Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 123 (2013) 4090–4110

stochastic processes and their applications

www.elsevier.com/locate/spa

# Karhunen–Loève expansion for additive Brownian motions

Jin V. Liu*

*College of Opto-Electronic Science and Engineering, National University of Defense Technology, China*

Received 20 February 2013; received in revised form 21 May 2013; accepted 1 June 2013
Available online 10 June 2013

**Abstract**

For Gaussian random fields defined as additive processes based on standard Brownian motions and Brownian bridges, we find their Karhunen–Loève expansions and make connections with related mean centered processes in distribution. Moreover, Pythagorean type distribution identities are established for additive Brownian motions and Brownian bridges. As applications, the corresponding Laplace transform and small deviation estimates are given.


*Keywords:* Karhunen–Loève expansion; Additive process; Small deviation; Laplace transform

## 1. Introduction

Let $X_j(t)$, $1 \le j \le d$, be independent real-valued stochastic processes with index set $E$ on the same probability space. Define the associated (real-valued) additive process (field)

$$\mathbb{X}(\mathbf{t}) = \mathbb{X}(t_1, \ldots, t_d) = \sum_{j=1}^{d} X_j(t_j), \quad \mathbf{t} = (t_1, \ldots, t_d) \in E^d. \tag{1.1}$$

There are various motivations for the study of the additive process $\mathbb{X}(\mathbf{t})$, $\mathbf{t} \in E^d$, and it has been actively investigated recently from different points of view. First of all, additive processes play a role in the study of other more interesting multiparameter processes. For example, the Brownian sheet (arguably the most fundamental multiparameter Gaussian process) is well-approximated by additive Brownian motion locally and with time suitable rescaled; see [10,11,8,9]. In

* Tel.: +86 13810954455.
*E-mail address:* liujin1344@gmail.com.

0304-4149/$ - see front matter 
http://dx.doi.org/10.1016/j.spa.2013.06.001

multiparameter potential theory, additive processes are connected with a natural class of energy forms and their corresponding capacities, and thus provide useful links to the study of fractal geometry of the associated processes; see [18,22,23,19–21] for detailed discussion and the bibliography for further works in this area. Additive processes also arise in the theory of intersections and self-intersections of Brownian processes; see [17,5,6]. From the functional analytic point of view, spectral asymptotics for the associated compact self-adjoint operators of the additive Gaussian process are considered by Karol et al. [16]. Moreover, recent progress has shown that additive processes are more amenable to analysis, as we will also see in this paper.

The main objective of this paper is the study of Karhunen–Loève expansion for additive Brownian motion

$$\mathbb{W}(\mathbf{t}) = \mathbb{W}(t_1, \dots, t_d) = \sum_{j=1}^{d} W_j(t_j), \quad \mathbf{t} = (t_1, \dots, t_d) \in [0, 1]^d \tag{1.2}$$

and the additive Brownian bridge

$$\mathbb{B}(\mathbf{t}) = \mathbb{B}(t_1, \dots, t_d) = \sum_{j=1}^{d} B_j(t_j), \quad \mathbf{t} = (t_1, \dots, t_d) \in [0, 1]^d \tag{1.3}$$

where $W_i(t)$ and $B_i(t)$ are independent standard Brownian motions and Brownian bridges, respectively, throughout this paper. We also consider a more general case of additive mixtures of Brownian motion and the Brownian bridge; see (3.1). Note that both additive Brownian motion $\mathbb{W}$ and additive Brownian bridge $\mathbb{B}$ are Gaussian fields on $[0, 1]^d$. The Karhunen–Loève expansions of Gaussian fields are of significant interest in probability and statistics. However, to our best knowledge, little is known for explicit expansion for non-tensored Gaussian fields (covariance functions are not of the product form of lower dimensional covariance functions). Thus it is natural to exam the additive Gaussian processes in order to gain insights on explicit Karhunen–Loève expansions and relevant consequences.

Before we state our results based on the Karhunen–Loève expansions developed in Section 3, we introduce the so called mean centered Brownian motion $\overline{W}$ and the mean centered Brownian bridge $\overline{B}$ on interval [0, 1] defined by

$$\overline{W}(t) = W(t) - \int_0^1 W(t)dt, \quad 0 \le t \le 1, \tag{1.4}$$

with covariance function $\mathbb{E}\overline{W}(t)\overline{W}(s) = t \wedge s + (t^2 + s^2)/2 - t - s + 1/3$, and

$$\overline{B}(t) = B(t) - \int_0^1 B(t)dt, \quad 0 \le t \le 1, \tag{1.5}$$

with covariance function $\mathbb{E}\overline{B}(t)\overline{B}(s) = t \wedge s - ts + (t^2 + s^2 - t - s)/2 + 1/12$.

**Theorem 1.1.** *For additive Brownian motion $\mathbb{W}$ defined in (1.2), we have*

$$\begin{aligned}\int_{[0,1]^d} \mathbb{W}^2(\mathbf{t})d\mathbf{t} &\overset{law}{=} \int_0^1 \overline{W}_1^2(t)dt + \cdots + \int_0^1 \overline{W}_{d-1}^2(t)dt + \int_0^1 Y^2(t)dt \\ &\overset{law}{=} \int_0^1 B_1^2(t)dt + \cdots + \int_0^1 B_{d-1}^2(t)dt + \int_0^1 Y^2(t)dt\end{aligned}$$

*where the mean zero Gaussian process $Y$ is defined by*

$$Y(t) = W_d(t) + (\sqrt{d} - 1)\int_0^1 W_d(s)ds, \quad 0 \le t \le 1 \tag{1.6}$$

*with covariance function* $\mathbb{E}Y(t)Y(s) = t \wedge s - ts - (\sqrt{d}-1)(t^2+s^2-t-s)/2 + (\sqrt{d}-1)^2/12.$

The process can also be replaced by $W_d(t) - (\sqrt{d}+1)\int_0^1 W_d(t)dt$. This follows from the simple observation as following

$$\begin{aligned}\int_0^1 (X(t) + (b-1)\int_0^1 X(s)ds)^2 dt &= \int_0^1 (X(t) - (b+1)\int_0^1 X(s)ds)^2 dt \\ &= \int_0^1 X^2(t)dt + (b^2-1)\left(\int_0^1 X(t)dt\right)^2,\end{aligned}$$

for any process $X$, and for any $b \in \mathbb{R}$.

The process can also be replaced by $W_d(t) + (\pm\sqrt{d} - 1)tW_d(1)$ which can be seen from the stochastic Fubini's theorem. See [15]. More precisely, for any $C^1$ function $f : [0, 1] \to \mathbb{R}$,

$$\begin{aligned}\int_0^1 \left(W(t) - \int_0^1 f'(s)W(s)ds\right)^2 dt &\overset{law}{=} \int_0^1 (W(t) - (1 - f(1) + f(t))W(1))^2\, dt \\ &\overset{law}{=} \int_0^1 (W(t) - (f(1-t) - f(1))W(1))^2\, dt.\end{aligned}$$

In particular, for $a \in \mathbb{R}$, take $f(t) = at$, we obtain

$$\int_0^1 (W(t) - atW(1))^2 dt \overset{law}{=} \int_0^1 (W(t) - a\int_0^1 W(s)ds)^2 dt.$$

The two Gaussian processes

$$Y_{\pm b}(t) = W(t) + (\pm b - 1)\int_0^1 W(s)ds \tag{1.7}$$

are the same under the $L_2$-norm, but the two are not the same Gaussian process for $b \ne 0$ since

$$\mathbb{E}\, Y_{\pm b}(t)Y_{\pm b}(s) = t \wedge s + (1 \mp b)(t^2/2 + s^2/2 - t - s) + (1 \mp b)^2/3. \tag{1.8}$$

Unlike $Y_{\pm b}$, the two Gaussian processes $W(t) + (\pm b - 1)tW(1)$ are the same process since for each $b \in \mathbb{R}$

$$\begin{aligned}\{W(t) - (b+1)tW(1) : 0 \le t \le 1\} &\overset{law}{=} \{W(t) + (b-1)tW(1) : 0 \le t \le 1\} \\ &\overset{law}{=} \{B(t) \pm btW(1) : 0 \le t \le 1\}\end{aligned} \tag{1.9}$$

where $W(t)$ is a standard Brownian motion, and the Brownian bridge $B(t)$ is independent of $W(1)$. One can easily check (1.9) by computing covariance for each of the Gaussian processes involved or using the well known representation between a Brownian motion and a Brownian bridge, $B(t) = W(t) - tW(1)$, $0 \le t \le 1$, independent of $W(1)$.

**Theorem 1.2.** *For additive Brownian bridge $\mathbb{B}$ defined in (1.3), we have*

$$\int_{[0,1]^d} \mathbb{B}^2(\mathbf{t})d\mathbf{t} \overset{law}{=} \int_0^1 \overline{B}_1^2(t)dt + \cdots + \int_0^1 \overline{B}_{d-1}^2(t)dt + \int_0^1 Z^2(t)dt$$

*where the mean zero Gaussian process $Z$ is defined by*

$$Z(t) = B_d(t) + (\sqrt{d} - 1) \int_0^1 B_d(s)ds, \quad 0 \le t \le 1 \tag{1.10}$$

*with covariance function* $\mathbb{E}Z(t)Z(s) = t \wedge s - ts - (\sqrt{d}-1)(t^2+s^2-t-s)/2 + (\sqrt{d}-1)^2/12.$

Similar to the remark after Theorem 1.1, the process $Z(t)$ in (1.10) can be replaced by $B_d(t) - (\sqrt{d}+1)\int_0^1 B_d(s)ds$. For each $b \in \mathbb{R}$, let us denote $Z_{\pm b}$ as the Gaussian process

$$Z_{\pm b}(t) := B(t) + (\pm b - 1) \int_0^1 B(s)ds, \quad 0 \le t \le 1, \tag{1.11}$$

with the covariance function

$$\mathbb{E}\, Z_{\pm b}(t) Z_{\pm b}(s) = t \wedge s - ts + (1 \mp b)(t^2 + s^2 - t - s)/2 + (1 \mp b)^2/12. \tag{1.12}$$

It is worth pointing out that both Theorems 1.1 and 1.2 are Pythagorean type theorems (sums of independent terms involving integrated squares) and it is natural to looking for direct proofs. However, it is not clear to us at this time why this is the case. One can easily rewrite for any additive process defined in (1.1),

$$\int_{[0,1]^d} \mathbb{X}^2(\mathbf{t})d\mathbf{t} = \sum_{i=1}^{d-1} \int_0^1 \overline{X}_i^2(t)dt + \int_0^1 \left( X_d(t) + \sum_{i=1}^{d-1} \int_0^1 X_i(t)dt \right)^2 dt \tag{1.13}$$

where $\overline{X}_i(t) = X_i(t) - \int_0^1 X_i(t)dt$ is the associated mean centered process. However, the process $X_d(t) + \sum_{i=1}^{d-1} \int_0^1 X_i(t)dt$ in the last term is *not* independent of the other terms. And so it is somewhat surprising that we indeed have the distribution identities in Theorems 1.1 and 1.2. Of course, one could try the well developed stochastic Fubini approach in [14], but we seem missing some ingredients; see Section 3 for more detailed discussion.

The remaining of the paper is organized as follows. Section 2 contains a short overview of Karhunen–Loève expansions, including some well-known ones such as the mean centered Brownian motion and Brownian bridge. Various distribution identities related to them are also discussed. We then provide KL expansions for processes $Y(t)$ and $Z(t)$ defined in (1.6) and (1.10). In Section 3, we obtain Karhunen–Loève expansions for additive processes associated with Brownian motions and Brownian bridges. The proof we present is much simpler than our original one and somehow it is strictly based on one dimensional structure. Section 4 establishes, as an application of Karhunen–Loève expansion, the small deviation estimates for the additive processes discussed.

## 2. The KL expansions for $Y$ and $Z$ in (1.6)–(1.10)

We start with a short overview of the Karhunen–Loève expansions and various known results needed including the mean centered Brownian motion and Brownian bridge and their associated distribution identities.

Let $\{X(\mathbf{t})\}$ denote a centered Gaussian processes defined on $[0,1]^d$ with $d \geq 1$. For convenience, we set $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$ and $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{R}^d$, and denote the covariance function of $X$ by

$$K(\mathbf{t}, \mathbf{s}) = \mathbb{E}(X(\mathbf{t})X(\mathbf{s})), \quad \text{for } \mathbf{t}, \mathbf{s} \in [0,1]^d,$$

which is a positive definite function. Assuming $K(\mathbf{t}, \mathbf{s})$ is continuous on $[0,1]^d \times [0,1]^d$, then by Mercer's theorem,

$$K(\mathbf{t}, \mathbf{s}) = \sum_{i \geq 1} \lambda_i e_i(\mathbf{t}) e_i(\mathbf{s}) \tag{2.1}$$

where $\{\lambda_i, i \geq 1\}$ and $\{e_i(\mathbf{t}), i \geq 1\}$ are the set of eigenvalues and normalized eigenvectors of the integral operator corresponding to the covariance function in the sense of

$$\lambda f(\mathbf{t}) = \int_{[0,1]^d} K(\mathbf{t}, \mathbf{s}) f(\mathbf{s}) d\mathbf{s}, \quad \mathbf{t} \in [0,1]^d. \tag{2.2}$$

Note that $\lambda_i \geq 0$ and the convergence in (2.1) is absolute and uniform. The well-known Karhunen–Loève (KL) expansion for Gaussian process $X(\mathbf{t})$ on $[0,1]^d$ is

$$X(\mathbf{t}) = \sum_{i \geq 1} \xi_i \sqrt{\lambda_i} e_i(\mathbf{t}),$$

where $\{\xi_i, i \geq 1\}$ is a sequence of i.i.d. standard normal random variables. Note that $\{e_i(t), i \geq 1\}$ forms an orthogonal base in $L^2([0,1]^d)$ and thus a natural consequence of the KL expansion is the distributional identity

$$\int_{[0,1]^d} X^2(\mathbf{t}) d\mathbf{t} \overset{law}{=} \sum_{i \geq 1} \lambda_i \xi_i^2. \tag{2.3}$$

Karhunen–Loève expansion of a Gaussian process is a favor subject of study in probability and statistics, and plays important role in various applications. For $d = 1$, the KL expansions have been computed explicitly for many Gaussian processes related to Brownian motion or Brownian bridge; see for example, [1–3,12,13] and the references cited therein. For the mean centered Brownian motion $\overline{W}(t)$ defined in (1.4), we know from [3]

$$\int_0^1 \overline{W}^2(t) dt \overset{law}{=} \int_0^1 B^2(t) dt \overset{law}{=} \sum_{i \geq 1} \frac{1}{\pi^2 i^2} \xi_i^2.$$

For the mean centered Brownian bridge $\overline{B}(t)$ defined in (1.5), we know from [30,28]

$$\int_0^1 \overline{B}^2(t) dt \overset{law}{=} \frac{1}{4} \left( \int_0^1 B_1^2(t) dt + \int_0^1 B_2^2(t) dt \right) \overset{law}{=} \sum_{i \geq 1} \frac{1}{4\pi^2 i^2} \xi_{1,i}^2 + \sum_{i \geq 1} \frac{1}{4\pi^2 i^2} \xi_{2,i}^2$$

where an elementary explanation of the first identity in law is given by Shi and Yor [27]. Pycke [26] gives a short proof based on the decomposition of an arbitrary function and the stochastic Fubini's theorem.

The KL expansion for $d \geq 2$ is much harder in general, unless we are dealing with the so-called tensored Gaussian fields with covariance function

$$K(\mathbf{t}, \mathbf{s}) = \prod_{j=1}^{d} K_j(t_j, s_j),$$

where $K_j(t, s)$ are covariance functions on $[0, 1] \times [0, 1]$ with known KL expansions. More precisely, assuming

$$K_j(t, s) = \sum_{i \geq 1} \lambda_i^{(j)} e_i^{(j)}(t) e_i^{(j)}(s), \quad 1 \leq j \leq d,$$

we have the representation

$$K(\mathbf{t}, \mathbf{s}) = \sum_{i_1 \geq 1, \ldots, i_d \geq 1} \prod_{j=1}^{d} \lambda_{i_j}^{(j)} e_{i_j}^{(j)}(t_j) e_{i_j}^{(j)}(s_j),$$

and the KL expansion

$$X(\mathbf{t}) = \sum_{i_1 \geq 1, \ldots, i_d \geq 1} \xi_{(i_1, \ldots, i_d)} \prod_{j=1}^{d} \sqrt{\lambda_{i_j}^{(j)}} e_{i_j}^{(j)}(t_j),$$

where $\{\xi_{(i_1, \ldots, i_d)}\}$ is a sequence of $d$-index i.i.d. $N(0, 1)$ random variables. Thus we have the distributional identity

$$\int_{[0,1]^d} X^2(\mathbf{t}) d\mathbf{t} \stackrel{law}{=} \sum_{i_1 \geq 1, \ldots, i_d \geq 1} \xi^2_{(i_1, \ldots, i_d)} \prod_{j=1}^{d} \lambda_{i_j}^{(j)}.$$

For non-tensored Gaussian fields (covariance functions are not of the product form of lower dimensional covariance functions), very little is known for explicit KL expansion. As far as we know, the additive Gaussian processes we considered in this paper are the first family of nontrivial examples.

Next we consider the KL expansions for $Y(t)$ and $Z(t)$ in (1.6), (1.10) and their extensions in (1.7) and (1.11).

**Proposition 2.1.** *Let $Y_b(t)$ be defined by (1.7) with covariance function (1.8), and let the process $\tilde{Y}_b(t)$ be defined by*

$$\tilde{Y}_b(t) = W(t) + (b - 1)tW(1), \quad 0 \leq t \leq 1.$$

*with covariance function*

$$\mathbb{E}\, \tilde{Y}_b(t) \tilde{Y}_b(s) = t \wedge s + (b^2 - 1)ts.$$

*Then*

$$\int_0^1 Y_b^2(t) dt \stackrel{law}{=} \int_0^1 \tilde{Y}_b^2(t) \stackrel{law}{=} \sum_{i \geq 1} \lambda_i \xi_i^2 \tag{2.4}$$

*where $\xi_i$ are i.i.d standard normal r.v's and the eigenvalues $\lambda_i$ are solutions of the equation*

$$(b^2 - 1)\sqrt{\lambda} \sin(1/\sqrt{\lambda}) - b^2 \cos(1/\sqrt{\lambda}) = 0. \tag{2.5}$$

*In addition, we have*

$$\mathbb{E}\exp\left(-\frac{\theta^2}{2}\int_0^1 Y_b^2(t)dt\right) = \left(b^2\cosh\theta - (b^2-1)\theta^{-1}\sinh\theta\right)^{-1/2}. \tag{2.6}$$

**Proof.** For the covariance function (1.8), we need to compute the integral equation

$$\lambda f(t) = \int_0^1 (t\wedge s + (1-b)(t^2/2 + s^2/2 - t - s) + (1-b)^2/3) f(s)ds, \quad 0\le t\le 1.$$

Taking derivatives, we obtain

$$\lambda f'(t) = \int_t^1 f(s)ds + (1-b)(t-1)\int_0^1 f(s)ds,$$

$$\lambda f''(t) = -f(t) + (1-b)\int_0^1 f(s)ds.$$

Then the general solutions are

$$f(t) = c_1\cos(t/\sqrt{\lambda}) + c_2\sin(t/\sqrt{\lambda}) + c_3 \tag{2.7}$$

where $c_1, c_2, c_3$ are constants with $c_3 = (1-b)\int_0^1 f(s)ds$. Substitute (2.7) into the expression of $c_3$, then we see that

$$(1-b)\sqrt{\lambda}\sin(1/\sqrt{\lambda})c_1 + (1-b)\sqrt{\lambda}(1-\cos(1/\sqrt{\lambda}))c_2 - bc_3 = 0. \tag{2.8}$$

Notice that

$$\lambda f'(1) = 0,$$

which implies

$$(1-b)\sqrt{\lambda}\sin(1/\sqrt{\lambda})c_1 + (1-b)\sqrt{\lambda}(1-\cos(1/\sqrt{\lambda}))c_2 - bc_3 = 0. \tag{2.9}$$

We also have

$$(1-b)\lambda f'(0) = b(1-b)\int_0^1 f(s)ds = bc_3,$$

we can see that this is just the linear combination of (2.8) and (2.9).

Thus we need one more boundary condition. Note that

$$\lambda f(0) = \int_0^1 [(1-b)(s^2/2 - s) + (1-b)^2/3] f(s)ds, \tag{2.10}$$

while

$$\begin{aligned}\int_0^1 (s^2/2 - s) f''(s)ds &= \int_0^1 (s^2/2 - s)df'(s)\\ &= -f'(1)/2 - \int_0^1 (s-1)f'(s)ds\\ &= -\int_0^1 (s-1)df(s)\\ &= -f(0) + \int_0^1 f(s)ds,\end{aligned}$$

on the other hand, from the differential equation associated with $f''(t)$, we see that

$$\begin{aligned}
&\lambda \int_0^1 (s^2/2 - s) f''(s) ds \\
&\quad = -\int_0^1 (s^2/2 - s) f(s) ds + (1-b) \int_0^1 (s^2/2 - s) ds \int_0^1 f(s) ds \\
&\quad = -\int_0^1 (s^2/2 - s) f(s) ds - \frac{1-b}{3} \int_0^1 f(s) ds.
\end{aligned}$$

From (2.10), and together with the above two equations, we see that

$$\begin{aligned}
&-(1-b)\lambda f(0) + (1-b)\lambda \int_0^1 f(s) ds \\
&\quad = -(1-b) \int_0^1 (s^2/2 - s) f(s) ds - \frac{(1-b)^2}{3} \int_0^1 f(s) ds \\
&\quad = -\lambda f(0),
\end{aligned}$$

which implies $bf(0) + c_3 = 0$, i.e.

$$bc_1 + (1+b)c_3 = 0. \tag{2.11}$$

To find nontrivial $c_i$, we need the determinant of the matrix corresponding to Eqs. (2.8), (2.9) and (2.11) to be zero. Calculating the determinant, we obtain (2.5).

As we point out, the first identity of (2.4) can be derived by Donati-Martin and Yor [15]. Here we can also show the identity by KL expansions.

We start with the integral equation

$$\lambda f(t) = \int_0^1 (t \wedge s + (b^2 - 1)ts) f(s) ds, \quad 0 \le t \le 1.$$

Taking derivatives, we obtain

$$\lambda f'(t) = \int_t^1 f(s) ds + (b^2 - 1) \int_0^1 s f(s) ds, \quad \text{and} \quad \lambda f''(t) + f(t) = 0 \tag{2.12}$$

with boundary conditions

$$f(0) = 0 \quad \text{and} \quad (b^2 - 1) f(1) = b^2 f'(1).$$

The general solution for the second equation in (2.12) is

$$f(t) = c_1 \cos(t/\sqrt{\lambda}) + c_2 \sin(t/\sqrt{\lambda})$$

for some constants $c_1, c_2$. The boundary condition $f(0) = 0$ implies $c_1 = 0$, and we need $c_2 \neq 0$. From the boundary condition $(b^2 - 1) f(1) = b^2 f'(1)$, we obtain (2.5). Hence we obtain the first identity of (2.4).

Equality (2.6) follows from the general fact that

$$\begin{aligned}
\mathbb{E} \exp\left( -\frac{\theta^2}{2} \int_0^1 Y_b^2(t) dt \right) &= \mathbb{E} \exp\left( -\frac{\theta^2}{2} \sum_{i=1}^{\infty} \lambda_i \xi_i^2 \right) \\
&= \prod_{i=1}^{\infty} (1 + \lambda_i \theta^2)^{-1/2} = (D(-\theta^2))^{-1/2}.
\end{aligned}$$

Here from (2.5) the Fredholm determinant $D(\lambda)$ can be written as

$$D(\lambda) = -(b^2 - 1)\lambda^{-1/2} \sin \sqrt{\lambda} + b^2 \cos \sqrt{\lambda}, \tag{2.13}$$

with $D(0) = 1$; see [29] for a similar argument. In particular, one needs the Taylor expansions of the sin and cos functions to ensure $D(0) = 1$.

The Laplace transform (2.6) agrees with the results given by Chan et al. [4] and Donati-Martin and Yor [15]. □

Note that we have from (1.9) and (2.6)

$$\mathbb{E} \exp\left(-\frac{\theta^2}{2} \int_0^1 (B(t) + btW(1))^2 dt\right) = \left(b^2 \cosh\theta - (b^2 - 1)\theta^{-1} \sinh\theta\right)^{-1/2}. \tag{2.14}$$

We can see the above Laplace transform from a different point of view. Since $B(t)$ and $W(1)$ are independent, by conditioning on $W(1)$, the Laplace transform (2.14) can be obtained from the following identity observing $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$.

$$\mathbb{E} \exp\left(-\frac{\theta^2}{2} \int_0^1 (B(t) + bt)^2 dt\right) = \sqrt{\frac{\theta}{\sinh\theta}} \exp\left(-\frac{1}{2} b^2 [\theta \coth\theta - 1]\right). \tag{2.15}$$

Naturally, we can obtain (2.15) by using the formula about the Laplace transform of the quadratic functional of the Brownian bridge given by Chiang et al. [7]. Here more simply, the expression (2.15) can be viewed as a direct consequence of the following equality which is known in [24]

$$\mathbb{E}\left(\exp\left(-\frac{\theta^2}{2} \int_0^1 W^2(t) dt\right) \middle| W(1) = b\right) = \sqrt{\frac{\theta}{\sinh\theta}} \exp\left(-\frac{1}{2} b^2 [\theta \coth\theta - 1]\right).$$

**Proposition 2.2.** *Let $Z_b(t)$ be defined by (1.11) with covariance function (1.12), then*

$$\begin{aligned}
\int_0^1 Z_b^2(t) dt &\overset{law}{=} \sum_{i \ge 1} \frac{1}{4\pi^2 i^2} \xi_{1,i}^2 + \frac{1}{4} \sum_{i \ge 1} \lambda_i \xi_{2,i}^2 \\
&\overset{law}{=} \frac{1}{4} \int_0^1 B^2(t) dt + \frac{1}{4} \int_0^1 Y_b^2(t) dt \\
&\overset{law}{=} \frac{1}{4} \int_0^1 B^2(t) dt + \frac{1}{4} \int_0^1 \tilde{Y}_b^2(t) dt
\end{aligned} \tag{2.16}$$

*where $\xi_{1,i}$, $\xi_{2,i}$ are i.i.d standard normal r.v's and the eigenvalues $\lambda_i$ are solutions of Eq. (2.5). In addition, we have*

$$\mathbb{E} \exp\left(-\frac{\theta^2}{2} \int_0^1 Z_b^2(t) dt\right) = \theta \left(b^2 \theta \sinh\theta + 2(b^2 - 1)(1 - \cosh\theta)\right)^{-1/2}. \tag{2.17}$$

**Proof.** For the covariance function (1.12), we need to compute the integral equation

$$\lambda f(t) = \int_0^1 (t \wedge s - ts + (1-b)(t^2 + s^2 - t - s)/2 + (1-b)^{2/12}) f(s) ds,$$
$$0 \le t \le 1. \tag{2.18}$$

Taking derivatives, we obtain

$$\lambda f'(t) = \int_t^1 f(s)ds - \int_0^1 sf(s)ds + \frac{1-b}{2}(2t-1)\int_0^1 f(s)ds, \tag{2.19}$$

$$\lambda f''(t) = -f(t) + (1-b)\int_0^1 f(s)ds. \tag{2.20}$$

Then the general solutions are

$$f(t) = c_1 \cos(t/\sqrt{\lambda}) + c_2 \sin(t/\sqrt{\lambda}) + c_3, \tag{2.21}$$

where $c_1, c_2, c_3$ are constants with $c_3 = (1-b)\int_0^1 f(s)ds$. Substitute (2.7) into the expression of $c_3$, then we see that

$$(1-b)\sqrt{\lambda}\sin(1/\sqrt{\lambda})c_1 + (1-b)\sqrt{\lambda}(1-\cos(1/\sqrt{\lambda}))c_2 - bc_3 = 0. \tag{2.22}$$

Notice that from (2.18) and (2.19),

$$\lambda f(0) = \int_0^1 (-(b-1)(s^2-s)/2 + (b-1)^2/12) f(s)ds;$$

$$\lambda f(1) = \int_0^1 (-(b-1)(s^2-s)/2 + (b-1)^2/12) f(s)ds;$$

$$\lambda f'(0) = \frac{1+b}{2}\int_0^1 f(s)ds - \int_0^1 sf(s)ds;$$

$$\lambda f'(1) = \frac{1-b}{2}\int_0^1 f(s)ds - \int_0^1 sf(s)ds.$$

We obtain the boundary condition

$$f(0) = f(1),$$

which implies

$$c_1(1-\cos(1/\sqrt{\lambda})) - c_2\sin(1/\sqrt{\lambda}) = 0. \tag{2.23}$$

We can also obtain the boundary condition

$$(1-b)\lambda(f'(0) - f'(1)) = b(1-b)\int_0^1 f(s)ds = bc_3,$$

which is equivalent to the condition (2.22). On the other hand, we have

$$\lambda\left(\frac{1-b}{2}f'(0) - \frac{1+b}{2}f'(1)\right) = b\int_0^1 sf(s)ds. \tag{2.24}$$

Substitute $f(s)$ with (2.21) into the right hand side of (2.24), then we can see that the condition is a linear combination of (2.22) and (2.23).

So one more condition is still needed. First we can check that

$$\begin{aligned}\int_0^1 (s^2-s)f''(s)ds &= \int_0^1 (s^2-s)df'(s)\\ &= -\int_0^1 (2s-1)f'(s)ds\end{aligned}$$

$$= -\int_0^1 (2s-1)df(s)$$
$$= -(f(1)+f(0)) + 2\int_0^1 f(s)ds,$$

on the other hand, from the differential equation associated with $f''(t)$, we see that

$$\lambda \int_0^1 (s^2-s)f''(s)ds = -\int_0^1 (s^2-s)f(s)ds + (1-b)\int_0^1 (s^2-s)ds \int_0^1 f(s)ds$$
$$= -\int_0^1 (s^2-s)f(s)ds - \frac{1-b}{6}\int_0^1 f(s)ds.$$

From the condition of $f(0)$, and together with the above two equations, we see

$$-\frac{1-b}{2}\lambda(f(1)+f(0)) + (1-b)\lambda\int_0^1 f(s)ds$$
$$= -\frac{1-b}{2}\int_0^1 (s^2-s)f(s)ds - \frac{(b-1)^2}{12}\int_0^1 f(s)ds$$
$$= -\lambda f(0) + \frac{(b-1)^2}{12}\int_0^1 f(s)ds - \frac{(b-1)^2}{12}\int_0^1 f(s)ds,$$

which implies

$$-\frac{1-b}{2}(f(0)+f(1)) + c_3 + f(0) = 0;$$

In other words

$$(1+b-(1-b)\cos(1/\sqrt{\lambda}))c_1 - (1-b)\sin(1/\sqrt{\lambda})c_2 + 2(1+b)c_3 = 0. \tag{2.25}$$

To find nontrivial $c_i$, we need the determinant of the matrix corresponding to Eqs. (2.22), (2.23) and (2.25) to be zero. Calculating the determinant, we obtain

$$2(b^2-1)\sqrt{\lambda}(1-\cos(1/\sqrt{\lambda})) - b^2\sin(1/\sqrt{\lambda}) = 0$$

simplifying

$$\sin(1/(2\sqrt{\lambda}))\left(2(b^2-1)\sqrt{\lambda}\sin(1/(2\sqrt{\lambda})) - b^2\cos(1/(2\sqrt{\lambda}))\right) = 0,$$

hence we can obtain (2.16).

The Laplace transform follows from similar arguments as Proposition 2.1.

The Laplace transform (2.17) agrees with the results given by Chan et al. [4]. □

At last, let us take a sight into the last term on the right hand side of (1.13). The process $W_d(t) + \sum_{i=1}^{d-1}\int_0^1 W_i(t)dt$ is a centered Gaussian process with covariance function $t\wedge s + (d-1)/3$. The process $B_d(t) + \sum_{i=1}^{d-1}\int_0^1 B_i(t)dt$ is a centered Gaussian process with covariance function $t\wedge s - ts + (d-1)/12$. More generally, let $M_a(t)$ denotes the centered Gaussian process with covariance function $t\wedge s + a^2$, and let $N_a(t)$ denotes the centered Gaussian process with covariance function $t\wedge s - ts + a^2$ for $a\in\mathbb{R}$. We give the following results without proof.

**Proposition 2.3.** *We have*

$$\int_0^1 M_a^2(t)dt \overset{law}{=} \sum_{i\geq 1} \lambda_i \xi_i^2$$

*where $\xi_i$ are i.i.d standard normal r.v's and the eigenvalues $\lambda_i$ are solutions of the equation*

$$a^2 \sin(1/\sqrt{\lambda}) - \sqrt{\lambda}\cos(1/\sqrt{\lambda}) = 0. \tag{2.26}$$

*In addition, we have*

$$\mathbb{E}\exp\left(-\frac{\theta^2}{2}\int_0^1 M_a^2(t)dt\right) = \left(\cosh\theta + a^2\theta\sinh\theta\right)^{-1/2}.$$

**Proposition 2.4.** *We have*

$$\begin{aligned}\int_0^1 N_a^2(t)dt &\overset{law}{=} \sum_{i\geq 1}\frac{1}{4\pi^2 i^2}\xi_{1,i}^2 + \frac{1}{4}\sum_{i\geq 1}\lambda_i \xi_{2,i}^2 \\ &\overset{law}{=} \frac{1}{4}\int_0^1 B^2(t)dt + \frac{1}{4}\int_0^1 M_{2a}^2(t)dt\end{aligned} \tag{2.27}$$

*where $\xi_i$ are i.i.d standard normal r.v's and the eigenvalues $\lambda_i$ are solutions of Eq. (2.26).*

*In addition, we have*

$$\mathbb{E}\exp\left(-\frac{\theta^2}{2}\int_0^1 N_a^2(t)dt\right) = \left(\theta^{-1}\sinh\theta - 2a^2(1-\cosh\theta)\right)^{-1/2}.$$

## 3. The KL expansions for mixed additive processes

Consider a more general case of additive mixtures of Brownian motions and Brownian bridges

$$\mathbb{X}_{m,d}(\mathbf{t}) := \sum_{i=1}^{m} W_i(t_i) + \sum_{i=m+1}^{d} B_i(t_i), \quad \mathbf{t} = (t_1,\ldots,t_d) \in [0,1]^d \tag{3.1}$$

for $0 \leq m \leq d$. Clearly, we have the additive Brownian motion in (1.2) for $m = d$ and the additive Brownian bridge in (1.3) for $m = 0$. The mixed additive process $\mathbb{X}_{m,d}$ is a centered Gaussian process with the covariance function

$$K(\mathbf{t},\mathbf{s}) = \sum_{i=1}^{m} t_i \wedge s_i + \sum_{i=m+1}^{d}[t_i \wedge s_i - t_i s_i]. \tag{3.2}$$

**Theorem 3.1.** *For $m \geq 1$, we have the distributional identity*

$$\int_{[0,1]^d} \mathbb{X}_{m,d}^2(\mathbf{t})d\mathbf{t} \overset{law}{=} \sum_{1\leq j\leq m-1}\sum_{i\geq 1}\frac{1}{\pi^2 i^2}\xi_{j,i}^2 + \sum_{1\leq j\leq 2(d-m)}\sum_{i\geq 1}\frac{1}{4\pi^2 i^2}\tilde{\xi}_{j,i}^2 + \sum_{i\geq 1}\lambda_i \xi_i^{*2};$$

*where* $\{\xi_{j,i}, \tilde{\xi}_{j,i}, \xi_i^*\}$ *are independent i.i.d* $N(0,1)$ *r.v's; the eigenvalues* $\lambda_i$ *are solutions of the equation*

$$(d-1)\sqrt{\lambda}\sin(1/\sqrt{\lambda}) - m\cos(1/\sqrt{\lambda}) - (d-m)(1+\cos(1/\sqrt{\lambda}))/2 = 0. \tag{3.3}$$

*While for* $m = 0$ *(the additive Brownian bridge), we have the distributional identity in Theorem* 1.2.

*In addition, we have for* $0 \le m \le d$

$$\mathbb{E}\exp\left(-\frac{\theta^2}{2}\int_{[0,1]^d}\mathbb{X}_{m,d}^2(\mathbf{t})d\mathbf{t}\right)$$
$$= \left[\frac{(\sinh\theta)^{m-1}(\sinh(\theta/2))^{2(d-m)}(-(d-1)\theta^{-1}\sinh\theta + \cosh\theta(d+m)/2 + (d-m)/2)}{2^{-2(d-m)}\theta^{2d-m-1}}\right]^{-1/2}. \tag{3.4}$$

**Proof.** For the covariance function (3.2) of $\mathbb{X}_{m,d}$, we need to compute the integral equation

$$\lambda f(t_1,\dots,t_d) = \int_0^1\cdots\int_0^1\left(\sum_{i=1}^m t_i\wedge s_i + \sum_{i=m+1}^d [t_i\wedge s_i - t_i s_i]\right) f(\mathbf{s})d\mathbf{s}, \quad 0\le t_i\le 1.$$

It is straightforward to claim that the function $f$ has the form:

$$f(t_1,\dots,t_d) = \sum_{i=1}^d f_i(t_i).$$

By differentiating both sides of the integral equation (3.4) with respect to $t_i$, we have for $1\le i\le m$

$$\lambda f_i'(t_i) = \int_0^1\cdots\int_{t_i}^1\cdots\int_0^1 f(s_1,\dots,s_d)ds_1\cdots ds_d, ; \tag{3.5}$$

and for $m+1\le i\le d$,

$$\lambda f'(t_i) = \int_0^1\cdots\int_{t_i}^1\cdots\int_0^1 f(\mathbf{s})d\mathbf{s} - \int_0^1\cdots\int_0^1 s_i f(\mathbf{s})d\mathbf{s}. \tag{3.6}$$

Differentiating again with respect to $t_i$ in $d$ Eqs. (3.5) and (3.6), we have

$$\lambda f_i''(t_i) = -\int_0^1\cdots\int_0^1 f(s_1,\dots,s_{i-1},t_i,s_{i+1},\dots,s_d)ds_1\cdots ds_{i-1}ds_{i+1}\cdots ds_d$$
$$= -f_i(t_i) - \sum_{j\ne i}\int_0^1 f_j(s)ds, \quad 1\le i\le d.$$

Then the general solutions for the above equations are

$$f_i(t) = c_{2i-1}\cos(t/\sqrt{\lambda}) + c_{2i}\sin(t/\sqrt{\lambda}) - \sum_{1\le j\ne i\le d}\int_0^1 f_j(s)ds, \quad 1\le i\le d \tag{3.7}$$

for some constants $c_i$, $1\le i\le 2d$.

From the original integral equation (3.4) we have the boundary condition

$$f_1(0)+\cdots+f_d(0) = 0. \tag{3.8}$$

So if $c_i = 0$ for all $1 \le i \le 2d$, then all the functions $f_i$ in (3.7) are constants, which yields $f(t_1, \dots, t_d) = \sum_{i=1}^{d} f_i(t_i) \equiv 0$. This forces us to find nontrivial $c_i$ such that $\sum_{i=1}^{2d} c_i^2 > 0$.

From (3.5) and (3.6) we have the boundary conditions

$$f_i'(1) = 0, \quad 1 \le i \le m; \tag{3.9}$$

$$f_i(0) = f_i(1), \quad m+1 \le i \le d; \tag{3.10}$$

$$f_1'(0) - f_1'(1) = \cdots = f_d'(0) - f_d'(1). \tag{3.11}$$

Note that by integrating both sides of (3.7), we have a sequence of conditions which are equivalent to (3.11).

The boundary conditions (3.8)–(3.11) imply

$$c_1 + c_3 + \cdots + c_{2d-1} - (d-1)\left(c_1\sqrt{\lambda}\sin(1/\sqrt{\lambda}) - c_2\sqrt{\lambda}[\cos(1/\sqrt{\lambda}) - 1]\right) = 0; \tag{3.12}$$

$$-c_{2i-1}\sin(1/\sqrt{\lambda}) + c_{2i}\cos(1/\sqrt{\lambda}) = 0, \quad 1 \le i \le m; \tag{3.13}$$

$$c_{2i-1}(1 - \cos(1/\sqrt{\lambda})) - c_{2i}\sin(1/\sqrt{\lambda}) = 0, \quad m+1 \le i \le d; \tag{3.14}$$

$$c_1\sin(1/\sqrt{\lambda}) - c_2[\cos(1/\sqrt{\lambda}) - 1] = \cdots = c_{2d-1}\sin(1/\sqrt{\lambda}) - c_{2d}[\cos(1/\sqrt{\lambda}) - 1]. \tag{3.15}$$

To find nontrivial $c_i$, $1 \le i \le 2d$, we need the determinant of the following matrix to be zero.

$$\begin{pmatrix}
1-(d-1)\sqrt{\lambda}\sin & (d-1)\sqrt{\lambda}(\cos-1) & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\
-\sin & \cos & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & & & & & & & & & & & \\
0 & 0 & 0 & 0 & \cdots & -\sin & \cos & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cos-1 & \sin & \cdots & 0 & 0 \\
\cdots & & & & & & & & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \cos-1 & \sin \\
\sin & 1-\cos & -\sin & \cos-1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & & & & & & & & & & & \\
\sin & 1-\cos & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\sin & \cos-1
\end{pmatrix}.$$

Let us denote the above matrix as $\Delta = (\delta_{i,j})$. Here for convenience, we write shortly cos (sin) to mean $\cos(1/\sqrt{\lambda})$ ($\sin(1/\sqrt{\lambda})$). To obtain $\det(\Delta)$, we expand the matrix by the elements of the first row through

$$\delta_k = \begin{pmatrix} \delta_{1,2k-1} & \delta_{1,2k} \\ \delta_{k+1,2k-1} & \delta_{k+1,2k} \end{pmatrix}$$

for $1 \le k \le d$. We denote the matrices left as $\Delta_k$. From the matrix theory, we know

$$\det(\Delta) = \sum (-1)^{1+(2k-1)+(k+1)+2k}\det(\delta_k)\det(\Delta_k) = \sum (-1)^{k+1}\det(\delta_k)\det(\Delta_k).$$

For the $2 \times 2$ matrices $\delta_k$, we see

$$\det(\delta_k) = \begin{cases} \cos - (d-1)\sqrt{\lambda}\sin, & k = 1; \\ \cos, & 2 \le k \le m; \\ \sin, & m+1 \le k \le d. \end{cases}$$

Now let us study the matrices $\Delta_k$. First we see

$$\Delta_1 = \begin{pmatrix} -\sin & \cos & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots \\ 0 & 0 & \cdots & -\sin & \cos & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cos -1 & \sin & \cdots & 0 & 0 \\ \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \cos -1 & \sin \\ -\sin & \cos -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\sin & \cos -1 \end{pmatrix}.$$

Each row has only two non zero elements, the first $(m-1)$ rows have the two non zero elements $(-\sin, \cos)$, the following $(d-m)$ rows have the two non zero elements $(\cos -1, \sin)$, and the last $(d-1)$ rows have the two non zero elements $(-\sin, \cos -1)$. More generally, let $\Delta^{x,y}$ be a $2(x+y) \times 2(x+y)$ matrix similar as $\Delta_1$, in which the first $x$ rows have the two non zero elements $(-\sin, \cos)$, while the following $y$ rows have the two non zero elements $(\cos -1, \sin)$, and the last $(x+y)$ rows have the two non zero elements $(-\sin, \cos -1)$. Notice that

$$\det \begin{pmatrix} -\sin & \cos \\ -\sin & \cos -1 \end{pmatrix} = \sin,$$

and

$$\det \begin{pmatrix} \cos -1 & \sin \\ -\sin & \cos -1 \end{pmatrix} = 2 - 2\cos,$$

we obtain the determinant:

$$\det(\Delta^{x,y}) = \sin^x (2 - 2\cos)^y.$$

In fact, the determinant is actually $\pm 1$ times of what we just obtained. Due to the form of the matrix, the constant $\pm 1$ only depends on $l + k$, thus we can omit the constant in our problem.

For $k \geq 2$,

$$\Delta_k = \begin{pmatrix} -\sin & \cos & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots \\ 0 & 0 & 0 & 0 & \cdots & -\sin & \cos & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cos -1 & \sin & \cdots & 0 & 0 \\ \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \cos -1 & \sin \\ \sin & 1-\cos & -\sin & \cos -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots \\ \sin & 1-\cos & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots \\ \sin & 1-\cos & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\sin & \cos -1 \end{pmatrix}.$$

The first $d-1$ rows still have only two non zero elements. It is slightly different for the last $d-1$ rows with $\Delta_1$, each row has four non zero elements except the $(d-1+k-1)$ row. All the last $d-1$ rows have the same first two elements $(\sin, 1-\cos)$, then let us take elementary row transformations

$$r_{d-1+i} \leftarrow r_{d-1+i} - r_{d-1+k-1}, \quad \text{for } 1 \leq i \leq d-1, i \neq k-1.$$

$$r_{d-1+k-1} \leftarrow (-1)r_{d-1+k-1}.$$

And move the $d-1+k-1$-th row to the $d$-th row, which means the original $d-1+i$-th row now become the new $d+i$-th row, for $1 \le i \le k-2$. We see

$$\det(\Delta_k) = \begin{cases} \Delta^{m-1,d-m}, & k=1; \\ (-1)(-1)^{k-2}\Delta^{m-1,d-m}, & 2 \le k \le m; \\ (-1)(-1)^{k-2}\Delta^{m,d-m-1}, & m+1 \le k \le d. \end{cases}$$

which implies

$$\det(\Delta_k) = \begin{cases} (-1)^{k+1}\sin^{m-1}(2-2\cos)^{d-m}, & 1 \le k \le m; \\ (-1)^{k+1}\sin^{m}(2-2\cos)^{d-m-1}, & m+1 \le k \le d. \end{cases}$$

Combining $\det(\delta_k)$ and $\det(\Delta_k)$, we now see that

$$\begin{aligned} \det(\Delta) &= \sum (-1)^{k+1}\det(\delta_k)\det(\Delta_k) = (\cos - (d-1)\sqrt{\lambda}\sin)(\sin)^{m-1}(2-2\cos)^{d-m} \\ &\quad + (m-1)\cos(\sin)^{m-1}(2-2\cos)^{d-m} + (d-m)\sin(\sin)^{m}(2-2\cos)^{d-m-1} \\ &= \sin^{m-1}(2-2\cos)^{d-m}\Big(-(d-1)\sqrt{\lambda}\sin + m\cos(1/\sqrt{\lambda}) + (d-m)(1+\cos)/2\Big). \end{aligned}$$

Hence the eigenvalues are the solutions of the equation

$$\begin{aligned} &(\sin(1/\sqrt{\lambda}))^{m-1}(1-\cos(1/\sqrt{\lambda}))^{d-m} \\ &\times \Big((d-1)\sqrt{\lambda}\sin(1/\sqrt{\lambda}) - m\cos(1/\sqrt{\lambda}) - (d-m)(1+\cos(1/\sqrt{\lambda}))/2\Big) = 0. \end{aligned} \tag{3.16}$$

It is remarkable that for $m=0$, the form we obtain is

$$(1-\cos(1/\sqrt{\lambda}))^{d-1}\Big((d-1)\sqrt{\lambda}(1-\cos(1/\sqrt{\lambda})) - d\sin(1/\sqrt{\lambda})/2\Big) = 0,$$

simplifying

$$(1-\cos(1/\sqrt{\lambda}))^{d-1}\sin(1/(2\sqrt{\lambda}))\Big((d-1)\sqrt{\lambda}\sin(1/(2\sqrt{\lambda})) - d\cos(1/(2\sqrt{\lambda}))/2\Big) = 0,$$

which is the same as (3.16) if we omit the confusion of $(\sin(1/\sqrt{\lambda}))^{-1}$. Since we cannot claim an eigenvalue with multiplicity $-1$, we need to state the case $m=0$ separately.

From the equations above, we obtain the distributional identities.

The Laplace transform follows from similar arguments as Proposition 2.1.

Due to the KL expansion for $\mathbb{X}_{m,d}$ (specially, take $m=0$ or $m=d$), the KL expansions for the Brownian bridge, mean centered Brownian motion, the mean centered Brownian bridge, and the KL expansions for $Y$, $Z$, we can complete the proof of Theorems 1.1 and 1.2. □

## 4. Applications

In this section, we study the small deviation estimates of the additive process we discussed.

In the small deviation estimates, we choose to provide a less precise description of several constants involved since they do not play a significant role in applications. We use the notation $K > 0$ to denote the constant involved which may change from line to line.

From formula (3.4) of [25], $\forall a > -1$, there exists a constant $K > 0$, such that as $\varepsilon \to 0$

$$\mathbb{P}\left(\sum_{i\ge 1}(i+a)^{-2}\xi_i^2 \le \varepsilon\pi^2\right) = (1+o(1))K\varepsilon^{-a}\exp\left(-\frac{1}{8\varepsilon}\right). \tag{4.1}$$

We have the small deviation corresponding to Brownian motion

$$\mathbb{P}\left(\int_0^1 W^2(t)dt \le \varepsilon\right) = \mathbb{P}\left(\sum_{i\ge1} \frac{1}{\pi^2(i-1/2)^2}\xi_i^2 \le \varepsilon\right) = (1+o(1))K\varepsilon^{1/2}\exp\left(-\frac{1}{8\varepsilon}\right),$$

the small deviation corresponding to Brownian bridge (mean centered Brownian motion)

$$\mathbb{P}\left(\sum_{i\ge1} \frac{1}{\pi^2 i^2}\xi_i^2 \le \varepsilon\right) = (1+o(1))K\exp\left(-\frac{1}{8\varepsilon}\right),$$

and the small deviation corresponding to the mean centered Brownian bridge

$$\mathbb{P}\left(\sum_{i\ge1} \frac{1}{4\pi^2 i^2}\xi_{1,i}^2 + \sum_{i\ge1} \frac{1}{4\pi^2 i^2}\xi_{2,i}^2 \le \varepsilon\right) = (1+o(1))K\varepsilon^{-1/2}\exp\left(-\frac{1}{8\varepsilon}\right).$$

Now let us first discuss the small deviations of the process $Y_b(t) = W(t) + (b-1)\int_0^1 W(s)ds$ and the process $Z_b(t) = B(t) + (b-1)\int_0^1 B(s)ds$, for $b \in \mathbb{R}$.

**Proposition 4.1.** *For $b \neq 0$, there exists some constant $K > 0$, such that as $\varepsilon \to 0$,*

$$\mathbb{P}\left(\int_0^1 Y_b^2(t)dt \le \varepsilon\right) = (K+o(1))\varepsilon^{1/2}\exp\left(-\frac{1}{8\varepsilon}\right).$$

*For $b = 0$, the process $Y_b(t)$ is in fact the mean centered Brownian motion.*

**Proposition 4.2.** *For $b \neq 0$, there exists some constant $K > 0$, such that as $\varepsilon \to 0$,*

$$\mathbb{P}\left(\int_0^1 Z_b^2(t)dt \le \varepsilon\right) = (K+o(1))\exp\left(-\frac{1}{8\varepsilon}\right).$$

*For $b = 0$, the process $Z_b(t)$ is in fact the mean centered Brownian bridge.*

**Proof.** We only need to show Proposition 4.1. Since the case $b = 0$ is trivial, we assume $b \neq 0$ here. The starting point is Theorem 2 of [25] that we recall here. Given any two sequences $a_i > 0$ and $b_i > 0$ with

$$\sum_{i\ge1} a_i < \infty, \qquad \sum_{i\ge1} b_i < \infty, \qquad \sum_{i\ge1} |1 - a_i/b_i| < \infty,$$

we have, as $\varepsilon \to 0$,

$$\mathbb{P}\left(\sum_{i\ge1} a_i\xi_i^2 \le \varepsilon\right) = (1+o(1))\left(\prod_{i\ge1} b_i/a_i\right)^{1/2}\mathbb{P}\left(\sum_{i\ge1} b_i\xi_i^2 \le \varepsilon\right).$$

For our setting, since $\sin x/x \sim 0$ for large $x$, the solutions $\lambda_i$ of Eq. (2.5) satisfy the approximate relation

$$\lambda_i^{-1/2} \sim (i-1/2)\pi, \tag{4.2}$$

for large $i$.

We set $a_i = \lambda_i$, let $b_i = l_i^{-2} = (i - 1/2)^{-2}\pi^{-2}$. Clearly, $\lambda_i^{-1/2} \to \infty$ as $i \to \infty$. Take $\lambda_i^{-1/2} = l_i + \delta_i$ for large $i$, where $\delta_i \to 0$ as $i \to \infty$, we have the equality

$$(b^2 - 1)\frac{\sin l_i \cos \delta_i + \cos l_i \sin \delta_i}{l_i + \delta_i} - b^2(\cos l_i \cos \delta_i - \sin l_i \sin \delta_i) = 0,$$

from which we have $\delta_i = O(i^{-1})$; hence we have $\sum_{i\geq 1} |1 - a_i/b_i| < \infty$. Then there exists a constant $K$ such that as $\varepsilon \to 0$,

$$\mathbb{P}\left(\sum_{i\geq 1} \lambda_i {\xi_i}^2 \leq \varepsilon\right) = (1 + o(1))K\mathbb{P}\left(\sum_{i\geq 1} b_i \xi_i^2 \leq \varepsilon\right) = (K + o(1))\varepsilon^{1/2} \exp\left(-\frac{1}{8\varepsilon}\right).$$

We now complete the proof. □

We next give the small deviations corresponding to the processes $M_a(t)$ (the centered Gaussian process with covariance function $t \wedge s + a^2$), and $N_a(t)$ (the centered Gaussian process with covariance function $t \wedge s - ts + a^2$) directly without proof.

**Proposition 4.3.** *For $a > 0$, there exists some constant $K > 0$, such that as $\varepsilon \to 0$,*

$$\mathbb{P}\left(\int_0^1 M_a^2(t)dt \leq \varepsilon\right) = (K + o(1)) \exp\left(-\frac{1}{8\varepsilon}\right).$$

*For $a = 0$, the process $M_a(t)$ is in fact the Brownian motion.*

**Proposition 4.4.** *For $a > 0$, there exists some constant $K > 0$, such that as $\varepsilon \to 0$,*

$$\mathbb{P}\left(\int_0^1 N_a^2(t)dt \leq \varepsilon\right) = (K + o(1))\varepsilon^{-1/2} \exp\left(-\frac{1}{8\varepsilon}\right).$$

*For $a = 0$, the process $N_a(t)$ is in fact the Brownian bridge.*

At last, we show the small deviation of the mixed additive process $\mathbb{X}_{m,d}$ defined in (3.1) for $0 \leq m \leq d$.

**Proposition 4.5.** *For $0 \leq m \leq d$, there exists some constant $K > 0$ such that as $\varepsilon \to 0$,*

$$\mathbb{P}\left(\int_{[0,1]^d} \mathbb{X}_{m,d}^2(\mathbf{t})d\mathbf{t} \leq \varepsilon\right) = (K + o(1))\varepsilon^{1-(2d-m)/2} \exp\left(-\frac{d^2}{8\varepsilon}\right).$$

**Proof.** To prove the result, we only need to show the case $m \geq 1$ since the case $m = 0$ follows from similar arguments. The solutions $\lambda_i$ of Eq. (3.3) satisfy the approximate relation

$$\lambda_i^{-1/2} \sim \begin{cases} (\alpha + i - 1)\pi, & i = 2k - 1; \\ (-\alpha + i)\pi, & i = 2k. \end{cases} \tag{4.3}$$

for large $i$, where $1/2 \leq \alpha \leq 1$ is the value such that $(d + m)\cos(\alpha\pi) + d - m = 0$. For $m = d$, we have $\alpha = 1/2$, while for $m = 0$, we have $\alpha = 1$. Following similar arguments we can show that there exists a constant $K$ such that as $\varepsilon \to 0$

$$\begin{aligned}\mathbb{P}\left(\sum_{i\geq 1}\lambda_i {\xi_i}^2 \leq \varepsilon\right) &= (1+o(1))K\mathbb{P}\left(\sum_{i=2k-1}\frac{1}{(\alpha+i-1)^2\pi^2}\xi_i^2\right. \\ &\qquad \left.+\sum_{i=2k}\frac{1}{(-\alpha+i)^2\pi^2}\xi_i^2 \leq \varepsilon\right) \\ &= (1+o(1))K\mathbb{P}\left(\frac{1}{4}\sum_{k\geq 1}\frac{1}{((\alpha-2)/2+k)^2\pi^2}\xi_{2k-1}^2\right. \\ &\qquad \left.+\frac{1}{4}\sum_{k\geq 1}\frac{1}{(-\alpha/2+k)^2\pi^2}\xi_{2k}^2 \leq \varepsilon\right) \\ &= (1+o(1))K\varepsilon^{1/2}\exp\left(-\frac{1}{8\varepsilon}\right), \end{aligned} \tag{4.4}$$

the last equality is ensured by Lemma 4.1 and the small deviation (4.1).

Together with the small deviations corresponding to Brownian bridge (mean centered Brownian motion) and mean centered Brownian bridge, then from Lemma 4.2, we have

$$\begin{aligned}&\mathbb{P}\left(\int_{[0,1]^d}\mathbb{X}_{m,d}^2(\mathbf{t})d\mathbf{t} \leq \varepsilon\right) \\ &= \mathbb{P}\left(\sum_{1\leq j\leq m-1}\sum_{i\geq 1}\frac{1}{\pi^2 i^2}\xi_{j,i}^2 + \sum_{1\leq j\leq 2(d-m)}\sum_{i\geq 1}\frac{1}{4\pi^2 i^2}\tilde{\xi}_{j,i}^2 + \sum_{i\geq 1}\lambda_i \xi_i^{*2} \leq \varepsilon\right) \\ &= (1+o(1))K\varepsilon^a\exp(-S^2\varepsilon^{-1}),\end{aligned} \tag{4.5}$$

where $a = 1/2 - (d-m)/2 - (\{m-1+d-m+1\}-1)/2 = 1-(2d-m)/2$, $S^2 = ((m-1+d-m+1)8^{-1/2})^2 = d^2/8$, from which we can complete the proof. □

The following lemma which we need is due to Lifshits; see [3].

**Lemma 4.1.** *Let $V_1, V_2 > 0$ be two independent random variables with known small deviation behavior. Namely, assume that*

$$\mathbb{P}(V_1 \leq \varepsilon) \sim c_1\varepsilon^{a_1}\exp(-b_1\varepsilon^{-r}),$$

*and*

$$\mathbb{P}(V_2 \leq \varepsilon) \sim c_2\varepsilon^{a_2}\exp(-b_2\varepsilon^{-r}),$$

*as $\varepsilon \to 0$. Then the sum has the small deviation asymptotic behavior:*

$$\mathbb{P}(V_1+V_2 \leq \varepsilon) \sim K\varepsilon^{a_1+a_2-r/2}\exp(-S^{r+1}\varepsilon^{-r}),$$

*where*

$$S = b_1^{1/(r+1)} + b_2^{1/(r+1)} \text{ and } K = c_1c_2\frac{2\pi r}{r+1}S^{r/2-1/2-a_1-a_2}b_1^{(2a_1+1)/2(r+1)}b_2^{(2a_2+1)/2(r+1)}.$$

Using induction by $n$, we have the following result.

**Lemma 4.2.** *Let $V_1, \ldots, V_n > 0$ be $n$ independent random variables with known behavior of small deviations. Namely, assume that*

$$\mathbb{P}(V_i \le \varepsilon) \sim c_i \varepsilon^{a_i} \exp(-b_i \varepsilon^{-r}),$$

*as $\varepsilon \to 0$. Then there exists some constant $K > 0$ such that as $\epsilon \to 0$, the sum has the following small deviation asymptotic behavior:*

$$\mathbb{P}(V_1 + \cdots + V_n \le \varepsilon) \sim K \varepsilon^{a_1 + \cdots + a_n - (n-1)r/2} \exp(-S^{r+1} \varepsilon^{-r}),$$

*where*

$$S = b_1^{1/(r+1)} + \cdots + b_n^{1/(r+1)}.$$

## Acknowledgments

This paper is a joint work with my advisor Wenbo V. Li. It is great sadness that Li passed away. This work is dedicated to and in memory of Wenbo V. Li.

I am grateful to the anonymous referees for their careful reading and helpful comments. I was supported in part by NSFC 10971253 and NSFC 11271220.

## References

[1] X. Ai, W.V. Li, Q. Liu, Karhunen–Loeve expansion for the detrended Brownian motion, Statist. Probab. Lett. 82 (2012) 1235–1241.
[2] M. Barczy, E. Igloi, Karhunen–Loeve expansions of $\alpha$-Wiener bridges, Cent. Eur. J. Math. 9 (2011) 65–84.
[3] L. Beghin, Ya. Nikitin, E. Orsingher, Exact small ball constants for some Gaussian processes under the $L_2$-norm, J. Math. Sci. 128 (1) (2005) 2493–2502.
[4] T. Chan, D.S. Dean, K.M. Jansons, L.C.G. Rogers, On polymer conformations in elongational flows, Commun. Math. Phys 160 (1994) 239–257.
[5] X. Chen, W.V. Li, Small deviation estimates for some additive processes, in: Proc. Conf. High Dimensional Probab. III, Progr. Probab. 55 (2003) 225–238.
[6] X. Chen, W.V. Li, Large and moderate deviations for intersection local times, Probab. Theory Related Fields 128 (2004) 213–254.
[7] T.S. Chiang, Y. Chow, Y.J. Lee, Exact formulas of certain functional integrals on Wiener spaces, Stoch. Stoch. Rep. 50 (1994) 211–223.
[8] R.C. Dalang, T. Mountford, Jordan curves in the level sets of additive Brownian motion, Trans. Amer. Math. Soc. 353 (2001) 3531–3545.
[9] R.C. Dalang, T. Mountford, Eccentric behaviors of the Brownian sheet along lines, Ann. Probab. 30 (1) (2002) 293–322.
[10] R. Dalang, J. Walsh, Geography of the level sets of the Brownian sheet, Probab. Theory Related Fields 96 (1993) 153–176.
[11] R. Dalang, J. Walsh, The structure of a Brownian bubble, Probab. Theory Related Fields 96 (1993) 475–501.
[12] P. Deheuvels, A Karhunen–Loeve expansion for a mean-centered Brownian bridge, Statist. Probab. Lett. 77 (2007) 1190–1200.
[13] P. Deheuvels, G.V. Martynov, A Karhunen–Loeve decomposition of a Gaussian process generated by independent pairs of exponential random variables, J. Funct. Anal. 255 (2008) 2363–2394.
[14] P. Deheuvels, G. Peccati, M. Yor, On quadratic functionals of the Brownian sheet and related processes, Stoch. Process. Appl. 116 (2006) 493–538.
[15] C. Donati-Martin, M. Yor, Fubini's theorem for double Wiener integrals and the variance of the Brownian path, Ann. Inst. Henri Poincaré Probab. Stat. 27 (1991) 181–200.
[16] A. Karol, A. Nazarov, Y. Nikitin, Small ball probabilities for Gaussian random fields and tensor products of compact operators, Trans. Amer. Math. Soc. 360 (3) (2008) 1443–1474.
[17] D. Khoshnevisan, Intersections of Brownian motions, Expo. Math. 21 (2) (2003) 97–114.
[18] D. Khoshnevisan, Z. Shi, Brownian sheet and capacity, Ann. Probab. 27 (1999) 1135–1159.

[19] D. Khoshnevisan, Y. Xiao, Level sets of additive Levy process, Ann. Probab. 30 (2002) 62–100.
[20] D. Khoshnevisan, Y. Xiao, Additive Levy Processes: Capacity and Hausdorff Dimension, in: Proc. of Inter. Conf. on Fractal Geometry and Stochastics III, Progr. Probab. 57 (2004) 151–170.
[21] D. Khoshnevisan, Y. Xiao, Harmonic analysis of additive Levy processes, Probab. Theory Related Fields 145 (2009) 459–515.
[22] D. Khoshnevisan, Y. Xiao, Y. Zhong, Measuring the range of an additive Lévy process, Ann. Probab. 31 (2003) 1097–1141.
[23] D. Khoshnevisan, Y. Xiao, Y. Zhong, Local times of additive Lévy processes, Stochastic Process. Appl. 104 (2003) 193–216.
[24] P. Lévy, Wiener's random function and other Laplacian random functions, in: Second Berkeley Symposium in Probability and Statistics, 1950, pp. 171–186.
[25] W.V. Li, Comparison results for the lower tail of Gaussian seminorms, J. Theoret. Probab. 5 (1992) 1–31.
[26] J.R. Pycke, Sur une identité en loi entre deux fonctionnelles quadratiques du pont Brownien, C. R. Acad. Sci. Paris, Ser. I 340 (2005) 373–376.
[27] Z. Shi, M. Yor, On an identity in law for the variance of the Brownian bridge, Bull. Lond. Math. Soc. 29 (1997) 103–108.
[28] G.R. Shorack, J.A. Wellner, Empirical Processes with Applications to Statistics, Wiley, New York, 1986.
[29] K. Tanaka, Time Series Analysis: Nonstationary and Noninvertible Distribution Theory, in: Wiley Series in Probability and Statistics, 1996.
[30] G.S. Watson, Goodness-of-fit tests on a circle, Biometrika 48 (1961) 109–114.