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Fluid limits of many-server queues with abandonments, general service and continuous patience time distributions

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Abstract

This paper extends the works of Kang and Ramanan (2010) and Kaspi and Ramanan (2011), removing the hypothesis of absolute continuity of the service requirement and patience time distributions. We consider a many-server queueing system in which customers enter service in the order of arrival in a non-idling manner and where reneging is considerate. Similarly to Kang and Ramanan (2010), the dynamics of the system are represented in terms of a process that describes the total number of customers in the system as well as two measure-valued processes that record the age in service of each of the customers being served and the "potential" waiting times. When the number of servers goes to infinity, fluid limit is established for this triple of processes. The convergence is in the sense of probability and the limit is characterized by an integral equation.

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1. Introduction

We study a many server queueing system in which customers are served in a non-idling, First-Come-First-Served manner according to i.i.d. (independent, identically distributed) *service requirement*. Customers can leave the system, without getting service, once they have been waiting in the queue for more than their *patience time*, which are assumed to be also i.i.d. random

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variables. The objective is to obtain fluid approximations or functional strong laws of large numbers of characteristic functionals of the model, when N, the number of servers goes to infinity.

Many server systems were treated first in the seminal paper of Halfin and Whitt [5]. After that, many authors have succeeded relaxing the assumption imposed in [5], on one hand, considering general service requirement distributions and, on the other hand, incorporating the reneging option in their models. Existing work includes Kang and Ramanan [8], Kaspi and Ramanan [10], Mandelbaum and Momcilović [12], Puhalskii and Reed [15], Reed [16], Zhang [19] and many other references therein.

Generalizations of [5] are motivated by statistical analysis made by Brown et al. [3] which suggests that the service requirement distribution does not obey an exponential law. In [10], under the assumption that the distribution of the service requirement is absolutely continuous with respect to the Lebesgue measure, with density g satisfying a mild condition, fluid limits where obtained for a pair of processes $(\bar{X}^{(N)}, \bar{\nu}^{(N)})$. The first one is a nonnegative integer-valued process that represents the scaled total number of customers in system and the second one, a scaled measure-valued process that keeps track of the ages of customers in service.

With the application to call centers in mind, customer abandonment plays an important role and therefore must be considered. Garnett et al. [4] explain how the performance of certain systems are very sensitive to the impact produced by the impatience of customers. In this direction, the work in [10] has been extended in [8], including another measure-valued process $\bar{\eta}^{(N)}$ that keeps track of the "potential" waiting time of customers in the queue and they have obtained fluid limits adding this new process. The cumulative reneging process can be then expressed in terms of the triplet $(\bar{X}^{(N)}, \bar{\nu}^{(N)}, \bar{\eta}^{(N)})$ and the patience time must satisfy the same assumptions on the service requirement, that is, the existence of a density with respect to the Lebesgue measure.

Related to [8] and tracking the residual service and patience times instead of ages and potential waiting times, [19] obtained fluid limits approximations for the model studied in [8]. The approach of [19] avoids using martingales techniques, the fluid equations are different from those of [8], the functionals expressed in terms of the hazard rates are not taken into account. In this way, the fluid limits require weaker assumption on the service time distributions. The assumption for the service time distribution is continuity and for the patience time distribution is Lipschitz continuity.

Our work extends the results of [8,10,19] to the case of completely general service requirement distribution and continuous patience time distribution. We consider the same kind of reneging as in [8], that is, we assume that the queue is invisible to waiting customers, which is very suitable for call centers models. We obtain fluid limits for the triplet of process $(X^{(N)}, \nu^{(N)}, \eta^{(N)})$ of [8]. The fluid equations are very close to those considered in [8]. The difficulty arises in finding substitutes for the terms in the fluid equations of [8] depending on the densities of the distributions. For this purpose, we consider two sequences $(\bar{q}^{(N)})$ and $(\bar{p}^{(N)})$ of L^1-valued process (where L^1 is the set of integrable functions with respect to the Lebesgue measure on $\mathbb{R}_+$) which represent respectively for any time t and number of servers N, the densities of the measure $\int_0^t \bar{\nu}_s^{(N)} \mathrm{d}s$ and $\int_0^t \bar{\eta}_s^{(N)} \mathrm{d}s$ with respect to the Lebesgue measure on $\mathbb{R}_+$. These sequences always exist and the fluid equations are written in terms of their limits $\bar{q}$ and $\bar{p}$.

To take into account the processes $(\bar{q}^{(N)})$, $(\bar{p}^{(N)})$, $\bar{q}$ and $\bar{p}$ is very useful even in the case where the densities of service and patience time distributions exists. On one hand, it facilitates the analysis of convergence of the sequences $(\bar{\nu}^{(N)})$ and $(\bar{\eta}^{(N)})$ and provides convergence in probability of the process and not only weak convergence. On the other hand, it simplifies the proof of the uniqueness of the fluid equations. Indeed following [10], the delicate part is to establish uniqueness of the so-called "age equation" defined in [10], but using the process $\bar{q}$, the problem can be easily translated to an integral equation, which is easier to treat.

Using martingale convergence methods originating from Krichagina and Puhalskii [11], fluid and diffusion limits for the process $\bar{X}^{(N)}$ were introduced in [15,16] in a general framework, that is, without assumptions about the distribution of the service requirement but excluding abandonment possibility. A related work on many-server queues with impatience is that in [12] for both general service distribution and general patience time distribution and where diffusion limit of customer-count processes and virtual waiting time processes are obtained. However, in the approach of [12] a detailed fluid analysis is not required.

We extended the results of [15,16] not only to include abandonment but also taking into consideration the measure valued processes $(\bar{\nu}^{(N)})$ and $(\bar{\eta}^{(N)})$. The importance of adding these sequences is to obtain a Markovian description of the dynamics, as was shown in [8]. Besides, since the fluid limit obtained here contains more information than just the limit of the scaled number of customers in the system, it is possible to obtain fluid limits of other functionals of interest, in an analogous way to [10]. As was pointed in [19], the number of customers in the system does not give much information because the remaining service times and patience times can affect considerably the evolution of the system.

A difference between our work and those of [8,10,15,16] is that we do not use a martingale approach in order to obtain the convergences of the processes. With a suitable property of continuity of the solutions of the fluid equations stated in Section 4, we will use results about convergence of non-decreasing functions, see for example Jacod and Shiryaev [6]. Without the existence of density functions, the compensators introduced in [8,10] are not well defined and therefore a martingale approach is only possible with the techniques of [15,16]. See Pang et al. [13] for a detailed discussion of this martingale method. Since these techniques are somewhat complicated, we opted to take an alternative approach for the convergence of the processes which is simpler according to our initial conditions. We remark that in the way that we prove the convergence of the cumulative reneging process, our initial assumptions must be imposed even if we use the martingales techniques of [15,16].

The paper is organized as follows. The model and the definition of the processes $(\bar{X}^{(N)}, \bar{\nu}^{(N)}, \bar{\eta}^{(N)})$ are introduced in Section 2, together with some additional notation and basic assumptions. In Section 3, we define the scaled version of the primitives introduced in Section 2 and specify the assumptions used throughout the paper. Next, we give the fluid model equations (FME) and finally we state the main result, that is, the convergence of the process $(\bar{X}^{(N)}, \bar{\nu}^{(N)}, \bar{\eta}^{(N)})$ to the unique solution of the FME. In Section 4 we establish uniqueness of solutions to the FME. Following [10], we started proving uniqueness of the so-called "age equation". We prove also a continuity property of the solutions of the FME when the initial assumptions are imposed. At the end of the section, we explain how the method adopted here can be used for the case treated in [8,10], when the distribution of service requirements is absolutely continuous with respect to the Lebesgue measure. In Section 5 we prove the main result. For simplify, we first introduce in Section 5 some results that we will use in the proof of the main result and that will be established in Section 6.

2. Description of the model and initial assumptions

In this paper we consider a general multi-server queue with customer abandonment. We use most of the notation of [8,10]. Consider a sequence of systems index by the number of servers N, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. We will use a superscript (N) to denote all processes and quantities associated with the Nth-system described as follows. Customers arrive according to a general arrival process $E^{(N)} = \{E^{(N)}(t) : t \in \mathbb{R}_+\}$ and enter service immediately if there is an available server, otherwise they wait in the queue. We assume that each $E^{(N)}$ is a non-decreasing, pure jump process with $E^{(N)}(0) = 0$ and $E^{(N)}(t) < \infty$ for all $t \in \mathbb{R}_+$.

Customers are served in the order of their arrival by the first available server (nonidling condition) and they have patience times when they are in the queue. They will abandon the system as soon as the amount of time they have spent in the queue reaches their patience times. Customers do not renege once they have entered service and in this case, they will exit the system when their service is finished.

In each N-system, customers are indexed by $\mathbb{Z}$, with non-positive indexes corresponding to customers present in the system at time 0 and ordered according to their arrival time (prior to time zero). The index $i \in \mathbb{N}$ is for the i-th customer who will enter into the system after time zero. Let $X_0^{(N)}$ be the number of customers in the Nth-system at time 0. At this time the queue $Q_0^{(N)}$ is given by $(X_0^{(N)} - N)^+$. For $i \in \mathbb{Z}$, the variables $\xi_i^{(N)}$ (resp. $\alpha_i^{(N)}$) represent respectively the time in which customer i entered or will enter into the system (resp. started or will start service), if such a customer exists. If customer i abandon the system then $\alpha_i^{(N)} = \infty$. In this way, $\xi_i^{(N)} \le 0$ if and only if $i = -X_0^{(N)} + 1, \ldots, 0$ and $\alpha_i^{(N)} \le 0$ if and only if $i \in \{-X_0^{(N)} + 1, \ldots, -Q_0^{(N)}\}$ because these indices correspond to customers in service at time 0. Note that $\xi_i^{(N)} \le \xi_j^{(N)}$ if $i \le j$ and for $i < j$, $\alpha_i^{(N)} \ge \alpha_j^{(N)}$ if and only if $\alpha_i^{(N)} = \infty$, that is, if customer i abandons the system. For $i \in \mathbb{N}$,

$$\xi_i^{(N)} = \inf\{t \ge 0 : E^{(N)}(t) \ge i\}.$$

The σ-algebra observable at time 0 is denoted by $\widetilde{\mathcal{F}}_0^{(N)}$ and it is generated by

$$\left\{X_0^{(N)}, \{\xi_i^{(N)} : i \in -X_0^{(N)} + 1, \ldots, 0\}, \{\alpha_i^{(N)} : i \in -X_0^{(N)} + 1, \ldots, -Q_0^{(N)}\}\right\}. \tag{2.1}$$

For $i \in \mathbb{N}$, r_i represents the patience time of customer i (the ith customer to enter into the system after time zero). For $i = -Q_0^{(N)} + 1, \ldots, 0$, $r_i^{(N)}$ represents the patience time of customers who are in the queue at time zero. The variables $\{v_i : i \in \mathbb{Z}\}$ are the service requirements of customers that will enter service after time zero, that is, those of indices $i \in \{-Q_0^{(N)} + 1, \ldots, 0\} \cup \mathbb{N}$. For $i = -X_0^{(N)} + 1, \ldots, -Q_0^{(N)}$, $v_i^{(N)}$ represents the service requirement of customers in service at time zero. Note that for $i = -Q_0^{(N)} + 1, \ldots, 0$ and $j = -X_0^{(N)} + 1, \ldots, -Q_0^{(N)}$ we have

$$\xi_i^{(N)} + r_i^{(N)} > 0 \quad \text{and} \quad \alpha_j^{(N)} + v_j^{(N)} > 0. \tag{2.2}$$

Assumption 2.1. The following assumptions on the variables introduced above are imposed throughout without explicit mention:

1. The variables $\{r_i : i \in \mathbb{N}\}$ and $\{v_i : i \in \mathbb{Z}\}$ do not depend on N, that is, they are in common for all the N-systems.
2. $\{r_i : i \in \mathbb{N}\}$ is an i.i.d. sequence with common cumulative distribution function G^{r} continuous on $[0, \infty)$.
3. $\{v_i : i \in \mathbb{Z}\}$ is an i.i.d. sequence with common cumulative distribution function G^{s} such that $G^{\mathsf{s}}(0) < 1$.

4. For any $N \in \mathbb{N}$, $\widetilde{\mathcal{G}}_0^{(N)}$, $E^{(N)}$, $\{r_i : i \in \mathbb{N}\}$ and $\{v_i : i \in \mathbb{Z}\}$ are independent, where $\widetilde{\mathcal{G}}_0^{(N)}$ is the σ-algebra generated by $\widetilde{\mathcal{F}}_0^{(N)}$, $\{r_i^{(N)} : i = \{-Q_0^{(N)}+1, \dots, 0\}\}$ and $\{v_i^{(N)} : i = \{-X_0^{(N)}+1, \dots - Q_0^{(N)}\}\}$.
5. Conditionally on $\widetilde{\mathcal{F}}_0^{(N)}$, the variables $\{r_i^{(N)}, v_j^{(N)} : i = -Q_0^{(N)}+1, \dots, 0;\ j = -X_0^{(N)}+1, \dots, -Q_0^{(N)}\}$ are independent and for any $x \in \mathbb{R}_+$,

$$\mathbb{P}(r_i^{(N)} > x | \widetilde{\mathcal{F}}_0^{(N)}) = \frac{1 - G^{\mathsf{r}}(x \wedge (-\xi_i^{(N)}))}{1 - G^{\mathsf{r}}(-\xi_i^{(N)})} \quad \text{for } i = -Q_0^{(N)}+1, \dots, 0,$$

$$\mathbb{P}(v_i^{(N)} > x | \widetilde{\mathcal{F}}_0^{(N)}) = \frac{1 - G^{\mathsf{s}}(x \wedge (-\alpha_i^{(N)}))}{1 - G^{\mathsf{s}}(-\alpha_i^{(N)})} \quad \text{for } i = X_0^{(N)}+1, \dots, -Q_0^{(N)}.$$

For any positive t, $X^{(N)}(t)$ represents the number of customers in the system at time t, including those in service and those in the queue, waiting to enter service. Let $D^{(N)}(t)$ be the total number of customers who have finished their service in the time interval $[0, t]$ and $R^{(N)}(t)$ the cumulative number of customers who have abandoned the system in the interval $[0, t]$. As a consequence of (2.2) and the fact that $E^{(N)}(0) = 0$, we have

$$D^{(N)}(0) = R^{(N)}(0) = 0, \tag{2.3}$$

in particular $X^{(N)}(0) = X_0^{(N)}$. Evidently, for any $t \geq 0$,

$$X^{(N)}(t) = X^{(N)}(0) + E^{(N)}(t) - D^{(N)}(t) - R^{(N)}(t). \tag{2.4}$$

For $i \in \{-Q_0^{(N)}+1, \dots, 0\} \cup \mathbb{N}$ and $j \in \mathbb{N}$ let $a_i^{(N)}$ and $w_j^{(N)}$ denote respectively the age process of customer i and the potential waiting time process of customer j defined for all $t \in \mathbb{R}_+$ by

$$a_i^{(N)}(t) := v_i \wedge (t - \alpha_i^{(N)})^+ \quad \text{and} \quad w_j^{(N)}(t) := r_j \wedge (t - \xi_j^{(N)})^+. \tag{2.5}$$

For $i \in \{-X_0^{(N)}+1, \dots, -Q_0^{(N)}\}$ and $j \in \{-Q_0^{(N)}+1, \dots, 0\}$ define $a_i^{(N)}$ and $w_i^{(N)}$ be given by the above equations with $v_i^{(N)}$ and $r_j^{(N)}$ replacing v_i and r_j respectively. For $t \in \mathbb{R}_+$ let $\eta_t^{(N)}$ be the discrete Borel measure on $\mathbb{R}_+$ that has a unit mass at the potential waiting time of each customer who has entered the system in the time interval $(0, t]$ or who was in the queue in time zero and whose potential waiting time has not yet reached its patience time. That is

$$\eta_t^{(N)} = \sum_{i=-Q_0^{(N)}+1}^{E^{(N)}(t)} \delta_{w_i^{(N)}(t)} 1_{\{w_i^{(N)}(t) < r_i\}},$$

where for $x \in \mathbb{R}$, δ_x represents the Dirac mass at x. Note that this measure valued process is slightly different from that defined in [8], indeed, at time zero we consider only customers in the queue and not all customers who entered into the system prior time zero. For $t \in \mathbb{R}_+$, $\nu_t^{(N)}$ denotes the discrete Borel measure on $\mathbb{R}_+$ that has a unit mass at the age of each of the customers in service at time t. That is

$$\nu_t^{(N)} = \sum_{i=-X_0^{(N)}+1}^{\infty} \delta_{a_i^{(N)}(t)} 1_{\{a_i^{(N)}(t) < v_i, \alpha_i^{(N)} \leq t\}} = \sum_{i=-X_0^{(N)}+1}^{E^{(N)}(t)} \delta_{a_i^{(N)}(t)} 1_{\{a_i^{(N)}(t) < v_i, \alpha_i^{(N)} \leq t\}},$$

where $v_i = v_i^{(N)}$ if $i < 0$. Hence, $\langle 1, \nu_t^{(N)} \rangle = \nu_t^{(N)}[0, \infty)$ represents the total number of customers in service at time t and the nonidling condition is written as

$$\langle 1, \nu_t^{(N)} \rangle = N - [N - X^{(N)}(t)]^+ \quad \text{for all } t \in \mathbb{R}_+. \tag{2.6}$$

For any $t \in \mathbb{R}_+$, $K^{(N)}(t)$ denotes the cumulative number of customers who have entered service in the interval $[0, t]$. For any $t \in \mathbb{R}_+$, $D^{(N)}(t)$ and $K^{(N)}(t)$ are given by

$$D^{(N)}(t) = \sum_{i=-X_0^{(N)}+1}^{E^{(N)}(t)} 1_{\{\alpha_i^{(N)}+v_i \le t\}} \quad \text{and} \quad K^{(N)}(t) = \sum_{i=-Q_0^{(N)}+1}^{E^{(N)}(t)} 1_{\{\alpha_i^{(N)} \le t\}}, \tag{2.7}$$

where $v_i = v_i^{(N)}$ if $i < 0$. For $t \in \mathbb{R}_+$, let $Q^{(N)}(t)$ be the number of customers waiting in the queue at time t, that is,

$$Q^{(N)}(t) = (X^{(N)}(t) - N)^+.$$

Let $\chi^{(N)}(t)$ be the waiting time of the customer in the head of the line at time t, or be 0 if the queue is empty. Then

$$\chi^{(N)}(t) := \inf\{x > 0 : \eta_t^{(N)}[0, x] \ge Q^{(N)}(t)\} \tag{2.8}$$

and the cumulative reneging process $R^{(N)}(t)$ admits the representation

$$R^{(N)}(t) = \sum_{i=-Q_0^{(N)}+1}^{E^{(N)}(t)} 1_{\{\chi_-^{(N)}(\xi_i^{(N)}+r_i) \ge r_i;\, \xi_i^{(N)}+r_i \le t\}}, \tag{2.9}$$

where we use the notation $\chi_-^{(N)}(t) := \chi^{(N)}(t-)$. The above equation holds if we assume that the customer at the head of the line abandons the system if reaches hit patience when a server comes available. Note however that this happens with probability zero, due to our assumptions of independence and the continuity of G^{r}.

3. Fluid limits

If (S, d) is a Polish space, we denote by $\mathcal{C}_{[0,\infty)}(S)$ (resp. $\mathcal{D}_{[0,\infty)}(S)$) the space of S-valued continuous (resp. right continuous with left limits) functions on $[0, \infty)$. The space $\mathcal{C}_{[0,\infty)}(S)$ is endowed with the J_U-topology of uniform convergence on compact sets of $\mathbb{R}_+$. The space $\mathcal{D}_{[0,\infty)}(S)$ can be endowed with the J_U-topology or with the Skorokhod J_1-topology (see [6], Chapter VI). We will use the notations $\mathcal{D}_{[0,\infty)}(S, J_U)$ and $\mathcal{D}_{[0,\infty)}(S, J_1)$ in order to specify the used topology. For $f \in \mathcal{D}_{[0,\infty)}(\mathbb{R})$ and $t \in \mathbb{R}_+$, we define $\|f\|_t := \sup\{|f(s)| : s \le t\}$. We denote by BV (resp. V^+) the subspace of $\mathcal{D}_{[0,\infty)}(\mathbb{R})$ of bounded variation (resp. non-decreasing) functions on compact sets. The subspace of functions in BV (resp. V^+) started at 0 is denoted by BV_0 (resp. V_0^+). For $f \in V^+$, we define its inverse f^{-1} by,

$$f^{-1}(x) := \inf\{t \ge 0 : f(t) \ge x\}.$$

The space of Radon measures on S equipped with the vague topology is denoted by $\mathcal{M}(S)$. The set of measures μ on $\mathcal{M}(S)$ satisfying $\mu(S) \le 1$ (resp. $\mu(S) < \infty$) and endowed with the weak topology is denoted by $\mathcal{M}_{\le 1}(S)$ (resp. $\mathcal{M}_F(S)$). It is well-known that $\mathcal{M}(S)$, $\mathcal{M}_F(S)$

and $\mathcal{M}_{\leq 1}(S)$ are Polish spaces. For a measure μ and a μ-integrable function f, the integral $\int f \mathrm{d}\mu$ is denoted by $\langle f, \mu \rangle$ or by $\mu(f)$. A function f is said to be μ-continuous if the set of its discontinuities is of μ-measure zero.

The scaled versions of the basic processes introduced in Section 2 are defined as follows:

$$\bar{X}^{(N)} := \frac{X^{(N)}}{N}, \qquad \bar{E}^{(N)} := \frac{E^{(N)}}{N}, \qquad \bar{D}^{(N)} := \frac{D^{(N)}}{N}, \qquad \bar{R}^{(N)} := \frac{R^{(N)}}{N},$$
$$\bar{K}^{(N)} := \frac{K^{(N)}}{N}, \qquad \bar{\nu}^{(N)} := \frac{\nu^{(N)}}{N}, \qquad \bar{\eta}^{(N)} := \frac{\eta^{(N)}}{N}.$$

Define $M^{\mathsf{s}} := (G^{\mathsf{s}})^{-1}(1)(=\inf\{t \geq 0 : G^{\mathsf{s}}(t) = 1\})$ and $M^{\mathsf{r}} = (G^{\mathsf{r}})^{-1}(1)$. Note that $\bar{\nu}^{(N)} \in \mathcal{M}_{\leq 1}([0, M^{\mathsf{s}}))$ and $\bar{\eta}^{(N)} \in \mathcal{M}_F([0, M^{\mathsf{r}}))$.

From now on, we make the following assumptions on the primitives of the scaled processes introduced above.

Assumption 3.1 (*Initial Conditions*)**.** There exists a random element $(\bar{X}(0), \bar{E}, \bar{\nu}_0, \bar{\eta}_0)$ in $\mathbb{R}_+ \times V_0^+ \times \mathcal{M}_{\leq 1}([0, M^{\mathsf{s}})) \times \mathcal{M}_F([0, M^{\mathsf{r}}))$ such that, as $N \to \infty$, the following limits hold for any $\omega \in \Omega$:

1. $\bar{E}^{(N)} \to \bar{E}$ in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$. Moreover $\bar{E}$ is continuous for all $\omega \in \Omega$.
2. $\bar{X}_0^{(N)} \to \bar{X}_0$.
3. $\bar{\nu}_0^{(N)} \to \bar{\nu}_0$ in $\mathcal{M}_{\leq 1}([0, M^{\mathsf{s}}))$. Moreover, if G^{s} is not continuous then $\bar{\nu}_0$ is a diffuse measure for all $\omega \in \Omega$.
4. $\bar{\eta}_0^{(N)} \to \bar{\eta}_0$ in $\mathcal{M}_F([0, M^{\mathsf{r}}))$ and $\bar{\eta}_0$ is a diffuse measure for all $\omega \in \Omega$. (We recall that we assume that G^{r} is continuous.)

We want to prove convergence of the quantities $(\bar{X}^{(N)}, \bar{\nu}^{(N)}, \bar{\eta}^{(N)})$ as $N \to \infty$ in similar way to [8,10]. Assume for the moment that the distribution function G of the service times is absolutely continuous with respect to the Lebesgue measure and denote by g its (weak-) derivative. Set $M := G^{-1}(1)$, it was shown in [8,10] that if $h := g/(1 - G)$ satisfies a mild condition, then $\bar{\nu}$, the law limit of $\bar{\nu}^{(N)}$ satisfies the following decomposition for any $\varphi \in \mathcal{C}_c^\infty([0, M) \times \mathbb{R}_+)$:

$$\begin{aligned}\langle \varphi(., t), \bar{\nu}_t \rangle = \langle \varphi(., 0), \bar{\nu}_0 \rangle &+ \int_0^t \langle \varphi_s(., s) + \varphi_x(., s), \bar{\nu}_s \rangle \mathrm{d}s \\ &- \int_0^t \langle h(.)\varphi(., s), \bar{\nu}_s \rangle \mathrm{d}s + \int_0^t \varphi(0, s)\mathrm{d}Z(s),\end{aligned} \tag{3.1}$$

where Z is an element of V_0^+ (which is the limit of $\bar{K}^{(N)}$). For general distribution function G, the third term on right hand side of the above equation is not well defined, nevertheless, still assuming the existence of g, thanks to Lemma 4.6, for any t, the measure $A \to \int_0^t \bar{\nu}_s(A)\mathrm{d}s$ is absolutely continuous with respect to the Lebesgue measure on $[0, M]$. We denote by q_t its density. In this case, the third term in the right hand side of the above equation can be rewritten as follows,

$$-\int_0^M \left(\int_0^t \varphi(x, s)\mathrm{d}_s q_s(x) \right) (1 - G(x))^{-1} \mathrm{d}G(x).$$

This is the term that we will use for general distribution G. Before introducing the fluid equations we need some definitions. For any $\nu \in \mathcal{D}_{[0,\infty)}(\mathcal{M}[0, M))$ and $t \in \mathbb{R}_+$, we denote by $\int_0^t \nu_s \mathrm{d}s$

the measure given by $A \to \int_0^t \nu_s(A)\mathrm{d}s$. Note that for any $f \in \mathcal{C}_c^\infty[0, M)$, $\langle f, \nu\rangle$ belongs to $\mathcal{D}_{[0,\infty)}(\mathbb{R})$ and then $\int_0^t \langle f, \nu_s\rangle \mathrm{d}s < \infty$ for any $t \in \mathbb{R}_+$. In particular, $\int_0^t \nu_s \mathrm{d}s$ is a Radon measure on $[0, M)$.

Definition 3.2. We denote by $\mathcal{D}^{\mathsf{abs}}_{[0,\infty)}(\mathcal{M}[0, M))$ the set of $\nu \in \mathcal{D}_{[0,\infty)}(\mathcal{M}[0, M))$ such that for any $t \geq 0$, the measure $\int_0^t \nu_s \mathrm{d}s$ is absolutely continuous with respect to the Lebesgue measure on $[0, M]$.

For any $\nu \in \mathcal{D}^{\mathsf{abs}}_{[0,\infty)}(\mathcal{M}[0, M))$ and $t \geq 0$ there exists $\tilde{q}_t \in L^1_{\mathrm{loc}}[0, M]$ such that $\int_0^t \nu_s(A)\mathrm{d}s = \int_A \tilde{q}(t, x)\mathrm{d}x$ for any Borel set $A \subset [0, M]$. Since different versions of $\tilde{q}_t$ can give different integrals with respect to $\mathrm{d}G$, we need to decide what version of $\tilde{q}_t$ to take. We say that q_t is a regular version of $\tilde{q}_t$ if for any $x \in [0, M)$:

$$q_t(x) := \liminf_{n\to\infty} n \int_{x-\frac{1}{n}}^{x} \tilde{q}_t(x)\mathrm{d}x = \liminf_{n\to\infty} n \int_0^t \nu_s((x - n^{-1}, x])\mathrm{d}s. \tag{3.2}$$

It follows from an elementary result of analysis (see e.g. Theorem 9, Chapter 5 of [17]) that the above limit define a version of the density function of $\int_0^t \nu_s \mathrm{d}s$, that is, $q_t(x) = \tilde{q}_t(x)$ for a.e. $x \in [0, M]$.

Remark 3.3. (1) From now on, we suppose that the density of $\int_0^t \nu_s \mathrm{d}s$ for an element ν of $\mathcal{D}^{\mathsf{abs}}_{[0,\infty)}(\mathcal{M}[0, M))$ and $t \geq 0$ is always given by its regular version.
(2) Note that $q := (t, x) \to q_t(x)$ is jointly measurable and non-decreasing on t, in particular, if for any x, t we define $q_t^+(x) := \lim_{s\downarrow t} q_s(x)$ then for any measurable function f, $\int_{\mathbb{R}_+} f(s)\mathrm{d}_s q_s^+(x)$ is defined as a Lebesgue–Stieltjes integral. In what follows, we suppress the $+$ and we use the notation: $\int_{\mathbb{R}_+} f(s)\mathrm{d}_s q_s(x) := \int_{\mathbb{R}_+} f(s)\mathrm{d}_s q_s^+(x)$.

Now we introduce the fluid equations. Define the measures $\mathrm{d}H^{\mathsf{s}}$ on $[0, M^{\mathsf{s}}]$ and $\mathrm{d}H^{\mathsf{r}}$ on $[0, M^{\mathsf{r}}]$ by:

$$\mathrm{d}H^{\mathsf{s}}(x) := 1_{\{x<M^{\mathsf{s}}\}}(1 - G^{\mathsf{s}}(x-))^{-1}\mathrm{d}G^{\mathsf{s}}(x) + 1_{\{G^{\mathsf{s}}(M^{\mathsf{s}}-)<1\}}\delta_{M^{\mathsf{s}}}(\mathrm{d}x),$$
$$\mathrm{d}H^{\mathsf{r}}(x) := 1_{\{x<M^{\mathsf{r}}\}}(1 - G^{\mathsf{r}}(x))^{-1}\mathrm{d}G^{\mathsf{r}}(x).$$

Definition 3.4 (*Fluid Equations*)**.** An element (X, η, ν) of the product space $\mathcal{D}_{[0,\infty)}(\mathbb{R}) \times \mathcal{D}^{\mathsf{abs}}_{[0,\infty)}(\mathcal{M}[0, M^s)) \times \mathcal{D}^{\mathsf{abs}}_{[0,\infty)}(\mathcal{M}[0, M^{\mathsf{r}}))$ is said to solve the fluid equations associated to $(X_0, E, \nu_0, \eta_0) \in \mathbb{R} \times BV \times \mathcal{M}_{\leq 1}[0, M^{\mathsf{s}}) \times \mathcal{M}_F[0, M^{\mathsf{r}})$ if q_t and p_t, the densities of $\int_0^t \nu_s \mathrm{d}s$ and $\int_0^t \eta_s \mathrm{d}s$ satisfy for any $t \in \mathbb{R}_+$, $\ell < M^{\mathsf{r}}$ and $m < M^{\mathsf{s}}$,

$$\int_{[0,m]} q_t(x)\mathrm{d}H^{\mathsf{s}}(x) < \infty, \qquad \int_{[0,\ell]} p_t(x)\mathrm{d}H^{\mathsf{r}}(x) < \infty, \tag{3.3}$$

there exists $K \in BV_0$ such that for all $\varphi \in C_c^\infty([0, M^{\mathsf{s}}) \times \mathbb{R}_+)$,

$$\langle \varphi(., t), \nu_t\rangle = \langle \varphi(., 0), \nu_0\rangle + \int_0^t \langle \varphi_s(., s) + \varphi_x(., s), \nu_s\rangle \mathrm{d}s$$
$$- \int_{[0,M^{\mathsf{s}})} \left(\int_0^t \varphi(x, s)\mathrm{d}_s q_s(x)\right) \mathrm{d}H^{\mathsf{s}}(x) + \int_0^t \varphi(0, s)\mathrm{d}K(s), \tag{3.4}$$

for every $\psi \in C_c^\infty([0, M^{\mathsf{r}}) \times \mathbb{R}_+)$:

$$\begin{aligned}\langle \psi(., t), \eta_t \rangle &= \langle \psi(., 0), \bar{\eta}_0 \rangle + \int_0^t \langle \psi_s(., s) + \psi_x(., s), \eta_s \rangle \mathrm{d}s \\ &\quad - \int_{[0, M^{\mathsf{r}})} \left(\int_0^t \psi(x, s) \mathrm{d}_s p_s(x) \right) \mathrm{d}H^{\mathsf{r}}(x) + \int_0^t \psi(0, s) \mathrm{d}E(s), \end{aligned} \tag{3.5}$$

$$Q(t) = X(t) - \langle 1, \nu_t \rangle, \tag{3.6}$$

$$Q(t) \le \langle 1, \eta_t \rangle, \tag{3.7}$$

$$R(t) = \int_{[0, M^{\mathsf{r}}]} \int_0^t 1_{\{x \le \chi(s)\}} \mathrm{d}_s p_s(x) \mathrm{d}H^{\mathsf{r}}(x), \tag{3.8}$$

where $\chi(t) := (F^{\eta_t})^{-1}(Q(t))$ and $F^{\eta_t}(x) := \eta_t[0, x]$,

$$X(t) = X_0 + E(t) - \int_{[0, M^{\mathsf{s}}]} q_t(x) \mathrm{d}H^{\mathsf{s}}(x) - R(t) \quad \text{and} \tag{3.9}$$

$$\langle 1, \nu_t \rangle = 1 - [1 - X(t)]^+. \tag{3.10}$$

We will prove uniqueness of the solution of the fluid equations in Section 4. In Section 5 we will prove the following theorem which is our main result.

Theorem 3.5. *Suppose Assumption 3.1, then $(\bar{X}^{(N)}, \bar{\nu}^{(N)}, \bar{\eta}^{(N)})$ converges in probability in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U) \times \mathcal{D}_{[0,\infty)}(\mathcal{M}_{\le 1}[0, M^{\mathsf{s}}), J_1) \times \mathcal{D}_{[0,\infty)}(\mathcal{M}_F[0, M^{\mathsf{r}}), J_1)$ as $N \to \infty$, to the unique solution of the fluid equations associated with $(\bar{X}(0), \bar{E}, \bar{\nu}_0, \bar{\eta}_0)$.*

Since a sequence of random variables converges in probability if and only if for any subsequence there is a further subsequence converging almost surely, Theorem 3.5 is also true if we assume that the convergences in Assumption 3.1 are in the sense of $\mathbb{P}$-probability. Using the Skorokhod representation theorem, it can be shown that convergence results in Theorem 3.5 continue to hold if, in Assumption 3.1, the limits are replaced by limits in the sense of weak convergence.

4. Uniqueness of the solution of the fluid equations

Following [10], in order to prove uniqueness of the solutions of the fluid equations, we need Lemma 4.1, which provides a unique solution of the "age equation" (3.4) in terms of $\bar{K}$ and $\bar{\nu}_0$. This result was proved in [10], Theorem 4.1 for Eq. (3.1), when the service times has distribution G admitting a hazard function $h = g/(1 - G)$. It has been a key step in the proofs of the main results obtained in [1,8–10] and therefore it is interesting in its own right. In view of Lemma 4.6, our proof of Lemma 4.1 is also a short way to prove Theorem 4.1 in [10].

Throughout of this section, G denotes a general distribution function on $\mathbb{R}_+$, $M = G^{-1}(1)$ and H is the measure define by $\mathrm{d}H(x) := 1_{\{x<M\}}(1 - G(x-))^{-1}\mathrm{d}G(x) + 1_{\{G(M-)<1\}}\delta_M(\mathrm{d}x)$.

Lemma 4.1. *Let μ be an element of $\mathcal{D}^{\mathsf{abs}}_{[0,\infty)}(\mathcal{M}[0, M))$ and $Z \in BV_0$. For $t \ge 0$, denote by q_t the density of $\int_0^t \mu_s \mathrm{d}s$. Suppose that for any $t > 0$ and $m < M$,*

$$\int_{[0,m]} q_t(x) \mathrm{d}H(x) < \infty. \tag{4.1}$$

Then, μ satisfies the equation

$$\langle \varphi(.,t), \mu_t\rangle = \langle \varphi(.,0), \mu_0\rangle + \int_0^t \langle \varphi_s(.,s) + \varphi_x(.,s), \mu_s\rangle \mathrm{d}s$$
$$- \int_{[0,M)} \left(\int_0^t \varphi(x,s)\mathrm{d}_s q_s(x) \right) \mathrm{d}H(x) + \int_0^t \varphi(0,s)\mathrm{d}Z(s) \tag{4.2}$$

for all $\varphi \in C_c^\infty([0,M) \times \mathbb{R}_+)$ if and only if

$$\langle f, \mu_t\rangle = \int_{[0,M)} f(x+t)\frac{1-G(x+t)}{1-G(x)}\mu_0(\mathrm{d}x) + \int_0^t f(t-s)(1-G(t-s))\mathrm{d}Z(s) \tag{4.3}$$

for all $f \in C_c^\infty[0,M)$. In this case, the density q_t is given by

$$q_t(x) = (1-G(x-))Z(t-x) + \int_{[x-t,x)} \frac{1-G(x-)}{1-G(y)}\mu_0(\mathrm{d}y). \quad x \in \mathbb{R}_+. \tag{4.4}$$

Remark 4.2. If $\mu \in \mathcal{D}^{\text{abs}}_{[0,\infty)}(\mathcal{M}[0,M))$ satisfies (4.2) and μ_0 belongs to $\mathcal{M}_F[0,M)$, then it follows by (4.3) that $\mu_t \in \mathcal{M}_F[0,M)$ for any $t \in \mathbb{R}_+$. Moreover, elementary integration by parts formula from (4.4) leads to

$$\int_{[0,M]} q_t(x)\mathrm{d}H(x) = \int_{0-}^t Z(t-x)\mathrm{d}G(x) + \int_{[0,M)} \frac{G(y+t)-G(y)}{1-G(y)}\mu_0(\mathrm{d}y) \tag{4.5}$$
$$= \int_0^t G(t-x)\mathrm{d}Z(x) + \int_{[0,M)} \frac{G(y+t)-G(y)}{1-G(y)}\mu_0(\mathrm{d}y)$$
$$= Z(t) - \langle 1, \mu_t\rangle + \langle 1, \mu_0\rangle. \tag{4.6}$$

In particular, $\int_{[0,M]} q_t(x)\mathrm{d}H(x) < \infty$ for any $t \in \mathbb{R}_+$.

Lemma 4.3. *Let (X_0, E, ν_0, η_0) be an element of $\in \mathbb{R} \times BV_0 \times \mathcal{M}_{\le 1}[0, M^{\mathsf{s}}) \times \mathcal{M}_F[0, M^{\mathsf{r}})$ and (X, ν, η) be a solution of its associated fluid equations. Let K, Q and R be defined in (3.4), (3.6) and (3.8) respectively. Denote by q_t the density of $\int_0^t \nu_s \mathrm{d}s$ and define D by $D(t) := \int_0^t q_s(x)\mathrm{d}H^{\mathsf{s}}(x)$. If E is continuous and ν_0 is a diffuse measure if G^{s} is not continuous, then all processes D, K, Q, R, ν, η and X are continuous.*

Using Lemmas 4.1 and 4.3 above, we will prove the uniqueness of the solutions of the fluid equations. We shall then prove Lemmas 4.1 and 4.3. Even if we do not assume that the process E is continuous, Theorem 4.4 below it is still true since the arguments used in the proof of Theorem 4.6 of [8] are also valid in our case. Nevertheless the continuity properties given by Lemma 4.3 simplify the computations on one hand and on the other hand, they will be needed in the proof of the main result (Theorem 3.5). We shall therefore prove uniqueness of the fluid equations only when the assumptions of Lemma 4.3 are satisfied.

Theorem 4.4. *Let (X_0, E, ν_0, η_0) be an element of $\in \mathbb{R} \times BV_0 \times \mathcal{M}_{\le 1}[0, M^{\mathsf{s}}) \times \mathcal{M}_F[0, M^{\mathsf{r}})$. Suppose that E is continuous and ν_0 is a diffuse measure if G^{s} is not continuous. Then there exists at most one solution to the fluid equations associated to (X_0, E, ν_0, η_0).*

Proof. The proof follows from the arguments used in Theorem 4.6 of [8]. Suppose that (X^1, ν^1, η^1) and (X^2, ν^2, η^2) are two solutions of the fluid equations. For each $i = 1, 2$, let

Q^i, K^i, R^i, ν^i and η^i be the processes associated with the solution (X^i, ν^i, η^i). Let q_t^i be the density of $\int_0^t \nu_s^i \mathrm{d}s$ and define,

$$D^i(t) := \int_{[0,M^{\mathsf{s}}]} q_t^i(x)\mathrm{d}H^{\mathsf{s}}(x).$$

Let ∂A denote $A^2 - A^1$ for $A = Q, K, D, R, \langle 1, \nu\rangle$.

It follows by Lemma 4.1 that $\eta^1 = \eta^2$. As a consequence of (3.6), (3.9) and (4.6) we have for $i = 1, 2$,

$$\begin{aligned} K^i(t) &= \langle 1, \nu_t^i\rangle - \langle 1, \nu_0^i\rangle + D^i(t) \\ &= \langle 1, \nu_t^i\rangle - X^i(t) - \langle 1, \nu_0^i\rangle + X^i(0) + E(t) - R^i(t) \\ &= Q^i(t) - R^i(t) + X^i(0) + E(t) \end{aligned}$$

and hence,

$$\partial K(t) = -\partial Q(t) - \partial R(t). \tag{4.7}$$

Choose $\delta > 0$ and define $\tau := \inf\{t \geq 0 : \partial K(t) \geq \delta\}$. We will show by contradiction that $\tau = \infty$. Suppose that $\tau < \infty$.

For $s < \tau, \partial K(s) < \delta$ and therefore $\int_0^\tau \partial K(\tau - s)\mathrm{d}G^{\mathsf{s}}(s) < \delta$. (recall that $G^{\mathsf{s}}(0) < 1$). Hence, by (4.5), (4.6) and the continuity of ∂K given by Lemma 4.3, we obtain,

$$\delta = \partial K(\tau) = \partial\langle 1, \nu_\tau\rangle + \partial D(\tau) = \partial\langle 1, \nu_\tau\rangle + \int_0^\tau \partial K(\tau - s)\mathrm{d}G^{\mathsf{s}}(s) < \partial\langle 1, \nu_\tau\rangle + \delta.$$

Then $0 < \partial\langle 1, \nu_\tau\rangle$ and therefore, $\langle 1, \nu_\tau^1\rangle < 1$. It follows by the continuity of $\langle 1, \nu^1\rangle$, given by Lemma 4.3, that $\langle 1, \nu^1\rangle < 1$ in a neighborhood of τ.

Thanks to (3.10) and (3.6), $Q^1(s) = 0$ if $\langle 1, \nu_s^1\rangle < 1$, then $Q^1 = 0$ in a neighborhood of τ. Therefore $r := \sup\{s < \tau : Q^2(s) < Q^1(s)\} \vee 0 < \tau$,

$$\partial Q(r) = 0 \quad \text{and} \quad \partial K(r) < \delta, \tag{4.8}$$

where the first equality is a consequence of the continuity of ∂Q given by Lemma 4.3. Equality (4.7) leads to,

$$\begin{aligned} \partial K(\tau) - \partial K(r) &= -(\partial Q(\tau) - \partial Q(r)) - (\partial R(\tau) - \partial R(r)) \\ &= -Q^2(\tau) + \partial R(r) - \partial R(\tau) \\ &\leq \partial R(r) - \partial R(\tau) \\ &= -\int_{[0,M^{\mathsf{r}}]} \int_r^\tau \partial 1_{\{x \leq \chi(s)\}} \mathrm{d}_s p_s(x)\mathrm{d}H^{\mathsf{r}}(x), \end{aligned}$$

where $\partial 1_{\{x\leq\chi(s)\}} := 1_{\{x\leq(F^{\eta_s})^{-1}(Q^2(s))\}} - 1_{\{x\leq(F^{\eta_s})^{-1}(Q^1(s))\}}$. But for any $s \in (r, \tau]$, $Q^2(s) \geq Q^1(s)$. Hence $0 \leq \partial 1_{\{x\leq\chi(s)\}}$ for any $x \in \mathbb{R}_+$ and then,

$$\partial K(\tau) \leq \partial K(r) < \delta,$$

which contradicts the definition of δ. This shows that $\tau = \infty$ and $\partial K(t) \leq \delta$ for all $t \in \mathbb{R}_+$. Since δ is arbitrary, $\partial K(t) \leq 0$ for all $t \in \mathbb{R}_+$. By symmetry we also have $-\partial K(t) \leq 0$ for all

$t \in \mathbb{R}_+$ and then,

$$\partial K(t) = 0 \quad \text{for all } t \in \mathbb{R}_+.$$

By (4.7), $\partial Q(t)+\partial R(t) = 0$ for all $t \in \mathbb{R}_+$. Suppose that there exists $t \in \mathbb{R}_+$ such that $\partial Q(t) > 0$ and let $s := \sup\{u < t : \partial Q(u) = 0\}$. It follows by the continuity of ∂Q that $\partial Q(s) = 0$ and $\partial Q(u) > 0$ for all $u \in (s,t]$. Hence for any $u \in (s,t]$ and $x \in \mathbb{R}_+$, $\partial 1_{\{x\leq\chi(u)\}} \geq 0$ and $\partial R(s) = -\partial Q(s) = 0$. This shows that $0 = \partial R(t) + \partial Q(t) \geq \partial R(s) + \partial Q(t) = \partial Q(t) > 0$. This contradiction leads to $\partial Q(t) \leq 0$ for all $t \in \mathbb{R}$. By symmetry, $\partial Q(t) \geq 0$ for all $t \in \mathbb{R}_+$. Therefore,

$$\partial Q(t) = 0 \quad \text{and} \quad \partial R(t) = 0 \quad \text{for all } t \in \mathbb{R}_+.$$

Since $K^1 = K^2$, it follows by Lemma 4.1 that $\nu^1 = \nu^2$ and by (4.6), $D^1 = D^2$. Finally, thanks to (3.9), $X^1 = X^2$. □

Proof of Lemma 4.3. In view of the expression for $p_t(x)$ (Eq. (4.4) for $Z = E, G = G^{\mathsf{r}}$ and $\mu_0 = \eta_0$) and the continuity of E and G^{r}, the process R defined by (3.8) is continuous.

For any function f, $\Delta f(t)$ denotes $f(t) - f(t-)$. For Eqs. (4.5) and (4.6), with $Z = K$ and $G = G^{\mathsf{s}}$, we have for all $t \in [0,\infty)$,

$$\Delta K(t) = \varDelta\langle 1, \nu_t\rangle + \Delta D(t) \tag{4.9}$$

$$\begin{aligned}\Delta D(t) &= \varDelta\left(\int_0^t G^{\mathsf{s}}(t-s)\mathrm{d}K(s)\right)\\ &= \sum_{s\leq t} \Delta G^{\mathsf{s}}(t-s)\Delta K(s).\end{aligned} \tag{4.10}$$

The second equality in the above equations is a consequence of the continuity of the second term in the right-hand side of (4.5) (with $\mu_0 = \nu_0$ and $G = G^{\mathsf{s}}$). This is due to the assumption that ν_0 is a diffuse measure if G^{s} is not continuous.

It follows by (3.2) that for any x, $q_t(x) \geq q_s(x)$ if $t \geq s$. Therefore, D is increasing. From (3.9) and the continuity of E and R, $\Delta X(t) = -\Delta D(t) \leq 0$. Thanks to (3.10), $\varDelta\langle 1,\nu_t\rangle = \varDelta(1\wedge X(t)) \leq 0$. Hence (4.9) leads to

$$\Delta K(t) \leq \Delta D(t) \quad \forall t \geq 0. \tag{4.11}$$

From the inequalities $-\varDelta(1\wedge X(t)) = |\varDelta(1\wedge X(t))| \leq |\Delta X(t)| = -\Delta X(t)$ we obtain, $\varDelta\langle 1,\nu_t\rangle = \varDelta(1\wedge X(t)) \geq \Delta X(t) = -\Delta D(t)$ and (4.9) leads to,

$$\Delta K(t) \geq 0 \quad \forall t \geq 0. \tag{4.12}$$

Set $M = 1\wedge M^{\mathsf{s}}$. We will show by recurrence that D and K are continuous in $[0, LM]$ for all $L \in \mathbb{N}$. For $L = 0$ it is evident since D and K are right continuous.

Suppose that D and K are continuous in $[0, LM]$, then from (4.10),

$$\begin{aligned}\sum_{LM<t\leq(L+1)M} \Delta D(t) &= \sum_{LM<t\leq(L+1)M}\left(\sum_{LM<t\leq t}\Delta G^{\mathsf{s}}(t-s)\Delta K(s)\right)\\ &= \sum_{LM<s\leq(L+1)M}\left(\sum_{s<t\leq(L+1)M}\Delta G^{\mathsf{s}}(t-s)\right)\Delta K(s)\end{aligned}$$

$$\leq \sum_{0<t<M} \Delta G^{\mathsf{S}}(t) \sum_{LM<s\leq(L+1)M} \Delta K(s)$$

$$\leq \sum_{0<t<M} \Delta G^{\mathsf{S}}(t) \sum_{LM<s\leq(L+1)M} \Delta D(s),$$

where the last inequality is a consequence of (4.11). But $\sum_{0<t<M} \Delta G^{\mathsf{S}}(t) < 1$, then $\sum_{LM<t\leq(L+1)M} \Delta D(t) = 0$. This shows that D is continuous in $[0,(L+1)M]$ and the same holds for K thanks to (4.11).

The continuity of ν and η is a consequence of Lemma 4.1. Finally, the continuity of X and Q follows by (3.9) and (3.6) respectively. □

The following result will be used in the proof of Lemma 4.1.

Lemma 4.5. *Let μ a measure in $\mathbb{R}_+$ and $T := \inf\{t \in \mathbb{R}_+ : \mu[0,t) = \infty\}$. If u is a Borel function and locally bounded in $[0,T)$ such that*

$$u(t) = -\int_{[0,t)} u(s)\mu(\mathrm{d}s) \quad \forall t < T,$$

then $u \equiv 0$ in $[0,T)$

Proof. We have $|u(t)| \leq \int_{[0,t)} |u(s)|\mu(\mathrm{d}s)$, then the lemma is a consequence of an extended Gronwall inequality (see e.g. [14]). □

Proof of Lemma 4.1. We will prove that (4.2) implies (4.3). The converse follows by elementary computations.

Take $f \in \mathcal{C}_c^\infty[0,M)$ and $T \geq 0$. Define $\varphi(x,t) := \int_0^{T-t} f(x+s)\mathrm{d}s$. Then for $(x,s) \in [0,M) \times [0,T]$ we have $\varphi_x(x,s) + \varphi_s(x,s) = -f(x)$ and then,

$$\int_0^T \langle \varphi_s(.,s) + \varphi_x(.,s), \mu_s\rangle \mathrm{d}s = -\int_0^\infty f(x)q(x,T)\mathrm{d}x, \tag{4.13}$$

where we used the notation $q(x,t) = q_t(x)$, $(x,t) \in [0,M) \times \mathbb{R}_+$.

Besides, for any $x \in [0,M)$, $t \in \mathbb{R}_+$ set $q_t^+(x) = q^+(x,t) := \lim_{s\downarrow t} q(s,x)$. Then for any $x \in \mathbb{R}_+$ we have (see Remark 3.3)

$$\begin{aligned}\int_0^T \varphi(x,s)\mathrm{d}_s q_s(x) &= -\varphi(x,0)q_0^+(x) - \int_0^T \varphi_s(x,s)q_s^+(x)\mathrm{d}s \\ &= -q_0^+(x)\int_x^{x+T} f(s)\mathrm{d}s + \int_x^{x+T} q^+(x,x+T-s)f(s)\mathrm{d}s\end{aligned}$$

and integrating the above equation with respect to $H(\mathrm{d}x)$ on $[0,M]$ we get

$$\begin{aligned}&\int_{[0,M]} \left(\int_0^T \varphi(x,s)\mathrm{d}_s q_s(x)\right)\mathrm{d}H(x) \\ &\quad = -\int_0^\infty \int_{[s-T,s)} [q^+(x,0) - q^+(x,T+x-s)]H(\mathrm{d}x)f(s)\mathrm{d}s.\end{aligned} \tag{4.14}$$

Elementary computations lead to

$$\int_0^T \varphi(0,s)\mathrm{d}Z(s) = \int_0^\infty Z(T-s)f(s)\mathrm{d}s,$$

$$\langle\varphi(.,0),\mu_0\rangle = \int_0^\infty \mu_0([s-T,s))f(s)\mathrm{d}s \quad \text{and} \\ \langle\varphi(.,T),\mu_T\rangle = 0. \tag{4.15}$$

Therefore, by (4.2) and (4.13)–(4.15) we have that for a.e. $x \in [0, M)$

$$q(x,T) = \mu_0([x-T,x)) + Z(T-x) + \int_{[x-T,x)} [q^+(y,0) - q^+(y,T-x+y)]\mathrm{d}H(y).$$

Note that the right hand side of the above equation is left continuous in x and hence, equals to the left hand side of (3.2) with μ instead of ν, that is, is a regular version of the density of $\int_0^T \mu_s \mathrm{d}s$. In particular, the above equation holds for any $(x,T) \in [0,M) \times \mathbb{R}_+$. Then q is left continuous in x, right continuous in t, $q_0(x) = 0$ for any x and for any $(x,t) \in [0,M) \times \mathbb{R}_+$

$$q(x,t) = \mu_0([x-t,x)) + Z(t-x) - \int_{[x-t,x)} q(y,t-x+y)\mathrm{d}H(y). \tag{4.16}$$

Now we will show that the above equation admits a unique solution left continuous in x and right continuous in t given by (4.4).

If $r(x,t)$ is left continuous in x, right continuous in t and solves (4.16), then $p(x,t) := q(x,t) - r(x,t)$ satisfies $p(x,t) = -\int_{[x-t,x)} p(y,t-x+y)\mathrm{d}H(y)$. Fix $t \in \mathbb{R}_+$ and define $\ell_t(x) := p(x,x+t)$. Then $\ell_t(x) = -\int_{[0,x)} p(y,t+y)\mathrm{d}H(y) = -\int_{[0,x)} \ell_t(y)\mathrm{d}H(y)$, then by Lemma 4.5, $\ell \equiv 0$ on $[0,M)$. This shows that $p \equiv 0$ on $\{(x,t) \in [0,M) \times \mathbb{R}_+ : x \le t\}$. For $x \in [0,M)$ define $u_x(t) := p(t+x,t)$, $t \in [0, M-x)$ and the measure $\mathrm{d}H_x$ by $\mathrm{d}H_x(A) := \mathrm{d}H(A+x)$ where $A + x := \{x+z : z \in A\}$ for a Borel set $A \subset [0, M-x)$. Then $u_x(t) = -\int_{[x,t+x)} p(y,y-x)\mathrm{d}H(y) = -\int_{[0,t)} p(y+x,y)\mathrm{d}H_x(y) = -\int_{[0,t)} u_x(y)\mathrm{d}H_x(y)$. Then, thanks to Lemma 4.5, $u_x \equiv 0$ on $[0, M-x)$. This shows that $p \equiv 0$ on $\{(x,t) \in [0,M) \times \mathbb{R}_+ : x > t\}$ and therefore $r \equiv q$ on $[0,M) \times \mathbb{R}_+$.

We have then shown that if μ satisfies (4.2) then the density of $\int_0^t \mu_s \mathrm{d}s$ is q_t. If we denote by π the process in $\mathcal{D}^{\mathrm{abs}}_{[0,\infty)}(\mathcal{M}[0,M))$ defined by (4.3), elementary computations show that the density of $\int_0^t \pi_s \mathrm{d}s$ is also q_t for any $t \ge 0$. Therefore $\int_0^t \pi_s \mathrm{d}s = \int_0^t \mu_s \mathrm{d}s$ for any $t \ge 0$ and by right-continuity, $\mu_t = \pi_t$ for any $t \ge 0$. $\square$

Lemma 4.6. *Suppose that the hazard function $h := g/(1-G)$ exists. If μ is an element of $\mathcal{D}_{[0,\infty)}(\mathcal{M}[0,M))$ that satisfies (3.1) for $Z \in BV_0$ and $\int_0^t \langle 1_{[0,m]}h, \mu_s\rangle \mathrm{d}s < \infty$ for any $m < M$, $t \in \mathbb{R}_+$, then the measure $\int_0^t \mu_s \mathrm{d}s$ is absolutely continuous with respect to the Lebesgue measure and its density q_t satisfies ess sup$\{q_t(x) : x \le m\} < \infty$ for any $m < M$. In particular, q_t satisfies (4.1) for any $t \in \mathbb{R}_+$ and μ satisfies (4.2).*

Proof. Evidently (3.1) can be extended to functions in $\mathcal{C}^1_c([0,M) \times \mathbb{R}_+)$. Fix $m < M$, for any $0 \le a < b < m$ and $\varepsilon > 0$ such that $b + \varepsilon < m$ define $\ell_\varepsilon(x) := 1 - \varepsilon^{-1}(\varepsilon \wedge d(x,[a,b]))$ where $d(x,[a,b])$ denotes the distance between x and $[a,b]$. Let r be a function in $C_c^\infty[0,M)$ such that $\|r\|_\infty = 1$ and $r(x) = 1$ for $x \in [0,m]$. Now define $\hat{f}_\varepsilon(x) = \int_0^x \ell_\varepsilon(z)\mathrm{d}z$ and $f_\varepsilon(x) = r(x)\hat{f}_\varepsilon(x)$. Note that $\ell_\varepsilon(x) = 0$ for $x > m$, then $f'_\varepsilon(x) = \ell_\varepsilon(x) + r'(x)\hat{f}_\varepsilon(x)$. We apply (3.1) to f_ε and obtain

$$\int_0^t \langle \ell_\varepsilon, \mu_s\rangle \mathrm{d}s = -\int_0^t \langle r'\hat{f}_\varepsilon, \mu_s\rangle \mathrm{d}s + \int_0^t \langle f_\varepsilon h, \mu_s\rangle \mathrm{d}s + \langle f_\varepsilon, \mu_t\rangle - \langle f_\varepsilon, \mu_0\rangle - f_\varepsilon(0)Z(t).$$

Since $\langle f_\varepsilon, \mu_0\rangle \geq 0$ and $f_\varepsilon(0) = 0$,

$$\int_0^t \langle \ell_\varepsilon, \mu_s\rangle \mathrm{d}s \leq \int_0^t \langle \hat{f}_\varepsilon |r'|, \mu_s\rangle \mathrm{d}s + \int_0^t \langle f_\varepsilon h, \mu_s\rangle \mathrm{d}s + \langle f_\varepsilon, \mu_t\rangle.$$

Note that $\hat{f}_\varepsilon(x) \leq b - a + 2\varepsilon$. If we define,

$$c_m := \int_0^t \langle |r'|, \mu_s\rangle \mathrm{d}s + \int_0^t \langle rh, \mu_s\rangle \mathrm{d}s + \langle r, \mu_t\rangle, \tag{4.17}$$

then $\int_0^t \langle \ell_\varepsilon, \mu_s\rangle \mathrm{d}s \leq c_m(2\varepsilon + b - a)$. The second and third terms on the right-hand side of (4.17) are finite by the hypothesis about μ and h. Since $r' \in \mathcal{C}_c^\infty([0, M))$, the function $t \to \langle |r'|, \mu_t\rangle$ belongs to $\mathcal{D}_{[0,\infty)}(\mathbb{R})$, then, the first term on the right-hand side of (4.17) is finite and therefore $c_m < \infty$. For any $s \in \mathbb{R}_+$, $\mu_s([a, b]) \leq \langle \ell_\varepsilon, \mu_s\rangle$ and when $\varepsilon \to 0$ we obtain

$$\int_0^t \mu_s([a, b])\mathrm{d}s \leq c_m(b - a) \quad \text{for all } 0 \leq a < b < m < M.$$

This shows that the measure $\int_0^t \mu_s \mathrm{d}s$ is absolutely continuous with respect to the Lebesgue measure and its density q_t satisfies $\operatorname{ess\,sup}\{q_t(x) : x \leq m\} \leq c_m$ for any $m < M$. □

5. Proof of the main theorem

The fluid equations admit at most one solution $(\bar{X}, \bar{\nu}, \bar{\eta})$. In order to establish Theorem 3.5 it suffices to prove tightness of the sequence $(\bar{X}^{(N)}, \bar{\nu}^{(N)}, \bar{\eta}^{(N)})$ and that the limit of any convergent subsequence satisfies the fluid equations. The approach of [8,10] cannot be used without the assumption of existence of densities of G^{s} and G^{r}. Besides, the convergence results of $(\bar{X}^{(N)})$ shown in [15,16] do not hold if we include reneging. Despite it seems possible to adapt many of the results of [15,16] to the method of [8], martingale techniques are complicated even if we do not consider reneging and then, we prefer to avoid them.

We will prove tightnesses of $(\bar{\nu}^{(N)})$ and $(\bar{\eta}^{(N)})$ in similar way as [8], that is, using a criterion due to Jakubowski [7]. We will not prove tightness of $(\bar{X}^{(N)})$. Since the terms on the right-hand side of (2.4) are increasing and uniformly bounded when t is fixed, we know that $(\bar{X}^{(N)})$ is relatively compact with respect to the topology of the pointwise convergence, this thanks to Helly's Theorem (Theorem 5.5). We will prove convergence of $(\bar{X}^{(N)}(t))$ to $\bar{X}(t)$ (along any convergent subsequence) for any t in a dense set of $\mathbb{R}_+$. The uniform convergence of $(\bar{X}^{(N)})$ to $\bar{X}$ will be a consequence of the continuity established in Lemma 4.3 and a result of convergence of increasing processes. (Theorem 5.5.)

To show convergence of $(\bar{\nu}^{(N)})$ to $\bar{\nu}$ is difficult even if we assume that G^{s} admits a density g. In fact, if we consider the integration by parts formula (3.1) and its analogue for $\bar{\nu}^{(N)}$ given in [10], the difficulty arises when we try to prove that if $(\bar{\nu}^{(N)})$ converges to ν then $\left(\int_0^t \langle hf, \bar{\nu}_s^{(N)}\rangle \mathrm{d}s\right)$ converges to $\int_0^t \langle hf, \nu_s\rangle \mathrm{d}s$ for any continuous function f and $h := g/(1 - G^{\mathsf{s}})$. This is evident only when h is continuous and it is proved in [10] with some additional assumptions of h, as for example, semi-continuity. For a general distribution G^{s} it will be necessary to show convergence of $\left(\int_0^t \bar{\nu}_s^{(N)} \mathrm{d}s\right)$ not only weakly but also in total variation. To this end, we will prove that the density $\bar{q}_t^{(N)}$ of $\int_0^t \bar{\nu}_s^{(N)} \mathrm{d}s$ with respect to the Lebesgue measure converges to $\bar{q}_t$, the density of $\int_0^t \bar{\nu}_s \mathrm{d}s$.

We will consider $(\bar{q}^{(N)})$ as a sequence of $\mathcal{C}_{[0,\infty)}(L^1(\mathbb{R}_+))$, where $L^1(\mathbb{R}_+)$ denotes the set of integrable functions with respect to the Lebesgue measure in $\mathbb{R}_+$. We will prove tightness of $(\bar{q}^{(N)})_{N\in\mathbb{N}}$ and that any convergent subsequence converges to $\bar{q}$. In an analogous way, we will obtain convergence of $\bar{\eta}^{(N)}$ to $\bar{\eta}$.

Set $\mathfrak{R}_t(x) = (1-G^{\mathsf{r}}(x))^{-1}\int_0^t 1_{\{x\le\chi(s)\}}\mathrm{d}_s p_s(x)$, where χ was defined in (3.8). Thus $R(t) = \int \mathfrak{R}_t(x)\mathrm{d}G^{\mathsf{r}}(x)$. By the end of the paper we will prove that for fixed t, $\bar{R}^{(N)}(t) \sim \int \mathfrak{R}_t^N(x)\mathrm{d}G^{\mathsf{r}}(x)$ for some function $\mathfrak{R}_t^N$ and that the sequence $(\mathfrak{R}_t^N)$ converges to $\mathfrak{R}_t$ outside of a countable set (the continuity points of $\mathfrak{R}_t$). Since we assume that G^{r} is continuous, the previous convergence holds G^{r}-a.e. and then, we obtain convergence of $\bar{R}^{(N)}$ to $\bar{R}$.

Before proving Theorem 3.5, we first introduce some additional notation and useful results that will be proved in Section 6.

5.1. Preliminary results

Straightforward computations show that the measure $\int_0^t \bar{\nu}_s^{(N)}\mathrm{d}s$ has a density with respect to the Lebesgue measure $\bar{q}_t^{(N)}$ given by

$$\bar{q}_t^{(N)}(x) := \frac{1}{N}\sum_{i=-Q_0^{(N)}+1}^{E^{(N)}(t-x)} 1_{\{v_i\ge x;\alpha_i^{(N)}\le t-x\}} + \frac{1}{N}\sum_{i=-X_0^{(N)}+1}^{-Q_0^{(N)}} 1_{\{x-t\le a_i^{(N)}(0)<x\le v_i^{(N)}\}}. \tag{5.1}$$

Lemma 5.1. *For any $x, t\in\mathbb{R}_+$ define*

$$\widehat{q}_t^{(N)}(x) := (1-G^{\mathsf{r}}(x-))\bar{K}^{(N)}(t-x) + \int_{[x-t,x)} \frac{(1-G^{\mathsf{s}}(x-))}{1-G^{\mathsf{s}}(y)}\bar{\nu}_0^{(N)}(\mathrm{d}y). \tag{5.2}$$

We have that $\bar{q}^{(N)} - \widehat{q}^{(N)} \in \mathcal{C}_{[0,\infty)}(L^1(\mathbb{R}_+))$ and if (N_k) is a sequence such that

$$\sum_{k=1}^{\infty}\frac{1}{N_k} < \infty, \tag{5.3}$$

then $(\bar{q}^{(N_k)} - \widehat{q}^{(N_k)})$ converges $\mathbb{P}$-a.e. to zero in $\mathcal{C}_{[0,\infty)}(L^1(\mathbb{R}_+))$.

For $K\in V_0^+$ define ν^K be the element of $\mathcal{D}^{\mathsf{abs}}_{[0,\infty)}(\mathcal{M}_F([0,M^{\mathsf{s}})))$ satisfying the Eq. (3.4), that is, by Lemma 4.1, for $f\in\mathcal{B}^+(\mathbb{R})$,

$$\begin{aligned}\langle f,\nu_t^K\rangle &:= \int_0^t f(t-s)[1-G^{\mathsf{s}}(t-s)]\mathrm{d}K(s)\\ &\quad + \int_{[0,M^{\mathsf{s}})} f(x+t)\frac{1-G^{\mathsf{s}}(x+t)}{1-G^{\mathsf{s}}(x)}\bar{\nu}_0(\mathrm{d}x).\end{aligned} \tag{5.4}$$

Let $\bar{\eta}$ be the element of $\mathcal{D}^{\mathsf{abs}}_{[0,\infty)}(\mathcal{M}_F([0,M^{\mathsf{r}})))$ satisfying the Eq. (3.5), then for $f\in\mathcal{B}^+(\mathbb{R})$,

$$\begin{aligned}\langle f,\bar{\eta}_t\rangle &:= \int_0^t f(t-s)(1-G^{\mathsf{r}}(t-s))\mathrm{d}\bar{E}(s)\\ &\quad + \int_{[0,M^{\mathsf{r}})} f(x+t)\frac{1-G^{\mathsf{r}}(x+t)}{1-G^{\mathsf{r}}(x)}\bar{\eta}_0(\mathrm{d}x).\end{aligned} \tag{5.5}$$

Proposition 5.2. *Let (N_k) be a sequence such that (5.3) holds. There exists $\tilde{\Omega}$, a subset of Ω of $\mathbb{P}$-probability one, satisfying for any $\omega \in \Omega$:*

1. *If $(\bar{K}^{(N_k)}(\omega), \bar{D}^{(N_k)}(\omega))$ converges to $(K(\omega), D(\omega))$ in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U) \times \mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$ for some $D, K \in V_0^+$, then $\bar{\nu}^{(N_k)}(\omega)$ converges to $\nu^K(\omega)$ in $\mathcal{D}_{[0,\infty)}(\mathcal{M}_{\leq 1}([0, M)), J_1)$.*
2. *$\bar{\eta}^{(N_k)}(\omega)$ converges to $\bar{\eta}(\omega)$ in $\mathcal{D}_{[0,\infty)}(\mathcal{M}_F([0, M)), J_1)$.*

We recall that $D^{(N)}$ and $K^{(N)}$ are given by (2.7) and that $\bar{D}^{(N)} = N^{-1}D^{(N)}$, $\bar{K}^{(N)} = N^{-1}K^{(N)}$. Due to the independence between the variables $\{r_i : i \in \mathbb{N}\}$ and the rest of variables in the model, we can follow the martingale arguments developed in [15], for example, and to show that $\bar{D}^{(N)} - \widehat{D}^{(N)}$ converges to zero in probability in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$, where

$$\begin{aligned}\widehat{D}^{(N)}(t) &:= \int_0^t G^{\mathsf{s}}(t-s)\mathrm{d}\bar{K}^{(N)}(s) + \int_{\mathbb{R}_+} \frac{G^{\mathsf{s}}(x+t) - G^{\mathsf{s}}(x)}{1 - G^{\mathsf{s}}(x)} \bar{\nu}_0^{(N)}(\mathrm{d}x) \\ &= \int_{0-}^t \bar{K}^{(N)}(t-s)\mathrm{d}G^{\mathsf{s}}(s) + \int_{\mathbb{R}_+} \frac{G^{\mathsf{s}}(x+t) - G^{\mathsf{s}}(x)}{1 - G^{\mathsf{s}}(x)} \bar{\nu}_0^{(N)}(\mathrm{d}x). \end{aligned} \tag{5.6}$$

Nevertheless, we will only prove (in Section 6) the following partial result that will be sufficient for our purposes. In the sequel, the abbreviation a.e. stands for "almost everywhere with respect to the Lebesgue measure".

Lemma 5.3. *Let (N_k) be a sequence satisfying (5.3). Then for $\mathbb{P}$-a.s. $\omega \in \Omega$,*

$$(\bar{D}^{(N_k)}(\omega, t) - \widehat{D}^{(N_k)}(\omega, t)) \to 0 \quad \text{as } k \to \infty \text{ for a.e. } t \in \mathbb{R}_+.$$

Recall that $\bar{R}^{(N)} = N^{-1}R^{(N)}$, where $R^{(N)}$ is given by (2.9). It follows by Lemma 4.1 that the measure $\int_0^t \bar{\eta}_s \mathrm{d}s$ have a density with respect to the Lebesgue measure $\bar{p}_t$ given by

$$\bar{p}_t(x) := (1 - G^{\mathsf{r}}(x))\bar{E}(t-x) + \int_{[x-t,x)} \frac{(1 - G^{\mathsf{r}}(x))}{1 - G^{\mathsf{r}}(y)} \bar{\eta}_0(\mathrm{d}y). \tag{5.7}$$

Proposition 5.4. *Let (N_k) be a sequence satisfying (5.3). There exists $\tilde{\Omega}$, a subset of Ω of $\mathbb{P}$-probability one, satisfying for any $\omega \in \Omega$:*

If $\bar{Q}^{(N_k)}(t, \omega)$ converges (along a subsequence) to $\bar{Q}(t, \omega)$ for a.e. $t \in \mathbb{R}_+$, for some $\bar{Q}(., \omega) \in \mathcal{D}_{[0,\infty)}(\mathbb{R})$, then $\bar{R}^{(N_k)}(\omega)$ converges (along such subsequence) to $\bar{R}(\omega)$ in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$, where $\bar{R}$ is defined by

$$\bar{R}(t) := \int_{[0,M^{\mathsf{r}}]} \int_0^t 1_{\{x \leq \chi(s)\}} \mathrm{d}_s \bar{p}_s(x) \mathrm{d}H^{\mathsf{r}}(x) \tag{5.8}$$

and $\chi(s) := (F^{\bar{\eta}_s})^{-1}(\bar{Q}(s))$.

Finally, in the proof of Theorem 3.5 and the proof of above results, we will use repeatedly Helly's selection Theorem (see [2]) and Theorem VI.2.15 of [6]. We recall them for reader's convenience. We use the French abbreviation càdlàg (resp. càglàd) for "right continuous with left limits" (resp. for "left continuous with right limits").

Theorem 5.5. (i) *Let* $(f^n)_{n\in\mathbb{N}}$ *be a sequence of functions such that, for each* n, f^n *is non-decreasing or non-increasing and càdlàg or càglàd. Moreover assume that* $\sup_{n\in\mathbb{N}} |f^n(x)| < \infty$ *for all* $x \in \mathbb{R}_+$. *Then for any sequence* $(n_k) \subset \mathbb{N}$, *there exists a subsequence* $(m_k) \subset (n_k)$ *and a càdlàg function* f *such that* $f^{m_k}(x) \to f(x)$ *for any continuity point of* f.

(ii) *Let* $(f^n)_{n\in\mathbb{N}}$ *be a sequence of functions in* V_0^+, *moreover, assume that there exists a continuous function* $f \in V_0^+$ *such that* $f^n(x) \to f(x)$ *for* x *in a dense subset of* $\mathbb{R}$. *Then* f^n *converges to* f *in* $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$ *and for any* $t \in \mathbb{R}_+$,

$$\sup_{s\leq t} |\Delta f^n(s)| \to 0 \quad \textit{as } n \to \infty.$$

Remark 5.6. The result in [2] is established for non-decreasing càdlàg functions. But from its proof, we can see that it is also true for sequences of non-decreasing càglàd functions and for sequences of non-increasing functions.

5.2. Proof of Theorem 3.5

Proof of Theorem 3.5. Denote by $(\bar{X}, \bar{\nu}, \bar{\eta})$ the unique solution of the fluid equations associated to $(\bar{X}_0, \bar{E}, \bar{\nu}_0, \bar{\eta}_0)$. To show convergence of $(\bar{X}^{(N)}, \bar{\nu}^{(N)}, \bar{\eta}^{(N)})$ to $(\bar{X}, \bar{\nu}, \bar{\eta})$ in probability is equivalent to show convergence $\mathbb{P}$-a.s. along any subsequence satisfying (5.3) or equivalently, to show convergence $\mathbb{P}$-a.s. along any subsequence (N_k) satisfying the conclusions of Propositions 5.2, 5.4 and Lemmas 5.1, 5.3. (This last claim is a consequence of these same results.) In order to simplify the notation, we assume without loss of generality that $(N_k) = \mathbb{N}$, that means, that there exists $\tilde{\Omega} \subset \Omega$ such that $\mathbb{P}(\tilde{\Omega}) = 1$ and for any $\omega \in \tilde{\Omega}$ we have,

$$\bar{\eta}^{(N)}(\omega) \to \bar{\eta}(\omega) \quad \text{in } \mathcal{D}_{[0,\infty)}(\mathcal{M}_F([0, M^{\mathsf{r}})), J_1),$$

if there exist $K(\omega), D(\omega) \in V_0^+$ such that $\bar{K}^{(N)}(\omega) \to K(\omega)$ and $\bar{D}^{(N)}(\omega) \to D(\omega)$ in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$ then

$$\bar{\nu}^{(N)}(\omega) \to \nu^K(\omega) \quad \text{in } \mathcal{D}_{[0,\infty)}(\mathcal{M}_{\leq 1}([0, M^{\mathsf{s}})), J_1), \tag{5.9}$$

where ν^K was defined in (5.4),

$$(\bar{D}^{(N)}(\omega, t) - \widehat{D}^{(N)}(\omega, t)) \to 0 \quad \text{as } N \to \infty \text{ for a.e. } t \in \mathbb{R}_+, \tag{5.10}$$

where $\widehat{D}^{(N)}$ was defined in (5.6),

$$(\widehat{q}^{(N)} - \bar{q}^{(N)}) \to 0 \quad \text{as } N \to \infty \text{ in } \mathcal{C}_{[0,\infty)}(L^1(\mathbb{R}_+)), \tag{5.11}$$

where $\bar{q}^{(N)}$ and $\widehat{q}^{(N)}$ are given by (5.1) and (5.2) respectively and finally, if $\bar{Q}^{(N)}(\omega, t)$ converges to $\bar{Q}(\omega, t)$ for a.e. $t \in \mathbb{R}_+$, for some $\bar{Q} \in \mathcal{D}_{[0,\infty)}(\mathbb{R})$, then

$$\bar{R}^{(N)}(\omega) \to \bar{R}(\omega) \quad \text{in } \mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U), \tag{5.12}$$

where $\bar{R}$ is defined by (5.8). From now on we fix $\omega \in \Omega$ satisfying all above convergences. All variables will depend on ω, but we suppress it in our notation. It is enough to show that for any sequence (N_j), there exists a further subsequence (N_{j_ℓ}) such that

$$(\bar{X}^{(N_{j_\ell})}, \bar{\nu}^{(N_{j_\ell})}) \to (\bar{X}, \bar{\nu}) \quad \text{in } \mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U) \times \mathcal{D}_{[0,\infty)}(\mathcal{M}_{\leq 1}([0, M^{\mathsf{s}})), J_1). \tag{5.13}$$

Given a sequence (N_j), by Theorem 5.5(i), there exists a further subsequence (N_{j_ℓ}) and $\bar{D}, \bar{K}, R^* \in V^+$ such that $(\bar{D}^{(N)}(t), \bar{K}^{(N)}(t), \bar{R}^{(N)}(t)) \to (\bar{D}(t), \bar{K}(t), R^*(t))$ (along N_{j_ℓ}) for any t outside of a countable set. We will show that (N_{j_ℓ}) satisfies (5.13). In order to simplify the notation, we assume again without loss of generality that $(N_{k_j}) = \mathbb{N}$, that is,

$$(\bar{D}^{(N)}(t), \bar{K}^{(N)}(t), \bar{R}^{(N)}(t)) \to (\bar{D}(t), \bar{K}(t), R^*(t)) \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{5.14}$$

Thanks to (2.4), $\bar{X}^{(N)}(t) \to X^*(t) := \bar{X}(0) + \bar{E}(t) - \bar{D}(t) - \bar{R}(t)$ for a.e. $t \in \mathbb{R}_+$.

Set $\bar{\nu}^* := \nu^{\bar{K}}$. We will show that $(\bar{X}^*, \bar{\nu}^*, \bar{\eta})$ satisfies the fluid equations for $(\bar{X}(0), \bar{E}, \bar{\nu}_0, \bar{\eta}_0)$, that is, $\bar{X} = \bar{X}^*$ and $\bar{\nu}^* = \bar{\nu}$.

It is clear that $\bar{Q}^{(N)}(t) \to \bar{Q}(t) := (X^*(t) - 1)^+$ for a.e. $t \in \mathbb{R}_+$, then it follows by (5.12) that $R^* = \bar{R}$ where $\bar{R}$ is defined by (5.8), that is, $(\bar{X}^*, \bar{\nu}^*)$ satisfies Eq. (3.8). We claim that

$$\bar{D}(t) = \int_{0-}^{t} \bar{K}(t-s)\mathrm{d}G^{\mathsf{S}}(s) + \int_{\mathbb{R}_+} \frac{G^{\mathsf{S}}(x) - G^{\mathsf{S}}(x+t)}{1 - G^{\mathsf{S}}(x)} \bar{\nu}_0(\mathrm{d}x). \tag{5.15}$$

In fact, for any t, continuity point of $t \mapsto \int_{0-}^{t} \bar{K}(t-s)\mathrm{d}G^{\mathsf{S}}$ we have that $\bar{K}^{(N)}(t-s) \to \bar{K}(t-s)$ for $\mathrm{d}G^{\mathsf{S}}$-a.e. $s \in [0, t]$, then by dominated convergence, the first term on the right hand side of (5.6) converges to the first term on the right hand side of (5.15). Besides, since $\bar{\nu}_0$ is a diffuse measure if G^{S} is not continuous, for any $t \in \mathbb{R}_+$ the function

$$x \mapsto \frac{G^{\mathsf{S}}(x) - G^{\mathsf{S}}(x+t)}{1 - G^{\mathsf{S}}(x)}$$

is $\bar{\nu}_0$-continuous and therefore, the second term on the right hand side of (5.6) converges to the second term on the right hand side of (5.15) for any $t \in \mathbb{R}_+$. We have hence proved, that for a.e. $t \in \mathbb{R}_+$, $\widehat{D}^{(N)}(t)$ converges to the right hand side of (5.15). Then (5.15) is the consequence of (5.10). As a consequence of (5.15),

$$\bar{D}(0) = \bar{K}(0)G^{\mathsf{S}}(0). \tag{5.16}$$

Note also that for any $t \in \mathbb{R}_+$ and $N \in \mathbb{N}$, $\bar{K}^{(N)}(t) \le \bar{D}^{(N)}(t) + \bar{E}^{(N)}(t)$. By letting $N \to \infty$, we obtain $\bar{K}(t) \le \bar{D}(t) + \bar{E}(t)$. Since $\bar{E}(0) = 0$, we have $\bar{K}(0) \le \bar{D}(0)$. This inequality, (5.16) and the fact that $G^{\mathsf{S}}(0) < 1$ show that $\bar{K}(0) = \bar{D}(0) = 0$. It follows by Lemma 4.1 that $\bar{\nu}^*$ satisfies (3.4) and $\bar{\nu}_0^* = \bar{\nu}_0$.

Denote by $\bar{q}_t(x)$ the density of $\int_0^t \bar{\nu}_s^* \mathrm{d}s$ which is given by (4.4) for $Z = \bar{K}$ and $\nu_0 = \bar{\nu}_0$. We get from (4.5) and (5.15),

$$\bar{D}(t) = \int_{[0, M^{\mathsf{S}}]} \bar{q}_t(x)\mathrm{d}H^{\mathsf{S}}(x) \quad \text{for all } t \in \mathbb{R}_+, \tag{5.17}$$

and then $\bar{X}^*$ satisfies (3.9). For any $t \in \mathbb{R}_+$ and $f \in \mathcal{C}_b(\mathbb{R}_+)$ we have,

$$\begin{aligned}
\int_0^t \langle f, \bar{\nu}_s^* \rangle \mathrm{d}s &= \int_{\mathbb{R}} f(x)\bar{q}_t(x)\mathrm{d}x \\
&= \int_0^{\infty} (1 - G^{\mathsf{S}}(x))\bar{K}(t-x) f(x)\mathrm{d}x \\
&\quad + \int_0^{\infty} \int_y^{y+t} (1 - G^{\mathsf{S}}(x)) f(x)\mathrm{d}x \frac{\bar{\nu}_0(\mathrm{d}y)}{1 - G^{\mathsf{S}}(\mathrm{d}y)}.
\end{aligned}$$

The function $y \to (1 - G^{\mathsf{S}}(y))^{-1} \int_y^{y+t} (1 - G^{\mathsf{S}}(x))\mathrm{d}x$ is $\bar{\nu}_0$-continuous and bounded by t. Therefore we have,

$$\begin{aligned}\int_0^t \langle f, \bar{\nu}^*_s \rangle \mathrm{d}s &= \lim_{N\to\infty} \Bigg(\int_0^\infty (1 - G^{\mathsf{S}}(x)) \bar{K}^{(N)}(t - x) f(x)\mathrm{d}x \\ &\quad + \int_0^\infty \int_y^{y+t} (1 - G^{\mathsf{S}}(x)) f(x)\mathrm{d}x \frac{\bar{\nu}_0^{(N)}(\mathrm{d}y)}{1 - G^{\mathsf{S}}(\mathrm{d}y)} \Bigg) \qquad (5.18)\\ &= \lim_{N\to\infty} \int_{\mathbb{R}_+} \widehat{q}^{(N)}(x) f(x)\mathrm{d}x \\ &= \lim_{N\to\infty} \int_{\mathbb{R}_+} \bar{q}^{(N)}(x) f(x)\mathrm{d}x \\ &= \lim_{N\to\infty} \int_0^t \langle f, \bar{\nu}_s^{(N)} \rangle \mathrm{d}s \qquad (5.19)\end{aligned}$$

where the second equality is a consequence of the definition of $\widehat{q}^{(N)}$ (given by (5.2)) and by iterating integrals in the second term in the right hand side of (5.18) and the third equality is a consequence of (5.11). In particular, if $f \equiv 1$, it follows by (2.6) that

$$\begin{aligned}\int_0^t \langle 1, \bar{\nu}^*_s \rangle \mathrm{d}s &= \lim_{N\to\infty} \int_0^t \langle 1, \bar{\nu}_s^{(N)} \rangle \mathrm{d}s \\ &= \lim_{N\to\infty} \int_0^t \left(1 - [1 - \bar{X}^{(N)}(s)]^+\right) \mathrm{d}s \\ &= \int_0^t \left(1 - [1 - X^*(s)]^+\right) \mathrm{d}s.\end{aligned}$$

By right continuity, $1 - [1 - X^*(t)]^+ = \langle 1, \bar{\nu}^*_t \rangle$ for all $t \in \mathbb{R}_+$. This shows (3.10) for $(X^*, \bar{\nu}^*)$. With this method, we can show that (3.6) and (3.7) also hold for $(X^*, \bar{\eta}, Q)$.

We have shown that $(X^*, \bar{\nu}^*, \bar{\eta})$ satisfies the fluid equations, that is, $(X^*, \bar{\nu}^*) = (\bar{X}, \bar{\nu})$.

Since $\bar{K}(0) = \bar{D}(0) = 0$, the continuity of $\bar{D}$ and $\bar{K}$ given by Lemma 4.3 and Theorem 5.5(ii), we have the convergence, $(\bar{D}^{(N)}, \bar{K}^{(N)}) \to (\bar{D}, \bar{K})$ in $\mathcal{D}_{[0,\infty)}(\mathbb{R}_+, J_U)$. Eq. (2.4) shows the convergence of $\bar{X}^{(N)}$ to $\bar{X}$ in $\mathcal{D}_{[0,\infty)}(\mathbb{R}_+, J_U)$. Finally, thanks to (5.9) we have the convergence of $\bar{\nu}^{(N)}$ to $\bar{\nu}$ in $\mathcal{D}_{[0,\infty)}(\mathcal{M}_{\le 1}([0, M^{\mathsf{S}}), J_1))$. □

6. Proof of the results in Section 5

The following lemma is useful in this section. We recall that $\widetilde{\mathcal{F}}_0^{(N)}$ was defined as the σ-algebra generated by (2.1).

Lemma 6.1. *For any $i \in \mathbb{Z}$, define by $\{\mathcal{G}_t^{N,i}\}$ the right continuous and $\mathbb{P}$-completed filtration generated by*

$$\Big\{\widetilde{\mathcal{F}}_0^{(N)}, \{\xi_j^{(N)} : j = 1 \ldots, i\}, \{r_j \wedge (t - \xi_j^{(N)})^+ : j = -Q_0^{(N)}, \ldots, i\}, \\ \{v_j : j = -X_0^{(N)} + 1, \ldots, i - 1\}\Big\}$$

where $r_j = r_j^{(N)}$ *for* $j \le 0$ *and* $v_j = v_j^{(N)}$ *for* $j = -X_0^{(N)} + 1, \dots, Q_0^{(N)}$. *Then for any* $t \in \mathbb{R}_+$,

$$\{\alpha_i^{(N)} \le t, i \ge -X_0^{(N)} + 1\} \in \mathcal{G}_t^{N,i}.$$

In particular, conditionally on $\widetilde{\mathcal{F}}_0^{(N)}$, v_j *is independent of* $(\alpha_i^{(N)}, \alpha_j^{(N)}, v_i)$ *if* $-Q_0^{(N)}+1 \le i < j$.

Proof. The second claim of the lemma is a consequence of the first one and Assumption 2.1. Fix $N \in \mathbb{N}$, for simplicity, we omit the dependence on N in our notation. For $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $\mathcal{G}_t^{n,i}$ be the σ-algebra of $\Omega \cap \{Q_0 = n\}$ given by $\{A \cap \{Q_0 = n\} : A \in \mathcal{G}_t^i\}$. Since $\mathcal{G}_t^{n,i} \subset \mathcal{G}_t^i$ for all $n \in \mathbb{N}$ and $\{\alpha_i \le t,\ i = -X_0 + 1, \dots, -n, Q_0 = n\} \in \mathcal{F}_0$, it is enough to show

$$\{\alpha_i \le t; Q_0 = n\} \in \mathcal{G}_t^{n,i} \quad \text{for all } i \ge -n + 1,$$

and thus, will assume, without of loss of generality that there exists $n \in \mathbb{N}$ such that $Q_0(\omega) = n$ for all $\omega \in \Omega$ and hence, we must show that for $i \ge -n + 1$, α_i is an $\mathcal{G}^i$-stopping time. We will prove this fact by recurrence.

Suppose that α_k is an $\mathcal{G}^i$-stopping time for all $k = -n + 1, \dots, i$. Define the processes,

$$Q^i(t) := \sum_{k=-n+1}^{i} 1_{\{\xi_k \le t < \xi_k + r_k; \alpha_k > t\}}, \qquad R^i(t) := \sum_{k=-n+1}^{i} 1_{\{\xi_k + r_k \le t, \alpha_k = \infty\}},$$

$$X^i(t) \wedge N := \sum_{k=-X_0+1}^{i} 1_{\{\alpha_k \le t < \alpha_k + v_k\}}, \qquad D^i(t) := \sum_{k=-X_0+1}^{i} 1_{\{\alpha_k + v_k \le t\}}.$$

The process D^i, for example, represents the departures of customers who arrived before i (with i included). The number of customers present in the system at time t and that arrived before i (with i included) is given by

$$X^i(t) := X^i(t) \wedge N + Q^i(t). \tag{6.1}$$

By our recurrence assumption, all above processes are $\mathcal{G}^i$-adapted and hence $\mathcal{G}^{i+1}$-adapted. In this case, define the following $\mathcal{G}^{i+1}$-stopping times,

$$T(s) := \inf\{t \ge s : D^i(t) > D^i(s)\}, \quad \text{for } s \in \mathbb{R}_+,$$
$$U(m, s) := \inf\{t \ge s : D^i(t) + R^i(t) > m + D^i(s) + R^i(s)\}, \quad \text{for } s \in \mathbb{R}_+,\ m \in \mathbb{N}.$$

We have

$$\{\alpha_{i+1} < t\} = \{\alpha_i < t, \alpha_{i+1} < t\} \cup \{\alpha_{i+1} < t, \alpha_i = \infty\}. \tag{6.2}$$

Since customer $i + 1$ cannot enter service before customer i,

$$\begin{aligned}\{\alpha_{i+1} < t, \alpha_i = \infty\} &= \{\alpha_i \wedge t = t, \xi_i + r_i \le \alpha_{i+1} < t\} \\ &= \{\alpha_i \wedge t = t, \xi_i + r_i \le \alpha_{i+1} < t, X^i(\xi_i + r_i) < N\} \\ &\quad \cup \bigcup_{\ell=N}^{\infty} \{\alpha_i \wedge t = t, \xi_i + r_i \le \alpha_{i+1} < t, X^i(\xi_i + r_i) = \ell\}.\end{aligned} \tag{6.3}$$

The first set in the right hand side of (6.3) happens when customer i abandons the system at the moment of somebody finishes service, hence it is an event of $\mathbb{P}$-probability 0 and therefore $\mathcal{G}_t^{i+1}$-measurable. Besides, for $\ell \ge N$,

$$\begin{aligned}&\{\alpha_i \wedge t = t, \xi_i + r_i \le \alpha_{i+1} < t, X^i(\xi_i + r_i) = \ell\} \\ &\quad = \{\alpha_i \wedge t = t, \xi_i + r_i < t, \xi_{i+1} < t, U(\ell - N, \xi_i + r_i) < (\xi_{i+1} + r_{i+1}) \wedge t\}\end{aligned}$$

which is $\mathcal{G}_t^{i+1}$-measurable. Hence the second term in the right hand side of (6.2) is $\mathcal{G}_t^{i+1}$-measurable. For the first term in the right hand side of (6.2) we have,

$$\begin{aligned}\{\alpha_i < t, \alpha_{i+1} < t\} &= \{\alpha_i < t, X^i(\alpha_i) < N, \alpha_{i+1} < t\} \cup \{\alpha_i < t, X^i(\alpha_i) = N, \alpha_{i+1} < t\} \\ &= \{X^i(\alpha_i) < N, \xi_{i+1} < t, \alpha_i < (\xi_{i+1} + r_{i+1}) \wedge t\} \\ &\quad \cup \{\alpha_i < t, X^i(\alpha_i) = N, \xi_{i+1} < t, T(\alpha_i) < (\xi_{i+1} + r_{i+1}) \wedge t\}. \quad (6.4)\end{aligned}$$

Hence $\{\alpha_i < t, \alpha_{i+1} < t\} \in \mathcal{G}_t^{i+1}$ and therefore $\{\alpha_{i+1} < t\} \in \mathcal{G}_t^{i+1}$ for all $t \in \mathbb{R}_+$, that is α_{i+1} is an $\mathcal{G}_t^{i+1}$-stopping time.

In order to finish the proof of lemma, it remains to show that α_{-n+1} is an $\mathcal{G}_t^{-n+1}$-stopping time, but it can be shown using the arguments of (6.4) with $i = -n$ and replacing α_i by zero. □

6.1. Proof of Lemma 5.1

Proof of Lemma 5.1. For any $N \in \mathbb{N}$ define $\bar{Y}^{(N)}$ and $\bar{Z}^{(N)}$ as follows, for any $t, x \in \mathbb{R}_+$,

$$\bar{Y}_t^{(N)}(x) := \frac{1}{N} \sum_{i=-Q_0^{(N)}+1}^{E^{(N)}(t-x)} 1_{\{\alpha_i^{(N)} \le t-x\}} \left(1_{\{v_i \ge x\}} - (1 - G^{\mathsf{s}}(x-))\right) \quad \text{and}$$

$$\bar{Z}_t^{(N)}(x) := \frac{1}{N} \sum_{i=-X_0^{(N)}+1}^{-Q_0^{(N)}} 1_{\{x-t \le a_i^{(N)}(0) < x\}} \left(1_{\{x \le v_i^{(N)}\}} - \frac{1 - G^{\mathsf{s}}(x-)}{1 - G^{\mathsf{s}}(a_i^{(N)}(0))}\right).$$

Let (N_k) be a subsequence satisfying (5.3). We need to show:

(i) $\mathbb{P}$-a.s., $(\bar{Y}^{(N_k)})_{k\in\mathbb{N}}$ converges to zero in $\mathcal{C}_{[0,\infty)}(L^1(\mathbb{R}_+))$.
(ii) $\mathbb{P}$-a.s. $(\bar{Z}^{(N_k)})_{k\in\mathbb{N}}$ converges to zero in $\mathcal{C}_{[0,\infty)}(L^1(\mathbb{R}_+))$.

(i) For any $t, x \in \mathbb{R}_+$ and $N \in \mathbb{N}$ define $\kappa_i^N(t, x) = 1_{\{\alpha_i^{(N)} \le t-x\}}(1_{\{v_i \ge x\}} - (1 - G^{\mathsf{s}}(x-)))$. For any $L \in \mathbb{N}$ set

$$\Lambda_L^N(t, x) = \frac{1}{N} \sum_{i=-L\wedge Q_0^{(N)}+1}^{L\wedge E^{(N)}(t-x)} \kappa_i^N(t, x).$$

It follows by Lemma 6.1 that for $j > i \ge -Q_0^{(N)} + 1$,

$$\mathbb{E}[\kappa_j^N(t, x) | \alpha_j^{(N)}, \alpha_i^{(N)}, v_i, \widetilde{\mathcal{F}}_0^{(N)}] = 0,$$

then $\mathbb{E}[(\Lambda_L^N(t, x))^2] \le \frac{2L}{N}$ and therefore,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^\infty |\Lambda_L^N(t, x)| \mathrm{d}x\right)^2\right] &\le t\mathbb{E}\left[\int_0^t |\Lambda_L^N(t, x)|^2 \mathrm{d}x\right] \\ &\le \frac{2Lt^2}{N}. \quad (6.5)\end{aligned}$$

Then by Borel–Cantelli Lemma, we have $\mathbb{P}$-a.s the following: For any $L \in \mathbb{N}$ and $t \in \mathbb{Q}^+$, $\Lambda_L^{N_k}(t, x)$ converges to zero in $L^1(\mathbb{R})$ as $k \to \infty$. But for $L > \sup_N \bar{E}^{(N)}(t) \vee \bar{Q}_0^{(N)}$, $\bar{Y}_t^{(N)}(x) = \Lambda_L^N(t, x)$ for all $N \in \mathbb{N}$. This shows that $\mathbb{P}$-a.s., $\bar{Y}_t^{(N_k)} \to 0$ in $L^1(\mathbb{R})$ for all $t \in \mathbb{Q}_+$. In order to show (i), it remains to prove that $\mathbb{P}$-a.s., $(\bar{Y}^{(N)})$ is relatively compact on $\mathcal{C}_{[0,\infty)}(L^1(\mathbb{R}_+))$ or equivalently, to show that for any $T > 0$,

$$\lim_{\delta \to 0} \sup_{N \in \mathbb{N}} w(\bar{Y}^{(N)}, \delta, T) = 0,$$

where w denotes the continuity module on $\mathcal{C}_{[0,\infty)}(L^1(\mathbb{R}))$ (see e.g. [2]). For any $r < t \le T$,

$$\begin{aligned}\int_0^\infty |\bar{Y}_t^{(N)}(x) - \bar{Y}_r^{(N)}(x)| \mathrm{d}x &\le \int_0^\infty (\bar{K}^{(N)}(t-x) - \bar{K}^{(N)}(r-x)) \mathrm{d}x \\ &= \int_r^t \bar{K}^{(N)}(x) \mathrm{d}x \\ &\le (t-r) \sup_{N \in \mathbb{N}} (\bar{E}^{(N)}(T) + \bar{Q}_0^{(N)}).\end{aligned}$$

Then for $\delta > 0$, $\sup_{N \in \mathbb{N}} w(\bar{Y}^{(N)}, T, \delta) \le \delta \sup_{N \in \mathbb{N}} (\bar{E}^{(N)}(T) + \bar{Q}_0^{(N)}) \to 0$ as $\delta \to 0$.

(ii) We start proving that for any $t \in \mathbb{R}_+$, $(\bar{Z}_t^{(N_k)})_{k \in \mathbb{N}}$ converges to zero in $L^1(\mathbb{R})$, $\mathbb{P}$-a.s. Toward this end we shall prove,

$$\mathbb{P}\text{-a.s., } (\bar{Z}^{(N_k)})_{N \in \mathbb{N}} \text{ converges to zero in } L^1_{\mathrm{loc}}(\mathbb{R}_+) \quad \text{and} \tag{6.6}$$

$$\mathbb{P}\text{-a.s., } (\bar{Z}^{(N)})_{N \in \mathbb{N}} \text{ is uniformly integrable in } L^1(\mathbb{R}_+). \tag{6.7}$$

In the same way that we have proved (6.5) we obtain

$$\mathbb{E}[(\bar{Z}_t^{(N)}(x))^2] \le N^{-1}, \quad \text{for all } x \in \mathbb{R}_+,$$

thus $\mathbb{E}[\|\bar{Z}_t^{(N)}\|^2_{L^1[0,\ell]}] \le \ell^2 N^{-1}$ for all $\ell, N \in \mathbb{N}$. Therefore (6.6) follows by Borel–Cantelli Lemma. Besides

$$\begin{aligned}\lim_{\ell \to \infty} \sup_{N \in \mathbb{N}} \int_\ell^\infty \bar{Z}^{(N)}(x) \mathrm{d}x &\le \lim_{\ell \to \infty} \sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{i=-X_0^{(N)}+1}^{-Q_0^{(N)}} \int_\ell^\infty 1_{\{x-t \le a_i^{(N)}(0) < x\}} \mathrm{d}x \\ &\le \lim_{\ell \to \infty} \sup_{N \in \mathbb{N}} t \bar{\nu}_0^{(N)}(\ell - t, \infty) \\ &= 0,\end{aligned} \tag{6.8}$$

where the last equality follows from the fact that $\bar{\nu}_0^{(N)}$ converges to $\bar{\nu}_0$ and therefore it is tight. This shows (6.7). We have shown that $\mathbb{P}$-a.s., $(\bar{Z}_t^{(N_k)})$ converges to zero in $L^1(\mathbb{R})$ for all $t \in \mathbb{Q}$. In order to finish the proof we need to show that for any $T > 0$,

$$\lim_{\delta \to 0} \sup_{N \in \mathbb{N}} w(\bar{Z}^{(N)}, T, \delta) = 0. \tag{6.9}$$

For $r < t$,

$$\begin{aligned}\|\bar{Z}_t^{(N)} - \bar{Z}_r^{(N)}\|_{L^1(\mathbb{R}_+)} \le \frac{1}{N} &\sum_{i=-X_0^{(N)}+1}^{-Q_0^{(N)}} \int_0^\infty 1_{\{a_i^{(N)}(0) < s \le v_i^{(N)}\}} 1_{\{s-t \le a_i^{(N)}(0) < s-r\}} \mathrm{d}s \\ &+ \frac{1}{N} \sum_{i=-X_0^{(N)}+1}^{-Q_0^{(N)}} \int_0^\infty 1_{\{a_i^{(N)}(0) < s\}}\end{aligned}$$

$$\times \frac{1 - G^{\mathsf{S}}(s)}{1 - G^{\mathsf{S}}(a_i^{(N)}(0)-)} 1_{\{s-t \le a_i^{(N)}(0) < s-r\}} \mathrm{d}s$$
$$\le 2(t-r).$$

Therefore $w(\widehat{Y}^{(N)}, T, \delta) \le 2\delta$ for any $T, \delta > 0$. $\square$

Proof of Lemma 5.3

Proof of Lemma 5.3. Let (N_k) be a sequence satisfying (5.3). For any $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$ define,

$$\rho_i^k(t) := \begin{cases} 1_{\{\alpha_i^{(N_k)} + v_i \le t\}} - G^{\mathsf{S}}(t - \alpha_i^{(N_k)}) \quad \text{if } i \ge -Q_0^{(N_k)} + 1 \\ 1_{\{v_i^{(N_k)} \le t + a_i^{(N_k)}(0)\}} - \dfrac{G^{\mathsf{S}}(a_i^{(N_k)}(0) + t) - G^{\mathsf{S}}(a_i^{(N_k)}(0))}{1 - G^{\mathsf{S}}(a_i^{(N_k)}(0))} \\ \quad \text{if } i = -X_0^{(N_k)} + 1, \ldots, -Q_0^{(N)}. \end{cases}$$

Then we have,

$$\bar{D}^{(N_k)}(t) - \widehat{D}^{(N_k)}(t) = \frac{1}{N_k} \sum_{i=-X_0^{(N_k)}+1}^{\infty} \rho_i^k(t).$$

It follows by Lemma 6.1 that $\mathbb{E}[\rho_j^k(t) | \alpha_i^{(N_k)}, \alpha_j^{(N_k)}, v_i, \widetilde{\mathcal{F}}_0^{(N_k)}] = 0$ for $-X_0^{(N)} + 1 \le i < j$. Then

$$\mathbb{E}[\rho_i^k(t)\rho_j^k(t) | \widetilde{\mathcal{F}}_0^{(N_k)}] = \mathbb{E}\left[\rho_i^k(t)\mathbb{E}[\rho_j^k(t) | \alpha_i^{(N_k)}, \alpha_j^{(N_k)}, v_i, \widetilde{\mathcal{F}}_0^{(N_k)}] | \widetilde{\mathcal{F}}_0^{(N_k)}\right] = 0.$$

Therefore for any $i < j \in \mathbb{Z}$, $\mathbb{E}[\rho_i^k \rho_j^k; i \ge -X_0^{(N_k)} + 1] = 0$ and for any $L \in \mathbb{N}$ we obtain,

$$\mathbb{E}\left[\int_0^\infty e^{-t} \sum_{L=1}^\infty L^{-3} \sum_{k=1}^\infty \left(\frac{1}{N_k} \sum_{i=-L \wedge X_0^{(N_k)}+1}^{LN_k} \rho_i^k(t)\right)^2\right] \le 2 \sum_{L=1}^\infty L^{-2} \sum_{k=1}^\infty \frac{1}{N_k} < \infty.$$

Therefore, for $\mathbb{P}$-a.s. $\omega \in \Omega$, there exists $\mathcal{A} \subset \mathbb{R}_+$ of Lebesgue measure zero, such that for all $t \in \mathbb{R}_+ \setminus \mathcal{A}$,

$$\sum_{L=1}^\infty L^{-3} \sum_{k=1}^\infty \left(\frac{1}{N_k} \sum_{i=-L \wedge X_0^{(N_k)}+1}^{LN_k} \rho_i^k(\omega, t)\right)^2 < \infty.$$

Hence, for all $L \in \mathbb{N}$, $\lim_{k\to\infty} \frac{1}{N_k} \sum_{i=-L \wedge X_0^{(N_k)}+1}^{LN_k} \rho_i^k(\omega, t) = 0$. But for

$$L > \sup_{k \in \mathbb{N}} [\bar{E}^{(N_k)}(\omega, t) \vee X_0^{(N_k)}(\omega)]$$

and for all $k \in \mathbb{N}$,

$$\frac{1}{N_k} \sum_{i=L \wedge X_0^{(N_k)}+1}^{LN_k} \rho_i^k(\omega, t) = \bar{D}_1^{(N_k)}(\omega, t) - \widehat{D}_1^{(N_k)}(\omega, t).$$

This finish the proof of lemma. $\square$

Proof of Proposition 5.2

In order to establish Proposition 5.2, we need the following result. Denote by $S^{(N)}$ the cumulative number of potential reneging, that is,

$$S^{(N)}(t) := \frac{1}{N} \sum_{j=-Q_0^{(N)}+1}^{E^{(N)}(t)} 1_{\{r_i+\xi_i^{(N)} \le t\}}$$

and define $\bar{S}^{(N)} := \frac{1}{N} S^{(N)}$.

Lemma 6.2. *Let (N_k) be a sequence satisfying (5.3). Then $(\bar{S}^{(N_k)})$ converges $\mathbb{P}$-a.s. to $\bar{S}$ in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$, where for any $t \in \mathbb{R}_+$,*

$$\bar{S}(t) := \int_0^t G^{\mathrm{r}}(t-s)\mathrm{d}\bar{E}(s) + \int_{\mathbb{R}_+} \frac{G^{\mathrm{r}}(x) - G^{\mathrm{r}}(x+t)}{1 - G^{\mathrm{r}}(x)} \bar{\eta}_0(\mathrm{d}x).$$

Proof. Define $\widehat{S}^{(N)}$ by

$$\widehat{S}^{(N)}(t) := \int_0^t G^{\mathrm{r}}(t-s)\mathrm{d}\bar{E}^{(N)}(s) + \int_{\mathbb{R}_+} \frac{G^{\mathrm{r}}(x) - G^{\mathrm{r}}(x+t)}{1 - G^{\mathrm{r}}(x)} \bar{\eta}_0^{(N)}(\mathrm{d}x). \tag{6.10}$$

In an analogous way to the proof of Lemma 5.3, we can show that there exists $\tilde{\Omega} \subset \Omega$ such that $\mathbb{P}(\tilde{\Omega}) = 1$ and for any $\omega \in \tilde{\Omega}$,

$$(\bar{S}^{(N_k)}(\omega, t) - \widehat{S}^{(N_k)}(\omega, t)) \to 0 \quad \text{as } k \to \infty \text{ for a.e. } t \in \mathbb{R}_+.$$

Besides, since $\bar{E}$ is continuous and $\bar{\eta}_0$ is a diffuse measure, $\widehat{S}^{(N)}(\omega, t) \to \bar{S}(\omega, t)$ for all $t \in \mathbb{R}_+$. Therefore, for any $\omega \in \tilde{\Omega}$, $\bar{S}^{(N)}(\omega, t)$ converges to $\bar{S}(\omega, t)$ for a.e. $t \in \mathbb{R}_+$. Since $\bar{S}(\omega, .)$ is continuous and $\bar{S}(\omega, 0) = 0$, thanks to Theorem 5.5(ii), the convergence holds in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$. $\square$

Proof of Proposition 5.2. (1) Let (N_k) be a sequence satisfying (5.3) and let $\tilde{\Omega}$ be the subset of $\omega \in \Omega$ such that $(\bar{q}^{(N_k)}(\omega) - \widehat{q}^{(N_k)}(\omega))_{k\in\mathbb{N}}$ converges to zero in $\mathcal{C}_{[0,\infty)}(L^1(\mathbb{R}))$. Thanks to Lemma 5.1, $\mathbb{P}(\tilde{\Omega}) = 1$. Fix $\omega \in \tilde{\Omega}$ such that $(\bar{K}^{(N_k)}(\omega), \bar{D}^{(N_k)})_{k\in\mathbb{N}}$ converges in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U) \times \mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$ for some $K(\omega), D(\omega) \in V_0^+$. We need to show that $\bar{\nu}^{(N)}(\omega)$ converges to $\nu^K(\omega)$ in $\mathcal{D}_{[0,\infty)}(\mathcal{M}_{\le 1}([0, M^{\mathrm{s}})), J_1)$. From now on, all variables will depend on such ω, however, in order to simplify the notation, we suppress it in our notation, moreover, we will assume without loss of generality that $(N_k) = \mathbb{N}$.

It is enough to show that for any $T > 0$, the sequence $(\{\bar{\nu}_t^{(N)} : t \in [0, T]\})_{N\in\mathbb{N}}$ converges to $\{\nu^K : t \in [0, T]\}$ on $\mathcal{D}_{[0,T]}(\mathcal{M}_{\le 1}(\mathbb{R}_+), J_1)$.

First, we will prove

$$\{\bar{\nu}^{(N)} : N \in \mathbb{N}\} \cup \{\nu^K\} \text{ is a closed set of } \mathcal{D}_{[0,T]}(\mathcal{M}_{\le 1}([0, M)), J_1). \tag{6.11}$$

Indeed, if a subsequence $(\bar{\nu}^{(N_k)})_{k\in\mathbb{N}}$ converges to μ in $\mathcal{D}_{[0,T]}(\mathcal{M}_{\le 1}([0, M)), J_1)$ then for any $f \in \mathcal{C}_b(\mathbb{R}_+)$ and $t \le T$, we can show in the same way as (5.19) that,

$$\int_0^t \nu_s^K(f)\mathrm{d}s = \lim_{k\to\infty} \int_0^t \bar{\nu}_s^{(N_k)}(f)\mathrm{d}s = \int_0^t \mu_s(f)\mathrm{d}s.$$

Then by right continuity $\nu^K = \mu$ and (6.11) hold. We have shown in fact that any convergent subsequence of $(\bar{\nu}^{(N)})$ must converge to ν^K. Therefore, $(\bar{\nu}^{(N)})$ converges to ν^K if and only if

$$\{\bar{\nu}^{(N)} : N \in \mathbb{N}\} \text{ is a relatively compact set of } \mathcal{D}_{[0,T]}(\mathcal{M}_{\leq 1}([0, M^{\mathsf{S}})), J_1). \tag{6.12}$$

In view of Lemma 3.3 of [7] and Remark 5.11 of [10], to show (6.12) it is sufficient to show (6.13) and (6.14).

$$\exists \mathcal{K} \subset \mathcal{M}_{\leq 1}([0, M^{\mathsf{S}})) \text{ relatively compact}: \ \bar{\nu}_t^{(N)} \subset \mathcal{K}, \quad \forall N \in \mathbb{N} \text{ and } t \leq T. \tag{6.13}$$

$$\{\bar{\nu}^{(N)}(f) : N \in \mathbb{N}\} \text{ is a relatively compact set of } \mathcal{D}_{[0,T]}(\mathbb{R}, J_1) \quad \forall f \in \mathcal{C}_b^1(\mathbb{R}_+). \tag{6.14}$$

We start showing (6.13). The following argument was used in the proof of Lemma 5.12 in [10]. Since $\bar{\nu}_0^{(N)}$ converges to $\bar{\nu}_0$, it follows by Prohorov's theorem that for any $\varepsilon > 0$, there exists $s(\varepsilon) < \infty$ such that

$$\bar{\nu}_0^{(N)}([s(\varepsilon), \infty[) < \varepsilon \quad \text{for all } N \in \mathbb{N}.$$

Then for any $N \in \mathbb{N}, t \in [0, T]$ and $\varepsilon > 0$, $\bar{\nu}_0^{(N)}([s(\varepsilon) - t + T, \infty[) < \varepsilon$ and therefore, $\bar{\nu}_t^{(N)}([s(\varepsilon) + T, \infty[) < \varepsilon$. Set

$$\mathcal{K} := \{\mu \in \mathcal{M}_{\leq F}(\mathbb{R}_+) : \forall \varepsilon > 0, \mu(]s(\varepsilon) + T, \infty[) \leq \varepsilon\}.$$

Then by Prohorov's theorem, the set $\mathcal{K}$ is relatively compact in $\mathcal{M}_{\leq 1}(\mathbb{R}_+)$. Moreover $\bar{\nu}_t^{(N)} \in \mathcal{K}$ for all $t \in [0, T]$ and $N \in \mathbb{N}$. This shows (6.13).

Now we will show (6.14). Using the classical criterion of tightness in $\mathcal{D}_{[0,T]}(\mathbb{R}, J_1)$, (6.14) is equivalent to show that for any $f \in \mathcal{C}_b^1(\mathbb{R}_+)$,

$$\sup_{N\in\mathbb{N}} \sup_{t\leq T} \bar{\nu}_t^{(N)}(f) < \infty \quad \text{and} \tag{6.15}$$

$$\lim_{\delta\to 0} \sup_{N\in\mathbb{N}} w'(\bar{\nu}^{(N)}(f), \delta, T) \tag{6.16}$$

where w' denotes the continuity module on $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_1)$ (see e.g. [2]). Since for any $t \leq T$ and $N \in \mathbb{N}$, $\bar{\nu}_t^{(N)}(f) \leq \|f\|_\infty$, (6.15) is evident. Besides, for any $s < t \leq T$ and $N \in \mathbb{N}$,

$$\begin{aligned}
|\bar{\nu}_t^{(N)}(f) - \bar{\nu}_s^{(N)}(f)| \leq \frac{1}{N} &\left| \sum_{i=-X_0^{(N)}+1}^{\infty} f(a_i^{(N)}(t)) 1_{\{a_i^{(N)}(t) < v_i, \alpha_i^{(N)} \leq t\}} \right. \\
&\left. - \sum_{i=-X_0^{(N)}+1}^{\infty} f(a_i^{(N)}(t)) 1_{\{a_i^{(N)}(s) < v_i, \alpha_i^{(N)} \leq s\}} \right| \\
&+ \frac{1}{N} \left| \sum_{i=-X_0^{(N)}+1}^{\infty} f(a_i^{(N)}(t)) 1_{\{a_i^{(N)}(s) < v_i, \alpha_i^{(N)} \leq s\}} \right. \\
&\left. - \sum_{i=-X_0^{(N)}+1}^{\infty} f(a_i^{(N)}(s)) 1_{\{a_i^{(N)}(s) < v_i, \alpha_i^{(N)} \leq s\}} \right|,
\end{aligned} \tag{6.17}$$

where $v_i = v_i^{(N)}$ if $i < 0$. The first term on the right-hand side of the above inequality is bounded by

$$\|f\|_\infty \frac{1}{N} \sum_{i=-X_0^{(N)}+1}^{\infty} \left(1_{\{\alpha_i^{(N)} \le t < \alpha_i^{(N)} + v_i, s < \alpha_i^{(N)}\}} + 1_{\{\alpha_i^{(N)} + v_i \le t, \alpha_i^{(N)} \le s < \alpha_i^{(N)} + v_i\}} \right)$$
$$\le \|f\|_\infty \left(\frac{1}{N} \sum_{i=-X_0^{(N)}+1}^{\infty} 1_{\{s < \alpha_i^{(N)} \le t\}} + \frac{1}{N} \sum_{i=-X_0^{(N)}+1}^{\infty} 1_{\{s < \alpha_i^{(N)} + v_i \le t\}} \right)$$
$$= \|f\|_\infty (\bar{K}^{(N)}(t) - \bar{K}^{(N)}(s) + \bar{D}^{(N)}(t) - \bar{D}^{(N)}(s)).$$

For any i, $|f(a_i^{(N)}(t)) - f(a_i^{(N)}(s))| \le \|f'\|_\infty |a_i^{(N)}(t) - a_i^{(N)}(s)| \le \|f'\|_\infty (t-s)$. Then the second term on the right hand side of (6.17) is bounded by $(\bar{K}^{(N)}(s) + 1)\|f'\|_\infty (t-s)$. Finally we obtain,

$$|\bar{v}_t^{(N)}(f) - \bar{v}_s^{(N)}(f)| \le \|f\|_\infty (\bar{K}^{(N)}(t) - \bar{K}^{(N)}(s) + \bar{D}^{(N)}(t) - \bar{D}^{(N)}(s)) + [\bar{K}^{(N)}(s) + 1]\|f'\|_\infty (t-s)$$

and therefore, for any δ,

$$w'(\bar{v}^{(N)}(f), T, \delta) \le \|f\|_\infty w'(\bar{K}^{(N)} - \bar{D}^{(N)}, T, \delta) + \|f'\|_\infty [\bar{K}^{(N)}(T) + 1]\delta.$$

Since $\bar{K}^{(N)} - \bar{D}^{(N)}$ converges to $K - D$, $(\bar{K}^{(N)} - \bar{D}^{(N)})_{N \in \mathbb{N}}$ is relatively compact and then

$$\lim_{\delta \to 0} \sup_{N \in \mathbb{N}} w'(\bar{D}^{(N)} + \bar{K}^{(N)}, T, \delta) = 0.$$

Besides, since $\sup_{N \in \mathbb{N}} \bar{K}^{(N)}(T) < \infty$, $[\bar{K}^{(N)}(T) + 1]\delta$ converges to 0 as $\delta \to 0$. This shows (6.16).

(2) This can be shown with the same arguments used in (1) with $(\bar{\eta}, \bar{E}, \bar{S})$ instead of (ν^K, K, D) and using Lemma 6.2 and an analogous result of Lemma 5.1 for the measure $(\bar{\eta}^{(N)})$. The only difference is to show an equivalence to (6.13). That means, to show,

$$\exists \mathcal{K}_r \subset \mathcal{M}_F([0, M^{\mathsf{r}})) \text{ relatively compact}: \ \bar{\eta}_t^{(N)} \subset \mathcal{K}_r, \quad \forall N \in \mathbb{N} \text{ and } t \le T. \tag{6.18}$$

But as in the proof of (6.13) it holds that for any $\varepsilon > 0$, there exists $r(\varepsilon) < \infty$ such that $\bar{v}_t^{(N)}([r(\varepsilon) + T, \infty[) < \varepsilon$. Set $a_T := \bar{E}(T) + 1$ and

$$\mathcal{K}_r := \{\mu \in \mathcal{M}_{\le F}(\mathbb{R}_+) : \langle 1, \mu \rangle \le a_t \text{ and } \mu(]r(\varepsilon) + T, \infty[) \le \varepsilon \ \forall \varepsilon > 0\}.$$

Then by Prohorov's theorem, the set $\mathcal{K}$ is relatively compact in $\mathcal{M}_F(\mathbb{R}_+)$. Moreover $\bar{\eta}_t^{(N)} \in \mathcal{K}$ for all $t \in [0, T]$ and $N \in \mathbb{N}$. This shows (6.18). □

Proof of Proposition 5.4

We recall that the filtration $\{\mathcal{G}_t^{N,i}\}$ was defined in Lemma 6.1. Denote by $\{\mathcal{H}_t^{N,i}\}$ the $\mathbb{P}$-completed right continuous filtration generated by $\mathcal{G}^{N,i}$ and v_i ($v_i^{(N)}$ if $i \in \{-X_0^{(N)} + 1, \ldots, -Q_0^{(N)}\}$). For any $i \in \{-X_0^{(N)} + 1, \ldots, 0\} \cup \mathbb{N}$, denote by $X^{N,i}(t)$, the number of customers present in the system at time zero and who have entered into the system before customer i (with i included). This process was already defined in (6.1) and it is adapted to the filtration $\{\mathcal{H}_t^{N,i}\}$. For $i \in \{-Q_0^{(N)}+1, \dots, 0\} \cup \mathbb{N}$ define

$$U_i^N := \inf\{s \geq \xi_i^{(N)} : X^{N,i-1}(s) < N\}.$$

The cumulative reneging process (2.9) can be represented as follows,

$$R^{(N)}(t) = \sum_{i=-Q_0^{(N)}+1}^{E^{(N)}(t)} 1_{\{r_i+\xi_i^{(N)} \leq t \wedge U_i^N\}},$$

where we use the notation $r_i = r_i^{(N)}$ if $i \in \{-Q_0^{(N)}+1, \dots, 0\}$. For $i \in \mathbb{N}$ set

$$\rho_i^N(t) := 1_{\{r_i+\xi_i^{(N)} \leq t \wedge U_i^N\}} - \int_0^\infty 1_{\{x+\xi_i^{(N)} \leq t \wedge U_i^N\}} \mathrm{d}G^{\mathrm{r}}(x)$$

and for $i \in \{-Q_0^{(N)}+1 \dots, 0\}$ set

$$\rho_i^N(t) := 1_{\{r_i^{(N)}+\xi_i^{(N)} \leq t \wedge U_i^N\}} - \frac{1}{1-G^{\mathrm{r}}(-\xi_i^{(N)})} \int_{-\xi_i^{(N)}}^\infty 1_{\{x+\xi_i^{(N)} \leq t \wedge U_i^N\}} \mathrm{d}G^{\mathrm{r}}(x).$$

For all $x, t \in \mathbb{R}_+$ let $\mathfrak{R}_t^N(x)$ and $\widehat{R}^{(N)}(t)$ be given by

$$\mathfrak{R}_t^N(x) := \frac{1}{N}\sum_{i=1}^{E^{(N)}(t)} 1_{\{x+\xi_i^{(N)} \leq t \wedge U_i^N\}} + \frac{1}{N}\sum_{i=-Q_0^{(N)}+1}^{0} \left(\frac{1_{\{x>-\xi_i^{(N)}\}}}{1-G^{\mathrm{r}}(-\xi_i^{(N)})} 1_{\{x+\xi_i^{(N)} \leq t \wedge U_i^N\}} \right),$$

$$\widehat{R}^{(N)}(t) := \int_0^\infty \mathfrak{R}_t^N(x) \mathrm{d}G^{\mathrm{r}}(x).$$

Then we have,

$$\bar{R}^{(N)}(t) - \widehat{R}^{(N)}(t) = \sum_{i=-Q_0^{(N)}+1}^{E^{(N)}(t)} \rho_i^N(t).$$

Lemma 6.3. *Let (N_k) be a sequence satisfying (5.3). Then for $\mathbb{P}$-a.s. $\omega \in \Omega$,*

$$(\bar{R}^{(N_k)}(\omega,t) - \widehat{R}^{(N_k)}(\omega,t)) \to 0 \quad \textit{as } k \to \infty \textit{ for a.e. } t \in \mathbb{R}_+.$$

Proof. Note that U_i^N is measurable with respect to the σ-algebra generated by $\mathcal{H}_\infty^{N,i-1}$ and $\xi_i^{(N)}$, hence conditionally on $\widetilde{\mathcal{F}}_0^{(N)}$, r_i is independent of $\{\xi_j^{(N)}, U_j^N : j = -Q_0^{(N)}, \dots, i\}$ and therefore $\mathbb{E}[\rho_i^N(t)\rho_j^N(t) | \widetilde{\mathcal{F}}_0^{(N)}] = 0$ if $-Q_0^{(N)}+1 \leq i < j$. Hence, the lemma can be established with the same arguments used in the proof of Lemma 5.3. □

For any $t, x \in \mathbb{R}_+$ define $\widehat{f}_t^N(x)$ and $f_t^N(x)$ by

$$\widehat{f}_t^N(x) := (1-G^{\mathrm{r}}(x))\mathfrak{R}_t^N(x) \quad \text{and}$$

$$f_t^N(x) := \frac{1}{N}\sum_{i=1}^{E^{(N)}(t)} 1_{\{x+\xi_i^{(N)} \leq t \wedge U_i^N\}} 1_{\{x \leq r_i\}} + \frac{1}{N}\sum_{i=-Q_0^{(N)}+1}^{0} 1_{\{x+\xi_i^{(N)} \leq t \wedge U_i^N\}} 1_{\{-\xi_i^{(N)} < x \leq r_i\}}.$$

The following lemma can be proved with the same arguments used in the proof of Lemma 6.3.

Lemma 6.4. *Let* (N_k) *be a sequence satisfying* (5.3). *We have* $\mathbb{P}$*-a.s.,*

$$(\widehat{f}_t^{N_k}(x) - f_t^{N_k}(x)) \to 0 \quad \textit{as } k \to \infty \textit{ for a.e. } x \in \mathbb{R}_+., \ \forall t \in \mathbb{Q}^+.$$

Proof of Proposition 5.4. It is enough to show the assertion of the proposition for any subsequence (N_k) that satisfies the conclusion of Proposition 5.2(2), Lemmas 6.3 and 6.4. We assume then, without loss of generality and in order to simplify the notation that $(N_k) = \mathbb{N}$, that is, $\mathbb{P}$-a.s.,

$$(\bar{\eta}^{(N)})_{N\in\mathbb{N}} \to \bar{\eta} \quad \text{in } \mathcal{D}_{[0,\infty)}(\mathcal{M}_F([0, M^{\mathsf{r}})), J_1), \tag{6.19}$$

$$(\bar{R}^{(N)}(t) - \widehat{R}^{(N)}(t)) \to 0 \quad \text{for a.e. } t \in \mathbb{R}_+ \text{ and} \tag{6.20}$$

$$(\widehat{f}_t^N(x) - f_t^N(x)) \to 0 \quad \text{for a.e. } x \in \mathbb{R}_+ \text{ and } t \in \mathbb{Q}_+. \tag{6.21}$$

From now on, we fix ω which satisfies all the above convergences and such that $\bar{Q}^{(N)}(t) \to \bar{Q}(t)$ for some $\bar{Q} \in \mathcal{D}_{[0,\infty)}(\mathbb{R})$, for a.e. $t \in \mathbb{R}_+$. All variables will depend on such ω.

For any $x, t \in \mathbb{R}_+$ define

$$\mathfrak{R}_t(x) := \int_0^t 1_{\{\chi(s)\geq x\}} \mathrm{d}_s \bar{E}(s-x) + \int_{x-t}^x 1_{\{\chi(x-s)\geq x\}} \frac{\bar{\eta}_0(\mathrm{d}s)}{1 - G^{\mathsf{r}}(s)}.$$

We want to prove the following. For any $t \in \mathbb{Q}_+$,

$$\mathfrak{R}_t^N(x) \to \mathfrak{R}_t(x) \quad \text{for all } x \in [0, M^{\mathsf{r}}) \text{ outside of a countable set.} \tag{6.22}$$

Note that $\mathfrak{R}_t^N(x) = \alpha_t^N(x) - \beta_t^N(x)$, where

$$\alpha_t^N(x) := \bar{E}^{(N)}(t) + \frac{1}{N} \sum_{i=Q_0^{(N)}+1}^{0} \frac{1_{\{x>-\xi_i^{(N)}\}}}{1 - G^{\mathsf{r}}(-\xi_i^{(N)})} \quad \text{and}$$

$$\beta_t^N(x) := \frac{1}{N} \sum_{i=1}^{E^{(N)}(t)} 1_{\{x+\xi_i^{(N)} > t\wedge U_i^N\}} + \frac{1}{N} \sum_{i=Q_0^{(N)}+1}^{0} \frac{1_{\{x>-\xi_i^{(N)}+t\wedge U_i^N\}}}{1 - G^{\mathsf{r}}(-\xi_i^{(N)})}.$$

Note that for any $t \in \mathbb{R}_+$ and $x < M^{\mathsf{r}}$,

$$\alpha_t^N(x) \to \alpha_t(x) := \bar{E}(t) + \int_0^x \frac{1}{1 - G^{\mathsf{r}}(y)} \bar{\eta}_0(\mathrm{d}y),$$

and for any $N \in \mathbb{N}$, β_t^N is a non-decreasing functions such that $\forall x < M^{\mathsf{r}}$,

$$\sup_{N\in\mathbb{N}} \beta_t^N(x) \leq \sup_{N\in\mathbb{N}} \alpha_t^N(x) \leq \sup_{N\in\mathbb{N}} \bar{E}^{(N)}(t) + \frac{1}{1 - G^{\mathsf{r}}(x)} \sup_{N\in\mathbb{N}} \bar{\eta}_0^{(N)}((0, x]) < \infty. \tag{6.23}$$

Then it follows by Theorem 5.5(i) that for any sequence (N_k) there exists a further subsequence (N_{k_j}) such that β_t^N converges along (N_{k_j}), to some non-decreasing function β_t for any $x \in [0, M^{\mathsf{r}})$ outside of a countable set. In order to prove (6.22) we need only to show that $\beta_t = \alpha_t - \mathfrak{R}_t$, because in this case, the set of x where the convergence holds, which is the set of continuity points of $\mathfrak{R}$, is not depending on the subsequence (N_{k_j}). The function β_t is determinate by the set of integrals

$$\left\{ \int_0^{M^{\mathsf{r}}} (1 - G^{\mathsf{r}}(x))\varphi(x)\beta_t(x)\mathrm{d}x : \varphi \in \mathcal{C}_c[0, M^{\mathsf{r}}) \right\},$$

where $\mathcal{C}_c[0, M^{\mathsf{r}})$ is the set of continuous functions with support in $[0, M^{\mathsf{r}})$. Therefore, if we define $f_t(x) = (1 - G^{\mathsf{r}}(x))\mathfrak{R}_t(x)$, (6.22) is equivalent to show that for any $t \in \mathbb{Q}_+$ and any $\varphi \in \mathcal{C}_c[0, M^{\mathsf{r}})$,

$$\lim_{N\to\infty} \int_{[0,M^{\mathsf{r}})} \widehat{f}_t^N(x)\varphi(x)\mathrm{d}x = \int_{[0,M^{\mathsf{r}})} f_t(x)\varphi(x)\mathrm{d}x. \tag{6.24}$$

It follows by (6.23) that $\sup_{x,N} \widehat{f}_t^N(x) < \infty$ and directly to the definition of f_t^N, $\sup_{x,N} f_t^N(x) < \sup_{N\in\mathbb{N}}(\bar{E}^{(N)}(t) + Q_0^{(N)}) < \infty$. Then thanks to (6.21) and dominated convergence, (6.24) is equivalent to

$$\lim_{N\to\infty} \int_{[0,M^{\mathsf{r}})} f_t^N(x)\varphi(x)\mathrm{d}x = \int_{[0,M^{\mathsf{r}})} f_t(x)\varphi(x)\mathrm{d}x. \tag{6.25}$$

For any $i \geq -Q_0^{(N)}+1$, $\{x : x+\xi_i^{(N)} \leq t\wedge U_i^N; x \leq r_i\} = \{x : \chi_-^{(N)}(\xi_i^{(N)}+r_i) \geq r_i; x \leq r_i\}$, where for any $s \geq 0$, $\chi_-^{(N)}(s)$ denotes the left-limit of the function $\chi^{(N)}(s)$ defined in (2.8). Then

$$\begin{aligned} f_t^N(x) = \frac{1}{N} &\sum_{i=1}^{E^{(N)}(t-x)} 1_{\{\chi_-^{(N)}(\xi_i^{(N)}+r_i)\geq r_i\}} 1_{\{x\leq r_i\}} \\ &+ \frac{1}{N} \sum_{i=-Q_0^{(N)}+1}^{0} 1_{\{\chi_-^{(N)}(\xi_i^{(N)}+r_i)\geq r_i\}} 1_{\{x-t\leq -\xi_i^{(N)}<x\leq r_i\}}. \end{aligned}$$

Elementary computations show that $\int_0^t \bar{\eta}_s^{(N)}\mathrm{d}s$ has a density with respect to the Lebesgue measure. This density is denoted by $\bar{p}_t^{(N)}$ and is given by

$$\bar{p}_t^{(N)}(x) := \frac{1}{N} \sum_{i=1}^{E^{(N)}(t-x)} 1_{\{r_i\geq x\}} + \frac{1}{N} \sum_{i=-Q_0^{(N)}+1}^{0} 1_{\{x-t\leq w_i^{(N)}(0)<x\leq r_i\}}.$$

The function f_t^N admits the representation

$$f_t^N(x) = \int_0^t 1_{\{\chi_-^{(N)}(s)\geq x\}}\mathrm{d}_s \bar{p}_s^{(N)}(x), \quad \text{for all } x \in \mathbb{R}_+,$$

and then, for any $\varphi \in \mathcal{C}_c[0, M^{\mathsf{r}})$,

$$\int_0^{M^{\mathsf{r}}} f_t^N(x)\varphi(x)\mathrm{d}x = \int_0^t \int_{\mathbb{R}_+} 1_{\{\chi^{(N)}(s)\geq x\}}\bar{\eta}_s^{(N)}(\mathrm{d}x)\mathrm{d}s.$$

For any $x \in \mathbb{R}_+$, $f_t(x) = \int_0^t 1_{\{\chi^{(N)}(s)\geq x\}}\mathrm{d}_s\bar{p}_s(x)$, where $\bar{p}_t$ is the density function of the measure $\int_0^t \bar{\eta}_s\mathrm{d}s$ given by (5.7). The for any $\varphi \in \mathcal{C}_c[0, M^{\mathsf{r}})$,

$$\int_0^{M^{\mathsf{r}}} f_t(x)\varphi(x)\mathrm{d}x = \int_0^t \int_{\mathbb{R}_+} 1_{\{\chi(s)\geq x\}}\bar{\eta}_s(\mathrm{d}x)\mathrm{d}s$$

and (6.25) can be written as follows,

$$\lim_{N\to\infty} \int_0^t \int_{\mathbb{R}_+} 1_{\{\chi^{(N)}(s)\geq x\}}\bar{\eta}_s^{(N)}(\mathrm{d}x)\mathrm{d}s = \int_0^t \int_{\mathbb{R}_+} 1_{\{\chi(s)\geq x\}}\bar{\eta}_s(\mathrm{d}x)\mathrm{d}s. \tag{6.26}$$

Define $\ell_0^N(s) := \int_{\mathbb{R}_+} 1_{\{\chi^{(N)}(s)\geq x\}}\varphi(x)\mathrm{d}\bar{\eta}_s^{(N)}(x)$ and $\ell_0(s) := \int_{\mathbb{R}_+} 1_{\{\chi(s)\geq x\}}\varphi(x)\mathrm{d}\bar{\eta}_s(x)$. Then (6.26) will be the consequence of the following fact.

$$\text{If } \bar{Q}^{(N)}(s) \to \bar{Q}(s) \text{ then } \ell_0^N(s) \to \ell_0(s). \tag{6.27}$$

From now on, for two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, $a_n \sim b_n$ means that $(a_n - b_n)_{n\in\mathbb{N}}$ converges to zero as $n \to \infty$. The proof of (6.27) will be done in the following steps. If $\bar{Q}^{(N)}(s) \to \bar{Q}(s)$then,

$$\ell_0^N(s) \sim \ell_1^N(s) := \int_{\mathbb{R}_+} 1_{\{\bar{Q}^{(N)}(s)\geq y\}}\varphi((F^{\bar{\eta}_s^{(N)}})^{-1}(y))\mathrm{d}y, \tag{6.28}$$

$$\ell_1^N(s) \sim \ell_2^N(s) := \int_{\mathbb{R}_+} 1_{\{\bar{Q}(s)\geq y\}}\varphi((F^{\bar{\eta}_s^{(N)}})^{-1}(y))\mathrm{d}y, \tag{6.29}$$

$$\ell_2^N(s) \sim \ell_3(s) := \int_{\mathbb{R}_+} 1_{\{\bar{Q}(s)\geq y\}}\varphi((F^{\bar{\eta}_s})^{-1}(y))\mathrm{d}y \quad \text{and} \tag{6.30}$$

$$\ell_3(s) = \ell_0(s) \tag{6.31}$$

where we use the convention $\varphi(\infty) = 0$.

We have thanks to the change of variables formula,

$$\ell_0^N(s) = \int_{\mathbb{R}_+} 1_{\{\chi^{(N)}(s)\geq (F^{\bar{\eta}_s^{(N)}})^{-1}(y)\}}\varphi((F^{\bar{\eta}_s^{(N)}})^{-1}(y))\mathrm{d}y,$$

and this leads to,

$$\begin{aligned}
\|\varphi\|_\infty^{-1}|\ell_0(s) - \ell_1^N(s)| &\leq \int_{\mathbb{R}_+} 1_{\{(F^{\bar{\eta}_s^{(N)}})^{-1}(\bar{Q}^{(N)}(s))\geq (F^{\bar{\eta}_s^{(N)}})^{-1}(y);\, \bar{Q}^{(N)}(s)<y\}}\mathrm{d}y \\
&= \int_{\mathbb{R}_+} 1_{[\bar{Q}^{(N)}(s),\, F^{\bar{\eta}_s^{(N)}}(F^{\bar{\eta}_s^{(N)}})^{-1}(\bar{Q}^{(N)}(s))]}(y)\mathrm{d}y \\
&= |F^{\bar{\eta}_s^{(N)}}(F^{\bar{\eta}_s^{(N)}})^{-1}(\bar{Q}^{(N)}(s)) - \bar{Q}^{(N)}(s)| \\
&\leq \bar{\eta}_s^{(N)}\left(\left\{(F^{\bar{\eta}_s^{(N)}})^{-1}(\bar{Q}^{(N)}(s))\right\}\right).
\end{aligned} \tag{6.32}$$

For any $T \in \mathbb{R}_+$, $\bar{\eta}_s^{(N)}\left(\left\{(F^{\bar{\eta}_s^{(N)}})^{-1}(\bar{Q}^{(N)}(s))\right\}\right) \leq \sup_{\{\tau\leq T\}} \Delta F^{\bar{\eta}_s^{(N)}}(\tau) + \bar{\eta}^{(N)}((T,\infty])$. By the definition of $\bar{\eta}^{(N)}$ given by (5.5) and the fact that $\bar{E}$ is continuous and $\bar{\nu}_0$ is a diffuse measure, the function $x \to F^{\bar{\eta}_s}(x)$ is continuous. Hence, thanks to Theorem 5.5(ii) and the fact that $\bar{\eta}_s^{(N)}$ converges to $\bar{\eta}_s$, the right hand side of (6.32) converges to zero. This shows (6.28). The convergence (6.29) is evident from $\bar{Q}^{(N)}(s) \to \bar{Q}(s)$. As a consequence of the fact that $F^{\bar{\eta}_s^{(N)}}$ converges to $F^{\bar{\eta}_s}$ we have that $(F^{\bar{\eta}_s^{(N)}})^{-1}(y)$ converges to $(F^{\bar{\eta}_s})^{-1}(y)$ for any $y < \langle 1, \bar{\eta}_s\rangle$ continuity point of $(F^{\bar{\eta}_s})^{-1}$ and converges to ∞ for any $y > \langle 1, \bar{\eta}_s\rangle$ (see Theorem 13.6.3 of [18]). Then (6.30) follows by dominated convergence. Since $F^{\bar{\eta}_s}$ is continuous, the function $(F^{\bar{\eta}_s})^{-1}$ is strictly increasing and then

$$1_{\{y\leq \bar{Q}(s)\}} = 1_{\{(F^{\bar{\eta}_s})^{-1}(y)\leq (F^{\bar{\eta}_s})^{-1}(\bar{Q}(s))\}} = 1_{\{(F^{\bar{\eta}_s})^{-1}(y)\leq \chi(s)\}}.$$

Then (6.31) is got by a change of variables formula. This finishes the proof of (6.26) and therefore (6.22) holds for any $t \in \mathbb{Q}_+$. Since G^{r} is continuous,

$$\mathfrak{R}_t^N \to \mathfrak{R}_t \ \mathrm{d}G^{\mathsf{r}}\text{-a.e.} \quad \text{for all } t \in \mathbb{Q}_+. \tag{6.33}$$

For all $N \in \mathbb{N}, t \in \mathbb{R}_+$ and $x \in [0, M^{\mathsf{r}})$ define

$$\begin{aligned}\mathfrak{S}_t^N(x) &:= \frac{1}{N}\sum_{i=1}^{E^{(N)}(t)} 1_{\{x+\xi_i^{(N)}\le t\}} + \frac{1}{N}\sum_{i=-Q_0^{(N)}+1}^{0} \frac{1}{1-G^{\mathsf{r}}(-\xi_i^{(N)})} 1_{\{x-t\le -\xi_i^{(N)}<x\}} \\ &= \bar{E}^{(N)}(t-x) + \int_{[x-t,x)} \frac{1}{1-G^{\mathsf{r}}(y)} \bar{\eta}_0^{(N)}(\mathrm{d}y) \quad \text{and} \\ \mathfrak{S}_t(x) &:= \bar{E}(t-x) + \int_{[x-t,x)} \frac{1}{1-G^{\mathsf{r}}(y)} \bar{\eta}_0(\mathrm{d}y).\end{aligned}$$

For any $N \in \mathbb{N}, t \in \mathbb{R}_+$ and $x \in [0, M^{\mathsf{r}}), \mathfrak{R}_t^N(x) \le \mathfrak{S}_t^N(x)$, $\int_0^\infty \mathfrak{S}_t^N(x)\mathrm{d}G^{\mathsf{r}}(x) = \widehat{S}^{(N)}(t)$ and $\int_0^\infty \mathfrak{S}_t(x)\mathrm{d}G^{\mathsf{r}}(x) = \bar{S}(t)$, where $\widehat{S}_t^{(N)}$ and $\bar{S}_t$ were defined in Lemma 6.2 and (6.10). As we have seen in the proof of Lemma 6.2, $\widehat{S}^{(N)}$ converges to $\bar{S}$ in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$ and therefore, for any $t \in \mathbb{R}_+$ $\lim_{N\to\infty} \int_0^\infty \mathfrak{S}_t^N(x)\mathrm{d}G^{\mathsf{r}}(x) = \int_0^\infty \mathfrak{S}_t(x)\mathrm{d}G^{\mathsf{r}}(x)$. From Fatou's Lemma we obtain

$$\begin{aligned}\int_0^\infty (\mathfrak{S}_t(x) - \mathfrak{R}_t(x))\mathrm{d}G^{\mathsf{r}}(x) &\le \liminf_{N\to\infty} \int_0^\infty (\mathfrak{S}_t^N(x) - \mathfrak{R}_t^N(x))\mathrm{d}G^{\mathsf{r}}(x) \\ &= \int_0^\infty \mathfrak{S}_t(x)\mathrm{d}G^{\mathsf{r}}(x) - \limsup_{N\to\infty} \int_0^\infty \mathfrak{R}_t^N(x)\mathrm{d}G^{\mathsf{r}}(x).\end{aligned}$$

Using the above inequality and Fatou's Lemma again we get for all $t \in \mathbb{Q}_+$,

$$\limsup_{N\to\infty} \int_0^\infty \mathfrak{R}_t^N(x)\mathrm{d}G^{\mathsf{r}}(x) \le \int_0^\infty \mathfrak{R}_t(x)\mathrm{d}G^{\mathsf{r}}(x) \le \liminf_{N\to\infty} \int_0^\infty \mathfrak{R}_t^N(x)\mathrm{d}G^{\mathsf{r}}(x).$$

Note that $\int_0^\infty \mathfrak{R}_t(x)\mathrm{d}G^{\mathsf{r}}(x) = \bar{R}(t)$ for $\bar{R}$ defined in (5.8) and therefore we have shown that $\widehat{R}^{(N)}(t) \to \bar{R}(t)$ as $N \to \infty$ for all $t \in \mathbb{Q}_+$. The convergence is in fact in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$ thanks to Theorem 5.5(ii) and the fact that $\bar{R}$ is continuous. For this reason and (6.20), we obtain the convergence $\bar{R}^{(N)} \to \bar{R}$ in $\mathcal{D}_{[0,\infty)}(\mathbb{R}, J_U)$. □

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