt - \mathbb{P}_B^{+(x_L)}(X_{2t} = 0) \right) dt \right| \\ &\quad + \int_{t_L}^{\infty} \mathbb{P}_B^{+(x_L)}(X_{2t} = 0) dt + \int_0^{t_L} (1 - e^{-\frac{\lambda t}{(2L)^d}}) \mathbb{P}_B^{+(x_L)}(X_{2t} = 0) dt. \end{aligned}$$

By Lemma 3.2, if  $S \in Q^L$ , the first term is smaller than  $\varepsilon/4$  provided that  $L$  is large.

Since either  $S \in I(0, t_L^2)$  or  $d_S(0, x_L) > t_L^2$ , the probabilities of a meeting before time  $t_L$  on  $S$  and on the big world differ only if the value of the underlying Poisson process  $N_t$  (recall Eq. (2.1)) at time  $2t_L$  is at least  $t_L^2$ : by Chebyshev's inequality the second term on the right-hand side is smaller than

$$2t_L \mathbb{P}(N_{2t_L} \geq t_L^2) \leq \frac{(2t_L)^2}{(t_L^2 - t_L)^2} \leq \varepsilon/4$$

if  $L$  is large enough.

Note that by (2.3) the integrands of the last two terms are both dominated by  $\mathbb{P}_B^0(X_{2t} = 0)$  which does not depend on  $L$  and is integrable. Thus by the Dominated Convergence Theorem they are both smaller than  $\varepsilon/4$  if  $L$  is sufficiently large.  $\square$

**Theorem 3.4.** *For all  $K \in \mathcal{K}$ ,  $\varepsilon > 0$  there exists  $\tilde{L}$  such that for all  $L \geq \tilde{L}$ ,  $x_L \in \Lambda(L)$ , and  $\lambda \in K$ ,*

$$\left| G^L(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - G_B^{ev}(x_L) \right| \leq \varepsilon.$$

**Proof.** Recall that

$$G^L(x_L, \lambda/(2L)^d) = \sum_S \mathbf{P}(S) G_S^L(x_L, \lambda/(2L)^d).$$

By Theorem 3.3 there exists  $\tilde{L}$  such that for all  $L \geq \tilde{L}$ ,

$$\left| \sum_{S \in Q^L \cap I(0, t_L^2)} \mathbf{P}(S) G_S^L(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - G_B^{ev}(x_L) \right| \leq \varepsilon/3.$$

Thus, since  $\mathbf{P}((Q^L)^c)$  and  $\mathbf{P}(I(0, t_L^2)^c)$  are both small if  $L$  is large, we may choose  $\tilde{L}$  such that for all  $\lambda \in K$  and  $L \geq \tilde{L}$

$$\sum_{S \in (Q^L)^c \cup I(0, t_L^2)^c} \mathbf{P}(S) \left( \frac{1}{\lambda} + G_B^{ev}(x_L) \right) \leq \varepsilon/3.$$

Now we only need to prove that

$$\sum_{S \in (Q^L)^c \cup I(0, t_L^2)^c} \mathbf{P}(S) G_S^L(x_L, \lambda/(2L)^d) \leq \varepsilon/3.$$

By Lemma 3.1 we know that  $\sum_{S \in (Q^L)^c} \mathbf{P}(S) G_S^L(x_L, \lambda/(2L)^d) \leq \varepsilon/6$  for all  $L \geq \tilde{L}$  and  $\lambda > 0$ . Finally, by Proposition 2.6 and Lemma 3.2, for some  $C > 0$  and  $L$  sufficiently large

$$\begin{aligned} & \sum_{S \in Q^L \cap I(0, t_L^2)^c} \mathbf{P}(S) G_S^L(x_L, \lambda/(2L)^d) \\ &= \sum_{S \in Q^L \cap I(0, t_L^2)^c} \mathbf{P}(S) \left( \int_0^{t_L} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) + \int_{t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) \right) \\ &\leq \left( t_L + \frac{1}{\lambda} + C \right) \mathbf{P}(I(0, t_L^2)^c) \leq \varepsilon/6. \quad \square \end{aligned}$$

### 3.2. From $G$ to $F$

We note that if  $x_L \neq 0$  then  $G_S^L(x_L, \lambda/(2L)^d)$  may be written as

$$\sum_z \int_0^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \int_0^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds, X_s = z).$$

while  $G_S^L(0, \lambda/(2L)^d)$  is equal to

$$1 + \sum_z \int_0^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \int_0^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{0, 0}(T_L \in ds, X_s = z).$$

Define  $H_1$ ,  $H_2$  and  $H_3$  (which depend on  $S$ ,  $x_L$  and  $L$ ) by

$$\begin{aligned} H_1 &:= \sum_z \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \int_0^{t_L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds, X_s = z) \\ H_2 &:= \sum_z \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds, X_s = z) \\ H_3 &:= \sum_z \int_{t_L}^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \int_0^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds, X_s = z). \end{aligned}$$



By Lemma 3.1, for all  $L$  sufficiently large and if the limit exists,

$$\lim_{L \rightarrow \infty} G^L \left( x_L, \frac{\lambda}{(2L)^d} \right) = \lim_{L \rightarrow \infty} \sum_{S \in Q^L} \mathbf{P}(S)(H_1 + H_2 + H_3). \quad (3.2)$$

Clearly if  $x_L = 0$  for all  $L$  sufficiently large we only need to add 1 to the previous limit. The same equality holds in discrete time, replacing the integral with the sum.

We now study each of the three summands separately, in order to obtain the limit of  $F^L$  as a function of the limit of  $G^L$ .

**Lemma 3.5.** *If  $S \in I(0, t_L^2)$  and  $x_L \in \Lambda(L)$ , for each  $\varepsilon > 0$  there exists  $\tilde{L}$  such that for each  $L > \tilde{L}$  then*

$$\left| H_1 - \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_B^0(X_{2q} = 0) dq \int_0^{t_L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds) \right| < \varepsilon. \quad (3.3)$$

*This inequality also holds whenever  $d_S(0, x_L) > t_L^2$ . Moreover, uniformly with respect to the choice of the sequence  $\{x_L\}_L$  and of  $\lambda$ ,*

$$\begin{aligned} \sum_{S \in (I(0, t_L^2))^c} \mathbf{P}(S) \sum_z \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \\ \int_0^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds, X_s = z) \xrightarrow{L \rightarrow \infty} 0. \end{aligned} \quad (3.4)$$

**Proof.** We first note that if we consider the discrete time random walk then at time  $t_L$  the walker is at a distance at most  $t_L$  from her starting site. Thus the sites  $z$  in the sum of  $H_1$  are those at distance at most  $t_L$  from 0 (all other terms being zero). Then, since  $q \leq t_L$  and  $S \in I(0, t_L^2)$  (actually, in discrete time  $S \in I(0, 2t_L)$  suffices), we have that  $\mathbb{P}_S^z(X_{2q} = z) = \mathbb{P}_B^0(X_{2q} = 0)$  and the difference in Eq. (3.3) is equal to zero. On the other hand, if  $d_S(0, x_L) > t_L^2$  it is not possible for two random walkers starting at 0 and at  $x_L$  respectively, to meet within time  $t_L$  and all quantities in Eq. (3.3) are zero.

In continuous time the walkers may take a large number of steps even in a small amount of time (though this is quite unlikely). Denote by  $N_t$  the Poisson process underlying the random walk from 0 (see the second integral in  $H_1$ ) and by  $N'_t$  the Poisson process underlying the random walk from  $z$  (see the first integral in  $H_1$ ).

Suppose that  $S \in I(0, t_L^2)$ : if  $(N_{t_L} < t_L^2/2)$  and  $(N'_{2t_L} < t_L^2)$  then the whole path from  $z$  to  $z$  of duration  $2q$  lies in the ball of radius  $t_L^2$  centered at 0 and the law of the walk coincides with the corresponding walk on the big world. By Chebyshev's inequality

$$\mathbb{P}(N_{t_L} \geq t_L^2/2) \leq \frac{C}{t_L^3} \quad \text{and} \quad \mathbb{P}(N'_{2t_L} \geq t_L^2) \leq \frac{C}{t_L^3}, \quad (3.5)$$

for some positive constant  $C$ . Then straightforward computation shows that  $H_1$  differs from

$$\begin{aligned} \sum_z \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z, N'_{2t_L} < t_L^2) dq \int_0^{t_L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0} \\ \times (T_L \in ds, X_s = z, N_{t_L} < t_L^2/2) \end{aligned} \quad (3.6)$$

at most by  $3t_L^2 \cdot C/t_L^3$ . The same argument that we used in discrete time gives

$$\mathbb{P}_S^z(X_{2q} = z, N'_{2t_L} < t_L^2) = \mathbb{P}_B^0(X_{2q} = 0, N'_{2t_L} < t_L^2)$$

and by Eq. (3.5), Eq. (3.3) follows (similarly one proves it in the case  $d_S(0, x_L) > t_L^2$ ).

In order to prove (3.4), note that for some  $C > 0$

$$\begin{aligned} \sum_{S \in I(0, t_L^2)^c} \mathbf{P}(S) \sum_z \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) \int_0^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L = s, X_s = z) \\ \leq Ct_L \mathbf{P}(I(0, t_L^2)^c) F^L(x_L, \lambda/(2L)^d), \end{aligned}$$

which, by Proposition 2.6 and since  $F^L(x, \lambda) \leq 1$  for all  $\lambda$  and  $x$ , goes to 0, uniformly in  $x_L$  and  $\lambda$ , as  $L$  goes to infinity.  $\square$

**Lemma 3.6.** For all  $K \in \mathcal{K}$ ,  $\varepsilon > 0$  there exists  $\tilde{L}$  such that for all  $L \geq \tilde{L}$ ,  $x_L$  and  $\lambda \in K$ ,

$$\left| \sum_{S \in Q^L} \mathbf{P}(S) H_2 - \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_B^0(X_{2q} = 0) dq \right. \\ \left. \sum_{S \in Q^L} \mathbf{P}(S) \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds) \right| \leq \varepsilon. \quad (3.7)$$

**Proof.** Note that  $\sum_{S \in Q^L} \mathbf{P}(S) H_2$  can be written as

$$\begin{aligned} \sum_z \sum_{S \in Q^L \cap I(z, t_L^2)} \mathbf{P}(S) \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds, X_s = z) \\ + \sum_z \sum_{S \in Q^L \cap I(z, t_L^2)^c} \mathbf{P}(S) \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0} \\ \times (T_L \in ds, X_s = z) \\ = H_{2,1} + H_{2,2}. \end{aligned}$$

We prove that  $H_{2,2} \rightarrow 0$ , indeed since  $\exp(-\lambda q/(2L)^d) \mathbb{P}_S^z(X_{2q} = z) \leq 1$  then

$$\begin{aligned} H_{2,2} \leq t_L \sum_z \sum_{S \in Q^L \cap I(z, t_L^2)^c} \mathbf{P}(S) \left\{ \int_{t_L}^{\log L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(X_s = Y_s = z) ds \right. \\ \left. + \int_{\log L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(X_s = Y_s = z) ds \right\} =: H_{2,2,1} + H_{2,2,2}. \end{aligned}$$

Note that  $H_{2,2,1}$  is smaller than or equal to

$$t_L \sum_{S \in Q^L} \mathbf{P}(S) \int_{t_L}^{\log L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^0(X_{2s} = x_L) ds.$$

We write  $\mathbb{P}_S^0(X_{2s} = x_L) \leq |\mathbb{P}_S^0(X_{2s} = x_L) - 1/(2L)^d| + 1/(2L)^d$ , which by (2.7) is smaller than or equal to  $e^{-\gamma s} + 1/(2L)^d$  (recall that  $\gamma$  depends only on the parameter  $\alpha$  which has been

fixed in  $Q^L$ ). It is thus only a matter of computation to show that  $H_{2,2,1}$  goes to zero (uniformly in  $x_L$  and  $\lambda$ ) as  $L$  goes to infinity.

Now we consider  $H_{2,2,2}$ . Note that  $\mathbb{P}_S^{x_L,0}(X_s = Y_s = z) = \mathbb{P}_S^z(X_s = 0)\mathbb{P}_S^z(Y_s = x_L)$ . Write

$$\mathbb{P}_S^z(X_s = 0) = \mathbb{P}_S^z(X_s = 0) - \frac{1}{(2L)^d} + \frac{1}{(2L)^d}$$

and do the same for  $\mathbb{P}_S^z(Y_s = x_L)$ . Using (2.7) and Proposition 2.6, we have that  $H_{2,2,2}$  is smaller than or equal to

$$\begin{aligned} t_L \sum_z \sum_{S \in Q^L \cap I(z, t_L^2)^c} \mathbf{P}(S) \int_{\log L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \left( e^{-\gamma s} + \frac{1}{(2L)^d} \right)^2 ds \\ \leq C t_L L^d \frac{M^{4t_L^2}}{L^d} \int_{\log L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \left( e^{-2\gamma s} + \frac{1}{(2L)^{2d}} + \frac{2e^{-\gamma s}}{(2L)^d} \right) ds \leq C' \frac{t_L M^{4t_L^2}}{L^{2\gamma}}, \end{aligned}$$

for some  $C' > 0$ . The last quantity goes to 0 (uniformly in  $x_L$  and  $\lambda \in K$  for each  $K \in \mathcal{K}$ ) as  $L \rightarrow \infty$  (the numerator contains only terms which are logarithmic in  $L$ ).

We now prove that Eq. (3.7) holds with  $H_{2,1}$  in place of  $\sum_{S \in Q^L} \mathbf{P}(S) H_2$ . The same arguments that we used to prove that  $H_{2,2}$  converges to 0 tell us that

$$\sum_z \sum_{S \in Q^L \cap I(z, t_L^2)^c} \mathbf{P}(S) \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_B^0(X_{2q} = 0) dq \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds, X_s = z)$$

converges to zero. Thus we are left with

$$\begin{aligned} \sum_z \sum_{S \in Q^L \cap I(z, t_L^2)} \mathbf{P}(S) \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \left| \mathbb{P}_S^z(X_{2q} = z) - \mathbb{P}_B^0(X_{2q} = 0) \right| dq \\ \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds, X_s = z) \end{aligned} \quad (3.8)$$

and we have to prove that it is small when  $L$  is large. By (3.5) we have  $\int_0^{t_L} |\mathbb{P}_S^z(X_{2q} = z) - \mathbb{P}_B^0(X_{2q} = 0)| dq \leq C/t_L^2$ . Thus the quantity in Eq. (3.8) is at most

$$\frac{C}{t_L^2} \sum_{S \in Q^L} \int_0^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds) \leq \frac{C}{t_L^2} F^L(x_L, \lambda/(2L)^d)$$

which can be taken as small as we want if  $L$  is large (recall that  $F^L(x_L, \lambda/(2L)^d) \leq 1$  and that all sums and integrals are interchangeable since all quantities are nonnegative).  $\square$

**Lemma 3.7.** *Let*

$$a_\lambda^L(S) := H_2 - \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_B^0(X_{2q} = 0) dq \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds). \quad (3.9)$$

*Then  $a_\lambda^L \geq 0$  and  $a_\lambda^L \rightarrow 0$  in probability (for each  $K \in \mathcal{K}$ , uniformly in  $x_L$  and  $\lambda \in K$ ), that is for all  $K \in \mathcal{K}$ ,  $\varepsilon > 0$  and  $\delta > 0$  there exists  $\tilde{L}$  such that for all  $L \geq \tilde{L}$  and  $x_L$*

$$\mathbf{P}(A_\varepsilon^L(K)) := \mathbf{P}(S : a_\lambda^L(S) \leq \varepsilon, \quad \forall \lambda \in K) \geq 1 - \delta.$$

**Proof.** We first note that for all  $z$  and  $S$ ,  $\mathbb{P}_S^z(X_{2q} = z) \geq \mathbb{P}_B^0(X_{2q} = 0)$ , hence  $a_\lambda^L(S) \geq 0$ . Suppose by contradiction that there exist  $K$ ,  $\varepsilon > 0$  and  $\delta > 0$  such that  $\mathbf{P}(A_\varepsilon^L(K)) \leq 1 - \delta$

infinitely often. Then infinitely often

$$\sum_S \mathbf{P}(S) a_\lambda^L(S) > \delta \varepsilon.$$

By Lemmas 3.6 and 3.1, there exists  $\tilde{L}$  such that  $\sum_S \mathbf{P}(S) a_\lambda^L(S) < \delta \varepsilon$  for each  $L \geq \tilde{L}$ ,  $x_L$ ,  $\lambda \in K$ , whence the contradiction.  $\square$

**Lemma 3.8.** *For all  $K \in \mathcal{K}$  and  $\varepsilon > 0$  there exists  $\tilde{L}$  such that for all  $L \geq \tilde{L}$ ,  $S \in \mathcal{Q}^L$ ,  $x_L$  and  $\lambda \in K$ ,*

$$\left| H_3 - \frac{1}{\lambda} F_S^L(x_L, \lambda/(2L)^d) \right| \leq \varepsilon. \quad (3.10)$$

**Proof.** Note that

$$\begin{aligned} H_3 &= \sum_z \int_{t_L}^\infty \left( \mathbb{P}_S^z(X_{2q} = z) - \frac{1}{(2L)^d} \right) e^{-\frac{\lambda q}{(2L)^d}} dq \int_0^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds, X_s = z) \\ &\quad + \int_{t_L}^\infty e^{-\frac{\lambda q}{(2L)^d}} \frac{1}{(2L)^d} F_S^L(x_L, \lambda/(2L)^d) dq, \end{aligned}$$

and the modulus of the first member does not exceed

$$\int_{t_L}^\infty e^{-\frac{\lambda q}{(2L)^d} - 2\gamma q} dq \int_0^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L, 0}(T_L \in ds) \leq C \exp(-\gamma t_L)$$

by (2.7) (recall that  $S \in \mathcal{Q}^L$  and the fact that  $F_S^L(x_L, \lambda/(2L)^d) \leq 1$ ). The claim follows since for  $L$  sufficiently large

$$\left| \frac{1}{(2L)^d} \int_{t_L}^\infty e^{-\frac{\lambda q}{(2L)^d}} dq - \frac{1}{\lambda} \right| < \varepsilon/2. \quad \square$$

**Theorem 3.9.** *Let*

$$b_\lambda^L(S) := \left| F_S^L\left(x_L, \frac{\lambda}{(2L)^d}\right) - \frac{G_B^{ev}(x_L) + \frac{1}{\lambda} - \mathbb{I}_{\{0\}}(+)(x_L)}{G_B^{ev}(0) + \frac{1}{\lambda}} \right|.$$

(a) *Then  $b_\lambda^L \rightarrow 0$  in probability, for each  $K \in \mathcal{K}$  such that  $\sup K < \infty$ , uniformly in  $x_L \in \Lambda(L)$  and  $\lambda \in K$ , namely for all  $\varepsilon > 0$  ( $S : b_\lambda^L(S) \leq \varepsilon, \forall \lambda \in K$ )  $\supset \mathcal{Q}^L \cap I(0, t_L^2) \cap A_{\varepsilon/2}^L(K)$  for all  $L$  sufficiently large ( $A_\varepsilon^L(K)$  was defined in Lemma 3.7).*

(b) *For all  $\varepsilon > 0$ , ( $S : b_\lambda^L(S) \leq \varepsilon, \forall \lambda \in K$ )  $\supset \mathcal{Q}^L \cap (S : d_S(0, x_L) > t_L^2) \cap A_{\varepsilon/2}^L(K)$  for all  $L$  sufficiently large.*

**Proof.** (a) Note that by the Dominated Convergence Theorem (recall (1.1)), since  $\sup K = \lambda_0 < \infty$ , for each  $\varepsilon > 0$  and  $L > \tilde{L}$  large enough

$$\left| \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_B^0(X_{2q} = 0) dq - G_B^{ev}(0) \right| < \varepsilon. \quad (3.11)$$

Consider

$$G_S^L(x_L, \lambda/(2L)^d) - \mathbb{I}_{\{0\}}(x_L) - \left( G_B^{ev}(0) + \frac{1}{\lambda} \right) F_S^L(x_L, \lambda/(2L)^d).$$

Writing  $G_S^L(x_L, \lambda/(2L)^d) = \mathbb{I}_{\{0\}}(x_L) + H_1 + H_2 + H_3$ , using Lemmas 3.5, 3.7 and 3.8 and (3.11) follows that the previous difference is smaller than  $\varepsilon$  when  $L$  is sufficiently large and  $S \in Q^L \cap I(0, t_L^2) \cap A_{\varepsilon/2}^L(K)$ . Note that all these three sets have probability which converges to 1. By Theorem 3.3 we conclude that  $b_\lambda^L$  goes to zero in probability.

(b) Since Eq. (3.3) holds also when  $d_S(0, x_L) > t_L^2$ , we have that  $S \in Q^L \cap (S : d_S(0, x_L) > t_L^2) \cap A_{\varepsilon/2}^L(K)$ .  $\square$

**Theorem 3.10.** For each  $\varepsilon > 0$  and  $K \in \mathcal{K}$  such that  $\sup K < \infty$ , there exists  $\tilde{L}$  such that for each  $L > \tilde{L}$ ,  $\lambda \in K$  and for each sequence  $\{x_L\}_L$  such that  $x_L \in \Lambda(L)$ ,

$$\left| F^L \left( x_L, \frac{\lambda}{(2L)^d} \right) - \frac{G_B^{ev}(x_L) + \frac{1}{\lambda} - \mathbf{1}_{\{0\}}(+x_L)}{G_B^{ev}(0) + \frac{1}{\lambda}} \right| \leq \varepsilon.$$

**Proof.** To keep notation simple we deal only with the case  $+x_L \neq 0$  (the case  $+x_L = 0$  is completely analogous). Let

$$Q_{\varepsilon, \lambda}^L = \left\{ S : b_\lambda^L \leq \varepsilon \right\}, \quad (3.12)$$

( $b_\lambda^L$  was defined in Theorem 3.9). By Theorem 3.9 there exists  $\tilde{L}$  such that for all  $L \geq \tilde{L}$  we have  $\mathbf{P}(Q_{\varepsilon, \lambda}^L) > 1 - \varepsilon$ .

Then since both  $F_S^L(x_L, \lambda/(2L)^d)$  and  $(G_B^{ev}(x_L) + 1/\lambda)/(G_B^{ev}(0) + 1/\lambda)$  are in  $[0, 1]$ , for all  $L \geq \tilde{L}$

$$\sum_S \mathbf{P}(S) \left| F_S^L \left( x_L, \frac{\lambda}{(2L)^d} \right) - \frac{G_B^{ev}(x_L) + \frac{1}{\lambda}}{G_B^{ev}(0) + \frac{1}{\lambda}} \right| \leq 2\mathbf{P}((Q_{\varepsilon, \lambda}^L)^c) + \varepsilon \leq 3\varepsilon. \quad \square$$

**Remark 3.11.** Clearly for all  $\lambda > 0$ , if

$$\frac{G_B^{ev}(x_L) + \frac{1}{\lambda} - \mathbf{1}_{\{0\}}(x_L)}{G_B^{ev}(0) + \frac{1}{\lambda}}$$

has a limit  $f(\lambda)$  then by Theorem 3.10 we have that  $F^L(x_L, \lambda/(2L)^d)$  has limit  $f(\lambda)$ .

**Remark 3.12.** In discrete time one can show the same results with  $2t_L$  instead of  $t_L^2$  and constant  $\tilde{G}_B^{ev}(x_L)$ . As seen in the proof of Lemma 3.5 the key of the proof in discrete time is that two random walkers cannot meet before a time smaller than half of their initial distance (while this is possible in continuous time, though it is unlikely that particles at initial distance  $t_L^2$  meet before time  $t_L$ ).

#### 4. Meeting and hitting time of random walks

It is clear that, if  $G_B^{ev}(x_L)$  has a limit as  $L$  goes to infinity, then Theorems 3.9 and 3.10 provide the limits of  $F_S^L(x_L, \lambda/(2L)^d)$  and  $F^L(x_L, \lambda/(2L)^d)$ . The limit of  $G_B^{ev}(x_L)$  exists for instance in two particular cases:  $x_L = x$  for all  $L$  sufficiently large, or  $|x_L| \rightarrow \infty$ . In the first case clearly  $\lim_{L \rightarrow \infty} G_B^{ev}(x_L) = G_B^{ev}(x)$ . In the second case,  $G_B^{ev}(x_L)$  converges to 0 by the Dominated Convergence Theorem.

**Proof of Theorem 1.1.** We prove the claim in continuous time. The proof in discrete time works in a similar way.

1. By [Theorem 3.10](#) we know that for all  $\lambda > 0$

$$F^L \left( x_L, \frac{\lambda}{(2L)^d} \right) \xrightarrow{L \rightarrow \infty} \frac{\lambda G_{\mathcal{B}}^{ev}(x) + 1 - \lambda \mathbf{1}_{\{0\}}(+)(x)}{\lambda G_{\mathcal{B}}^{ev}(0) + 1}. \quad (4.1)$$

Since for each  $L$ ,  $F^L$  is a monotone function of  $\lambda$  and so is the right-hand side of (4.1), which is also continuous in  $\lambda$ , it follows that (4.1) holds uniformly in  $\lambda \geq 0$ .

Thus, if  $x \neq 0$ ,  $T_L/(2L)^d$  converges in law (with respect to  $\mathbb{P}^{x,0}$ ) to

$$\frac{G_{\mathcal{B}}^{ev}(x)}{G_{\mathcal{B}}^{ev}(0)} \delta_0 + \left( 1 - \frac{G_{\mathcal{B}}^{ev}(x)}{G_{\mathcal{B}}^{ev}(0)} \right) \exp \left( \frac{1}{G_{\mathcal{B}}^{ev}(0)} \right),$$

while if  $x = 0$  then it converges to

$$\left( 1 - \frac{1}{G_{\mathcal{B}}^{ev}(0)} \right) \delta_0 + \frac{1}{G_{\mathcal{B}}^{ev}(0)} \exp \left( \frac{1}{G_{\mathcal{B}}^{ev}(0)} \right).$$

Then (1.2) holds, and by monotonicity it holds uniformly in  $t \geq 0$ .

2. It follows as in the previous step using the fact that  $G_{\mathcal{B}}^{ev}(x_L) \rightarrow 0$  uniformly in  $\{x_L\}_L$  such that  $|x_L| \geq \alpha_L$ . Indeed  $G_{\mathcal{B}}^{ev}(x_L) = \int_0^\infty \mathbb{P}_{\mathcal{B}}^0(X_{2t} = +(x_L)) dt$  goes to 0 by the Dominated Convergence Theorem since  $\mathbb{P}_{\mathcal{B}}^0(X_{2t} = +(x_L)) \leq \mathbb{P}_{\mathcal{B}}^0(X_{2t} = 0)$  and  $\int_0^\infty \mathbb{P}_{\mathcal{B}}^0(X_{2t} = 0) dt \leq G_{\mathcal{B}}(0) < \infty$ .
3. By [Theorem 3.9](#) we know that for all  $n$  there exists  $L_n$  such that  $\mathbf{P}(S : b_\lambda^L(S) \leq 1/n, \forall \lambda \in [1/n, n]) \geq 1 - 1/n$  for all  $L \geq L_n$ . Clearly the sequence  $\{L_n\}_{n \geq 1}$  is nondecreasing and for any  $L \in [L_n, L_{n+1})$  we may define

$$H^L := (S : b_\lambda^L(S) \leq 1/n, \forall \lambda \in [1/n, n]).$$

By [Theorem 3.9](#) we have that  $\mathbf{P}(H^L) \xrightarrow{L \rightarrow \infty} 1$ . If for all  $L$  we choose  $S \in H^L$  then for all  $\lambda > 0$

$$F_S^L \left( x_L, \frac{\lambda}{(2L)^d} \right) \xrightarrow{L \rightarrow \infty} \frac{\lambda G_{\mathcal{B}}^{ev}(x) + 1}{\lambda G_{\mathcal{B}}^{ev}(0) + 1}.$$

This, by an argument as in step 1, proves that

$$\mathbb{P}_S^{x_L,0} \left( \frac{T_L}{(2L)^d} > t \right) \xrightarrow{L \rightarrow \infty} \left( 1 - \frac{G_{\mathcal{B}}^{ev}(x)}{G_{\mathcal{B}}^{ev}(0)} \right) \exp \left( -\frac{t}{G_{\mathcal{B}}^{ev}(0)} \right),$$

uniformly in  $t \geq 0$ . Since this convergence holds whenever we choose for all  $L$ ,  $S \in H^L$ , the assertion follows.

4. Choosing  $S \in H^L$  as in previous step, uniformly with respect to  $\{x_L\}_L$  such that either  $|x_L| \geq \alpha_L$  or  $d_S(0, x_L) \geq \alpha_L$  we get

$$\mathbb{P}_S^{x_L,0} \left( \frac{T_L}{(2L)^d} > t \right) \xrightarrow{L \rightarrow \infty} \exp \left( -\frac{t}{G_{\mathcal{B}}^{ev}(0)} \right),$$

uniformly in  $t \geq 0$ . This proves the claim.  $\square$

**Remark 4.1.** [Theorem 1.1.4](#) holds if we fix  $0 \in \Lambda(L)$  and we consider the supremum over all possible  $x_L \in \Lambda(L)$  such that  $d_S(x_L, 0) \geq \alpha_L$ . We can repeat the same proof to show that the result still holds if we take the supremum over all possible pairs  $(x_L, y_L) \in \Lambda(L) \times \Lambda(L)$  such

that  $d_S(x_L, y_L) \geq \alpha_L$ . Namely, let  $\alpha_L > t_L^2$  then for all  $\varepsilon > 0$

$$\mathbf{P} \left( S : \sup_{(x_L, y_L) \in \Lambda(L)^2 : d_S(x_L, y_L) \geq \alpha_L} \left| \mathbb{P}_S^{x_L, y_L} \left( \frac{T_L}{(2L)^d} > t \right) - g(t) \right| < \varepsilon, \forall t \geq 0 \right) \xrightarrow{L \rightarrow \infty} 1.$$

We observe that the same technique we employed to determine the asymptotic behaviour of the first encounter time of two random walkers, one starting at  $x_L$  and the other at 0, may be used to obtain similar results for the first time that a single random walker starting at  $x_L$  hits 0.

**Theorem 4.2.** *Let  $W_L$  be the first time that a random walk starting at  $x_L$  hits 0 either in discrete or in continuous time. Then Theorem 1.1 still holds with constant  $G_B(x)$  instead of  $G_B^{ev}(x)$ .*

**Proof.** (Discrete time) The proof is analogous to the one of Theorem 1.1 but easier, since we consider the return time of one single walk. Notice that the constant is the expected number of visits to 0 of the discrete time random walk on the big world starting at 0.

(Continuous time) A standard approach (for instance use Slutsky theorems) allows us to get the result starting from the one in discrete time.  $\square$

**Remark 4.3.** As a corollary of Theorems 1.1 and 4.2 one can get a similar convergence result for random walkers starting from the stationary distribution  $\pi$ . The key is that the initial distance between the random walk and the origin (respectively between two random walks) is larger than  $t_L^2$  with probability which converges to 1 as  $L$  goes to infinity, so that we are under hypothesis of Theorem 1.1 either (2), in the annealed case, or (4) in the quenched one.

## 5. Coalescing random walk on small world

The goal of this section is to prove a convergence result for coalescing random walk of  $n$  particles on the small world. From now on we work on the continuous time process.

Let  $\mathcal{I}(n) = \{\{x_1, \dots, x_n\} : x_i \in \Lambda(L), x_i \neq x_j\}$ . Given  $A \in \mathcal{I}(n)$ , let  $\{(X_t^S(x_i))_{t \geq 0}\}_{x_i \in A}$  be a family of independent random walks on small world  $S \in \mathcal{S}^L$  such that  $X_0^S(x_i) = x_i$  and transition ruled by  $P_S$  (recall Definition 2.3). In what follows we will drop the superscript  $S$  and simply write  $X_t(x_i)$ . We define for each  $(x_i, x_j) \in \Lambda(L) \times \Lambda(L)$  and  $S \in \mathcal{S}^L$

$$\tau(i, j) := \inf\{s > 0 : X_s^S(x_i) = X_s^S(x_j)\}$$

and for each  $A \in \mathcal{I}(n)$

$$\tau(A) := \inf_{\{x_i, x_j\} \subseteq A} \{\tau(i, j)\}.$$

Let  $\{\xi_t^S(A)\}_{t \geq 0}$  be the coalescing random walk starting from  $A \in \mathcal{I}(n)$  on  $S \in \mathcal{S}^L$ , that is the process of  $n$  independent random walks subjected to the rule that when two particles reach the same site they coalesce to one particle. Given a probability measure  $\mu$  on  $\Lambda(L)^n$ , we denote by  $\mathbb{P}_S^\mu$  the law of the coalescing random walk on  $S$  with initial probability  $\mu$  and transitions ruled by  $P_S$ . If  $\mu = \delta_A$  with  $|A| = n$ , we write  $\mathbb{P}_S^A$ .

Let  $|\xi_t^S(A)|$  be the number of particles of  $\xi_t^S(A)$  at time  $t$ . When not necessary we omit the dependence on  $S$  and we simply write  $\{\xi_t(A)\}_{t \geq 0}$ ,  $X_t(x_i)$ ,  $\tau(i, j)$  and  $\tau(A)$ .

Kingman's coalescent is a Markov process  $(D_t)_{t \geq 0}$  on  $\{0, 1, \dots, n\}$  with transition mechanism

$$n \rightarrow n-1 \text{ at rate } \binom{n}{2}.$$

The law  $P_n(D_t = k) = q_{n,k}(t)$  is given by

$$q_{n,k}(t) = \sum_{j=k}^n \frac{(-1)^{j+k} (2j-1)(j+k-2)! \binom{n}{j}}{k!(k-1)!(j-k)! \binom{n+j-1}{j}} \exp\left(-t \binom{j}{2}\right);$$

$$q_{\infty,k}(t) = \sum_{j=k}^{\infty} \frac{(-1)^{j+k} (2j-1)(j+k-2)!}{k!(k-1)!(j-k)!} \exp\left(-t \binom{j}{2}\right).$$

See for instance [6,14].

We define

$$\mathcal{A}^L(h, n) := \{A \in \mathcal{I}_n : d(x_i, x_j) > h, \text{ for all } i \neq j\} \quad (5.1)$$

$$\mathcal{A}_S^L(h, n) := \{A \in \mathcal{I}_n : d_S(x_i, x_j) > h, \text{ for all } i \neq j\} \quad (5.2)$$

the set of  $n$ -tuples with distance larger than  $h$  respectively on  $\Lambda(L)$  and on a fixed small world  $S$ . Notice that  $\mathcal{A}_S^L(h, n) \subseteq \mathcal{A}^L(h, n)$  for all  $S \in \mathcal{S}^L$ . Given  $A \in \mathcal{A}^L(h, n)$ , we introduce

$$\mathcal{D}(A) := \{S \in \mathcal{E}^L : A \in \mathcal{A}^L(h, n) \setminus \mathcal{A}_S^L(h, n)\}. \quad (5.3)$$

Remember that we focus on the nearest neighbour case, but all results can be extended to the case with neighbourhood structure given by  $\mathcal{N}_m^\infty$ .

We begin from  $n$  particles in  $A \in \mathcal{A}^L(h, n)$ . We prove that by taking a particular  $h := h_L$  and  $L$  large we get that  $A \in \mathcal{A}_S^L(h, n)$  with large probability. We assume that

$$(i) \ h_L \geq t_L^2 \quad (ii) \ \lim_{L \rightarrow \infty} \frac{M^{4h_L}}{(2L)^d} = 0 \quad (5.4)$$

where  $M = (2m+1)^d$  or  $M = 2d+1$  depending on the neighbourhood structure we work with. Note that hypothesis (5.4) are satisfied if  $h_L = t_L^2$ .

**Lemma 5.1.** *If (5.4) holds, for each  $n < \infty$ ,  $\varepsilon > 0$  there exists  $\tilde{L}$  such that for each  $L > \tilde{L}$  and  $A \in \mathcal{A}^L(h_L, n)$ ,*

$$\mathbf{P}(\mathcal{D}(A)) < \varepsilon.$$

**Proof.** Let  $A \in \mathcal{A}^L(h_L, n)$ . If  $S \in \mathcal{D}(A)$ , then there exists at least one pair of elements  $(x_i, x_j) \in A \times A$ ,  $i \neq j$ , such that  $d_S(x_i, x_j) < h_L$ . By (5.3) and (2.5)

$$\mathbf{P}(\mathcal{D}(A)) = \mathbf{P}\left(S \in \mathcal{E}^L : \exists (x_i, x_j) \in A \times A : d_S(x_i, x_j) \leq h_L\right) \leq n^2 \frac{CM^{4h_L}}{L^d}.$$

Since  $n$  is fixed, the claim follows by (5.4)(ii).  $\square$

Therefore given  $A \in \mathcal{A}^L(h_L, n)$  with large probability  $A \in \mathcal{A}_S^L(h_L, n)$ .

By Remark 4.1, if  $\alpha_L \geq t_L^2$ , there exists a sequence  $\{\tilde{H}^L\}_L$  with  $\tilde{H}^L \subseteq \mathcal{S}^L$  such that  $\mathbf{P}(\tilde{H}_L) \xrightarrow{L \rightarrow \infty} 1$  and for each sequence  $\{S^L\}_L$  with  $S^L \in \tilde{H}^L$

$$\sup_{(x_L, y_L) : d_S(x_L, y_L) \geq \alpha_L} \left| \mathbb{P}_S^{x_L, y_L} \left( \frac{T_L}{(2L)^d} > t \right) - \exp \left( -\frac{t}{G_B^{ev}(0)} \right) \right| \xrightarrow{L \rightarrow \infty} 0. \quad (5.5)$$

Note that (5.5) still holds for the sequence  $\{Q^L \cap \tilde{H}^L\}_L$  and  $\mathbf{P}(Q^L \cap \tilde{H}_L) \xrightarrow{L \rightarrow \infty} 1$ . Let  $H^L := \tilde{H}^L \cap Q^L$ .



The following lemma states that, starting from 4 particles in a set of small world with large probability, when two particles meet the others are distant.

**Lemma 5.2.** Assume (5.4). For each  $\varepsilon > 0$  there exists  $\tilde{L}$  such that for each  $L > \tilde{L}$ ,  $S \in H^L$  and  $A \in \mathcal{A}_S^L(h_L, 4)$ ,

$$\int_0^\infty \mathbb{P}_S^A(\tau(1, 2) \in ds, d_S(X_s(x_1), X_s(x_3)) \leq h_L) < \varepsilon, \quad (5.6)$$

$$\int_0^\infty \mathbb{P}_S^A(\tau(1, 2) \in ds, d_S(X_s(x_3), X_s(x_4)) \leq h_L) < \varepsilon. \quad (5.7)$$

**Proof.** We prove (5.6); (5.7) can be proved in a similar way. We split the integral into two parts. By Theorem 1.1.4 and by (5.4)(ii), for each  $\varepsilon > 0$  there exists  $\tilde{L}$  such that for each  $L > \tilde{L}$

$$\begin{aligned} \int_0^{\frac{d}{\gamma} \log(2L)} \mathbb{P}_S^A(\tau(1, 2) \in ds, d_S(X_s(x_1), X_s(x_3)) \leq h_L) &\leq \int_0^{\frac{d}{\gamma} \log(2L)} \mathbb{P}_S^A(\tau(1, 2) \in ds) \\ &= 1 - \exp\left(-\frac{d \log(2L)}{\gamma G_B^{ev}(0)(2L)^d}\right) + \varepsilon/6 < \varepsilon/3, \end{aligned} \quad (5.8)$$

where  $\gamma$  is given by (2.7) and does not depend on  $S$  since we are choosing  $S \in Q^L$ . The second part is

$$\begin{aligned} &\int_{\frac{d}{\gamma} \log(2L)}^\infty \mathbb{P}_S^A(\tau(1, 2) \in ds, d_S(X_s(x_1), X_s(x_3)) \leq h_L) \\ &\leq \int_{\frac{d}{\gamma} \log(2L)}^\infty \sum_{y \in \Lambda(L)} \mathbb{P}_S^A(\tau(1, 2) \in ds, X_s(x_1) = y) \sum_{z: d_S(y, z) \leq h_L} \left| \mathbb{P}_S^{x_3}(X_s = z) - \frac{1}{(2L)^d} \right| \\ &\quad + \int_{\frac{d}{\gamma} \log(2L)}^\infty \sum_{y \in \Lambda(L)} \mathbb{P}_S^A(\tau(1, 2) \in ds, X_s(x_1) = y) \sum_{z: d_S(y, z) \leq h_L} \frac{1}{(2L)^d} \\ &:= I(1) + I(2). \end{aligned}$$

Since the number of sites  $z$  such that  $d_S(y, z) \leq h_L$  is at most  $M^{h_L}$  for each  $y \in \Lambda(L)$ , for each  $L$  large enough we get

$$\begin{aligned} I(2) &\leq \int_{\frac{d}{\gamma} \log(2L)}^\infty \mathbb{P}_S^A(\tau(1, 2) \in ds) \frac{M^{h_L}}{(2L)^d} \\ &= \mathbb{P}_S^{x_1, x_2} \left( T_L > \frac{d}{\gamma} \log(2L) \right) \frac{M^{h_L}}{(2L)^d} \leq \varepsilon/3 \end{aligned} \quad (5.9)$$

by (5.4)(ii). Note that if  $s \geq \frac{d}{\gamma} \log(2L)$  then  $e^{-\gamma s} \leq \frac{1}{(2L)^d}$ ; therefore by (2.7) then  $I(1)$  is smaller than or equal to

$$\begin{aligned} &\int_{\frac{d}{\gamma} \log(2L)}^\infty \sum_{y \in \Lambda(L)} \mathbb{P}_S^A(\tau(1, 2) \in ds, X_s(x_1) = y) \sum_{z: d_S(y, z) \leq h_L} e^{-\gamma s} \\ &\leq \int_{\frac{d}{\gamma} \log(2L)}^\infty \sum_{y \in \Lambda(L)} \mathbb{P}_S^A(\tau(1, 2) \in ds, X_s(x_1) = y) \frac{M^{h_L}}{(2L)^d} < \varepsilon/3 \end{aligned} \quad (5.10)$$

and the claim follows by (5.8)–(5.10).  $\square$

**Remark 5.3.** Since  $S \in H^L$ , Lemma 5.2 still holds if for all  $A \in \mathcal{A}^L(h_L, n)$  we choose  $S \in H^L \cap \mathcal{D}(A)^c$ . Moreover, by Lemma 5.1 and (5.5) such a set has probability which converges to 1 as  $L$  goes to infinity.

We prove that the number of particles in the rescaled coalescing random walk converges in law to the number of particles of a Kingman's coalescent. A similar approach has been used for [6, Theorem 5] and in [8].

We work by induction on the number of particles  $n$ . If  $n = 2$ , the induction basis is given by Theorem 1.1.4. The following lemma shows that the assertion is true before the first collision of two particles.

**Lemma 5.4.** Assume (5.4). For each  $n \in \mathbb{N}$ ,  $T > 0$ ,  $A \in \mathcal{A}^L(h_L, n)$ , and  $\varepsilon > 0$  there exists  $\tilde{L}$  such that for each  $L > \tilde{L}$ ,  $S \in H^L \cap \mathcal{D}(A)^c$  and  $0 \leq t \leq T$ ,

$$\left| \mathbb{P}_S(|\xi_{s_L t}(A)| = n) - \exp\left(-\binom{n}{2} t\right) \right| < \varepsilon$$

where  $s_L := (2L)^d G_B^{ev}(0)$ .

**Proof.** Note that  $\mathbb{P}_S(|\xi_{s_L t}(A)| = n)$  and  $\exp\left(-\binom{n}{2} t\right)$  are non-increasing monotone  $t$  functions. We define, for each pair  $\{i, j\} \subseteq \{1, 2, \dots, n\}$ ,

$$H_t(i, j) := \{\tau(i, j) \leq s_L t\}; \quad q_t = q_t(A) := \mathbb{P}(\tau(A) \leq s_L t).$$

For all  $S \in H^L \cap \mathcal{D}(A)^c$ ,

$$\begin{aligned} \mathbb{P}_S^A(H_t(i, j)) &= \mathbb{P}_S^A(\tau = \tau(i, j) \leq s_L t) \\ &+ \sum_{\{k, l\} \neq \{i, j\}} \int_0^{s_L t} \mathbb{P}_S^A(\tau = \tau(k, l) \in ds, \tau(i, j) \leq s_L t). \end{aligned} \quad (5.11)$$

Each term of the sum on the right-hand side is equal to

$$\int_0^{s_L t} \sum_{y, z} \mathbb{P}_S^A(\tau = \tau(k, l) \in ds, X_s(x_i) = y, X_s(x_j) = z, \tau(i, j) \leq s_L t).$$

By Lemma 5.2 for all  $L$  sufficiently large

$$\begin{aligned} &\int_0^{s_L t} \sum_y \sum_{z: d_S(y, z) \leq h_L} \mathbb{P}_S^A(\tau = \tau(k, l) \in ds, X_s(x_i) = y, X_s(x_j) = z, \tau(i, j) \leq s_L t) \\ &\leq \int_0^\infty \mathbb{P}_S^A(\tau = \tau(k, l) \in ds, d_S(X_s(x_i), X_s(x_j)) \leq h_L) \leq \varepsilon / (8n^4) \end{aligned} \quad (5.12)$$

for all choices of  $S \in H^L \cap \mathcal{D}(A)^c$ ,  $\{i, j\} \subseteq \{1, \dots, n\}$  and  $t \geq 0$ . We are left with evaluating

$$\begin{aligned} &\int_0^{s_L t} \sum_y \sum_{z: d_S(y, z) > h_L} \mathbb{P}_S^A(\tau = \tau(k, l) \in ds, X_s(x_i) = y, X_s(x_j) = z) \\ &\times \mathbb{P}_S^{y, z}(T_L \leq s_L t - s). \end{aligned} \quad (5.13)$$

By Theorem 1.1.2,  $|\mathbb{P}_S^{y, z}(T_L \leq s_L t - s) - 1 + \exp(-t + s/s_L)| < \varepsilon / (8n^4)$  for all  $L$  sufficiently large and for all choices of  $S \in H^L \cap \mathcal{D}(A)^c$ ,  $y$  and  $z$  such that  $d_S(y, z) \geq h_L$ ,  $0 \leq s \leq t$ . Then

Eq. (5.13) does not differ by more than  $\varepsilon/(4n^4)$  from

$$\int_0^{s_L t} \mathbb{P}_S^A(\tau = \tau(k, l) \in ds) (1 - \exp(-t + s/s_L)). \quad (5.14)$$

Indeed the difference between (5.13) and (5.14) is not larger than the sum of (5.12) and

$$\int_0^\infty |\mathbb{P}_S^{y,z}(T_L \leq s_L t - s) - 1 + \exp(-t + s/s_L)| \mathbb{P}_S^A(\tau = \tau(k, l) \in ds),$$

which is not larger than  $\varepsilon/(8n^4)$  if  $L$  is sufficiently large. Integrating by parts and changing variables, we get

$$\begin{aligned} & \int_0^{s_L t} \mathbb{P}_S^A(\tau = \tau(k, l) \in ds) (1 - \exp(-t + s/s_L)) \\ &= \int_0^{s_L t} \mathbb{P}_S^A(\tau = \tau(k, l) \leq s) \frac{1}{s_L} \exp(-t + s/s_L) ds \\ &= \int_0^t \mathbb{P}_S^A(\tau = \tau(k, l) \leq s_L u) \exp(-(t - u)) du. \end{aligned} \quad (5.15)$$

By Theorem 1.1.2, for all  $L$  sufficiently large  $|\mathbb{P}_S^A(H_t(i, j) \leq t) - (1 - e^{-t})| \leq \varepsilon/(4n^2)$  for all  $S \in H^L \cap \mathcal{D}(A)^c$ ,  $(i, j) \subseteq \{1, \dots, n\}$  and  $t \geq 0$ . Summing over all pairs of  $i$  and  $j$  on (5.11) and using (5.15)

$$\begin{aligned} q_t &= \sum_{\{i,j\}} \mathbb{P}_S^A(\tau = \tau(i, j) \leq s_L t) \\ &= \sum_{i,j} \mathbb{P}_S^A(H_t(i, j)) - \sum_{\{i,j\}} \sum_{\{k,l\} \neq \{i,j\}} \int_0^{s_L t} \mathbb{P}_S^A(\tau = \tau(k, l) \in ds, \tau(i, j) \leq s_L t) \\ &= \binom{n}{2} (1 - e^{-t}) - \left( \binom{n}{2} - 1 \right) e^{-t} \int_0^t q_s e^s ds + R \end{aligned}$$

where the modulus of  $R$ , for all  $L$  sufficiently large for all choices of  $S \in H^L \cap \mathcal{D}(A)^c$ ,  $y$  and  $z$  such that  $d_S(y, z) \geq h_L$  and for all  $0 \leq t \leq T$  is smaller than  $\varepsilon/2$ . We know (see [8, Lemma 2]) that if

$$u^L(t) = \binom{n}{2} (1 - e^{-t}) - \left( \binom{n}{2} - 1 \right) e^{-t} \int_0^t u^L(s) e^s ds + R$$

then for  $L$  large enough  $u^L(t)$  does not differ by more than  $\varepsilon/2$  from  $u(t)$ , the solution of

$$u(t) = \binom{n}{2} (1 - e^{-t}) - \left( \binom{n}{2} - 1 \right) e^{-t} \int_0^t u(s) e^s ds$$

which is

$$u(t) = 1 - \exp\left(-\binom{n}{2} t\right)$$

and the claim follows.  $\square$

We are now ready to prove the final result.

**Proof of Theorem 1.2.** We fix  $A \in \mathcal{A}^L(h_L, n)$  and we show (1.7) by induction on  $n$ . Theorem 1.1 gives the result when  $n = 2$  for all  $k$  (that is  $k = 2$ ) and Lemma 5.4 gives the result for  $n$  and  $k = n$ .

Suppose the result holds for  $n - 1$  for all  $k$ . We have to prove it for  $n$  and  $k < n$ .

$$\begin{aligned}\mathbb{P}_S^A(|\xi_{sLt}(A)| < k) &= \int_0^{sLt} \mathbb{P}_S^A(\tau \in ds, |\xi_{sLt}(A)| < k) \\ &= \int_0^{sLt} \sum_{B \in \mathcal{I}(n-1)} \mathbb{P}_S^A(\tau \in ds, \xi_s(A) = B) \mathbb{P}_S^B(|\xi_{sLt-s}(B)| < k). \quad (5.16)\end{aligned}$$

Using Lemma 5.2, if  $B \notin \mathcal{A}_S^L(h_L, n-1)$ , for all  $L$  sufficiently large

$$\begin{aligned}&\int_0^{sLt} \sum_{B \notin \mathcal{A}_S^L(h_L, n-1)} \mathbb{P}_S^A(\tau \in ds, \xi_s(A) = B) \mathbb{P}_S^B(|\xi_{sLt-s}(B)| < k) \\ &\leq \sum_{\{i,j\}} \sum_{\{k,l\} \neq \{i,j\}} \int_0^{sLt} \mathbb{P}_S^A(\tau(i,j) \in ds, d_S(X_s(x_k), X_s(x_l)) \leq h_L) < \varepsilon/3\end{aligned}$$

since  $n$  is fixed, for each  $S \in H^L \cap \mathcal{D}(A)^c$ ,  $t \geq 0$ .

Changing variables, setting  $s = s_L v$ , then (5.16) is equal to

$$\int_0^t \sum_{B \in \mathcal{A}_S^L(h_L, n-1)} \mathbb{P}_S^A(\tau \in s_L dv, \xi_{s_L v}(A) = B) \mathbb{P}_S^B(|\xi_{s_L(t-v)}(B)| < k) + R$$

where the modulus of  $R$  is smaller than  $\varepsilon/3$  for all  $L$  sufficiently large for all choices of  $A \in \mathcal{A}^L(h_L, n)$ ,  $S \in H^L \cap \mathcal{D}(A)^c$ ,  $0 \leq t \leq T$ . By induction hypothesis, for all  $L$  sufficiently large

$$\left| \mathbb{P}_S^B(|\xi_{s_L(t-s)}(B)| < k) - P_{n-1}(D_{t-s} < k) \right| < \varepsilon/3$$

for  $B \in \mathcal{A}_S^L(h_L, n-1)$  and for each  $S \in H^L \cap \mathcal{D}(A)^c$  and  $0 \leq s \leq t$ . Thus the last term of the previous integral differs at most by  $\varepsilon$  from

$$\int_0^t \mathbb{P}_S^A\left(\frac{\tau}{s_L} \in dv\right) P_{n-1}(D_{t-v} < k) = - \int_0^t \mathbb{P}_S^A\left(\frac{\tau}{s_L} \leq v\right) \frac{d}{dv} P_{n-1}(D_{t-v} < k) dv$$

after an integration by parts. Note that  $v \rightarrow P_{n-1}(D_{t-v} = k)$  is a continuous function; therefore by the definition of Kingman's coalescent and because the right-hand side  $P_n(D_t < k)$  is finite, we get (see [6])

$$\begin{aligned}\mathbb{P}_S^A(|\xi_{sLt}(A)| < k) &= \sum_{i=1}^{k-1} \int_0^t \binom{n}{2} \exp\left(-\binom{n}{2} v\right) P_{n-1}(D_{t-v} = k) dv + R \\ &= \sum_{i=1}^{k-1} P_n(D_t = k) + R = P_n(D_t < k) + R\end{aligned}$$

where the modulus of  $R$ , for all  $L$  sufficiently large, for all choices of  $S \in H^L \cap \mathcal{D}(A)^c$  and  $0 \leq t \leq T$  is smaller than  $\varepsilon$ .  $\square$

**Remark 5.5.** In Theorem 1.2 we fix  $A \in \mathcal{A}^L(h_L, n)$  and the result holds in a sequence of small world graphs depending on  $A$ . One can prove that the same result holds for the sequence  $(H^L)_L$  uniformly in  $\mathcal{A}_S^L(h_L, n)$  and  $S \in H^L$ .

**Remark 5.6.** By summing over all realizations of the small world graph, one can get the annealed result as a corollary of Theorem 1.2.

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## Appendix. Comparison with the $d$ -dimensional torus

As observed in the Introduction, if we consider the usual neighbourhood structure on  $\mathbb{Z}^d$ , then the big world is the Cayley graph of  $\mathbb{Z}^d * \mathbb{Z}_2$ . Recall that we are given a transition matrix  $\Delta$  which defines a random walk on  $\mathbb{Z}^d$  and a positive number  $\beta$  which gives the probability of moving in the  $\mathbb{Z}_2$  direction on the big world (one moves with probability  $1 - \beta$  in the  $\mathbb{Z}^d$  component). In order to compare the asymptotic behaviour of the meeting time of two walkers on the small world and on the torus, we need to compare  $G_{\mathbb{Z}^d}^{ev}(0)$  with  $G_B^{ev}(0)$ . Proposition A.1 gives some information in this direction.

**Proposition A.1.** *Let  $d \geq 3$ ,  $\Delta$  be the transition matrix of an adapted, translation invariant symmetric random walk on  $\mathbb{Z}^d$  and  $\beta > 0$ .*

- (i) *There exists  $\beta_1 > 0$  such that  $G_{\mathbb{Z}^d}^{ev}(0) < G_B^{ev}(0)$  for each  $\beta \in [\beta_1, 1]$ .*
- (ii) *There exists  $\beta_2 > 0$  such that  $G_{\mathbb{Z}^d}^{ev}(0) > G_B^{ev}(0)$  for each  $\beta \in (0, \beta_2]$ .*

**Proof.** Since  $G_B^{ev}(0) = G_B(0)/2$  (we defined  $G_B(0)$  in Section 2.2) and  $G_{\mathbb{Z}^d}^{ev}(0) = G_{\mathbb{Z}^d}(0)/2$  (where  $G_{\mathbb{Z}^d}(0)$  is the expected time spent at 0 by the walk on  $\mathbb{Z}^d$ ), we prove that  $G_{\mathbb{Z}^d}(0)$  is smaller (respectively larger) than  $G_B(0)$  for  $\beta$  large (respectively small) enough.

(i) Since  $\mathbb{P}_B^0(X_{2n} = 0) \geq \beta^{2n}$  (note that one possible trajectory of the walk is the one which from 0 takes the long range edge and back  $n$  times) we get

$$G_B(0) \geq \sum_{n=0}^{\infty} \beta^{2n} = \frac{1}{1-\beta^2}$$

and the claim follows by taking  $\beta$  close to 1 since  $G_{\mathbb{Z}^d}(0) < \infty$  if  $d \geq 3$ .

(ii) Let  $\widehat{G}$  be the Green function of the Markov chain  $Y$  on  $\mathbb{Z}^d$  which has transition matrix  $\Delta$ , and  $\widehat{F}$  be the generating function of its first time returns

$$\widehat{G}(z) = \sum_{n=0}^{\infty} \mathbb{P}^0(Y_n = 0) z^n; \quad \widehat{F}(z) = \sum_{n=0}^{\infty} \mathbb{P}^0(Y_n = 0, Y_k \neq 0 \text{ for all } k < n) z^n.$$

Note that  $\widehat{G}(1) = G_{\mathbb{Z}^d}(0)$ . By [16, Proposition 9.10], there exists  $r > 0$  and a function  $\Phi(\cdot)$  such that

$$\widehat{G}(z) = \Phi(z\widehat{G}(z)), \quad z \in [0, r). \quad (\text{A.1})$$

Moreover,  $\Phi \in \mathcal{C}^2$  and it is strictly increasing and strictly convex.

Let  $P$  be the transition matrix on the big world. We denote by  $\Phi_{\mathbb{Z}^d * \mathbb{Z}_2}$ ,  $\Phi_{\mathbb{Z}^d}$  and  $\Phi_{\mathbb{Z}_2}$  the functions which satisfy (A.1) respectively for the Markov chains  $X$  on the big world,  $Y$  on  $\mathbb{Z}^d$

and the simple random walk on  $\mathbb{Z}_2$ . The function  $\Phi_{\mathbb{Z}_2}(t)$  can be computed explicitly,

$$\Phi_{\mathbb{Z}_2}(t) = \frac{1}{2} \left( 1 + \sqrt{1 + 4t^2} \right).$$

By [16, Theorem 9.19]

$$\Phi_{\mathbb{Z}^d * \mathbb{Z}_2}(t) = \frac{1}{2} \left( 1 + \sqrt{1 + 4\beta^2 t^2} \right) + \Phi_{\mathbb{Z}^d}((1 - \beta)t) - 1. \quad (\text{A.2})$$

We denote by  $\widehat{G}_\beta = \widehat{G}_{\mathbb{Z}^d * \mathbb{Z}_2}(1) = G_{\mathcal{B}}(0)$ , and by  $\widehat{G} = \widehat{G}_{\mathbb{Z}^d}(1) = G_{\mathbb{Z}^d}(0)$ . Note that, by (A.1)  $\widehat{G}_\beta$  is a fixed point of  $\Phi_{\mathbb{Z}^d * \mathbb{Z}_2}$ , while  $\widehat{G}$  is a fixed point of  $\Phi_{\mathbb{Z}^d}$ . We write (A.2) with  $t = \widehat{G}_\beta$ :

$$\widehat{G}_\beta = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\beta^2 \widehat{G}_\beta^2} + \Phi_{\mathbb{Z}^d}((1 - \beta)\widehat{G}_\beta). \quad (\text{A.3})$$

Our goal is to write the second member of (A.3) in a neighbourhood of  $\beta = 0$ , as a function of  $\widehat{G}$ .

We note that  $\lim_{\beta \rightarrow 0} \widehat{G}_\beta = \widehat{G}$ : to prove this denote by  $X$  and  $Y$  the random walks on  $\mathbb{Z}^d * \mathbb{Z}_2$  and on  $\mathbb{Z}^d$  respectively, both starting from the identity 0 of the group. Then, since every trajectory from 0 to 0 on  $\mathbb{Z}^d * \mathbb{Z}_2$  projects onto a trajectory from 0 to 0 in  $\mathbb{Z}^d$ , we have

$$\mathbb{P}(X_{2n} = 0) \leq \sum_{k=0}^n \mathbb{P}(Y_{2n-2k} = 0) \beta^{2k}.$$

Recognizing in the second term the general term of the product of two series, we get

$$\widehat{G}_\beta = \sum_{n=0}^{\infty} \mathbb{P}(X_{2n} = 0) \leq \sum_{n=0}^{\infty} \mathbb{P}(Y_{2n} = 0) \sum_{k=0}^{\infty} \beta^{2k} = \frac{\widehat{G}}{1 - \beta^2},$$

whence  $\limsup_{\beta \rightarrow 0} \widehat{G}_\beta \leq \widehat{G}$ .

Let  $A_m$  be the event that the trajectory of  $X$  up to step  $2m$  lies entirely in the first copy of  $\mathbb{Z}^d$ : for all  $m$

$$\begin{aligned} \widehat{G}_\beta &\geq \sum_{n=0}^m \mathbb{P}(X_{2n} = 0) = \sum_{n=0}^m \mathbb{P}(X_{2n} = 0 | A_m) (1 - \beta)^{2m} \\ &\quad + \sum_{n=0}^m \mathbb{P}(X_{2n} = 0 | A_m^c) (1 - (1 - \beta)^{2m}). \end{aligned}$$

Note that  $\sum_{n=0}^m \mathbb{P}(X_{2n} = 0 | A_m^c) \leq m$ , thus for all  $m$ ,

$$\liminf_{\beta \rightarrow 0} \widehat{G}_\beta \geq \liminf_{\beta \rightarrow 0} \sum_{n=0}^m \mathbb{P}(X_{2n} = 0 | A_m) = \sum_{n=0}^m \mathbb{P}(Y_{2n} = 0),$$

and  $\liminf_{\beta \rightarrow 0} \widehat{G}_\beta \geq \widehat{G}$ .

Notice that as  $\beta \rightarrow 0$

$$\sqrt{1 + 4\beta^2 \widehat{G}_\beta^2} = 1 + 2\beta^2 \widehat{G}_\beta^2 + o(\beta^3 \widehat{G}_\beta^3)$$

and by the Taylor expansion of  $\Phi_{\mathbb{Z}^d}$  centered at  $\widehat{G}$  with the Lagrange form of the remainder:

$$\Phi_{\mathbb{Z}^d}((1 - \beta)\widehat{G}_\beta) = \widehat{G} + \Phi'_{\mathbb{Z}^d}(\widehat{G}) [(1 - \beta)\widehat{G}_\beta - \widehat{G}] + \frac{1}{2} \Phi''_{\mathbb{Z}^d}(y) (y - \widehat{G})^2,$$

where  $y$  is between  $\widehat{G}$  and  $(1 - \beta)\widehat{G}_\beta$ . Two useful formulas for  $\Phi'$  and  $\Phi''$  can be found in [16, p. 99]:

$$\Phi'(t) = 1/(z + \widehat{G}(z)/\widehat{G}'(z)), \quad \Phi''(t) = (\widehat{G}(z)/(\widehat{G}(z) + z\widehat{G}'(z)))^3 \widehat{F}''(z),$$

where  $z$  is such that  $t = z\widehat{G}(z)$ . If  $t = \widehat{G}$  then

$$\Phi'_{\mathbb{Z}^d}(\widehat{G}) = \frac{\widehat{G}'}{\widehat{G}' + \widehat{G}}, \quad \Phi''_{\mathbb{Z}^d}(\widehat{G}) = \left( \frac{\widehat{G}}{\widehat{G} + \widehat{G}'} \right)^3 \widehat{F}''(1),$$

where  $\widehat{G}' = \frac{d}{dz}\widehat{G}_{\mathbb{Z}^d}(z)|_{z=1^-}$ . Therefore we may write (A.3) as

$$\widehat{G}_\beta - \widehat{G} = \beta^2 \widehat{G}_\beta^2 + \frac{\widehat{G}'}{\widehat{G}' + \widehat{G}} [\widehat{G}_\beta - \widehat{G} - \beta \widehat{G}_\beta] + \frac{1}{2} \Phi''_{\mathbb{Z}^d}(y)(y - \widehat{G})^2 + o(\beta^3 \widehat{G}_\beta^3).$$

Since  $(y - \widehat{G})^2 \leq (\widehat{G}_\beta - \widehat{G} - \beta \widehat{G}_\beta)^2$  we get

$$\begin{aligned} (\widehat{G}_\beta - \widehat{G}) \frac{\widehat{G}}{\widehat{G}' + \widehat{G}} &\leq \beta^2 \widehat{G}_\beta^2 - \beta \widehat{G}_\beta \frac{\widehat{G}'}{\widehat{G}' + \widehat{G}} \\ &\quad + \frac{1}{2} \Phi''_{\mathbb{Z}^d}(y) \left[ (\widehat{G}_\beta - \widehat{G})^2 + \beta^2 \widehat{G}_\beta^2 - 2\beta \widehat{G}_\beta (\widehat{G}_\beta - \widehat{G}) + o(\beta^3 \widehat{G}_\beta^3) \right]. \end{aligned}$$

Thus

$$\begin{aligned} (\widehat{G}_\beta - \widehat{G}) \left[ \frac{\widehat{G}}{\widehat{G}' + \widehat{G}} + \Phi''_{\mathbb{Z}^d}(y) \beta \widehat{G}_\beta - \frac{1}{2} \Phi''_{\mathbb{Z}^d}(y) (\widehat{G}_\beta - \widehat{G}) \right] \\ \leq -\beta \widehat{G}_\beta \left[ \frac{\widehat{G}'}{\widehat{G}' + \widehat{G}} + \beta \widehat{G}_\beta (1 + \frac{1}{2} \Phi''_{\mathbb{Z}^d}(y) + o(\beta \widehat{G}_\beta)) \right]. \end{aligned}$$

Note that by convexity  $\Phi'' > 0$ , and by continuity we get that  $\Phi''_{\mathbb{Z}^d}(y) \xrightarrow{\beta \rightarrow 0} \Phi''_{\mathbb{Z}^d}(\widehat{G}) > 0$ . Then the coefficient of  $(\widehat{G}_\beta - \widehat{G})$  on the left-hand side is strictly positive when  $\beta$  is small; while the coefficient of  $\beta \widehat{G}_\beta$  on the right-hand side is strictly negative when  $\beta$  is small. Whence  $\widehat{G}_\beta - \widehat{G}$  has to be negative for  $\beta$  sufficiently small and the claim follows.  $\square$

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