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Stochastic Processes and their Applications 121 (2011) 1720–1748

stochastic processes and their applications

www.elsevier.com/locate/spa

On the rate of convergence of weak Euler approximation for nondegenerate SDEs driven by Lévy processes

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Received 28 February 2010; received in revised form 14 March 2011; accepted 11 April 2011
Available online 22 April 2011

Abstract

The paper studies the rate of convergence of the weak Euler approximation for solutions to SDEs driven by Lévy processes, with Hölder-continuous coefficients. It investigates the dependence of the rate on the regularity of coefficients and driving processes. The equation considered has a nondegenerate main part driven by a spherically symmetric stable process.

Keywords: Lévy processes; Stochastic differential equations; Weak Euler approximation

1. Introduction

The paper studies the weak Euler approximation for solutions to SDEs driven by Lévy processes with a nondegenerate main part. The goal is to investigate the dependence of the convergence rate on the regularity of coefficients and driving processes.

1.1. Nondegenerate SDEs driven by Lévy processes

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ of σ-algebras satisfying the usual conditions and $\alpha \in (0, 2]$ be fixed. Consider the following model in $\mathbf{R}^d$:

$$X_t = X_0 + \int_0^t a(X_s)\mathrm{d}s + \int_0^t b(X_{s-})\mathrm{d}U_s^\alpha + \int_0^t G(X_{s-})\mathrm{d}Z_s, \quad t \in [0, T], \tag{1}$$

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2011.04.004

where $a(x) = (a^i(x))_{1\le i\le d}$, $b(x) = (b^{ij}(x))_{1\le i,j\le d}$, $G(x) = (G^{ij}(x))_{1\le i\le d,1\le j\le m}$, $x \in \mathbf{R}^d$ are measurable and bounded, with $a = 0$ if $\alpha \in (0, 1)$ and b being nondegenerate. The main part of the equation is driven by $U^\alpha = \{U^\alpha_t\}_{t\in[0,T]}$, a standard d-dimensional spherically symmetric α-stable process:

$$U^\alpha_t = \int_0^t \int (1 - \bar{\chi}_\alpha(y)) y p_0(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int \bar{\chi}_\alpha(y) y q_0(\mathrm{d}s, \mathrm{d}y), \quad \alpha \in (0, 2),$$

where $\bar{\chi}_\alpha(y) = \mathbf{1}_{\{\alpha\in(1,2)\}} + \mathbf{1}_{\{\alpha=1\}}\chi_{\{|y|\le 1\}}$ and $p_0(\mathrm{d}t, \mathrm{d}y)$ is a Poisson point measure on $[0, \infty) \times \mathbf{R}^d_0$ ($\mathbf{R}^d_0 = \mathbf{R}^d \setminus \{0\}$) with

$$\mathbf{E}[p_0(\mathrm{d}t, \mathrm{d}y)] = \frac{\mathrm{d}t\mathrm{d}y}{|y|^{d+\alpha}}, \qquad q_0(\mathrm{d}t, \mathrm{d}y) = p_0(\mathrm{d}t, \mathrm{d}y) - \frac{\mathrm{d}t\mathrm{d}y}{|y|^{d+\alpha}}.$$

If $\alpha = 2$, U^α is the standard Wiener process in $\mathbf{R}^d$. The last term is driven by $Z = \{Z_t\}_{t\in[0,T]}$, an m-dimensional Lévy process whose characteristic function is $\exp\{t\eta(\xi)\}$ with

$$\eta(\xi) = \int_{\mathbf{R}^m_0} \left[e^{\mathrm{i}(\xi,y)} - 1 - \mathrm{i}(\xi, y)\chi_{\{|y|\le 1\}}\mathbf{1}_{\{\alpha\in(1,2]\}}\right]\pi(\mathrm{d}y).$$

Hence,

$$Z_t = \int_0^t \int (1 - \chi_\alpha(y)) y p(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int \chi_\alpha(y) y q(\mathrm{d}s, \mathrm{d}y),$$

where $\chi_\alpha(y) = \mathbf{1}_{\{\alpha\in(1,2]\}}\chi_{\{|y|\le 1\}}$, $p(\mathrm{d}t, \mathrm{d}y)$ is a Poisson point measure on $[0, \infty) \times \mathbf{R}^m_0$ with $\mathbf{E}[p(\mathrm{d}t, \mathrm{d}y)] = \pi(\mathrm{d}y)\mathrm{d}t$, and $q(\mathrm{d}t, \mathrm{d}y) = p(\mathrm{d}t, \mathrm{d}y) - \pi(\mathrm{d}y)\mathrm{d}t$ is the centered Poisson measure. It is assumed that

$$\int (|y|^\alpha \wedge 1)\pi(\mathrm{d}y) < \infty.$$

1.2. Motivation

The process defined in (1) is used as a mathematical model for random dynamic phenomena in applications arising from fields such as finance and insurance, to capture continuous and discontinuous uncertainty. For many applications, the practical computation of functionals of the type $F = \mathbf{E}[g(X_T)]$ and $F = \mathbf{E}\left[\int_0^T f(X_s)\mathrm{d}s\right]$ plays an important role. For instance in finance, derivative prices can be expressed in terms of such functionals. However in reality, a stochastic differential equation does not always have a closed-form solution. In such cases, in order to evaluate F, an alternative option is to numerically approximate the Itô process X by a discrete-time Monte Carlo simulation, an approach which has been widely applied. The simplest and the most commonly used scheme is the weak Euler approximation.

Let the time discretization $\{\tau_i,\ i = 0, \ldots, n_T\}$ of the interval $[0, T]$ with maximum step size $\delta \in (0, 1)$ be a partition of $[0, T]$ such that $0 = \tau_0 < \tau_1 < \cdots < \tau_{n_T} = T$ and $\max_i(\tau_i - \tau_{i-1}) \le \delta$. The Euler approximation of X is an $\mathbb{F}$-adapted stochastic process $Y = \{Y_t\}_{t\in[0,T]}$ defined by the stochastic equation

$$Y_t = X_0 + \int_0^t a(Y_{\tau_{i_s}})\mathrm{d}s + \int_0^t b(Y_{\tau_{i_s}})\mathrm{d}U^\alpha_s + \int_0^t G(Y_{\tau_{i_s}})\mathrm{d}Z_s, \quad t \in [0, T], \tag{2}$$

where $\tau_{i_s} = \tau_i$ if $s \in [\tau_i, \tau_{i+1})$, $i = 0, \dots, n_T - 1$. In contrast to those in (1), the coefficients in (2) are piecewise constants in each time interval of $[\tau_i, \tau_{i+1})$.

The weak Euler approximation Y is said to converge with order $\kappa > 0$ if for each bounded smooth function g with bounded derivatives, there exists a constant C, depending only on g, such that

$$|\mathbf{E}[g(Y_T)] - \mathbf{E}[g(X_T)]| \le C\delta^\kappa,$$

where $\delta > 0$ is the maximum step size of the time discretization.

In the literature, the weak Euler approximation of stochastic differential equations with smooth coefficients has been consistently studied. For diffusion processes ($\alpha = 2$), Milstein was one of the first to investigate the order of weak convergence and derived $\kappa = 1$ [13,14]. Talay considered a class of the second-order approximations for diffusion processes [18,19]. For Itô processes with jump components, Mikulevičius & Platen showed the first-order convergence in the case in which the coefficient functions possess fourth-order continuous derivatives [7]. Platen, and Kloeden & Platen studied not only Euler approximation but also higher order approximations [4,15]. Protter & Talay analyzed the weak Euler approximation for

$$X_t = X_0 + \int_0^t G(X_{s-})\mathrm{d}Z_s, \quad t \in [0, T], \tag{3}$$

where $Z_t = (Z_t^1, \dots, Z_t^m)$ is a Lévy process and $G = (G^{ij})_{1\le i\le d, 1\le j\le m}$ is a measurable and bounded function [17]. They showed the order of convergence $\kappa = 1$, provided that G and g are smooth and the Lévy measure of Z has finite moments of sufficiently high order. Because of this, the main theorems in [17] do not apply to (1). On the other hand, (1) with a nondegenerate matrix b does not cover (3), which can degenerate completely.

In general, the coefficients and the test function g do not always have the smoothness properties assumed in the papers cited above. Mikulevičius & Platen proved that there still exists some order of convergence of the weak Euler approximation for nondegenerate diffusion processes under Hölder conditions on the coefficients and g [8]. Kubilius & Platen, and Platen & Bruti-Liberati considered a weak Euler approximation in the case of a nondegenerate diffusion process with a finite number of jumps in finite time intervals [6,16].

In this paper, we investigate the dependence of the rate of convergence on the Hölder regularity of coefficients and the driving processes. For a driving process, the variation of the process can be regarded as a part of its regularity. In this sense, the Wiener process is the worse, most "chaotic", among α-stable processes. Also, as pointed out in [17], the tails of Lévy processes influence the convergence rate as well.

1.3. Examples

For $\beta > 0$, denote as $C^\beta(\mathbf{R}^d)$ the Hölder–Zygmund space (see Section 3.1.1 for a definition). Let us look at two examples.

Example 1 (*See* Corollary 4)**.** Assume that $\beta < \alpha, \beta \notin \mathbf{N}$, the coefficients $a^i, b^{ij} \in C^\beta(\mathbf{R}^d)$, $G^{ij} \in C^{\frac{\beta}{\alpha\wedge 1}}(\mathbf{R}^d)$, $\inf_x |\det B(x)| > 0$, and

$$\int_{\mathbf{R}^m} |y|^\alpha \pi(\mathrm{d}y) < \infty,$$

where π is the Lévy measure of the driving process Z. Then it holds that

$$|\mathbf{E}[g(Y_T)] - \mathbf{E}[g(X_T)]| \le C|g|_{\alpha+\beta}\delta^{\frac{\beta}{\alpha}},$$
$$\left|\mathbf{E}\left[\int_0^T f(Y_{\tau_{i_s}})\mathrm{d}s\right] - \mathbf{E}\left[\int_0^T f(X_s)\mathrm{d}s\right]\right| \le C|f|_\beta\delta^{\frac{\beta}{\alpha}}.$$

Example 2 (*See* Corollary 5)**.** Consider the jump-diffusion case ($\alpha = 2$)

$$X_t = X_0 + \int_0^t a(X_s)\mathrm{d}s + \int_0^t b(X_s)\mathrm{d}W_s + \int_0^t G(X_{s-})\mathrm{d}Z_s, \quad t \in [0, T],$$

where $W = \{W_t\}_{t\in[0,T]}$ is a standard Wiener process. Assume that $\beta \notin \mathbf{N}, a, b^{ij} \in C^\beta(\mathbf{R}^d)$, $\inf_x |\det b(x)| > 0$, and there exists $\mu \in (0, 2]$ such that

$$\int_{|y|\le 1} |y|^2\pi(\mathrm{d}y) + \int_{|y|>1} |y|^\mu\pi(\mathrm{d}y) < \infty.$$

Let G^{ij} be Lipschitz if $\beta = \mu < 1$ and suppose that $G^{ij} \in C^{\frac{\beta}{\mu\wedge 1}}(\mathbf{R}^d)$ otherwise. Then it holds that

$$|\mathbf{E}[g(Y_T)] - \mathbf{E}[g(X_T)]| \le C|g|_{\alpha+\beta}\delta^{\frac{\beta\wedge\mu}{2}},$$
$$\left|\mathbf{E}\left[\int_0^T f(Y_{\tau_{i_s}})\mathrm{d}s\right] - \mathbf{E}\left[\int_0^T f(X_s)\mathrm{d}s\right]\right| \le C|f|_\beta\delta^{\frac{\beta\wedge\mu}{2}}.$$

If $\mu > 2$ and $\beta > 2$, the order of convergence is 1. The assumption that $G^{ij} \in C^{\frac{\beta}{\mu\wedge 1}}(\mathbf{R}^d)$ shows that if $\mu < 1$, the heavy tail of π can be balanced by a higher regularity of G^{ij}.

As in [8], this paper employs the idea of Talay (see [18]) and uses the solution to the backward Kolmogorov equation associated with X_t, Itô's formula, and one-step estimates (see Section 2.2 for the outline of the proof).

The paper is organized as follows. In Section 2, the main result is stated and the proof is outlined. In Section 3, we present the essential technical results, and these are followed by the proof of the main theorem in Section 4.

2. Notation and the main result

2.1. The main result and notation

The main result of this paper is the following statement.

Theorem 3. *Suppose that* $\beta \in (0, 3), \beta \notin \mathbf{N}, 0 < \beta \le \mu < \alpha + \beta$, *and*

$$\int_{|y|\le 1} |y|^\alpha\pi(\mathrm{d}y) + \int_{|y|>1} |y|^\mu\pi(\mathrm{d}y) < \infty.$$

Assume that $\inf_x |\det b(x)| > 0$ *and* $a^i, b^{ij} \in C^\beta(\mathbf{R}^d)$. *Let* G^{ij} *be Lipschitz if* $\beta = \mu < 1$ *and suppose that* $G^{ij} \in C^{\frac{\beta}{\mu\wedge 1}}(\mathbf{R}^d)$ *otherwise. Then there exists a constant C such that for all*

$g \in C^{\alpha+\beta}(\mathbf{R}^d)$, $f \in C^{\beta}(\mathbf{R}^d)$,

$$|\mathbf{E}[g(Y_T)] - \mathbf{E}[g(X_T)]| \le C|g|_{\alpha+\beta}\delta^{\kappa(\alpha,\beta)},$$
$$\left|\mathbf{E}\left[\int_0^T f(Y_{\tau_{i_s}})\mathrm{d}s\right] - \mathbf{E}\left[\int_0^T f(X_s)\mathrm{d}s\right]\right| \le C|f|_\beta\delta^{\kappa(\alpha,\beta)},$$

where

$$\kappa(\alpha,\beta) = \begin{cases} \dfrac{\beta}{\alpha}, & \beta < \alpha, \\ 1, & \beta > \alpha. \end{cases}$$

Applying Theorem 3 to the case $\alpha = \mu$ and the case of heavier tails results in Corollaries 4 and 5, respectively.

Corollary 4. *Suppose that $\beta \in (0, 3)$, $\beta \notin \mathbf{N}$, $\beta < \alpha$, and*

$$\int |y|^\alpha \pi(\mathrm{d}y) < \infty.$$

Assume that $a^i, b^{ij} \in C^\beta(\mathbf{R}^d)$, $G^{ij} \in C^{\frac{\beta}{\alpha\wedge 1}}(\mathbf{R}^d)$, and $\inf_x |\det B(x)| > 0$. Then there exists a constant C such that for all $g \in C^{\alpha+\beta}(\mathbf{R}^d)$, $f \in C^\beta(\mathbf{R}^d)$,

$$|\mathbf{E}[g(Y_T)] - \mathbf{E}[g(X_T)]| \le C|g|_{\alpha+\beta}\delta^{\frac{\beta}{\alpha}},$$
$$\left|\mathbf{E}\left[\int_0^T f(Y_{\tau_{i_s}})\mathrm{d}s\right] - \mathbf{E}\left[\int_0^T f(X_s)\mathrm{d}s\right]\right| \le C|f|_\beta\delta^{\frac{\beta}{\alpha}}.$$

Corollary 5. *Suppose that $\beta \in (0, 3)$, $\beta \notin \mathbf{N}$, $0 < \beta \le \mu < \alpha$, and*

$$\int_{|y|\le 1} |y|^\alpha \pi(\mathrm{d}y) + \int_{|y|>1} |y|^\mu \pi(\mathrm{d}y) < \infty.$$

Suppose that $\inf_x |\det b(x)| > 0$, $a^i, b^{ij} \in C^\beta(\mathbf{R}^d)$. Assume that G^{ij} is Lipschitz if $\beta = \mu < 1$ and suppose that $G^{ij} \in C^{\frac{\beta}{\mu\wedge 1}}(\mathbf{R}^d)$ otherwise. Then there exists a constant C such that for all $g \in C^{\alpha+\beta}(\mathbf{R}^d)$, $f \in C^\beta(\mathbf{R}^d)$,

$$|\mathbf{E}[g(Y_T)] - \mathbf{E}[g(X_T)]| \le C|g|_{\alpha+\beta}\delta^{\frac{\beta\wedge\mu}{\alpha}},$$
$$\left|\mathbf{E}\left[\int_0^T f(Y_{\tau_{i_s}})\mathrm{d}s\right] - \mathbf{E}\left[\int_0^T f(X_s)\mathrm{d}s\right]\right| \le C|f|_\beta\delta^{\frac{\beta\wedge\mu}{\alpha}}.$$

Define $H = [0, T] \times \mathbf{R}^d$, $\mathbf{N} = \{0, 1, 2, \ldots\}$, $\mathbf{R}^d_0 = \mathbf{R}^d \setminus \{0\}$. For $x, y \in \mathbf{R}^d$, write $(x, y) = \sum_{i=1}^d x_i y_i$. For $(t, x) \in H$, the multi-index $\gamma \in \mathbf{N}^d$ with $D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\ldots\partial x_d^{\gamma_d}}$, and $i, j = 1, \ldots, d$, define

$$\partial_t u(t,x) = \frac{\partial}{\partial t}u(t,x), \qquad D^k u(t,x) = \left(D^\gamma u(t,x)\right)_{|\gamma|=k}, \quad k \in \mathbf{N},$$
$$\partial_i u(t,x) = u_{x_i}(t,x) = \frac{\partial}{\partial x_i}u(t,x), \qquad \partial^2_{ij}u(t,x) = u_{x_i x_j}(t,x) = \frac{\partial^2}{\partial x_i x_j}u(t,x),$$

$$\partial_x u(t,x) = \nabla u(t,x) = \big(\partial_1 u(t,x), \dots, \partial_d u(t,x)\big),$$
$$\partial^2 u(t,x) = \Delta u(t,x) = \sum_{i=1}^{d} \partial_{ii}^2 u(t,x).$$

$C = C(\cdot, \dots, \cdot)$ denotes constants depending only on quantities appearing in parentheses. In a given context the same letter is (generally) used to denote different constants depending on the same set of arguments.

2.2. An outline of the proof

Due to the lack of regularity, standard techniques such as the stochastic flows method cannot be applied to prove Theorem 3. Instead, as in [8], the solution to the backward Kolmogorov equation associated with X_t is used. In the following, the operators of the Kolmogorov equation associated with X_t are first defined.

For $u \in C^{\alpha+\beta}(H)$, define

$$\begin{aligned} A_z u(t,x) \;&=\; \mathbf{1}_{\{\alpha=1\}}(a(z), \nabla_x u(t,x)) + \mathbf{1}_{\{\alpha=2\}} \sum_{i,j=1}^{d} D^{ij}(z)\partial_{ij}^2 u(t,x) \\ &\quad + \mathbf{1}_{\{\alpha\in(0,2)\}} \int [u(t,x+b(z)y) - u(t,x) - (\nabla u(t,x), b(z)y)\chi_\alpha(y)] \frac{\mathrm{d}y}{|y|^{d+\alpha}}, \\ Au(t,x) &= A_x u(t,x) = A_z u(t,x)|_{z=x}, \end{aligned}$$

with $\chi_\alpha(y) = \mathbf{1}_{\{\alpha\in(1,2)\}} + \mathbf{1}_{\{\alpha=1\}}\chi_{\{|y|\le 1\}}$, $D = b^* b$, and

$$\begin{aligned} B_z u(t,x) \;&=\; \mathbf{1}_{\{\alpha\in(1,2]\}}(a(z), \nabla_x u(t,x)) + \int_{\mathbf{R}_0^m} \big[u(t,x+G(z)y) - u(t,x) \\ &\quad - \mathbf{1}_{\{\alpha\in(1,2]\}}\mathbf{1}_{\{|y|\le 1\}}(\nabla_x u(t,x), G(z)y)\big]\pi(\mathrm{d}y), \\ Bu(t,x) &= B_x u(t,x) = B_z u(t,x)|_{z=x}. \end{aligned}$$

Applying Itô's formula to X_t and $u \in C_0^\infty(\mathbf{R}^d)$, we find that

$$u(X_t) - \int_0^t Au(X_s)\mathrm{d}s - \int_0^t Bu(X_s)\mathrm{d}s, \quad t \in [0,T]$$

is a martingale.

Remark 6. More precisely, under the assumptions of Theorem 3, there exists a unique weak solution to Eq. (1) and the stochastic process

$$u(X_t) - \int_0^t (A+B)u(X_s)\mathrm{d}s, \quad \forall u \in C^{\alpha+\beta}(\mathbf{R}^d)$$

is a martingale [10]. The operator $\mathcal{L} = A + B$ is the generator of X_t defined in (1); A is the principal part of $\mathcal{L}$ and B is the lower order or subordinate part of $\mathcal{L}$.

If $v(t,x)$, $(t,x) \in H$ satisfies the backward Kolmogorov equation

$$\begin{aligned} &\big(\partial_t + A + B\big)v(t,x) = 0, \quad 0 \le t \le T, \\ &v(T,x) = g(x), \end{aligned}$$

then as interpreted in Section 4, by Itô's formula

$$\mathbf{E}[g(Y_T)] - \mathbf{E}[g(X_T)] = \mathbf{E}[v(T, Y_T) - v(0, Y_0)] = \mathbf{E}\left[\int_0^T (\partial_t + \mathcal{L}_{Y_{\tau_{i_s}}})v(s, Y_s)\mathrm{d}s\right].$$

The regularity of v determines the one-step estimate and the rate of convergence of the approximation. For $\beta \in (0, 1)$, the results for the Kolmogorov equation in Hölder classes are available [9,11]. In a standard way the results can be extended to the case $\beta > 1$. The main difficulty is deriving the one-step estimates (see Lemma 15).

3. The backward Kolmogorov equation

In Hölder–Zygmund spaces, consider the backward Kolmogorov equation associated with X_t:

$$\begin{aligned} &\big(\partial_t + A + B\big)v(t, x) = f(t, x), \\ &v(T, x) = 0. \end{aligned} \tag{4}$$

The regularity of its solution is essential for the one-step estimate which determines the rate of convergence.

Definition 7. Let f be a measurable and bounded function on $\mathbf{R}^d$. We say that $u \in C^{\alpha+\beta}(H)$ is a solution to (4) if

$$u(t, x) = \int_t^T \big[\mathcal{L}u(s, x) - f(s, x)\big]\mathrm{d}s, \quad \forall (t, x) \in H. \tag{5}$$

The following theorem is the main result of this section.

Theorem 8. *Suppose that $\beta \in (0, 3)$, $\beta \notin \mathbf{N}$, $0 < \beta \le \mu < \alpha + \beta$, and*

$$\int_{|y|\le 1} |y|^\alpha \pi(\mathrm{d}y) + \int_{|y|>1} |y|^\mu \pi(\mathrm{d}y) < \infty.$$

Assume that $a^i, b^{ij} \in C^\beta(\mathbf{R}^d)$, $\inf_x |\det b(x)| > 0$. Let G^{ij} be Lipschitz if $\beta = \mu < 1$ and suppose that $G^{ij} \in C^{\frac{\beta}{\mu\wedge 1}}(\mathbf{R}^d)$ otherwise. Then for each $f \in C^\beta(\mathbf{R}^d)$, there exist a unique solution $v \in C^{\alpha+\beta}(H)$ to (4) and a constant C independent of f such that $|u|_{\alpha+\beta} \le C|f|_\beta$.

An immediate consequence of Theorem 8 is the following statement.

Corollary 9. *Suppose that $\beta \in (0, 3)$, $\beta \notin \mathbf{N}$, $0 < \beta \le \mu < \alpha + \beta$, and*

$$\int_{|y|\le 1} |y|^\alpha \pi(\mathrm{d}y) + \int_{|y|>1} |y|^\mu \pi(\mathrm{d}y) < \infty.$$

Assume that $a^i, b^{ij} \in C^\beta(\mathbf{R}^d)$, $\inf_x |\det b(x)| > 0$. Let G^{ij} be Lipschitz if $\beta = \mu < 1$ and suppose that $G^{ij} \in C^{\frac{\beta}{\mu\wedge 1}}(\mathbf{R}^d)$ otherwise. Then for each $f \in C^\beta(\mathbf{R}^d)$ and $g \in C^{\alpha+\beta}(\mathbf{R}^d)$, there exists a unique solution $v \in C^{\alpha+\beta}(H)$ to the Cauchy problem

$$\begin{aligned} &\big(\partial_t + A + B\big)v(t, x) = f(x), \\ &v(T, x) = g(x) \end{aligned} \tag{6}$$

and $|v|_{\alpha+\beta} \le C(|f|_\beta + |g|_{\alpha+\beta})$ with a constant C independent of f and g.

To prove Theorem 8 and Corollary 9, in a standard way, the equation with constant coefficients is first solved. Then variable coefficients are handled by using a partition of unity and deriving a priori Schauder estimates in Hölder–Zygmund spaces. Finally, the continuation by parameter method is applied to extend the solvability of an equation with constant coefficients to (4).

3.1. The Kolmogorov equation with constant coefficients

It is convenient to rewrite the principal operator A by changing the variable of integration in the integral part:

$$\begin{aligned}A_z u(t,x) &= \mathbf{1}_{\{\alpha=1\}}(a(z),\nabla_x u(t,x)) + \mathbf{1}_{\{\alpha=2\}}\sum_{i,j=1}^{d} D^{ij}(z)\partial_{ij}^2 u(t,x)\\ &\quad + \mathbf{1}_{\{\alpha\in(0,2)\}}\int [u(t,x+y)-u(t,x)-(\nabla u(t,x),y)\chi_\alpha(y)]m(z,y)\frac{\mathrm{d}y}{|y|^{d+\alpha}},\end{aligned}$$

where $D=b^*b$,

$$m(z,y)=\frac{1}{|\det b(z)|}\frac{1}{\left|b(z)^{-1}\frac{y}{|y|}\right|^{d+\alpha}},\quad \alpha\in(0,2). \tag{7}$$

Obviously,

$$\int_{S^{d-1}} y m(\cdot,y)\mu_{d-1}(\mathrm{d}y)=0. \tag{8}$$

Here S^{d-1} is the unit sphere in $\mathbf{R}^d$ and μ_{d-1} is the Lebesgue measure.

For $z_0\in\mathbf{R}^d$, define $A^0u(x)=A_{z_0}u(x)$. Consider a backward Kolmogorov equation with constant coefficients and $\lambda\geq 0$,

$$\begin{aligned}&\left(\partial_t + A^0-\lambda\right)v(t,x)=f(x), \qquad (9)\\ &v(T,x)=0.\end{aligned}$$

Proposition 10. *Suppose that $\beta>0$, $\beta\notin\mathbf{N}$, and $f\in C^\beta(\mathbf{R}^d)$. Assume that there are constants $c_1,K>0$ such that for all $z\in\mathbf{R}^d$,*

$$|\det b(z)|\geq c_1,\qquad \mathbf{1}_{\{\alpha=1\}}|a(z)|+|b(z)|\leq K.$$

Then there exists a unique solution $u\in C^{\alpha+\beta}(H)$ to (9) and

$$|u|_{\alpha+\beta}\leqslant C|f|_\beta, \tag{10}$$

where the constant C depends only on α,β,T,d,c_1,K. Moreover,

$$|u|_\beta\leq C(\alpha,d)(\lambda^{-1}\wedge T)|f|_\beta \tag{11}$$

and there exists a constant C such that for all $s\leq t\leq T$,

$$|u(t,\cdot)-u(s,\cdot)|_{\frac{\alpha}{2}+\beta}\leq C(t-s)^{\frac12}|f|_\beta. \tag{12}$$

To derive Proposition 10, some auxiliary results are presented first.

3.1.1. Continuity of the operator A^0 in Hölder–Zygmund spaces

To show that operator A^0 is continuous in Hölder–Zygmund spaces $C^\beta(\mathbf{R}^d)$, first recall their definition.

For $\beta = [\beta]^- + \{\beta\}^+ > 0$, where $[\beta]^- \in \mathbf{N}$ and $\{\beta\}^+ \in (0, 1]$, let $C^\beta(H)$ denote the space of measurable functions u on H such that the norm

$$|u|_\beta = \sum_{|\gamma|\le[\beta]^-} \sup_{(t,x)\in H} |D_x^\gamma u(t,x)| + \mathbf{1}_{\{\{\beta\}^+<1\}} \sup_{\substack{|\gamma|=[\beta]^-,\\ t,x\neq\tilde{x}}} \frac{|D_x^\gamma u(t,x) - D_x^\gamma u(t,\tilde{x})|}{|x-\tilde{x}|^{\{\beta\}^+}}$$
$$+ \mathbf{1}_{\{\{\beta\}^+=1\}} \sup_{\substack{|\gamma|=[\beta]^-,\\ t,x,h\neq 0}} \frac{|D_x^\gamma u(t,x+h) + D_x^\gamma u(t,x-h) - 2D_x^\gamma u(t,x)|}{|h|^{\{\beta\}^+}}$$

is finite. Accordingly, $C^\beta(\mathbf{R}^d)$ denotes the corresponding space of functions on $\mathbf{R}^d$. The classes C^β coincide with Hölder spaces if $\beta \notin \mathbf{N}$ (see 1.2.2 of [20]).

For $v \in C^\beta(\mathbf{R}^d)$ with $\beta \in (0, 1)$, define

$$|v|_0 = \sup_x |v(x)|, \qquad [v]_\beta = \sup_{x\neq y} \frac{|v(x)-v(y)|}{|x-y|^\beta}.$$

For $\alpha \in (0, 2)$, define for $v \in C^{\alpha+\beta}(\mathbf{R}^d)$ the fractional Laplacian

$$\partial^\alpha v(x) = \int [v(x+y) - v(x) - (\nabla v(x), y)\, \chi_\alpha(y)] \frac{\mathrm{d}y}{|y|^{d+\alpha}}, \quad x \in \mathbf{R}^d. \tag{13}$$

For various estimates, the following representation of the difference is useful.

Lemma 11 (*Lemma 2.1 in [5]*)**.** *For $\delta \in (0, 1)$ and $u \in C_0^\infty(\mathbf{R}^d)$,*

$$u\big(x+y\big) - u(x) = K \int k^{(\delta)}(y,z)\partial^\delta u(x-z)\mathrm{d}z,$$

where $K = K(\delta, d)$ is a constant,

$$k^{(\delta)}(y,z) = |z+y|^{-d+\delta} - |z|^{-d+\delta},$$

and there exists a constant C such that

$$\int |k^{(\delta)}(y,z)|\mathrm{d}z \le C|y|^\delta, \quad \forall y \in \mathbf{R}^d.$$

On taking the pointwise limit (∂^δ is defined by (13)) and applying the dominated convergence theorem, the statement can be extended to $u \in C^\delta(\mathbf{R}^d)$.

Let $m(y)$ be a measurable and bounded function on $\mathbf{R}^d$. Define

$$L^m u(x) = \int_{\mathbf{R}^d} [u(x+y) - u(x) - (\nabla u(x), y)\chi_\alpha(y)]\, m(y) \frac{\mathrm{d}y}{|y|^{d+\alpha}}, \quad u \in C^{\alpha+\beta}.$$

The following statement is proved in [12] for $\beta \in (0, 1)$. It is presented here for the sake of completeness and is extended to any $\beta > 0$, $\beta \notin \mathbf{N}$.

Lemma 12. *Suppose that $\alpha \in (0,2)$, $\beta > 0$, $\beta \notin \mathbf{N}$, $u \in C^{\alpha+\beta}(\mathbf{R}^d)$, and $|m| \le K$. Assume that if $\alpha = 1$,*

$$\int_{r<|y|\le 1} ym(y)\frac{\mathrm{d}y}{|y|^{d+1}} = 0, \quad \forall r \in (0,1). \tag{14}$$

Then there exists a constant C independent of u such that

$$|L^m u|_\beta \le CK|u|_{\alpha+\beta}.$$

Proof. Define $L = L^m$. For $\beta \in (0,1)$, consider three cases.

Case I: $\alpha \in (0,1)$. For $u \in C^{\alpha+\beta}(\mathbf{R}^d)$, let $\beta' \in (0,1)$ be such that $\alpha+\beta' < 1$. Then

$$\sup_x |L^m u(x)| \le CK|u|_{a+\beta'}\int (|y|^{\alpha+\beta'} \wedge 1)\frac{\mathrm{d}y}{|y|^{d+\alpha}} \le CK|u|_{\alpha+\beta}.$$

Suppose that $x, \bar{x} \in \mathbf{R}^d$ and $a = |x-\bar{x}|$. Write

$$Lu(x) - Lu(\bar{x}) = \int_{|y|\le a} \cdots + \int_{|y|>a} \cdots = I_1 + I_2.$$

Let $\beta' \in (0,\beta)$ be such that $\alpha+\beta' < 1$ and $\beta-\beta' < 1$. Then by Lemma 11,

$$\begin{aligned}|I_1| &\le K\int_{|y|\le a} |\partial^{\alpha+\beta'}u(x-z) - \partial^{\alpha+\beta'}u(\bar{x}-z)|\,|k^{(\alpha+\beta')}(y,z)|\frac{\mathrm{d}y}{|y|^{d+\alpha}} \\ &\le CK|u|_{\alpha+\beta}a^{\beta-\beta'}\int_{|y|\le a}|y|^{\alpha+\beta'}\frac{\mathrm{d}y}{|y|^{d+\alpha}} = CKa^{\beta-\beta'}a^{\beta'} = CKa^\beta.\end{aligned}$$

Let $\alpha' < \alpha$ be such that $\beta+\alpha-\alpha' < 1$. By Lemma 11,

$$\begin{aligned}|I_2| &\le K\int_{|y|>a} |\partial^{\alpha'}u(x-z) - \partial^{\alpha'}u(\bar{x}-z)|\,|k^{(\alpha')}(y,z)|\frac{\mathrm{d}y}{|y|^{d+\alpha}} \\ &\le CK|u|_{\alpha+\beta}a^{\alpha+\beta-\alpha'}\int_{|y|>a}|y|^{\alpha'}\frac{\mathrm{d}y}{|y|^{d+\alpha}} = CK|u|_{\alpha+\beta}a^\beta.\end{aligned}$$

Case II: $\alpha = 1$. For $u \in C^{1+\beta}(\mathbf{R}^d)$,

$$\begin{aligned}Lu(x) &= \int_{|y|\le 1}[u(x+y) - u(x) - (\nabla u(x), y)]\,m(y)\frac{\mathrm{d}y}{|y|^{d+1}} \\ &\quad + \int_{|y|>1}[u(x+y)-u(x)]\,m(y)\frac{\mathrm{d}y}{|y|^{d+1}} \\ &= L_1u(x) + L_2u(x).\end{aligned}$$

Since

$$L_1u(x) = \int_{|y|\le 1}\int_0^1 \big(\nabla u(x+sy) - \nabla u(x), y\big)m(y)\frac{\mathrm{d}y}{|y|^{d+1}}, \tag{15}$$

it follows that

$$\sup_x |L_1u(x)| \le CK|u|_{1+\beta}\int_{|y|\le 1}|y|^{1+\beta}\frac{\mathrm{d}y}{|y|^{d+1}} < \infty.$$

Suppose that $x, \bar{x} \in \mathbf{R}^d$ and $a = |x - \bar{x}|$. Write

$$L_1u(x) - L_1u(\bar{x}) = \int_{|y|\leq a} \cdots + \int_{|y|>a} \cdots = B_1 + B_2.$$

By (15),

$$|B_1| \leq CK|u|_{1+\beta} \int_{|y|\leq a} |y|^{1+\beta} \frac{\mathrm{d}y}{|y|^{d+1}} = CK|u|_{1+\beta} a^{\beta}.$$

Let $\alpha' < 1$ be such that $\alpha + \beta - \alpha' < 1$. By Lemma 11 and (14),

$$\begin{aligned}|B_2| &\leq K \int_{|y|>a} \int |\partial^{\alpha'} u(x - z) - \partial^{\alpha'} u(\bar{x} - z)|\,|k^{(\alpha')}(y, z)|\mathrm{d}z \frac{\mathrm{d}y}{|y|^{d+1}}\\ &\leq CK|u|_{\alpha+\beta} a^{\alpha+\beta-\alpha'} \int_{|y|>a} |y|^{\alpha'} \frac{\mathrm{d}y}{|y|^{d+\alpha}} = CK|u|_{\alpha+\beta} a^{\alpha+\beta-\alpha'} a^{\alpha'-\alpha}\\ &= CK|u|_{\alpha+\beta} a^{\beta}.\end{aligned}$$

The estimates of L_2u are obvious.

Case III: $\alpha \in (1, 2)$. For $u \in C^{\alpha+\beta}(\mathbf{R}^d)$,

$$Lu(x) = \iint_0^1 \big(\nabla u(x + sy) - \nabla u(x), y\big) m(y) \frac{\mathrm{d}y}{|y|^{d+\alpha}}.$$

Let $\beta' < 1$ be such that $\alpha + \beta' < 2$ and $0 < \beta - \beta' < 1$. By Lemma 11,

$$\begin{aligned}\sup_x |Lu(x)| &\leq CK|u|_{\alpha+\beta'} \int_{|y|\leq 1} |y|^{\alpha+\beta'-1}|y| \frac{\mathrm{d}y}{|y|^{d+\alpha}} + \sup_x |\nabla u(x)| K \int_{|y|>1} |y| \frac{\mathrm{d}y}{|y|^{d+\alpha}}\\ &\leq CK|u|_{\alpha+\beta'}.\end{aligned}$$

Carry out the splitting

$$\begin{aligned}Lu(x) &= \int_{|y|\leq a} \int_0^1 \big(\nabla u(x + sy) - \nabla u(x), y\big) m(y) \frac{\mathrm{d}s\mathrm{d}y}{|y|^{d+\alpha}}\\ &\quad + \int_{|y|>a} \int_0^1 \big(\nabla u(x + sy) - \nabla u(x), y\big) m(y) \frac{\mathrm{d}s\mathrm{d}y}{|y|^{d+\alpha}}\\ &= L_1u(x) + L_2u(x).\end{aligned}$$

Let $x, \bar{x} \in \mathbf{R}^d$, $a = |x - \bar{x}|$, and $\beta' < 1$ be such that $\alpha + \beta' < 2$ and $0 \leq \beta - \beta' < 1$. By Lemma 11,

$$\begin{aligned}|L_1u(x) - L_1(\bar{x})| &\leq K \int_{|y|\leq a} \int_0^1 \int |\partial^{\alpha+\beta'-1} \nabla u(x - z) - \partial^{\alpha+\beta'-1} \nabla u(x - z)|\\ &\qquad \times |k^{(\alpha+\beta'-1)}(sy, z)|\,|y| \frac{\mathrm{d}s\mathrm{d}z\mathrm{d}y}{|y|^{d+\alpha}}\\ &\leq CK|u|_{\alpha+\beta} a^{\beta-\beta'} \int_{|y|\leq a} |y|^{\alpha+\beta'} \frac{\mathrm{d}y}{|y|^{d+\alpha}} = CK|u|_{\alpha+\beta} a^{\beta}.\end{aligned}$$

Finally, let $1 < \alpha' < \alpha$ be such that $\alpha - \alpha' + \beta < 1$. By Lemma 11,

$$|L_2u(x) - L_2(\bar{x})|$$

$$= \left| \int_{|y|>a} \int_0^1 \int [\partial^{\alpha'-1} \nabla u(x-z) - \partial^{\alpha'-1} \nabla u(\bar{x}-z)] k^{(\alpha'-1)}(sy, z) y m(y) \frac{\mathrm{d}z\mathrm{d}s\mathrm{d}y}{|y|^{d+\alpha}} \right|$$
$$\leq CKa^{\alpha+\beta-\alpha'} \int_{|y|>a} |y|^{\alpha'} \frac{\mathrm{d}z\mathrm{d}s\mathrm{d}y}{|y|^{d+\alpha}} = CKa^{\beta}.$$

Therefore, the result holds for $\beta \in (0, 1)$. If $\beta > 0$, $\beta \notin \mathbf{N}$, and $u \in C^{\alpha+\beta}(\mathbf{R}^d)$, then for any multi-index $|\gamma| = [\beta]$, $D^{\gamma} u \in C^{\alpha+\beta-[\beta]}$, and

$$|D^{\gamma} \left(L^m u \right) |_{\beta-[\beta]} = |L^m \left(D^{\gamma} u \right) |_{\beta-[\beta]} \leq CK |D^{\gamma} u|_{\alpha+\beta-[\beta]}.$$

The statement follows. □

3.1.2. Proof of Proposition 10

The statement is proved by induction. Given $\alpha \in (0, 2]$ and $f \in C^{\beta}(H)$, for $\beta \in (0, 1)$, there exists a unique solution $u \in C^{\alpha+\beta}(H)$ to the Kolmogorov equation (9) such that (10)–(12) hold [11].

Assume that the result holds for $\beta \in \bigcup_{l=0}^{n-1} (l, l+1)$, $n \in \mathbf{N}$. Suppose that $\beta \in (n, n+1)$, $\tilde{\beta} = \beta - 1$, and $f \in C^{\beta}(H)$. Then $\tilde{\beta} \in (n-1, n)$, $f \in C^{\tilde{\beta}}(H)$, and there exists a unique solution $v \in C^{\alpha+\tilde{\beta}}(H)$, $\alpha \in (0, 2]$ to the Cauchy problem such that (10)–(12) hold for v with $\tilde{\beta}$. For $h \in \mathbf{R}$ and $k = 1, \ldots, d$, define

$$v_k^h(t, x) = \frac{v(t, x + he_k) - v(t, x)}{h},$$

where $\{e_k, \ k = 1, \ldots, d\}$ is the canonical basis in $\mathbf{R}^d$. Obviously, $v_k^h \in C^{\alpha+\tilde{\beta}}(H)$ and

$$\begin{aligned} &\left(\partial_t + A^0 - \lambda\right) v_k^h(t, x) = f_k^h(x), \quad x \in \mathbf{R}^d, \\ &v_k^h(T, x) = 0. \end{aligned} \tag{16}$$

Since $f \in C^{\beta}(H)$ and

$$f_k^h(t, x) = \int_0^1 \partial_k f(t, x + he_k s) \mathrm{d}s, \quad \forall h \neq 0,$$

then

$$|f_k^h|_{\tilde{\beta}} \leq C |\nabla f|_{\beta-1} \leq C |f|_{\beta} \tag{17}$$

with a constant C independent of h. Since $v \in C^{\alpha+\tilde{\beta}}(H)$, then $v_k^h \in C^{\alpha+\tilde{\beta}}(H)$. By (17) and the induction assumption, the estimates (10)–(12) hold for v_k^h with a constant independent of h. Hence $v_k^h(t, x)$ are equicontinuous in (t, x). By the Arzelà–Ascoli theorem, for each $h_n \to 0$, there exist a subsequence $\{h_{n_j}\}$ and continuous functions $v_k(t, x)$, $(t, x) \in H$, $k = 1, \ldots, d$, such that $v_k^{h_{n_j}}(t, x) \to v_k(t, x)$ uniformly on compact subsets of H as $j \to \infty$. Therefore, $v_k \in C^{\alpha+\tilde{\beta}}$ and $|v_k|_{\alpha+\tilde{\beta}} \leq C|f|_{\beta}$, $k = 1, \ldots, d$.

It then follows from passing to the limit in the integral form of (16) (see (5)) and the dominated convergence theorem that u_k is the unique solution to

$$\begin{aligned} &\left(\partial_t + A^0 - \lambda\right) v_k(t, x) = \partial_k f(t, x), \\ &v_k(T, x) = 0, \quad k = 1, \ldots, d \end{aligned}$$

and so $v_k^{h_n}(t,x) \to v_k(t,x),\ \forall h_n \to 0$. Hence,

$$v_k(t,x) = \lim_{h\to 0} v_k^h(t,x) = \lim_{h\to 0} \frac{v(t,x+he_k) - v(t,x)}{h} = \partial_k v(t,x),$$

$\partial_k v \in C^{\alpha+\tilde{\beta}}(H),\ k = 1,\ldots,d$, and $|\nabla v|_{\alpha+\tilde{\beta}} \leq C|f|_\beta$. Therefore, $v \in C^{\alpha+\beta}(H)$ and the statement follows.

3.2. The Kolmogorov equation with variable coefficients

In this section, an estimate is derived to show that Bu is a lower order operator, which is essential in deriving Schauder estimates in the case of variable coefficients. To prove Theorem 8, in a standard way we use a partition of unity and the estimates for constant coefficients, which allow us to obtain a priori estimates. Then the continuation by parameter method is applied to transfer from constant to variable coefficients.

3.2.1. Estimates of Bf, $f \in C^{\alpha+\beta}$

Proposition 13. *Suppose that $\beta \in (0,3), \beta \notin \mathbf{N},\ 0 < \beta \leq \mu < \alpha+\beta$, and*

$$\int_{|y|\leq 1} |y|^\alpha \pi(\mathrm{d}y) + \int_{|y|>1} |y|^\mu \pi(\mathrm{d}y) < \infty.$$

Assume that $a \in C^\beta(\mathbf{R}^d)$. Let G^{ij} be Lipschitz if $\beta = \mu < 1$ and suppose that $G^{ij} \in C^{\frac{\beta}{\mu\wedge 1}}(\mathbf{R}^d)$ otherwise. Then for each $\varepsilon > 0$, there exists a constant C_ε such that

$$|Bf|_\beta \leq \varepsilon|f|_{\alpha+\beta} + C_\varepsilon|f|_0, \quad f \in C^{\alpha+\beta}(\mathbf{R}^d).$$

Proof. Since the estimates involving the term with $a(x)$ are obvious, in the following estimates, assume $a = 0$.

Case I: $\beta \in (0,1),\ \beta \leq \mu < \alpha+\beta$. In what follows, $|G|_1$ is G Lipschitz constant. Carry out the splitting

$$\begin{aligned} B_z f(x) &= \int \left[f(x+G(z)y) - f(x) - \mathbf{1}_{\{\alpha\in(1,2]\}}(\nabla f(x), G(z)y)\chi_{\{|y|\leq 1\}}\right]\pi(\mathrm{d}y) \\ &= \int_{|y|\leq\delta} \cdots + \int_{|y|>\delta} \cdots = B_z^1 f(x) + B_z^2 f(x) \end{aligned}$$

and $B_z^2 f(x) = -B_z^{21} f(x) + B_z^{22} f(x)$ with

$$\begin{aligned} B_z^{21} f(x) &= f(x)\int_{|y|>\delta} \pi(\mathrm{d}y) + \mathbf{1}_{\{\alpha\in(1,2]\}}\left(\nabla f(x), \int_{\delta<|y|\leq 1} G(z)y\pi(\mathrm{d}y)\right), \\ B_z^{22} f(x) &= \int_{|y|>\delta} f(x+G(z)y)\pi(\mathrm{d}y). \end{aligned}$$

It follows by the assumptions that there exists β' such that $\mu < \alpha+\beta' < \alpha+\beta$ and

$$\begin{aligned} &|B_z^{21} f(\cdot)|_\beta + |B_z^{22} f(\cdot)|_\beta \leq C[|f|_\beta + \mathbf{1}_{\{\alpha\in(1,2]\}}|\nabla f|_\beta], \quad z \in \mathbf{R}^d, \\ &|B_\cdot^{21} f(x)|_\beta \leq C\mathbf{1}_{\{\alpha\in(1,2]\}}|\nabla f|_0|G|_\beta, \\ &|B_\cdot^{22} f(x)|_\beta \leq C\int_{|y|\geq\delta} |y|^\mu\pi(\mathrm{d}y)|f|_{\alpha+\beta'}|G|_{\frac{\beta}{\mu\wedge 1}}, \quad x \in R^d. \end{aligned} \tag{18}$$

Consider different scenarios for values of α to show that

$$|B_z^1 f(\cdot)|_\beta \le C|f|_{\alpha+\beta} \int_{|y|\le\delta} |y|^\alpha \mathrm{d}\pi, \quad z \in \mathbf{R}^d. \tag{19}$$

For $\alpha \in (0,1]$, by Lemma 11,

$$B_z^1 f(x) = \int_{|y|\le\delta} \int \partial^\alpha f(x-z) k^{(\alpha)}(C(z)y, z)\mathrm{d}z\pi(\mathrm{d}y) \quad \text{if } \alpha < 1,$$

$$B_z^1 f(x) = \int_{|y|\le\delta} \int_0^1 (\nabla f(x+sC(z)y), y)\mathrm{d}s\pi(\mathrm{d}y) \quad \text{if } \alpha = 1.$$

Hence, (19) follows.

For $\alpha = 2$, (19) follows since

$$B_z^1 f(x) = \int_{|y|\le\delta} \left[\int_0^1 \Big(D^2 f(x+sC(z)y)C(z)y, C(z)y\Big)(1-s)\mathrm{d}s\right] \quad \text{if } \alpha = 2.$$

For $\alpha \in (1,2)$, (19) follows since by Lemma 11

$$\begin{aligned} B_z^1 f(x) &= \int_{|y|\le\delta} \left[\int_0^1 \Big(\nabla f(x+sC(z)y) - \nabla f(x), C(z)y\Big)\mathrm{d}s\right]\mathrm{d}\pi \\ &= \int_{|y|\le\delta} \left[\int_0^1 \Big(\int \partial^{\alpha-1}\nabla f(x-t)k^{(\alpha-1)}(sC(z)y, t)\mathrm{d}t, C(z)y\Big)\mathrm{d}s\right]\mathrm{d}\pi. \end{aligned}$$

Similarly, to estimate $|B_\cdot^1 f(x)|_\beta$, consider different scenarios for values of α.

For $\alpha \in (0,1)$,

$$|B_\cdot^1 f(x)|_\beta \le |f|_{(\alpha+\beta)} \int_{|y|\le\delta} |y|^\alpha \pi(\mathrm{d}y)|G|_{\frac{\beta}{\mu\wedge 1}}, \quad x \in \mathbf{R}^d. \tag{20}$$

For $\alpha \in [1,2]$, suppose that $\beta \le \mu < \alpha + \beta' < \alpha + \beta$.

If $\alpha \in (1,2]$, for $|y| \le 1$, $z, \bar{z} \in \mathbf{R}^d$,

$$\begin{aligned} &|[f(x+G(z)y) - f(x) - (\nabla f(x), G(z)y)] \\ &\qquad - [f(x+G(\bar{z})y) - f(x) - (\nabla f(x), C(\bar{z})y)]| \\ &= |[f(x+G(z)y) - f(x+G(\bar{z})y) - (\nabla f(x), (G(z)-G(\bar{z}))y)]| \\ &\le \int_0^1 |(\nabla f(x+(1-s)G(\bar{z})y + sG(z)y) - \nabla f(x), G(z)y - G(\bar{z})y)\,\mathrm{d}s| \\ &\le |f|_\alpha \left(|G|_0^{\alpha-1}|y|^{\alpha-1} + |G(\bar{z}) - G(z)|^{\alpha-1}|y|^{\alpha-1}\right)|G(z) - G(\bar{z})|\,|y| \\ &\le C|f|_\alpha |G|_0^{\alpha-1}|G(\bar{z}) - G(z)|\,|y|^\alpha \end{aligned}$$

and if $\alpha = 1$,

$$\begin{aligned} &|[f(x+G(z)y) - f(x)] - [f(x+G(\bar{z})y) - f(x)]| \\ &\le \int_0^1 |(\nabla f(x+(1-s)G(\bar{z})y + sG(z)y), G(z)y - G(\bar{z})y)\,\mathrm{d}s| \\ &\le |\nabla f|_0 |G(z) - G(\bar{z})|\,|y|. \end{aligned}$$

It then follows that

$$|B^1_\cdot f(x)|_\beta \leq C|f|_{(\alpha+\beta')}|G|^\alpha_\beta, \quad x \in R^d. \tag{21}$$

By (18)–(21), for each $\varepsilon > 0$ there exists a constant C_ε such that

$$|Bf|_\beta \leq \varepsilon|f|_{\alpha+\beta} + C_\varepsilon|f|_0. \tag{22}$$

Case II: $\beta \in (1,2),\ \beta \leq \mu < \alpha+\beta$. Note that

$$\partial_j(Bf(x)) = \left(\frac{\partial}{\partial_{z_j}} B_z f(x)\right)\Bigg|_{z=x} + B_z f_{x_j}|_{z=x} = \left(\frac{\partial}{\partial_{z_j}} B_z f(x)\right)\Bigg|_{z=x} + Bf_{x_j}.$$

For the second term, apply estimate (22) of *Case* I: $f_{x_j} \in C^{\alpha+\beta-1}$, the tail moment is 1 and $\beta - 1 \leq \mu - 1 < \alpha + \beta - 1$. Also note that $|G|_{\frac{\beta-1}{(\mu-1)\wedge 1}} \leq C|G|_\beta < \infty$. Hence,

$$|Bf_{x_j}|_{\beta-1} \leq \varepsilon|f_{x_j}|_{\alpha+\beta-1} + C_\varepsilon|f_{x_j}|_0.$$

Only the first term needs to be estimated:

$$\begin{aligned} B^j_z f(x) &= \frac{\partial}{\partial z_j} B_z f(x) \\ &= \int \left[\nabla f(x+G(z)y)G_{z_j}(z)y - \mathbf{1}_{\{\alpha\in(1,2]\}}\nabla f(x)G_{z_j}(z)y\chi_{\{|y|\leq 1\}}\right] \mathrm{d}\pi. \end{aligned} \tag{23}$$

Suppose that $B^j f(x) = B^j_z f(x)|_{z=x},\ x \in R^d$. Consider different scenarios for values of α to show that for each $\varepsilon > 0$ there exists a constant C_ε such that

$$|B^j f|_{\beta-1} \leq \varepsilon|f|_{\alpha+\beta} + C_\varepsilon|f|_0, \quad f \in C^{\alpha+\beta}(\mathbf{R}^d). \tag{24}$$

For $\alpha \in (0,1]$, since

$$B^j_z f(x) = \int \nabla f(x+G(z)y)G_{z_j}(z)y\mathrm{d}\pi,$$

then for $\mu < \alpha + \beta' < \alpha + \beta$,

$$\begin{aligned} &|B^j_z f(\cdot)|_{\beta-1} \leq C|\nabla f|_{\beta-1}|G_{z_j}|_0, \quad z \in \mathbf{R}^d, \\ &[B^j_\cdot f(x)]_{\beta-1} \leq C[|\nabla f|_0|G_{z_j}|_{\beta-1} + |f|_{\alpha+\beta'}|G_{z_j}|_0|\nabla G|_0], \quad x \in \mathbf{R}^d. \end{aligned}$$

For $\alpha \in (1,2]$, carry out the splitting

$$B^j_z f(x) = \int_{|y|\leq 1} \cdots + \int_{|y|>1} \cdots = B^{j,1}_z f(x) + B^{j,2}_z f(x).$$

Since by Lemma 11

$$B^{j,1}_z f(x) = \int_{|y|\leq 1} \int \partial^{\alpha-1}\nabla f(x-t)k^{(\alpha-1)}(G(z)y,t)G_{z_j}(z)y\mathrm{d}t\mathrm{d}\pi,$$

then $|B^{j,1}_z f(\cdot)|_{\beta-1} \leq C|f|_{\alpha+\beta'}|G|^\alpha_\beta,\ z \in \mathbf{R}^d$.

For $|y| \leq 1,\ z, \bar{z} \in \mathbf{R}^d$,

$$\begin{aligned} &[\nabla f(x+G(z)y) - \nabla f(x)]G_{z_j}(z)y - [\nabla f(x+G(\bar{z})y) - \nabla f(x)]G_{z_j}(\bar{z})y \\ &\quad \leq |\nabla f(x+G(z)y) - \nabla f(x+G(\bar{z})y)|\,|G_{z_j}(z)y| \end{aligned}$$

$$+|\nabla f(x+G(\bar{z})y)-\nabla f(x)|\,|G_{z_j}(z)-G_{z_j}(\bar{z})|\,|y|$$
$$\leq |\partial^2 f|_0|G|_0|y|^2|G_{z_j}(z)-G_{z_j}(\bar{z})|$$

and

$$[B_\cdot^{j,1}f]_{\beta-1}\leq C|D^2 f|_0|G|_\beta|G_0|.$$

Since

$$B_z^{j,2}f(x)=\int_{|y|>1}\nabla f(x+G(z)y)G_{z_j}(z)y\mathrm{d}\pi,$$

then

$$|B_z^{j,2}f(\cdot)|_{\beta-1}\leq C|\nabla f|_{\beta-1}|G_{z_j}|_0,$$
$$[B_\cdot^{j,2}f(x)]_{\beta-1}\leq C|\nabla f|_0|G_{z_j}|_{\beta-1}+C\int_{|y|>1}|y|^\mu\mathrm{d}\pi\,|D^2 f|_0|\nabla G_{z_j}|_0^2.$$

It is hence proved that (24) holds.

Case III: $\beta\in(2,3)$, $\beta\leq\mu<\alpha+\beta$. Since

$$\partial_j(Bf(x))=\left(\frac{\partial}{\partial_{z_j}}B_z f(x)\right)\bigg|_{z=x}+B_z f_{x_j}|_{z=x},$$

then

$$\begin{aligned}\frac{\partial^2}{\partial x_i\partial x_j}(Bf(x)) &= B_z f_{x_ix_j}(x)|_{z=x}+\frac{\partial}{\partial z_i}\left(B_z f_{x_j}\right)\bigg|_{z=x}+\frac{\partial}{\partial_{z_j}}B_z f_{x_i}(x)\bigg|_{z=x}\\ &\quad+\frac{\partial^2}{\partial z_i\partial z_j}B_z f(x)\bigg|_{z=x}\\ &= B\partial_{ij}^2 f+B^i\partial_j f+B^j\partial_i f+B^{ij}f. \end{aligned}\tag{25}$$

The estimate (22) of *Case* I can be used for the first term ($\beta-2\leq\mu-2<\alpha+\beta-2$ with $\beta-2\in(0,1)$). For each ε', there exists a constant $C_{\varepsilon'}$ such that

$$|Bf_{x_ix_j}|_{\beta-2}\leq\varepsilon'|D^2 f|_{\alpha+\beta-2}+C_{\varepsilon'}|D^2 f|_0.$$

For the second and third terms in (25), estimate (24) of *Case* II is applied. Indeed, $f_{x_j}\in C^{\alpha+\beta-1}(\mathbf{R}^d)$, $\beta-1\in(1,2)$, $\beta-1\leq\mu-1<\alpha+\beta-1$. Hence, for each ε', there exists a constant $C_{\varepsilon'}$ such that

$$|B^i f_{x_j}|_{\beta-2}+|B^j f_{x_i}|_{\beta-2}\leq\varepsilon'|\nabla f|_{\alpha+\beta-1}+C_{\varepsilon'}|\nabla f|_0.$$

Therefore, only the last term is new. By (23),

$$\begin{aligned}\frac{\partial^2}{\partial z_i\partial z_j}B_z f(x) &= \int(\partial^2 f(x+G(z)y)G_{z_j}(z)y,G_{z_i}(z)y)\mathrm{d}\pi\\ &\quad+\mathbf{1}_{\{\alpha\in(0,1]\}}\int\nabla f(x+G(z)y)G_{z_iz_j}(z)y\mathrm{d}\pi\\ &\quad+\mathbf{1}_{\{\alpha\in(1,2]\}}\int\left(\nabla f(x+G(z)y)-\nabla f(x),G_{z_iz_j}(z)y\right)\mathrm{d}\pi\\ &= B_z^{ij,1}f(x)+B_z^{ij,2}f(x)+B_z^{ij,3}f(x),\end{aligned}$$

and for $\alpha \in (1, 2]$,

$$B_z^{ij,3} f(x) = \int \int_0^1 (D^2 f(x + sG(z)y)G(z)y, G_{z_i z_j}(z)y)\mathrm{d}s\mathrm{d}\pi.$$

It then follows that for $z \in \mathbf{R}^d$,

$$|B_z^{ij,1} f(\cdot)|_{\beta-2} \le |\partial^2 f|_{\beta-2}|\nabla G|_0^2, |B_z^{ij,2} f(\cdot)|_{\beta-2} \le |\nabla f|_{\beta-2}|\nabla G|_0|\partial^2 G|_0,$$
$$|B_z^{ij,3} f(\cdot)|_{\beta-2} \le |D^2 f|_{\beta-2}|D^2 G|_0|G|_0.$$

Suppose that $\beta \le \mu < \alpha + \beta' < \alpha + \beta$. Then for $x \in \mathbf{R}^d$,

$$|B_\cdot^{ij,1} f(x)|_{\beta-2} \le |D^2 f|_0|\nabla G|_{\beta-2}^2 + |D^2 f|_{\alpha+\beta'}|\nabla G|_0^3 \int |y|^\mu \mathrm{d}\pi,$$
$$|B_\cdot^{ij,2} f(x)|_{\beta-2} \le C(|D^2 f|_0|G|_\beta^2 + |\nabla f|_0|G|_\beta),$$
$$|B_\cdot^{ij,3} f(x)|_{\beta-2} \le |D^3 f|_0|G|_\beta^3 + |D^2 f|_0|G|_\beta^2.$$

The statement follows. □

3.2.2. *Proof of Theorem 8*

The proof follows that of Theorem 5 in [11], with some simple changes.

It is well known that for an arbitrary but fixed $\delta > 0$, there exist a family of cubes $D_k \subseteq \tilde{D}_k \subseteq \mathbf{R}^d$ and a family of deterministic functions $\eta_k \in C_0^\infty(\mathbf{R}^d)$ with the following properties:

1. For all $k \ge 1$, D_k and $\tilde{D}_k$ have a common center x_k, diam $D_k \le \delta$, dist $(D_k, \mathbf{R}^d \setminus \tilde{D}_k) \le C\delta$ for a constant $C = C(d) > 0$, $\cup_k D_k = \mathbf{R}^d$, and $1 \le \sum_k \mathbf{1}_{\tilde{D}_k} \le 2^d$.
2. For all k, $0 \le \eta_k \le 1$, $\eta_k = 1$ in D_k, $\eta_k = 0$ outside of $\tilde{D}_k$ and for all multi-indices γ with $|\gamma| \le 3$,
$$|\partial^\gamma \eta_k| \le C(d)\delta^{-|\gamma|}.$$

For $\alpha \in (0, 2)$, $\lambda \ge 0$, $k \ge 1$, define

$$Au(t,x) = A_x u(t,x), \quad Bu(t,x) = B_x u(t,x), A_k u(t,x) = A_{x_k} u(t,x),$$
$$E_k u(t,x) = \int [u(t,x+y) - u(t,x)][\eta_k(x+y) - \eta_k(x)]m(x_k, y)\frac{\mathrm{d}y}{|y|^{d+\alpha}},$$
$$E_{k,1} u(t,x) = \int [u(t,x+y) - u(t,x)][\eta_k(x+y) - \eta_k(x)]\frac{\mathrm{d}y}{|y|^{d+\alpha}},$$
$$F_k u(t,x) = u(t,x)A_k \eta_k(x), \qquad F_{k,1} u(t,x) = u(t,x)\partial^\alpha \eta_k(x).$$

By Lemma 24 in [11], for each $\varepsilon > 0$, there exists a constant C_ε such that

$$\sup_k \big(|E_k^{(\alpha)} u(t,\cdot)|_\beta + |E_{k,1}^{(\alpha)} u(t,\cdot)|_\beta\big) \le \varepsilon|\partial^\alpha u|_\beta + C_\varepsilon|u|_0$$

and there exists a constant $C = C(\alpha, \beta, d, \delta, M^{(\alpha)})$ such that

$$\sup_k \big(|F_k^{(\alpha)} u(t,\cdot)|_\beta + |F_{k,1}^{(\alpha)} u(t,\cdot)|_\beta\big) \le C|u|_\beta.$$

As can be easily seen (see (50)–(51) in [11]), for any $u \in C^{\alpha+\beta}(\mathbf{R}^d)$,

$$|u|_0 \leq \sup_k \sup_x |\eta_k(x)u(x)|,$$
$$|u|_\beta \leq \sup_k |\eta_k u|_\beta + C|u|_0,$$
$$\sup_k |\eta_k u|_\beta \leq |u|_\beta + C|u|_0,$$

and

$$|u|_{\alpha+\beta} \leq C \sup_k |\eta_k u|_{\alpha+\beta}. \tag{26}$$

Let $u \in C^{\alpha+\beta}(H)$ be a solution to (4). Then $\eta_k u$ satisfies the equation

$$\partial_t(\eta_k u) = A_k(\eta_k u) - \lambda(\eta_k u) + \eta_k[Au - A_k u] + \eta_k Bu + \eta_k f + F_k u + E_k u, \tag{27}$$

and by Proposition 10,

$$|\eta_k u|_{\alpha+\beta} \leq C[|\eta_k[Au - A_k u]|_\beta + |\eta_k Bu|_\beta + |\eta_k f|_\beta + |F_k u|_\beta + |E_k u|_\beta].$$

Hence,

$$|u|_{\alpha+\beta} \leq C[\sup_k |\eta_k f|_\beta + I], \tag{28}$$

where

$$\begin{aligned} I \leq\ & C_1 \sup_k [|\eta_k[Au - A_k u]|_\beta + |\eta_k Bu|_\beta + |F_k u|_\beta + |E_k u|_\beta + |F_{k,1} u|_\beta \\ & + |E_{k,1} u|_\beta] + C_2 |u|_0. \end{aligned}$$

By Lemma 12,

$$|\eta_k[Au - A_k u]|_\beta \leq C[M_\beta(1 + |\nabla\eta|_0 \delta^\beta)|u|_{\alpha+\frac{\beta}{2}} + \delta^\beta |u|_{\alpha+\beta}].$$

Therefore, for each $\varepsilon > 0$, there exists a constant C_ε such that

$$|\eta_k[Au - A_k u]|_\beta \leq \varepsilon|u|_{\alpha+\beta} + C_\varepsilon|u|_0.$$

By the estimates of Proposition 13 and Lemma 12, it follows that for each $\varepsilon > 0$, there exists a constant C_ε such that

$$I \leq \varepsilon|u|_{\alpha+\beta} + C_\varepsilon|u|_0.$$

By (28),

$$|u|_{\alpha,\beta;p} \leq C[|f|_{0,\beta;p} + |u|_{0;p}]. \tag{29}$$

On the other hand, (27) holds and by Proposition 10,

$$\begin{aligned} |u|_0 &\leq \sup_k |\eta_k u|_\beta \\ &\leq \mu(\lambda) \sup_k [|f|_\beta + |\eta_k[Au - A_k]|_\beta + |\eta_k Bu|_\beta + |F_k u|_\beta + |E_k u|_\beta], \end{aligned}$$

where $\mu(\lambda) \to 0$ as $\lambda \to \infty$. Thus,

$$|u|_0 \leq C\mu(\lambda)[|f|_\beta + |u|_{\alpha+\beta}]. \tag{30}$$

The inequalities (29) and (30) imply that there exist $\lambda_0 > 0$ and a constant C independent of u such that if $\lambda \geq \lambda_0$,

$$|u|_{\alpha+\beta} \leq C|f|_\beta. \tag{31}$$

In a standard way (see [11]), it can be verified that (31) holds for all $\lambda \geq 0$. Again by Proposition 10 and (26), there exists a constant C such that for all $s \leq t \leq T$,

$$\begin{aligned}|u(t,\cdot) - u(s,\cdot)|_{\frac{\alpha}{2}+\beta} &\leq \sup_k |\eta_k u(t,\cdot) - \eta_k u(s,\cdot)|_{\frac{\alpha}{2}+\beta} \\ &\leq C(t-s)^{\frac{1}{2}} \left(|f|_\beta + |u|_{\alpha+\beta}\right).\end{aligned}$$

Therefore there exists a constant C such that for all $s \leq t \leq T$,

$$|u(t,\cdot) - u(s,\cdot)|_{\frac{\alpha}{2}+\beta} \leq C(t-s)^{\frac{1}{2}}|f|_\beta.$$

To finish the proof, apply the continuation by parameter argument. Suppose that $\tau \in [0,1]$, $L_\tau u = \tau L u + (1-\tau)\,\partial^\alpha u$ and introduce the space $\hat{C}^{\alpha+\beta}(H)$ of functions $u \in C^{\alpha+\beta}(H)$ such that for each (t,x), $u(t,x) = \int_t^T F(s,x)\,\mathrm{d}s$, where $F \in C^\beta(H)$. It is a Banach space with respect to the norm

$$|u|_{\alpha,\beta} = |u|_{\alpha+\beta} + |F|_\beta\,.$$

Consider the mappings $T_\tau : \hat{C}^{\alpha+\beta}(H) \to C^\beta(H)$ defined by $u(t,x) = -\int_t^T F(s,x)\,\mathrm{d}s \longmapsto F + L_\tau u$. By Lemma 12 and Proposition 13, for some constant C independent of τ, $|T_\tau u|_\beta \leq C\,|u|_{\alpha,\beta}$. On the other hand, there exists a constant C independent of τ such that for all $u \in \hat{C}^{\alpha+\beta}(H)$,

$$|u|_{\alpha,\beta} \leq C\,|T_\tau u|_\beta\,. \tag{32}$$

Indeed,

$$u(t,x) = -\int_t^T F(s,x)\,\mathrm{d}s = \int_t^T (L_\tau u - (F + L_\tau u))\,\mathrm{d}s.$$

According to the estimate (31), there exists a constant C independent of τ such that

$$|u|_{\alpha+\beta} \leq C\,|T_\tau u|_\beta = C\,|F + L_\tau u|_\beta\,. \tag{33}$$

Hence, by Lemma 12, Proposition 13 and (33),

$$\begin{aligned}|u|_{\alpha,\beta} &= |u|_{\alpha+\beta} + |F|_\beta \leq |u|_{\alpha+\beta} + |F + L_\tau u|_\beta + |L_\tau u|_\beta \\ &\leq C(|u|_{\alpha+\beta} + |F + L_\tau u|_\beta) \leq C\,|F + L_\tau u|_\beta = C\,|T_\tau u|_\beta\,,\end{aligned}$$

and (32) follows. Since T_0 is an onto map, by Theorem 5.2 in [3], all the T_τ are onto maps and the statement follows.

3.2.3. *Proof of Corollary 9*

By Lemma 12 and Proposition 13, for $g \in C^{\alpha+\beta}(\mathbf{R}^d)$, $|Ag|_\beta \leq C|g|_{\alpha+\beta}$ and $|Bg|_\beta \leq C|g|_{\alpha+\beta}$ with a constant C independent of f and g. It then follows from (4) that there exists a unique solution $\tilde{v} \in C^{\alpha+\beta}(H)$ to the Cauchy problem

$$\begin{aligned}&\left(\partial_t + A_x + B_x\right)\tilde{v}(t,x) = f(t,x) - A_x g(x) - B_x g(x), \\ &\tilde{v}(T,x) = 0\end{aligned} \tag{34}$$

and $|\tilde{v}|_{\alpha+\beta} \le C\big(|g|_{\alpha+\beta} + |f|_\beta\big)$ with C independent of f and g. Define $v(t,x) = \tilde{v}(t,x) + g(x)$, where $\tilde{v}$ is the solution to problem (34). Then v is the unique solution to the Cauchy problem (6) and $|v|_{\alpha+\beta} \le C(|g|_{\alpha+\beta} + |f|_\beta)$.

Remark 14. If the assumptions of Corollary 9 hold and $v \in C^{\alpha+\beta}(H)$ is the solution to (6), then $\partial_t v = f - A_x v - B_x v$, and by Lemma 12 and Proposition 13, $|\partial_t v|_\beta \le C(|g|_{\alpha+\beta} + |f|_\beta)$.

4. The one-step estimate and proof of the main result

The following lemma provides a one-step estimate of the conditional expectation of an increment of the Euler approximation.

Lemma 15. *Suppose that $\beta \in (0,3)$, $\beta \notin \mathbf{N}$, $0 < \beta \le \mu < \alpha + \beta$, and*

$$\int_{|y|\le 1} |y|^\alpha \pi(\mathrm{d}y) + \int_{|y|>1} |y|^\mu \pi(\mathrm{d}y) < \infty.$$

Assume that $a^i, b^{ij} \in C^\beta(\mathbf{R}^d)$, and $\inf_x |\det B(x)| > 0$. Let G^{ij} be Lipschitz if $\beta = \mu < 1$ and suppose that $G^{ij} \in C^{\frac{\beta}{\mu\wedge 1}}(\mathbf{R}^d)$ otherwise. Then there exists a constant C such that for all $f \in C^\beta(\mathbf{R}^d)$,

$$|\mathbf{E}[f(Y_s) - f(Y_{\tau_{i_s}})|\widetilde{\mathcal{F}}\tau_{i_s}]| \le C|f|_\beta \delta^{\kappa(\alpha,\beta)}, \quad \forall s \in [0,T],$$

where $i_s = i$ if $\tau_i \le s < \tau_{i+1}$ and $\kappa(\alpha,\beta)$ is as defined in Theorem 3.

The proof of Lemma 15 is based on applying Itô's formula to $f(Y_s) - f(Y_{\tau_{i_s}})$, $f \in C^\beta(\mathbf{R}^d)$. If $\beta > \alpha$, by Remark 6 and Itô's formula, the inequality holds. If $\beta < \alpha$, f is first smoothed by using $w \in C_0^\infty(\mathbf{R}^d)$, a nonnegative smooth function with support on $\{|x| \le 1\}$ such that $w(x) = w(|x|)$, $x \in \mathbf{R}^d$, and $\int w(x)\mathrm{d}x = 1$ (see (8.1) in [2]). Note that, due to the symmetry,

$$\int_{\mathbf{R}^d} x^i w(x)\mathrm{d}x = 0, \quad i = 1,\ldots,d. \tag{35}$$

For $x \in \mathbf{R}^d$ and $\varepsilon \in (0,1)$, define $w^\varepsilon(x) = \varepsilon^{-d} w\left(\frac{x}{\varepsilon}\right)$ and the convolution

$$f^\varepsilon(x) = \int f(y) w^\varepsilon(x-y)\mathrm{d}y = \int f(x-y) w^\varepsilon(y)\mathrm{d}y, \quad x \in \mathbf{R}^d. \tag{36}$$

4.1. Some auxiliary estimates

For the estimates of $A_z f^\varepsilon$, the following simple integral estimates are needed. Recall that $m(z,y)$ in the definition of operator A_z (see (7)) is bounded, smooth, and 0-homogeneous and symmetric in y.

Lemma 16. *Suppose that $v \in C_0^\infty(\mathbf{R}^d)$.*

(i) *For $\alpha \in (0,2)$,*

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}_0^d} |v(y+y') - v(y) - \chi^{(\alpha)}(y')(\nabla v(y), y')| \frac{\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha}} < \infty,$$

where $\chi^{(\alpha)}(y) = \mathbf{1}_{\{|y|\le 1\}}\mathbf{1}_{\{\alpha=1\}} + \mathbf{1}_{\{\alpha\in(1,2)\}}$;

(ii) *For* $\beta \in (0,1)$, $\beta < \alpha$, $z \in \mathbf{R}^d$,

$$\sup_z \int_{\mathbf{R}^d} |(A_z w)(y)|\,|y|^\beta \mathrm{d}y < \infty;$$

(iii) *For* $1 < \beta < \alpha < 2$,

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}_0^d} \int_0^1 |w(y+sy') - w(y)|\,|y|^{\beta-1} \frac{\mathrm{d}s\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha-1}} < \infty.$$

Proof. (i) Indeed,

$$\begin{aligned}
&|v(y+y') - v(y) - \chi^{(\alpha)}(y')(\nabla v(y), y')| \\
&\quad \le \mathbf{1}_{\{|y'|\le 1\}} \left\{ \int_0^1 [\max_{i,j} |\partial_{ij}^2 v(y+sy')|\,|y'|^2 + \mathbf{1}_{\{\alpha\in(0,1)\}} |\nabla v(y+sy')|\,|y'|]\mathrm{d}s \right\} \\
&\qquad + \mathbf{1}_{\{|y'|>1\}} \big\{ |v(y+y')| + |v(y)| + \mathbf{1}_{\{\alpha\in(1,2)\}} |\nabla v(y)|\,|y'| \big\}, \quad y, y' \in \mathbf{R}^d.
\end{aligned}$$

The claim follows.

(ii) For $\beta \in (0,1)$, $\beta < \alpha$, $z \in \mathbf{R}^d$,

$$\begin{aligned}
\int_{\mathbf{R}^d} |(A_z w)(y)|\,|y|^\beta \mathrm{d}y &\le \int_{\mathbf{R}^d} \int_{|y'|>1} |w(y+y')|\,|y|^\beta \frac{\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha}} \\
&\quad + \int_{\mathbf{R}^d} \int_{|y'|>1} |w(y)|\,|y|^\beta \frac{\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha}} \\
&\quad + \max_{i,j} \int_{\mathbf{R}^d} \int_{|y'|\le 1} \int_0^1 |\partial_{ij}^2 w(y+sy')|\,|y'|^2 |y|^\beta \frac{\mathrm{d}s\mathrm{d}y'\mathrm{d}y}{|y'|^{d+\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbf{R}^d} \int_{|y'|>1} |w(y+y')|\,|y|^\beta \frac{\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha}} &\le C \Bigg[\int_{\mathbf{R}^d} \int_{|y'|>1} |w(y+y')|\,|y+y'|^\beta \frac{\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha}} \\
&\quad + \int_{\mathbf{R}^d} \int_{|y'|>1} |w(y+y')|\,|y'|^\beta \frac{\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha}} \Bigg].
\end{aligned}$$

Part (ii) follows.

(iii) For $1 < \beta < \alpha < 2$,

$$\begin{aligned}
&\int_{\mathbf{R}^d} \int_{\mathbf{R}_0^d} \int_0^1 |w(y+sy') - w(y)|\,|y|^{\beta-1} \frac{\mathrm{d}y\mathrm{d}y'\mathrm{d}s}{|y'|^{d+\alpha-1}} \\
&\quad \le \int_{\mathbf{R}^d} \int_{|y'|>1} \int_0^1 |w(y+sy')|\,|y|^{\beta-1} \frac{\mathrm{d}y\mathrm{d}y'\mathrm{d}s}{|y'|^{d+\alpha-1}} \\
&\qquad + \int_{\mathbf{R}^d} \int_{|y'|>1} \int_0^1 |w(y)|\,|y|^{\beta-1} \frac{\mathrm{d}y\mathrm{d}y'\mathrm{d}s}{|y'|^{d+\alpha-1}} \\
&\qquad + \int_{\mathbf{R}^d} \int_{|y'|\le 1} \int_0^1 \int_0^1 |\nabla w(y+s\tau y')|\,|y|^{\beta-1} \frac{\mathrm{d}s\mathrm{d}\tau\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha-2}}.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\mathbf{R}^d}\int_{|y'|>1}\int_0^1 |w(y+sy')|\,|y|^{\beta-1}\frac{\mathrm{d}y\mathrm{d}y'\mathrm{d}s}{|y'|^{d+\alpha-1}}\\
&\quad\le C\left[\int_{\mathbf{R}^d}\int_{|y'|>1}\int_0^1 |w(y+sy')|\,|y+sy'|^{\beta-1}\frac{\mathrm{d}y\mathrm{d}y'\mathrm{d}s}{|y'|^{d+\alpha-1}}\right.\\
&\qquad\left.+\int_{\mathbf{R}^d}\int_{|y'|>1}\int_0^1 |w(y+sy')|\,|y'|^{\beta-1}\frac{\mathrm{d}y\mathrm{d}y'\mathrm{d}s}{|y'|^{d+\alpha-1}}\right]
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_{\mathbf{R}^d}\int_{|y'|\le 1}\int_0^1\int_0^1 |\nabla w(y+s\tau y')|\,|y|^{\beta-1}\frac{\mathrm{d}s\mathrm{d}\tau\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha-2}}\\
&\quad\le C\left[\int_{\mathbf{R}^d}\int_{|y'|\le 1}\int_0^1\int_0^1 |\nabla w(y+s\tau y')|\,|y+s\tau y'|^{\beta-1}\frac{\mathrm{d}s\mathrm{d}\tau\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha-2}}\right.\\
&\qquad\left.+\int_{\mathbf{R}^d}\int_{|y'|\le 1}\int_0^1\int_0^1 |\nabla w(y+s\tau y')|\,|y'|^{\beta-1}\frac{\mathrm{d}s\mathrm{d}\tau\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha-2}}\right]
\end{aligned}$$

are finite, part (iii) follows. □

Now we prove some estimates for Af^{ε} and Bf^{ε}.

Lemma 17. *Suppose that* $\alpha\in(0,2)$, $\beta<\alpha$, $\beta\ne 1$, *and* $\varepsilon\in(0,1)$. *Then:*

(i) *there exists a constant* C *such that for all* $f\in C^{\beta}(\mathbf{R}^d)$, $x\in\mathbf{R}^d$,

$$|f^{\varepsilon}(x)-f(x)|\le C\varepsilon^{\beta}|f|_{\beta};$$

(ii) *there exists a constant* C *such that for all* $z,x\in\mathbf{R}^d$,

$$|A_z f^{\varepsilon}(x)|\le C\varepsilon^{-\alpha+\beta}|f|_{\beta} \tag{37}$$

and in particular, for all $f\in C^{\beta}(\mathbf{R}^d)$, $z,x\in\mathbf{R}^d$,

$$|\partial^{\alpha}f^{\varepsilon}(x)|\le C\varepsilon^{-\alpha+\beta}|f|_{\beta}; \tag{38}$$

(iii) *for* $k,l=1,\ldots,d$, $x\in\mathbf{R}^d$,

$$\begin{aligned}
&|\partial_k f^{\varepsilon}(x)|\le C\varepsilon^{-1+\beta}|f|_{\beta},\quad \text{if } \beta<1, \qquad (39)\\
&|f^{\varepsilon}|_1\le C|f|_1,\\
&|\partial^2_{kl}f^{\varepsilon}(x)|\le C\varepsilon^{-2+\beta}|f|_{\beta},\quad \text{if } \beta<2,
\end{aligned}$$

and

$$|f^{\varepsilon}|_{\alpha}\le C\varepsilon^{-\alpha+\beta}|f|_{\beta},\quad \text{if } \beta\in(0,1],\ \alpha\in(1,2), \tag{40}$$

$$|\partial^{\alpha-1}\nabla f^{\varepsilon}(x)|\le C\varepsilon^{-\alpha+\beta}|f|_{\beta},\quad \text{if } \beta\in(1,\alpha),\ \alpha\in(1,2). \tag{41}$$

Proof. (i) For $\beta\in(1,2)$, by (35),

$$f^{\varepsilon}(x)-f(x)=\int[f(x-y)-f(x)]w^{\varepsilon}(y)\mathrm{d}y$$

$$= \int [f(x+y) - f(x) - (\nabla f(x), y)] w^{\varepsilon}(y) \mathrm{d}y$$
$$= \int\int_0^1 (\nabla f(x+sy) - \nabla f(x), y) \mathrm{d}s w^{\varepsilon}(y) \mathrm{d}y$$

and

$$|f^{\varepsilon}(x) - f(x)| \leq C|\nabla f|_{\beta-1} \int |y|^{1+(\beta-1)} w^{\varepsilon}(y) \mathrm{d}y \leq C|f|_{\beta} \varepsilon^{\beta}.$$

For $\beta \in (0, 1]$,

$$f^{\varepsilon}(x) - f(x) = \int [f(x-y) - f(x)] w^{\varepsilon}(y) \mathrm{d}y$$
$$= \int [f(x+y) - f(x)] w^{\varepsilon}(y) \mathrm{d}y$$

and

$$f^{\varepsilon}(x) - f(x) = \frac{1}{2} \int [f(x+y) + f(x-y) - 2f(x)] w^{\varepsilon}(y) \mathrm{d}y.$$

Hence, for $\beta \in (0, 1]$,

$$|f^{\varepsilon}(x) - f(x)| \leq C|f|_{\beta} \varepsilon^{\beta}.$$

(ii) For $z, x \in \mathbf{R}^d$, by changing the variable of integration with $\bar{y} = \frac{y}{\varepsilon}$ and using (8) for $\alpha = 1$,

$$\begin{aligned} A_z w^{\varepsilon}(x) &= \mathbf{1}_{\{\alpha=1\}} (a_1(z), \nabla w^{\varepsilon}(x)) \\ &\quad + \int [w^{\varepsilon}(x+y) - w^{\varepsilon}(x) - \bar{\chi}_{\alpha}(y)(\nabla w^{\varepsilon}(x), y)] m(z, y) \frac{\mathrm{d}y}{|y|^{d+\alpha}} \\ &= \varepsilon^{-\alpha} \varepsilon^{-d} (A_z w) \left(\frac{x}{\varepsilon} \right), \end{aligned} \tag{42}$$

where $\bar{\chi}_{\alpha}(y) = \mathbf{1}_{\{|y| \leq 1\}} \mathbf{1}_{\{\alpha=1\}} + \mathbf{1}_{\{\alpha \in (1,2)\}}, y \in \mathbf{R}^d$. It follows from Lemma 16(i), the Fubini theorem, and (42), changing the variable of integration with $\bar{y} = \frac{y}{\varepsilon}$ as well, that

$$\begin{aligned} A_z f^{\varepsilon}(x) &= \int_{\mathbf{R}^d} \varepsilon^{-\alpha} \varepsilon^{-d} (A_z w) \left(\frac{x-y}{\varepsilon} \right) f(y) \mathrm{d}y \\ &= \int \varepsilon^{-\alpha} \varepsilon^{-d} (A_z w) \left(\frac{y}{\varepsilon} \right) f(x-y) \mathrm{d}y \\ &= \int \varepsilon^{-\alpha} (A_z w)(y) f(x - \varepsilon y) \mathrm{d}y, \quad x, z \in \mathbf{R}^d. \end{aligned}$$

By Lemma 16(i) and the Fubini theorem,

$$\int_{\mathbf{R}^d} A_z w(y) \mathrm{d}y = 0.$$

Hence, if $\beta \in (0, 1),\ \beta < \alpha$,

$$\begin{aligned} A_z f^{\varepsilon}(x) &= \int \varepsilon^{-\alpha} (A_z w)(y) f(x - \varepsilon y) \mathrm{d}y \\ &= \int \varepsilon^{-\alpha} (A_z w)(y) [f(x - \varepsilon y) - f(x)] \mathrm{d}y \end{aligned}$$

and

$$|A_z f^\varepsilon(x)| \le C\varepsilon^{-\alpha+\beta}|f|_\beta \int_{\mathbf{R}^d} |(A_z w)(y)|\,|y|^\beta \mathrm{d}y.$$

(37) then follows in this case by Lemma 16(ii).

Assume that $1 < \beta < \alpha < 2$. By Theorem 2.27 in [2], differentiation and integration can be switched:

$$\begin{aligned} A_z w(y) &= \int [w(y+y') - w(y) - (\nabla w(y), y')] m(z, y') \frac{\mathrm{d}y'}{|y'|^{d+\alpha}} \\ &= \iint_0^1 \big(\nabla_y w(y+sy') - \nabla_y w(y), y'\big) \mathrm{d}s\, m(z, y') \frac{\mathrm{d}y'}{|y'|^{d+\alpha}} \\ &= \sum_{i=1}^d \frac{\partial}{\partial y_i} \int \int_0^1 [w(y+sy') - w(y)] y_i' \mathrm{d}s\, m(z, y') \frac{\mathrm{d}y'}{|y'|^{d+\alpha}}. \end{aligned}$$

By integrating by parts,

$$\begin{aligned} A_z f^\varepsilon(x) &= \int \varepsilon^{-\alpha} A_z w(y) f(x-\varepsilon y)\mathrm{d}y \\ &= \varepsilon^{-\alpha+1} \iint_0^1 [w(y+sy') - w(y)](\nabla f(x-\varepsilon y), y') m(z, y') \frac{\mathrm{d}s\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha}}, \quad x \in \mathbf{R}^d. \end{aligned} \tag{43}$$

Since

$$\int_{\mathbf{R}_0^d} \int_{\mathbf{R}^d} \int_0^1 |w(y+sy') - w(y)| \frac{\mathrm{d}s\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha}} < \infty,$$

the Fubini theorem applies and $\int [w(y+sy') - w(y)]\mathrm{d}y = 0$. Rewrite (43) as

$$\begin{aligned} A_z f^\varepsilon(x) = \varepsilon^{-\alpha+1} &\int_{\mathbf{R}^d} \int_{\mathbf{R}_0^d} \int_0^1 [w(y+sy') - w(y)] \\ &\times (\nabla f(x-\varepsilon y) - \nabla f(x), y') m(z, y') \frac{\mathrm{d}s\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha}}, \quad x, z \in \mathbf{R}^d. \end{aligned}$$

Hence,

$$\begin{aligned} |A_z f^\varepsilon(x)| &\le C\varepsilon^{-\alpha+1}\varepsilon^{\beta-1}|\nabla f|_{\beta-1} \iint \int_0^1 |w(y+sy') - w(y)|\,|y|^{\beta-1} \frac{\mathrm{d}s\mathrm{d}y\mathrm{d}y'}{|y'|^{d+\alpha-1}} \\ &\le C\varepsilon^{-\alpha+\beta}|f|_\beta, x, z \in \mathbf{R}^d, \end{aligned}$$

and by Lemma 16(iii), (37) is proved. On taking $m = 1$, (38) follows. Finally, the case of $\alpha \in (1, 2)$, $\beta = 1$ is obtained by interpolation.

Fix $\alpha \in (1, 2), z \in \mathbf{R}^d$. Let E be the Banach space of continuous bounded functions on $\mathbf{R}^d$ with supremum norm. Consider the operator $T(f) = A_z f^\varepsilon(x)$. We prove that $T : C^{1\pm\frac{\alpha-1}{2}}(\mathbf{R}^d) \to E$ is bounded:

$$\begin{aligned} |T(f)| &= \sup_x |A_z f^\varepsilon(x)| \le C\varepsilon^{-\alpha+\frac{3-\alpha}{2}}|f|_{\frac{3-\alpha}{2}} = C\varepsilon^{\frac{3}{2}(1-\alpha)}|f|_{\frac{3-\alpha}{2}}, \ f \in C^{\frac{3-\alpha}{2}}(\mathbf{R}^d), \\ |T(f)| &= \sup_x |A_z f^\varepsilon(x)| \le C\varepsilon^{\frac{1}{2}(1-\alpha)}|f|_{\frac{1+\alpha}{2}}, \ f \in C^{\frac{1+\alpha}{2}}(\mathbf{R}^d). \end{aligned}$$

Therefore, by interpolation, $T : C^1(\mathbf{R}^d) \to E$ is bounded and

$$|T(f)| = \sup_x |A_z f^\varepsilon(x)| \le C\varepsilon^{-\alpha+1}|f|_1, \quad f \in C^1(\mathbf{R}^d).$$

(iii) If $\beta < 1$, by changing the variable of integration,

$$\begin{aligned}\partial_k f^\varepsilon(x) &= \varepsilon^{-1}\int_{\mathbf{R}^d} \varepsilon^{-d}\partial_k w\left(\frac{x-y}{\varepsilon}\right) f(y)\mathrm{d}y \\ &= \varepsilon^{-1}\int_{\mathbf{R}^d} \varepsilon^{-d}\partial_k w\left(\frac{y}{\varepsilon}\right) f(x-y)\mathrm{d}y \\ &= \varepsilon^{-1}\int_{\mathbf{R}^d} \partial_k w(y)[f(x-\varepsilon y)-f(x)]\mathrm{d}y.\end{aligned}$$

If $\beta = 1$, then

$$\begin{aligned}&f^\varepsilon(x+h)+f^\varepsilon(x-h)-2f^\varepsilon(x) \\ &\quad = \frac{1}{2}\int w_\varepsilon(y)[f(x-y+h)+f(x-y-h)-2f(x-y)]\mathrm{d}y\end{aligned}$$

and $|f^\varepsilon|_1 \le |f|_1$. Also, since $\partial^2_{kl}w(y) = \partial^2_{kl}w(-y),\ k,l = 1,\ldots,d,\ y \in \mathbf{R}^d$,

$$\begin{aligned}\partial^2_{kl} f^\varepsilon(x) &= \varepsilon^{-2}\int_{\mathbf{R}^d} \varepsilon^{-d}\partial^2_{kl} w\left(\frac{x-y}{\varepsilon}\right) f(y)\mathrm{d}y \\ &= \varepsilon^{-2}\int_{\mathbf{R}^d} \varepsilon^{-d}\partial^2_{kl} w\left(\frac{y}{\varepsilon}\right) f(x-y)\mathrm{d}y \\ &= \varepsilon^{-2}\int_{\mathbf{R}^d} \partial^2_{kl} w(y)[f(x-\varepsilon y)-f(x)]\mathrm{d}y \\ &= \frac{1}{2}\varepsilon^{-2}\int_{\mathbf{R}^d} \partial^2_{kl} w(y)[f(x+\varepsilon y)+f(x-\varepsilon y)-2f(x)]\mathrm{d}y.\end{aligned}$$

Thus, for all $x \in \mathbf{R}^d$,

$$\begin{aligned}&|\partial_k f^\varepsilon(x)| \le C\varepsilon^{-1+\beta}|f|_\beta \quad \text{if } \beta \in (0,1), \\ &|\partial^2_{kl} f^\varepsilon(x)| \le C\varepsilon^{-2+\beta}|f|_\beta \quad \text{if } \beta \in (0,1],\end{aligned}$$

and

$$|\partial^2_{kl} f^\varepsilon(x)| \le C\varepsilon^{-2+2}|f|_2 = C|f|_2 \quad \text{if } \beta = 2.$$

Similarly, if $1 < \beta < 2$,

$$\begin{aligned}\partial_k f^\varepsilon(x) &= \int \varepsilon^{-d} w\left(\frac{y}{\varepsilon}\right)\partial_k f(x-y)\mathrm{d}y \\ &= \int \varepsilon^{-d} w\left(\frac{x-y}{\varepsilon}\right)\partial_k f(y)\mathrm{d}y\end{aligned}$$

and

$$\begin{aligned}\partial^2_{kl} f^\varepsilon(x) &= \varepsilon^{-1}\int \varepsilon^{-d}\partial_l w\left(\frac{y}{\varepsilon}\right)\partial_k f(x-y)\mathrm{d}y \\ &= \varepsilon^{-1}\int \partial_l w(y)[\partial_k f(x-\varepsilon y)-\partial_k f(x)]\mathrm{d}y.\end{aligned}$$

Hence,

$$|\partial_{kl}^2 f^\varepsilon(x)| \le C\varepsilon^{-1}\varepsilon^{\beta-1}|f|_\beta.$$

To prove (40), apply (39) and the interpolation theorem. Suppose that $\beta \in (0,1]$. Consider an operator on C^β defined by $T^\varepsilon(f) = f^\varepsilon$. According to (39), $T^\varepsilon : C^\beta(\mathbf{R}^d) \to C^k(\mathbf{R}^d),\ k = 1, 2,$ is bounded,

$$|T^\varepsilon(f)|_k \le C\varepsilon^{-k+\beta}|f|_\beta, \quad k = 1, 2,\ f \in C^\beta(\mathbf{R}^d).$$

By Theorem 6.4.5 in [1], $T^\varepsilon : C^\beta(\mathbf{R}^d) \to C^\alpha(\mathbf{R}^d)$ is bounded and

$$|T^\varepsilon(f)|_\alpha \le C\varepsilon^{(-1+\beta)(2-\alpha)}\varepsilon^{(-2+\beta)(\alpha-1)}|f|_\beta = C\varepsilon^{-\alpha+\beta}|f|_\beta, \quad f \in C^\beta(\mathbf{R}^d).$$

If $\beta \in (1,\alpha)$, $\partial^{\alpha-1}\nabla f^\varepsilon = \partial^{\alpha-1}(\nabla f)^\varepsilon$ and by (38),

$$|\partial^{\alpha-1}\nabla f^\varepsilon(x)| = |\partial^{\alpha-1}(\nabla f)^\varepsilon(x)| \le C\varepsilon^{1-\alpha+(\beta-1)}|\nabla f|_{\beta-1} \le C\varepsilon^{-\alpha+\beta}|f|_\beta,$$

(41) follows. □

Corollary 18. *Assume that $\varepsilon \in (0,1)$, $a(x)$ is bounded, and*

$$\int \big(|y|^\alpha \wedge 1\big)\pi(\mathrm{d}y) < \infty.$$

Then there exists a constant C such that for all $z, x \in \mathbf{R}^d$, $f \in C^\beta(\mathbf{R}^d)$,

$$|B_z f^\varepsilon(x)| \le C\varepsilon^{-\alpha+\beta}|f|_\beta.$$

Proof. If $\beta < \alpha < 1$, by Lemma 11,

$$f^\varepsilon(x+y) - f^\varepsilon(x) = \int k^{(\alpha)}(y,y')\partial^\alpha f^\varepsilon(x-y')\mathrm{d}y',$$

and by (38),

$$|f^\varepsilon(x+y) - f^\varepsilon(x)| \le C\varepsilon^{-\alpha+\beta}|f|_\beta(|y|^\alpha \wedge 1), \quad x, y \in \mathbf{R}^d$$

and

$$\begin{aligned}|f^\varepsilon(x+G(x)y) - f^\varepsilon(x)| &\le C\varepsilon^{-\alpha+\beta}|f|_\beta(|G(x)y|^\alpha \wedge 1)\\ &\le C\varepsilon^{-\alpha+\beta}|f|_\beta\big[\mathbf{1}_{\{|y|\le 1\}}|G(x)y|^\alpha + \mathbf{1}_{\{|y|>1\}}(|G(x)y|^\alpha \wedge 1)\big].\end{aligned}$$

If $\beta < \alpha = 1$, by Lemma 17(ii) and (39),

$$\begin{aligned}|f^\varepsilon(x+y) - f^\varepsilon(x)| &\le C\sup_x[|f(x)| + |\nabla f^\varepsilon(x)|](|y| \wedge 1)\\ &\le C\varepsilon^{-1+\beta}|f|_\beta(|y| \wedge 1), \quad x, y \in \mathbf{R}^d\end{aligned}$$

and

$$\begin{aligned}|f^\varepsilon(x+G(x)y) - f^\varepsilon(x)| &\le C\varepsilon^{-1+\beta}|f|_\beta(|G(x)y| \wedge 1)\\ &\le C\varepsilon^{-1+\beta}|f|_\beta[\mathbf{1}_{|y|\le 1}|G(x)y| + \mathbf{1}_{|y|>1}(|G(x)y| \wedge 1)].\end{aligned}$$

Assume that $\alpha \in (1,2)$; then for $x, y \in \mathbf{R}^d$,

$$f^\varepsilon(x+y) - f^\varepsilon(x) - (\nabla f^\varepsilon(x), y) = \int_0^1 \big(\nabla f^\varepsilon(x+sy) - \nabla f^\varepsilon(x), y\big)\mathrm{d}s. \tag{44}$$

If $\beta \in (1, \alpha)$, by Lemmas 11 and 17 and (40), for $x, y' \in \mathbf{R}^d$,

$$|\nabla f^\varepsilon(x + y') - \nabla f^\varepsilon(x)| \leq C \sup_x |\partial^{\alpha-1}\nabla f^\varepsilon(x)|\, |y'|^{\alpha-1} \leq C\varepsilon^{-\alpha+\beta}|f|_\beta|y'|^{\alpha-1}. \tag{45}$$

If $\beta > \alpha > 1$, then directly

$$|\nabla f^\varepsilon(x + y') - \nabla f^\varepsilon(x)| \leq C|f|_\beta|y'|^{\alpha-1}.$$

If $\beta \in (0, 1]$, $\alpha \in (1, 2)$, then by (41),

$$|\nabla f^\varepsilon(x + y') - \nabla f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta}|y'|^{\alpha-1}|f|_\beta. \tag{46}$$

By applying (45), (46) to (44), it follows for $x, y \in \mathbf{R}^d$ that

$$|f^\varepsilon(x + y) - f^\varepsilon(x) - (\nabla f^\varepsilon(x), y)| \leq C\varepsilon^{-\alpha+\beta}|y|^\alpha|f|_\beta.$$

Hence, for $|y| \leq 1$,

$$|f^\varepsilon(x + G(x)y) - f^\varepsilon(x) - (\nabla f^\varepsilon(x), G(x)y)| \leq C\varepsilon^{-\alpha+\beta}|G(x)y|^\alpha|f|_\beta.$$

Also, for $\alpha > 1$, $\beta \in (1, \alpha)$, $|y| > 1$,

$$|f^\varepsilon(x + G(x)y) - f^\varepsilon(x)| \leq C|f|_\beta\big(|G(x)y| \wedge 1\big).$$

Therefore, the statement follows by the assumptions and Lemma 17. □

4.2. *Proof of Lemma 15*

If $\beta < \alpha$, define f^ε by (36) for $\varepsilon \in (0, 1)$ and apply Itô's formula (see Remark 6): for $s \in [0, T]$,

$$\mathbf{E}[f^\varepsilon(Y_s) - f^\varepsilon(Y_{\tau_{i_s}})|\mathcal{F}_{\tau_{i_s}}] = \mathbf{E}\left[\int_{\tau_{i_s}}^s \big(A_{Y_{\tau_{i_s}}} f^\varepsilon(Y_r) + B_{Y_{\tau_{i_s}}} f^\varepsilon(Y_r)\big)\mathrm{d}r|\mathcal{F}_{\tau_{i_s}}\right].$$

Hence, by Lemma 17 and Corollary 18, for $\varepsilon \in (0, 1)$,

$$\begin{aligned}|\mathbf{E}[f(Y_s) - f(Y_{\tau_{i_s}})|\mathcal{F}_{\tau_{i_s}}]| &\leq |\mathbf{E}[(f - f^\varepsilon)(Y_s) - (f - f^\varepsilon)(Y_{\tau_{i_s}})|\mathcal{F}_{\tau_{i_s}}]| \\ &\quad + |\mathbf{E}[f^\varepsilon(Y_s) - f^\varepsilon(Y_{\tau_{i_s}})|\mathcal{F}_{\tau_{i_s}}]| \\ &\leq C(\varepsilon^\beta + \delta\varepsilon^{-\alpha+\beta})|f|_\beta,\end{aligned}$$

with a constant C independent of ε, f. Minimizing $\varepsilon^\beta + \delta\varepsilon^{-\alpha+\beta}$ in $\varepsilon \in (0, 1)$, we obtain

$$|\mathbf{E}[f(Y_s) - f(Y_{\tau_{i_s}})|\mathcal{F}_{\tau_{i_s}}]| \leq C\delta^{\frac{\beta}{\alpha}}|f|_\beta.$$

If $\beta > \alpha$, apply Itô's formula directly (see Remark 6):

$$\mathbf{E}[f(Y_s) - f(Y_{\tau_{i_s}})|\mathcal{F}_{\tau_{i_s}}] = \mathbf{E}\left[\int_{\tau_{i_s}}^s \big(A^{(\alpha)}_{Y_{\tau_{i_s}}} f(Y_r) + B^{(\alpha)}_{Y_{\tau_{i_s}}} f(Y_r)\big)\mathrm{d}r|\mathcal{F}_{\tau_{i_s}}\right].$$

Hence, by Lemmas 12 and 17,

$$|\mathbf{E}[f(Y_s) - f(Y_{\tau_{i_s}})|\mathcal{F}_{\tau_{i_s}}]| \leq C\delta|f|_\beta.$$

The statement of Lemma 15 follows.

4.3. Proof of Theorem 3

Let $v \in C^{\alpha+\beta}(H)$ be the unique solution to (6) (see Corollary 9). By Itô's formula (see Remark 6) and (6),

$$\mathbf{E}[v(0, X_0)] = \mathbf{E}[v(T, X_T)] - \mathbf{E}\left[\int_0^T \left(\partial_t v(s, X_s) + A_{X_s} v(s, X_s) + B_{X_s} v(s, X_s)\right) \mathrm{d}s\right]$$
$$= \mathbf{E}\left[g(X_T) - \int_0^T f(X_s) \mathrm{d}s\right]$$

and

$$\mathbf{E}[v(0, X_0)] = \mathbf{E}[v(0, Y_0)]. \tag{47}$$

By Proposition 13, Corollary 9, Remark 14, and Lemma 12,

$$\begin{aligned} &|A_z v(s, \cdot)|_\beta + |B_z v(s, \cdot)|_\beta \le C|v|_{\alpha+\beta} \le C|g|_{\alpha+\beta}, \\ &|\partial_t v(s, \cdot)|_\beta \le C|g|_{\alpha+\beta}, \quad s \in [0, T]. \end{aligned} \tag{48}$$

Then, by Itô's formula (Remark 6) and Corollary 9, with (47) and (48), it follows that

$$\begin{aligned} &\mathbf{E}[g(Y_T)] - \mathbf{E}[g(X_T)] - \mathbf{E}\left[\int_0^T f(Y_{\tau_{i_s}}) \mathrm{d}s\right] + \mathbf{E}\left[\int_0^T f(X_s) \mathrm{d}s\right] \\ &= \mathbf{E}[v(T, Y_T)] - \mathbf{E}[v(0, Y_0)] - \mathbf{E}\left[\int_0^T f(Y_{\tau_{i_s}}) \mathrm{d}s\right] + \mathbf{E}\left[\int_0^T f(X_s) \mathrm{d}s\right] \\ &= \mathbf{E}\Bigg[\int_0^T \Big\{\left[\partial_t v(s, Y_s) - \partial_t v(s, Y_{\tau_{i_s}})\right] + \left[A_{Y_{\tau_{i_s}}} v(s, Y_s) - A_{Y_{\tau_{i_s}}} v(s, Y_{\tau_{i_s}})\right] \\ &\quad + \left[B_{Y_{\tau_{i_s}}} v(s, Y_s) - B_{Y_{\tau_{i_s}}} v(s, Y_{\tau_{i_s}})\right]\Big\} \mathrm{d}s\Bigg]. \end{aligned}$$

Hence, by (48) and Lemma 15, there exists a constant C independent of g such that

$$|\mathbf{E}[g(Y_T)] - \mathbf{E}[g(X_T)]| \le C\delta^{\kappa(\alpha,\beta)}|g|_{\alpha+\beta}.$$

The statement of Theorem 3 follows.

5. Conclusion

The paper studies weak Euler approximation of SDEs driven by Lévy processes. The dependence of the rate of convergence on the regularity of coefficients and driving processes is investigated under the assumption of β-Hölder continuity of the coefficients. It is assumed that the main term of the SDE is driven by a spherically symmetric α-stable process and that the tail of the Lévy measure of the lower order term has a μ-order finite moment ($\mu \in (0, 3)$). The resulting rate depends on β, α and μ. In order to estimate the rate of convergence, the existence of a unique solution to the corresponding backward Kolmogorov equation in Hölder space is first proved. The assumptions on the regularity of coefficients and test functions are different than those in the existing literature.

One possible improvement could be to consider the asymptotics of the tails at infinity instead of the tail moment μ. Besides this, the stochastic differential equations considered so far are associated with nondegenerate Lévy operators. A further step could be to study the case with degenerate operators. That is, consider Eq. (1) without assuming $\det b \neq 0$. For example, suppose that $\alpha \in [1, 2]$ and $\beta \in (\alpha, 2\alpha]$. Assume that the coefficients are in C^{β} and

$$\int_{|y|\le 1} |y|^{\alpha} \mathrm{d}\pi + \int_{|y|>1} |y|^{2\alpha} \mathrm{d}\pi < \infty.$$

In this case, a plausible convergence rate could be $\kappa = \frac{\beta}{\alpha} - 1$. With $\det b = 0$ being allowed, a higher regularity of coefficients and lighter tails of π would be required.

Acknowledgments

We are very grateful to the referees for their valuable suggestions, comments and patience.

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