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On exact sampling of the first passage event of a Lévy process with infinite Lévy measure and bounded variation

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Abstract

Exact sampling of the first passage event (FPE) of a Lévy process with infinite Lévy measure is challenging due to lack of analytic formulas. We present an approach to the sampling for processes with bounded variation. The idea is to embed a process for which we wish to sample the FPE into another process whose FPE can be sampled based on analytic formulas, and once the latter FPE is sampled, extract from it the part belonging to the former process. We obtain general procedures to sample the FPE across a regular nonincreasing boundary or out of an interval. Concrete algorithms are given for two important classes of Lévy processes. The approach is based on distributional results that appear to be new.

MSC: primary 60G51; secondary 60E07

Keywords: First passage; Lévy process; Bounded variation; Subordinator; Creeping; Dirichlet distribution

1. Introduction

The first passage event (FPE) of a Lévy process is an important subject in applied probability [2–4,14–16,23,25,34]. In many cases, it is crucial to sample the FPE as completely as possible, in particular, to sample the first passage time (FPT) and the values of the process just before and

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at the FPT ([35]; unless otherwise mentioned, by sampling we mean exact sampling). For processes with infinite Lévy measures, except for a few, the distribution of the FPE is unavailable analytically, which poses a serious hurdle to the sampling. In practice, there are various methods to sample the FPE approximately [19] or to evaluate some of its parameters [36,37]. However, it remains challenging and tantalizing to sample the FPE exactly.

This paper presents an approach to sampling the FPE for a rather wide range of real-valued Lévy processes with infinite Lévy measure and bounded variation. Each such process is the difference of two independent subordinators, i.e., nondecreasing Lévy processes, which also have bounded variation. Both types of processes are important in theory and application [3,13,21,24, 30,32,35]. The approach allows one to sample (1) the FPE of a subordinator across a nonincreasing boundary that has certain regularity, (2) the FPE of a process with nonpositive drift across a positive constant level, and (3) the first exit event (FEE) of a process with no drift out of a closed interval that has 0 in its interior. A useful by-product of the approach is a sampling method for infinitely divisible (ID) random variables alternative to the one in [8].

It should be noted that the approach as presented is not very practical due to its high computational complexity; see however [35] for an efficient application of its simplified version to real data. Rather, the motivation to attack the sampling issue at a conceptual level is to gain insights into the distributional properties and structure of the FPE, which would help develop more practical algorithms. The work is therefore more about theory than about application.

In Section 2, general procedures to sample the FPE or FEE in several scenarios are presented. The main idea is "embed and extract". That is, given a process for which we wish to sample the FPE, embed it into another process whose FPE can be sampled based on analytic formulas, and once the FPE of the latter process is sampled, conditional on the event, sample its part that belongs to the former process. The validity of the procedures is established in Section 4 by considering the distribution of the triple $(\tau, X(\tau-), X(\tau))$, where $X = (X_1, \dots, X_k)$, with X_i being independent subordinators, and τ is certain FPT of $X_1 + \cdots + X_k$. The results are related to [23], which however does not consider τ and the case of creeping; also see [3,4,13,14,16,30]. In Section 3, after a brief general discussion, the procedures are applied to two important types of Lévy measures of the form $\varphi(x)\,\mathrm{d}x + \chi(\mathrm{d}x)$, with φ a Lévy density and χ a finite Lévy measure. The first type has

$$\varphi(x) = \mathbf{1}_{\{0<x\le r\}} e^{-qx} \gamma x^{-1-\alpha}, \quad \text{with } q \ge 0,\ \gamma > 0,\ r > 0,\ \alpha \in (0, 1), \tag{1.1}$$

which will be referred to as an exponentially tilted upper truncated stable Lévy density. This type coincides with the one that has $\varphi(x) = (\gamma + O(x))x^{-\alpha-1}$ as $x \to 0+$, and gives rise to many interesting processes [6,26–29,31], e.g. Lamperti-stable that has $\varphi(x) = \mathbf{1}_{\{x>0\}} e^{\beta x}(e^x - 1)^{-\alpha-1}$ with $\beta < \alpha + 1$ and $\chi = 0$. For this particular process, we can set $r = \infty$ in (1.1). However, in general, r is finite. The second type has

$$\varphi(x) = \mathbf{1}_{\{0<x\le r\}} \gamma e^{-qx} x^{-1}, \quad \text{with } q > 0,\ \gamma > 0,\ r > 0, \tag{1.2}$$

which will be referred to as an upper truncated Gamma Lévy density. This type coincides with the one that has $\varphi(x) = (\gamma + O(x))x^{-1}$ as $x \to 0+$. The famous Vervaat perpetuity is of this type [11,18]. More importantly, subordinators with this type of Lévy measures play a prominent role in Bayesian survival analysis that uses $F(t) = 1 - e^{-Z(t)}$ as a prior on the distribution function of failure time, with Z a nondecreasing additive process [12,17,21,35]. To estimate the main parameters of a survival model, multiple failure times need be sampled from F. In the standard inversion method, F is fully specified and failure times are sampled as $\inf\{t : F(t) \ge u_i\}$ with $u_1, \dots, u_n$ i.i.d. uniformly distributed on $(0, 1)$ [10]. However, for Z with an infinite Lévy measure, F cannot be fully specified as it has infinitely many random jumps in every interval in $(0, \infty)$. Instead, one need first draw all of $u_1, \ldots, u_n$ and then, given their values, sample $t_i = \inf\{t : Z(t) \ge a_i\}$ for a single path of Z, where $a_i = -\ln(1 - u_i)$. To do so, start with $a_i = \min a_j$ and sample t_i and $Z(t_i)$. For the rest a_j's, let $t_j = t_i$ if $a_j \le Z(t_i)$. Next, find the smallest a_j greater than $Z(t_i)$, use the renewal property of Z to sample t_j and $Z(t_j)$, and so on, until no a_j is left (cf. [35]). Due to the jumps in Z, the probability to have equal t_j's is positive, a property very useful for modeling data that have tied failure times. The application in survival analysis highlights the importance of exact sampling of the FPE and strongly motivates the paper.

1.1. Notation and preliminaries

We will consider the FPE across a boundary that has certain regularity. A function c on $(0, \infty)$ is said to be regular if it is absolutely continuous and nonincreasing with $c(0+) > 0$, and is differentiable on $(0, \infty) \setminus F$, with F a closed set of Lebesgue measure 0. If c is regular, then for any constant $a > 0$, $c \wedge a$ is also regular.

For a Lévy process X, denote $X \sim \mathrm{BV}(\Pi, d)$ if it has Lévy measure Π, drift d, and bounded variation, i.e., $\int(|x| \wedge 1)\Pi(\mathrm{d}x) < \infty$ and X has no Brownian component. Denote by $\mathrm{ID}(\Pi, d)$ the distribution of $X(1)$. If $d = 0$, then write $\mathrm{BV}(\Pi)$ and $\mathrm{ID}(\Pi)$ for brevity. Denote by $|\Pi|$ the total mass of Π. For $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, denote $\|x\| = \sum |x_i|$. Define $\inf \emptyset = \infty$. A probability density function with respect to the Lebesgue measure will be simply referred to as a pdf. For $a, b > 0$, denote by $\mathrm{Gamma}(a, b)$ and $\mathrm{Beta}(a, b)$ the distributions with pdfs $\mathbf{1}_{\{x>0\}} x^{a-1} e^{-x/b}/[b^a \Gamma(a)]$ and $\mathbf{1}_{\{0<x<1\}} x^{a-1}(1-x)^{b-1}/B(a, b)$, respectively, where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Denote $\mathrm{Exp}(b) = \mathrm{Gamma}(1, b)$. Denote by $U(0, 1)$ the uniform distribution on $(0, 1)$ and δ the probability measure concentrated at 0. For $k > 1$, the Dirichlet distribution $\mathrm{Di}(a_1, \ldots, a_k)$ with parameters $a_i > 0$ is a distribution on $\mathbb{R}^k$, such that for any Borel function $g \ge 0$ on $\mathbb{R}^k$ and $\omega \sim \mathrm{Di}(a_1, \ldots, a_k)$,

$$\mathrm{E}[g(\omega)] = \frac{\Gamma(a_1 + \cdots + a_k)}{\Gamma(a_1) \cdots \Gamma(a_k)} \int \mathbf{1}_{\{\text{all} x_i \ge 0\}} g(x) \prod_{i=1}^k x_i^{a_i - 1} \, \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1},$$

where in the integral $x = (x_1, \ldots, x_k)$ with $x_k \equiv 1 - \sum_{i=1}^{k-1} x_i$. Its pdf with respect to the measure $\sigma_k(\mathrm{d}x) = \mathbf{1}_{\{\text{all } x_i \ge 0\}} \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1} \, \delta(\mathrm{d}x_k - 1 + \sum_{i=1}^{k-1} x_i)$ is $\Gamma(a_1 + \cdots + a_k) \prod_{i=1}^k x_i^{a_i - 1} / \prod_{i=1}^k \Gamma(a_i)$, and will be referred to as its pdf. If $k = 1$, then for $a > 0$, define $\mathrm{Di}(a) = \delta(\mathrm{d}x - 1)$ and its pdf (with respect to $\sigma_1(\mathrm{d}x) = \delta(\mathrm{d}x - 1)$) to be constant 1.

Let $\nu = \mathrm{ID}(\Lambda)$ with Λ concentrated on $(0, \infty)$. Given $q > 0$, $\Lambda_q(\mathrm{d}x) = e^{-qx}\Lambda(\mathrm{d}x)$ is called an exponentially tilted version of Λ. If $\nu_q = \mathrm{ID}(\Lambda_q)$, then $\nu_q(\mathrm{d}x) \propto e^{-qx}\nu(\mathrm{d}x)$ [1,3,5,22] and can be sampled by rejection sampling [10,19]. In general, if $\tilde{\nu}$ and ν are two probability measures satisfying $\mathrm{d}\tilde{\nu}/\mathrm{d}\nu \propto \varrho \le C$, with $\varrho \ge 0$ a function and $C > 0$ a constant, then the rejection sampling of $\tilde{\nu}$ proceeds by keeping sampling $\xi \sim \nu$ and $U \sim U(0, 1)$ (independently) until $CU \le \varrho(\xi)$.

2. Sampling of FPE and FEE

2.1. FPE of subordinator

Let $Z \sim \mathrm{BV}(\Pi, d)$ be a subordinator with $|\Pi| = \infty$ and c a regular function. The FPE of Z across c is characterized by the FPT $\tau_c^Z = \{t > 0 : Z(t) > c(t)\}$ as well as $Z(\tau_c^Z-)$ and

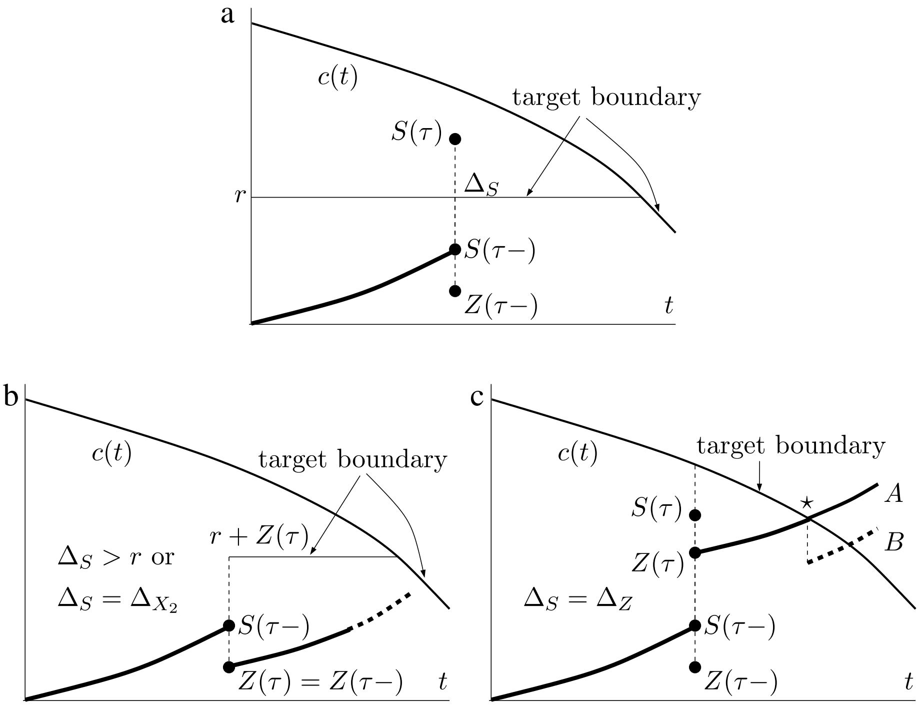

Fig. 1. Sampling the FPE of a subordinator by embedding.

$Z(\tau_c^Z)$. By the transformations $Z(t) \to Z(t) - dt$ and $c(t) \to c(t) - dt$, assume without loss of generality that $d = 0$. The main assumption here is

$$\Pi(\mathrm{d}x) = e^{-qx}\mathbf{1}_{\{x\le r\}}\Lambda(\mathrm{d}x) + \chi(\mathrm{d}x), \quad q \ge 0,\ 0 < r \le \infty,\ |\chi| < \infty, \tag{2.1}$$

such that, letting $S \sim \mathrm{BV}(\Lambda)$, the FPE of S across any regular function can be sampled. As seen later, this is the case if $e^{-qx}\mathbf{1}_{\{x\le r\}}\Lambda(\mathrm{d}x) = \varphi(x)\,\mathrm{d}x$, with φ as in (1.1) or (1.2). Let $X_1 \sim \mathrm{BV}(e^{-qx}\mathbf{1}_{\{x\le r\}}\Lambda(\mathrm{d}x))$, $X_2 \sim \mathrm{BV}(\mathbf{1}_{\{0<x\le r\}}(1-e^{-qx})\Lambda(\mathrm{d}x))$, $X_3 \sim \mathrm{BV}(\mathbf{1}_{\{x>r\}}\Lambda(\mathrm{d}x))$, and $Q \sim \mathrm{ID}(\chi)$ be independent. All the processes are subordinators, and all but X_1 are compound Poisson (CP). Since $X_1 + Q \sim \mathrm{BV}(\Pi)$, we identify it with Z.

We first explain the scheme informally. For now let us ignore the CP process Q by assuming $\chi = 0$, so that $Z = X_1$. As said in Section 1, the scheme is "embed and extract". Indeed, by $Z + X_2 + X_3 \sim \mathrm{BV}(\Lambda)$, S is identified with $Z + X_2 + X_3$. In this sense, Z is embedded in S. As for the "extract" part, let $b(t) = c(t) \wedge r$ be the "target boundary" for S to cross and $\tau = \tau_b^S$; see Fig. 1. By assumption, we can sample $(\tau, S(\tau-), S(\tau))$. Clearly, $S(\tau-) \le b(\tau) \le r$. Given $\tau = t$ and $S(\tau-) = s \le r$, we need to sample $Z(\tau-)$. If $q = 0$, then $X_2 \equiv 0$ and so $Z(\tau-) = s$. If $q > 0$, the sampling is possible for three reasons. First, the conditional distribution of $Z(\tau-)$ is the same as that of $Z(t)$ given $S(t) = s$, as if t is fixed beforehand. Second, the following simple but crucial fact holds: since X_3 only has jumps greater than r, $S(t) = s \iff Z(t) + X_2(t) = s$. Third, by using properties of exponential tilting for ID distributions, $Z(t)$ can be sampled conditional on $Z(t) + X_2(t) = s$. In Fig. 1(a), we have $Z(\tau-) < S(\tau-)$. However, as X_2 is CP, we may have $Z(\tau-) = S(\tau-)$. Next, we need to sample $Z(\tau)$. The jump of S at τ is $\Delta_S = S(\tau) - S(\tau-)$. By independence, only one of Z,

Table 1

Sampling of $(\tau, Z(\tau-), \Delta_Z(\tau))$, where $\tau = \tau_c^Z \wedge K$, c is regular or ∞, $0 < K \le \infty$ (finite if $c \equiv \infty$). To start with, $T = H = D = 0$, $A = K$.

1. If $D = 0$, then sample $(D, J) \sim (\tilde\tau \wedge A, \Delta_Q(\tilde\tau \wedge A))$ with $\tilde\tau = \inf\{t : Q(t) \ne 0\}$.
2. Set $b = c \wedge r$. Sample $t_1 \sim \tau_b^S$ and set $t = t_1 \wedge D$.
3. If $t = t_1 < D$, then sample $(s, v) \sim (S(t-), \Delta_S(t))$ conditional on $\tau_b^S = t$.
4. If $t = D < t_1$, then sample $s \sim S(t)$ conditional on $S(t) < b(t)$ and set $v = 0$.
5. Sample $x \sim X_1(t)$ conditional on $S(t) = s$.
6. If $v > 0$, then sample $U \sim U(0, 1)$ and reset $v \leftarrow v\mathbf{1}_{\{v\le r,\, U\le e^{-qv}\}}$.
7. Update $T \leftarrow T + t$. Set $\Delta = v + \mathbf{1}_{\{t=D\}} J$, $z = x + \Delta$, and update $H \leftarrow H + z$.
8. If $z < c(t)$ and $t < A$, then update $A \leftarrow A - t$, $D \leftarrow D - t$, $c(\cdot) \leftarrow c(\cdot + t) - z$, and go back to step 1; else output $(T, H - \Delta, \Delta)$ and stop.

X_2 and X_3 can jump at τ, so $\Delta_Z \in \{0, \Delta_S\}$. Fig. 1 illustrates two scenarios. If $\Delta_S > r$, then it must belong to X_3, giving $\Delta_Z = 0$. If $\Delta_S \le r$, then by comparing the Lévy measures of Z and X_2, with probability $e^{-q\Delta_S}$ (resp. $1 - e^{-q\Delta_S}$), Δ_S belongs to Z (resp. X_2), giving $\Delta_Z = \Delta_S$ (resp. $\Delta_Z = 0$). This gives $Z(\tau) = Z(\tau-) + \Delta_Z$. If $Z(\tau) < c(\tau)$, then by strong Markov property, the procedure is renewed for S with starting point $(t_0, S_0) = (\tau, Z(\tau))$. The procedure eventually stops, yielding a sample of $(\tau_c^Z, Z(\tau_c^Z-), Z(\tau_c^Z))$.

Note that, if c is decreasing, then S may creep across c, i.e., $\Delta_S = 0$, as marked by $\star$ in panel (c). In this case, $\Delta_Z = 0$ and moreover, if $q > 0$, we may have $Z(\tau) < S(\tau)$ and the procedure has to continue; see the scenario marked by B in (c). The characterization of creeping across a linear boundary is known [3,20]. Here, as c may be nonlinear, we need further results on this regard.

To implement the scheme, one may first sample τ, then $(S(\tau-), S(\tau))$ conditional on τ, and finally $(Z(\tau-), Z(\tau))$ conditional on $(\tau, S(\tau-), S(\tau))$. The sampling of τ is the simplest. The sampling of the rest will require several distributional properties of the FPE obtained in Section 4.

Formal procedure. Denote $\tau_c^S = \inf\{t > 0 : S(t) > c(t)\}$ and Δ_Q the jump process of Q. In the formal procedure in Table 1, the jumps in Q are taken into account in steps 1 and 7. In addition, a terminal point $0 < K \le \infty$ is introduced so that the final output is a sample of $(\tau, Z(\tau-), \Delta_Z(\tau))$, where $\tau = \tau_c^Z \wedge K$. In particular, if $c = \infty$ and $K = 1$, then $Z(1-) + \Delta_Z(1) \sim \mathrm{ID}(\Pi)$.

Theorem 2.1. *Let c be a regular function. The procedure in Table 1 stops w.p. 1, and its random output follows the distribution of $(\tau, Z(\tau-), \Delta_Z(\tau))$. The claim still holds if $c \equiv \infty$ and $K < \infty$.*

2.2. Extensions to Lévy processes with bounded variation

Let $Z \sim \mathrm{BV}(\Pi, d)$ take values in $\mathbb{R}$ with $|\Pi| = \infty$ and $d \le 0$. Then $Z = Z^+ - Z^-$, where $Z^+ \sim \mathrm{BV}(\Pi^+)$ and $Z^- \sim \mathrm{BV}(\Pi^-, -d)$ are independent subordinators with $\Pi^\pm(\mathrm{d}x) = \Pi(\pm \mathrm{d}x)$. Suppose that for $\sigma = \pm$,

$$\Pi^\sigma(\mathrm{d}x) = \exp(-q^\sigma x)\mathbf{1}_{\{x\le r^\sigma\}}\Lambda^\sigma(\mathrm{d}x) + \chi^\sigma(\mathrm{d}x), \quad q^\sigma \ge 0,\ 0 < r^\sigma \le \infty,\ |\chi^\sigma| < \infty, \tag{2.2}$$

such that, letting $S^\sigma \sim \mathrm{BV}(\Lambda^\sigma)$, the FPE of S^σ across any positive constant level can be sampled. Note that at most one Z^σ is CP. If, say Z^+ is CP, the set $S^+ \equiv 0$, so that the FPT of S^+ across any positive boundary is ∞.

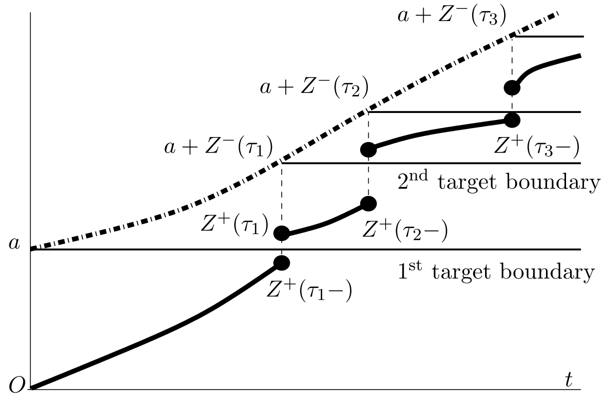

Fig. 2. Sampling the FPE of Z across $a > 0$.

We consider the sampling of the FPE of Z across a constant level $a > 0$ and, assuming $d = 0$, the FEE of Z out of an interval $I = [-a^-, a^+]$ with constants $a^\pm > 0$. Denote $\tau_a^Z = \inf\{t > 0 : Z(t) > a\}$ and $\tau_I^Z = \inf\{t > 0 : Z(t) \notin I\}$.

As in the subordinator case, we first give an informal description for the sampling schemes. The scheme for the FPE of Z across $a > 0$ can be presented in a more general setting. As Fig. 2 shows, it can be thought of as having Z^+ to "catch up" with $a + Z^-$. To start with, let a be the target boundary for Z^+ to cross and $\tau = \tau_1$ the corresponding FPT. It is evident that before τ, Z^+ stays below $a + Z^-$. However, at τ, since Z^+ has a jump, it is possible for Z^+ to pass $a + Z^-$. We can use the procedure in Section 2.1 to sample $Z^+(\tau-)$ and $Z^+(\tau)$. Meanwhile, as Z^- is independent of τ, we can also use the procedure to sample $Z^-(\tau-) = Z^-(\tau)$. If $a + Z^-(\tau) < Z^+(\tau)$, then $\tau = \tau_a^Z$. Otherwise, set $a + Z^-(\tau)$ as the new target boundary for Z^+ to cross, with starting time point and value $(t_0, Z_0^+) = (\tau, Z^+(\tau))$. As long as $\overline{\lim}_{t\to\infty} Z(t) = \infty$ w.p. 1, the procedure eventually stops. For the scheme, the assumption that Z has $d \le 0$ is necessary, as otherwise Z^+ can creep across $a + Z^-$ with positive probability. If this happens, the FPTs $\tau_1, \tau_2, \dots$ of Z^+ shown in Fig. 2 will converge to but never reach τ_a^Z, causing the procedure to go on forever.

To sample the FEE of Z out of I, a modified version of the scheme in Section 2.1 can be used. It is convenient to use the "phase plot" of the Lévy process $W = (Z^-, Z^+)$, which shows the trajectory of W on the plane. The FEE of Z out of I can be depicted as the FEE of W out of the band $\{(x, y) : -a^- \le y - x \le a^+\}$. To start with, let $b^- = a^- \wedge r^-$, $b^+ = a^+ \wedge r^+$, and set the top and right sides of the rectangle $[0, b^-] \times [0, b^+]$ to be the target boundary. In Fig. 3(a), $r^- > a^-$ and $r^+ < a^+$, resulting in the rectangle as shown. Now sample the FPE of $S = (S^-, S^+)$ across the target boundary. First, independently sample the FPTs of $S^\pm$ across $b^\pm$. If, as shown in panel (a), S^- makes a crossing at time τ before S^+, then sample $(S^-(\tau-), S^-(\tau))$, and sample $S^+(\tau-) = S^+(\tau)$ conditional on $S^+(\tau) < b^+$. We next can use the scheme in Section 2.1 to recover $Z^\pm(\tau)$. In the scenario shown in Fig. 3, since the jump of S^- at τ is greater than r^-, it is not part of Z^-, so we end up with $W(\tau)$ as in panel (b). The procedure is then renewed. As long as $Z \not\equiv 0$, the procedure stops eventually and produces a sample FEE of Z.

Formal procedures. As in the subordinator case, let $0 < K \le \infty$ be a terminal point. First, let $\tau = \tau_a^Z \wedge K$. If $\tau_a^Z < \infty$, then Z makes a positive jump at τ_a^Z [3, Exercise VI.9]. On the other hand, Z makes no jump at K. Therefore the only possible jump that Z can make at τ is positive, giving $\Delta_Z(\tau) = \Delta_{Z^+}(\tau)$, $Z^+(\tau) = Z^+(\tau-) + \Delta_Z(\tau)$, and $Z^-(\tau-) = Z^-(\tau)$. A procedure

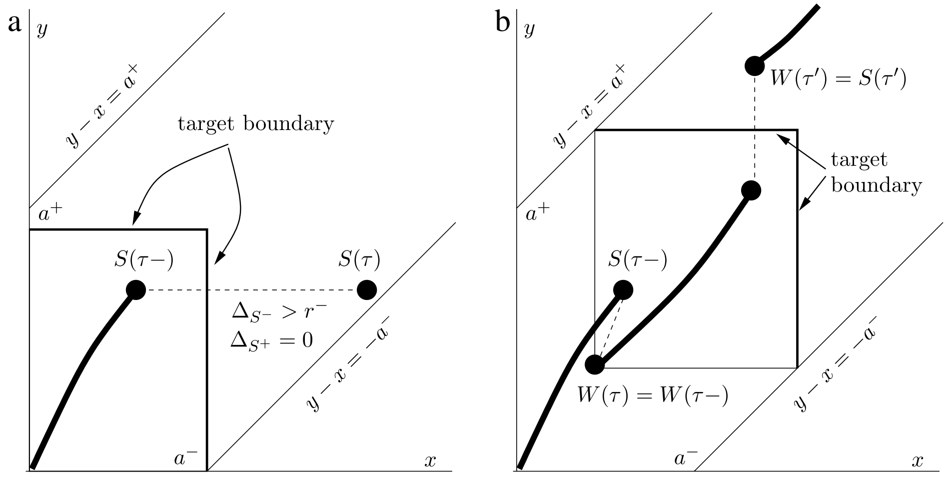

Fig. 3. Sampling the FEE of Z out of $[-a^-, a^+]$ with $a^\pm > 0$.

Table 2

(a) Sampling of $(\tau, Z^+(\tau-), Z^-(\tau), \Delta_Z(\tau))$, where $\tau = \tau_a^Z \wedge K$. To start with, $T = H^+ = H^- = 0$, $A = K$, $b = a$.

1. Sample $(t, z^+, v) \sim (\tau^+, Z^+(\tau^+-), \Delta_{Z^+}(\tau^+))$, where $\tau^+ = \tau_b^{Z+} \wedge A$. Set $x = z^+ + v$.
2. Sample $z^- \sim Z^-(t)$.
3. Update $T \leftarrow T + t$, $H^+ \leftarrow H^+ + x$, $H^- \leftarrow H^- + z^-$.
4. If $x - z^- < b$ and $t < A$, then update $A \leftarrow A - t$, $b \leftarrow b + z^- - x$, and go back to step 1; else output $(T, H^+ - v, H^-, v)$ and stop.

(b) Sampling of $(\tau, Z^+(\tau-), Z^-(\tau-), \Delta_{Z^+}(\tau), \Delta_{Z^-}(\tau))$, where $\tau = \tau_I^Z \wedge K$. To start with, $T = H^+ = H^- = D = 0$, $A = K$, $b^+ = a^+$, $b^- = a^-$.

1. If $D = 0$, then sample $(D, J) \sim (\tilde\tau \wedge A, \Delta_Q(\tilde\tau \wedge A))$, with $\tilde\tau = \inf\{t : Q(t) \neq 0\}$, and set $J^\pm = (\pm J) \vee 0$.
2. For $\sigma = \pm$, sample $t^\sigma \sim \tau^S_{b \wedge r}(\sigma)$. Set $t = t^+ \wedge t^- \wedge D$. (Note: w.p. 1, t^+, t^-, and D are different from each other.)
3. For $\sigma = \pm$, sample $(x^\sigma, v^\sigma) \sim (X_1^\sigma(t-), \Delta_{X_1^\sigma}(t))$ conditional on $\tau^+ \wedge \tau^- \wedge D = t$, by applying steps 3–6 in Table 1 to X^σ.
4. Update $T \leftarrow T + t$. For $\sigma = \pm$, set $\Delta^\sigma = v^\sigma + \mathbf{1}_{\{t=D\}} J^\sigma$, $z^\sigma = x^\sigma + \Delta^\sigma$, and update $H^\sigma \leftarrow H^\sigma + z^\sigma$.
5. If $z^+ - z^- \in (-b^-, b^+)$ and $t < A$, then update $A \leftarrow A - t$, $D \leftarrow D - t$, $b^+ \leftarrow b^+ + z^- - z^+$, $b^- \leftarrow b^- + z^+ - z^-$, and go back to step 1; else output $(T, H^+ - \Delta^+, H^- - \Delta^-, \Delta^+, \Delta^-)$ and stop.

to jointly sample τ, $Z^+(\tau-)$, $Z^-(\tau)$, and $\Delta_Z(\tau)$ is presented in Table 2(a). By assumption, $Z^\pm$ cannot be both CP. If Π^+ (resp. Π^-) can be decomposed as in (2.1), then the procedure in Table 1 can be called in step 1 (resp. 2) in Table 2. On the other hand, if one of $Z^\pm$ is CP, the corresponding step is straightforward.

Theorem 2.2. *Suppose* $\overline{\lim}_{t\to\infty} Z(t) = \infty$ *w.p.* 1 *or* $K < \infty$. *The procedure in* Table 2(a) *stops w.p.* 1, *and its random output follows the distribution of* $(\tau, Z^+(\tau-), Z^-(\tau), \Delta_Z(\tau))$.

Now let $\tau = \tau_I^Z \wedge K$ and suppose Z has no drift. Then $Z^\pm$ have no drift, so Z makes a positive jump if it first exits I at a^+, and a negative jump if it first exists I at $-a^-$. Thus τ, $Z^\pm(\tau-)$, and $\Delta_{Z^\pm}(\tau)$ characterize the FEE. A procedure to sample the random variables is shown in Table 2(b). In it, for $\sigma = \pm$, X_i^σ are defined by the same rule in Section 2.1, $Q \sim \mathrm{BV}(\chi^+ + \chi^-)$, and all the processes are independent. In each iteration, for $\sigma = \pm$, we have to monitor when

$S^\sigma = X_1^\sigma + X_2^\sigma + X_3^\sigma$ crosses $b^\sigma \wedge r^\sigma$, where b^σ is a constant obtained from a^σ. For brevity, denote by $\tau^S_{b\wedge r}(\sigma)$ the FPT of S^σ across $b^\sigma \wedge r^\sigma$.

Theorem 2.3. *Suppose* $Z \not\equiv 0$ *and* $0 < a^\pm < \infty$. *The procedure in Table 2(b) stops w.p. 1, and its random output follows the distribution of* $(\tau, Z^+(\tau-), Z^-(\tau-), \Delta_{Z^+}(\tau), \Delta_{Z^-}(\tau))$.

3. Examples

3.1. Sampling issues involved

For the procedures in Section 2, the main task is to sample random variables of the following types:

(1) the first jump of $Q \sim \mathrm{BV}(\chi)$ with $0 < |\chi| < \infty$;
(2) $\tau^S_c = \inf\{t : S(t) > c(t)\}$ with c being regular;
(3) $(S(t-), \Delta_S(t))$, conditional on $\tau^S_c = t$;
(4) $S(t)$, conditional on $S(t) < c(t)$; and
(5) $X_1(t)$, conditional on $S(t) = s \le r$.

For (1), the time and size of the first jump of Q are independent following $\mathrm{Exp}(1/|\chi|)$ and $\chi/|\chi|$, respectively [3,34]. If $|\chi|$ is unavailable or $\chi/|\chi|$ cannot be directly sampled, the following rejection sampling known as thinning can be used. Let μ be a finite Lévy measure such that $\mathrm{d}\chi = \varrho\,\mathrm{d}\mu$ for some function $\varrho \le 1$. To start with, set $t = 0$.

1. Sample $s \sim \mathrm{Exp}(1/|\mu|)$ and $x \sim \mu/|\mu|$. Update $t \leftarrow t + s$.
2. Sample $U \sim U(0, 1)$. If $U \le \varrho(x)$, then stop and output (t, x); else go back to step 1.

For (2), since S is strictly increasing w.p. 1 and c is nonincreasing,

$$\Pr\{\tau^S_c \le t\} = \Pr\{S(t) \ge c(t)\}, \tag{3.1}$$

which is continuous and strictly increasing in $t > 0$. If the distribution of $S(t)$ is analytically available for each $t > 0$, then τ^S_c may be sampled by the inversion method. Alternatively, if S has scaling property, it can be utilized to sample τ^S_c. Both possibilities will be demonstrated later. The sampling for (3) heavily relies on the distributional properties of the FPE obtained in Section 4. The sampling for (4) has a generic solution, which is to keep sampling $x \sim S(t)$ until $x \le a$. However, by utilizing the structure of $S(t)$, it is possible to make the sampling more efficient.

Finally, for (5), in the nontrivial case $q > 0$, if $S(t)$ has a bounded pdf g_t, then in principle rejection sampling can be used. Indeed, as $X_1(t) + X_2(t) \sim \mathrm{BV}(\nu)$ and $X_1(t) \sim \mathrm{BV}(e^{-qx}\nu(\mathrm{d}x))$ with $\nu(\mathrm{d}x) = t\mathbf{1}_{\{x\le r\}}\Lambda(\mathrm{d}x)$, $\Pr\{X_1(t) \in \mathrm{d}x\} \propto e^{-qx}\Pr\{X_1(t) + X_2(t) \in \mathrm{d}x\}$. On the other hand, for $x \in (0, r]$, $\Pr\{S(t) \in \mathrm{d}x\} = \Pr\{X_3(t) = 0\}\Pr\{X_1(t) + X_2(t) \in \mathrm{d}x\}$ as $X_3(t)$ is either 0 or $>r$. Then $X_1(t)$ has a pdf on $(0, r]$ which is in proportion to $e^{-qx}g_t(x)$, giving

$$\Pr\{X_1(t) \in \mathrm{d}x \mid S(t) = s\} \propto e^{-qx}g_t(x)\Pr\{s - X_2(t) \in \mathrm{d}x\}, \quad s \in (0, r]. \tag{3.2}$$

Thus, we may keep sampling $x \sim s - X_2(t)$ and $U \sim U(0, 1)$ until $U \le e^{-qx}g_t(x)/\sup g_t$ and then output x. Since $X_2(t)$ is CP, its sampling is standard. However, g_t can be hard to evaluate. To get around the problem, the structure of $S(t)$ may be exploited.

3.2. Exponentially tilted upper truncated stable Lévy density

Let $Z \sim \mathrm{BV}(\varphi(x)\,\mathrm{d}x + \chi(\mathrm{d}x))$ with φ as in (1.1). We next apply Table 1 to devise an algorithm to sample the FPE of Z across a regular boundary c. Using Table 2, an algorithm can be devised to sample the FPE of a process with nonpositive drift across a constant level, or the FEE of a process with no drift out of an interval, when the Lévy measure is $\mathbf{1}_{\{0<x\le r^+\}}\gamma^+ e^{-q^+x}x^{-1-\alpha^+}\,\mathrm{d}x + \mathbf{1}_{\{-r^-\le x<0\}}\gamma^- e^{-q^-x}|x|^{-1-\alpha^-}\,\mathrm{d}x + \chi(\mathrm{d}x)$ with $\alpha^\pm \in (0,1)$. The details are omitted for brevity. The algorithm can be extended to finite mixtures of upper truncated stable Lévy densities as well; see [9] for details.

To start with, $S = X_1 + X_2 + X_3$ is stable with $\mathrm{E}[e^{-\lambda S(t)}] = \exp\{-t\gamma\Gamma(1-\alpha)\alpha^{-1}\lambda^\alpha\}$, $\lambda \ge 0$. By scaling t, assume $\gamma = \alpha/\Gamma(1-\alpha)$. Then the pdf of $S(1)$ is

$$f(x) = \frac{\alpha}{(1-\alpha)\pi}\int_0^\pi h(x,\theta)\,\mathrm{d}\theta \tag{3.3}$$

where for $\theta \in (0,\pi)$ and $x > 0$, letting $h_0(\theta) = \sin[(1-\alpha)\theta][\sin(\alpha\theta)]^{\alpha/(1-\alpha)}(\sin\theta)^{-1/(1-\alpha)}$, $h(x,\theta) = \mathbf{1}_{\{x>0\}}h_0(\theta)\,x^{-1/(1-\alpha)}\exp\{-h_0(\theta)\,x^{-\alpha/(1-\alpha)}\}$. The sampling of $S(1)$ is well known [7,10,38]. Define $\psi(x) = \mathbf{1}_{\{x\ne 0\}}x^{-1}(1-e^{-x}) + \mathbf{1}_{\{x=0\}}$ and $M_\alpha = (1-\alpha)^{1-1/\alpha}\alpha^{-1-1/\alpha}e^{-1/\alpha}$. Given $0 < K \le \infty$, the following algorithm samples $\tau = \tau_c^Z \wedge K$, $Z(\tau-)$, and $\Delta_Z(\tau)$ jointly. To begin with, set $T = H = D = 0$, $A = K$.

1. Sample (D, J) as in step 1 in Table 1.
2. Set $b = c \wedge r$. Sample $S(1)$. Set t_1 such that $t_1^{1/\alpha}S(1) = b(t_1)$, $t = t_1 \wedge D$, and $z = b(t)$.
3. If $t = t_1 < D$, then set $w_0 = -b'(t)$, $w_1 = \gamma z^{1-\alpha}/[\alpha(1-\alpha)]$, and do the following steps. (Note: w.p. 1, b is differentiable at t with a nonpositive derivative.)
   (a) Sample $\vartheta \sim U(0,\pi)$ and $\iota \in \{0,1\}$ with $\Pr\{\iota = i\} = w_i/(w_0+w_1)$. If $\iota = 0$, then set $s = z$, $v = 0$; else sample $\beta_1 \sim \mathrm{Beta}(1, 1-\alpha)$, $\beta_2 \sim \mathrm{Beta}(\alpha, 1)$, and set $s = \beta_1 z$, $v = (z-s)/\beta_2$.
   (b) Sample $U \sim U(0,1)$. If $U > h(t^{-1/\alpha}s, \vartheta)/M_\alpha$, then go back to step 3(a).
4. If $t = D < t_1$, then sample $S(t)$ conditional on $S(t) \le z$. Set $s = S(t)$ and $v = 0$.
5. Set $m = s^{1-\alpha}t\gamma q\Gamma(1-\alpha)$, $d_k = k!\Gamma(1+k(1-\alpha))$, $C_k = m^k/d_k$, and $C = \sum_{k=0}^\infty C_k$, and do the following steps.
   (a) Sample $\vartheta \sim U(0,\pi)$ and $\kappa \in \{0,1,2\ldots\}$, such that $\Pr\{\kappa = k\} = C_k/C$.
   (b) If $\kappa = 0$, set $x = s$, $\varrho = 1$; else sample $\beta \sim \mathrm{Beta}(1, k(1-\alpha))$ and $(\omega_1,\ldots,\omega_\kappa) \sim \mathrm{Di}(1-\alpha,\ldots,1-\alpha)$, and set $x = s\beta$, $\varrho = \prod_i \psi(q(s-x)\omega_i)$.
   (c) Sample $U \sim U(0,1)$. If $M_\alpha U > \varrho e^{-qx}h(t^{-1/\alpha}x, \vartheta)$, then go back to step 5(a); else go to the next step.
6. The rest is the same as steps 6–8 in Table 1.

The steps of the algorithm correspond one-to-one to those in Table 1. Only the details of steps 2, 3 and 5 need to be verified. With t_1 being the unique solution to $t^{1/\alpha}S(1) = b(t)$, from (3.1) and the scaling property of S, $\Pr\{\tau_b^S \le t\} = \Pr\{t^{1/\alpha}S(1) \ge b(t)\} = \Pr\{t_1 \le t\}$. Thus the (t_1, t, z) in step 2 is a sample of $(\tau_b^S, \tau^*, b(\tau^*))$, where $\tau^* = \tau_b^S \wedge D$. Given $\tau_b^S = t_1$ and $\tau^* = t$, by step 3 in Table 1, if $t = t_1 < D$, then we need to sample $(S(t-), \Delta_S(t))$ conditional on $\tau_b^S = t$. From Theorem 4.4 in the next section, letting $z = b(t)$, $w_0 = |b'(t)|$, $w_1 = \gamma z^{1-\alpha}/[\alpha(1-\alpha)]$,

$$\Pr\{S(t-) \in \mathrm{d}s,\ \Delta_S(t) \in \mathrm{d}v \mid \tau_b^S = t\} \propto g_t(s)\,[w_0\delta(\mathrm{d}s - z)\,\delta(\mathrm{d}v) + w_1\rho(s,v)\,\mathrm{d}s\,\mathrm{d}v],$$

where $\rho(s,v) = \mathbf{1}_{\{0\le z-s<v\}}\alpha(1-\alpha)z^{-1+\alpha}v^{-1-\alpha}$ is a pdf. Define (ι, ζ, V) such that $\Pr\{\iota = 0\} = 1 - \Pr\{\iota = 1\} = w_0/(w_0+w_1)$, $\Pr\{\zeta = z, V = 0 \mid \iota = 0\} = 1$, and

$\Pr\{\zeta \in \mathrm{d}s,\ V \in \mathrm{d}v \mid \iota = 1\} = \rho(s, v)\,\mathrm{d}s\,\mathrm{d}v$. Let $\vartheta \sim U(0, \pi)$ be independent of (ι, ζ, V). Then by $g_t(s) = t^{-1/\alpha} f(t^{-1/\alpha} s)$,

$$\Pr\{S(t-) \in \mathrm{d}s,\ \Delta_S(t) \in \mathrm{d}v \mid \tau_b^S = t\}$$
$$\propto \int h(t^{-1/\alpha} s, \theta) \Pr\{\vartheta \in \mathrm{d}\theta,\ \iota \in \mathrm{d}i,\ \zeta \in \mathrm{d}s,\ V \in \mathrm{d}v\}, \tag{3.4}$$

with θ and i being integrated. Step 3(a) samples $(\vartheta, \iota, \zeta, V)$. To see this, note that $(\zeta, V) \sim \rho$ if $\zeta = (1 - U_1^{1/(1-\alpha)})z$ and $V = (z - \zeta)U_2^{-1/\alpha}$, with U_i i.i.d. $\sim U(0, 1)$. Next, for $x > 0$ and $\theta \in (0, \pi)$,

$$h(x, \theta) \le \sup_{\theta\in(0,\pi)} \left[h_0(\theta) \times \sup_{z>0}(z^{1/\alpha} e^{-h_0(\theta) z}) \right] = \alpha^{-1/\alpha} e^{-1/\alpha} \sup_{\theta\in(0,\pi)} h_0(\theta)^{1-1/\alpha}.$$

By $\sin(t\theta)/\sin(\theta) \ge t$ for $\theta \in (0, \pi)$ and $t \in (0, 1)$, $h_0(\theta) \ge (1-\alpha)\alpha^{\alpha/(1-\alpha)}$, so $h(x, \theta) \le M_\alpha$. Thus step 3 is rejection sampling of the distribution in proportion to $h(t^{-1/\alpha} s, \theta) \Pr\{\vartheta \in \mathrm{d}\theta,\ \iota \in \mathrm{d}i,\ \zeta \in \mathrm{d}s,\ V \in \mathrm{d}v\}$, so by (3.4), (s, v) is a sample of $(S(t-), \Delta_S(t))$ conditional on $\tau_b^S = t$.

By step 5 in Table 1, given $(\tau^*, S(\tau^*-)) = (t, s)$ with $s \in (0, r]$, we need to sample $X_1(t)$ conditional on $S(t) = s$. To this end we shall use (3.2). Since $X_2(t)$ is CP with Lévy density $\lambda(x) = \mathbf{1}_{\{0<x\le r\}} t\gamma(1 - e^{-qx})x^{-1-\alpha}$,

$$\Pr\{X_2(t) \in s - \mathrm{d}x\} \propto \mathbf{1}_{\{0\le x\le s\}} \left[\delta(\mathrm{d}x - s) + \sum_{k=1}^{\infty} \frac{\lambda^{*k}(s - x)\,\mathrm{d}x}{k!} \right],$$

where λ^{*k} is the k-fold convolution of λ. For $w > 0$ and $k \ge 1$, $\lambda^{*k}(w) = \int w^{k-1} \prod_{i=1}^k \lambda(wv_i)\,\sigma_k(\mathrm{d}v)$ with σ_k the measure specified in Section 1.1. Since $0 \le w \le s \le r$, by definition of ψ and Dirichlet distribution, for any $v = (v_1, \ldots, v_k)$ with $v_i \ge 0$ and $\|v\| = 1$,

$$w^{k-1} \prod_{i=1}^k \lambda(wv_i) = w^{k-1}(t\gamma)^k \prod_{i=1}^k \frac{1 - e^{-qwv_i}}{(wv_i)^{1+\alpha}} = w^{k(1-\alpha)-1}(t\gamma)^k q^k \prod_{i=1}^k \psi(qwv_i) \prod_{i=1}^k \frac{1}{v_i^\alpha}$$
$$= w^{k(1-\alpha)-1} \frac{[t\gamma q\Gamma(1-\alpha)]^k}{\Gamma(k(1-\alpha))} f_k(v) \prod_{i=1}^k \psi(qwv_i),$$

with f_k the pdf of $\mathrm{Di}(1 - \alpha, \ldots, 1 - \alpha)$. Let $\omega_k \sim f_k$. For $x \in [0, s]$ and $w = (w_1, w_2, \ldots) \in \cup_{j\ge1} \mathbb{R}^j$, define $\varrho(x, w) = \prod_{i=1}^k \psi(q(s - x)w_i)$, with k the dimension of w. Then

$$\lambda^{*k}(s - x) = (s - x)^{k(1-\alpha)-1} \times \frac{[t\gamma q\Gamma(1-\alpha)]^k}{\Gamma(k(1-\alpha))} \mathrm{E}[\varrho(x, \omega_k)].$$

Since $\mathbf{1}_{\{0\le x\le s\}} k(1-\alpha)(s-x)^{k(1-\alpha)-1}/s^{k(1-\alpha)}$ is the pdf of $s\beta_k$ with $\beta_k \sim \mathrm{Beta}(1, k(1-\alpha))$, then $\lambda^{*k}(s - x)\,\mathrm{d}x = k! C_k \Pr\{s\beta_k \in \mathrm{d}x\}\mathrm{E}[\varrho(x, \omega_k)]$, where C_k is as in the algorithm. This combined with (3.2) and (3.3) yields

$$\Pr\{X_1(t) \in \mathrm{d}x \mid X_1(t) + X_2(t) = s\}$$
$$\propto \int_0^\pi e^{-qx} h(t^{-1/\alpha} x, \theta)\,\mathrm{d}\theta \left\{ \delta(\mathrm{d}x - s) + \sum_{k=1}^{\infty} C_k \Pr\{s\beta_k \in \mathrm{d}x\}\mathrm{E}[\varrho(x, \omega_k)] \right\}.$$

Now the treatment is similar to step 3. Define random vector (κ, ζ, ω), such that $\kappa \in \{0, 1, 2, \ldots\}$ with $\Pr\{\kappa = k\} \propto C_k$, conditional on $\kappa = 0$, $\zeta = s$, $\omega = 0$, and conditional on $\kappa = k \ge 1$,

$\zeta \sim s\beta_k$ and $\omega \sim \omega_k$ are independent. Finally, let $\vartheta \sim U(0, \pi)$ be independent from (κ, ζ, ω). Then

$$\begin{aligned}&\Pr\{X_1(t) \in \mathrm{d}x \mid X_1(t) + X_2(t) = s\}\\&\quad\propto \int e^{-qx} h(t^{-1/\alpha}x, \theta)\varrho(x, w) \Pr\{\vartheta \in \mathrm{d}\theta,\ \kappa \in \mathrm{d}k,\ \zeta \in \mathrm{d}x,\ \omega \in \mathrm{d}w\},\end{aligned}$$

with θ, k, and w being integrated. It is then seen step 5 is rejection sampling of $X_1(t)$ conditional on $X_1(t) + X_2(t) = s$.

3.3. *Upper truncated Gamma Lévy density*

Let $Z \sim \mathrm{BV}(\Pi)$ be a subordinator with $\Pi(\mathrm{d}x) = \mathbf{1}_{\{0<x\le r\}} e^{-x} x^{-1}\,\mathrm{d}x + \chi(\mathrm{d}x)$ and c be regular. For Lévy measures $\tilde{\Pi}(\mathrm{d}x) = \varphi(x)\,\mathrm{d}x + \tilde{\chi}(\mathrm{d}x)$ with $\varphi(x) = \mathbf{1}_{\{0<x\le\tilde{r}\}} \gamma e^{-qx} x^{-1}$, $q > 0$, $\gamma > 0$, the sampling of the FPE can be reduced to that for Π. Indeed, if $\tilde{Z} \sim \mathrm{BV}(\tilde{\Pi})$, then letting $r = q\tilde{r}$ and $\chi(\mathrm{d}x) = \tilde{\chi}(\mathrm{d}x/q)$, $Z(t) = q\tilde{Z}(t/\gamma) \sim \mathrm{BV}(\Pi)$ and so the FPE of $\tilde{Z}$ across c can be obtained from that of Z across $qc(t/\gamma)$.

The sampling here is somewhat simpler than the one in Section 3.2, as exponential tilting itself is part of the Gamma Lévy measure. Let $X_1 \sim \mathrm{BV}(\mathbf{1}_{\{0<x\le r\}} e^{-x} x^{-1}\,\mathrm{d}x)$, $X_2 \sim \mathrm{BV}(\mathbf{1}_{\{x>r\}} e^{-x} x^{-1}\,\mathrm{d}x)$, and $Q \sim \mathrm{BV}(\chi)$ be independent. Then $S = X_1 + X_2$ is a Gamma process with $S(1) \sim \mathrm{Exp}(1)$. Given $0 < K \le \infty$, an algorithm to sample $\tau = \tau_c^Z \wedge K$, $Z(\tau-)$, and $\Delta_Z(\tau)$ jointly is as follows. To begin with, set $T = H = D = 0$, $A = K$.

1. Sample (D, J) as in step 1 in Table 1.
2. Set $b = c \wedge r$. Sample $U \sim U(0, 1)$. Set t_1 such that $\int_{b(t_1)}^\infty x^{t_1-1} e^{-x}\,\mathrm{d}x/\Gamma(t_1) = U$, $t = t_1 \wedge D$, and $z = b(t)$.
3. If $t = t_1 < D$, then set $w_0 = -b'(t)$, $w_1 = 2B(t, 1/2)z/e$, $w_2 = 1/t$,
$$\begin{aligned}h_1(x, v) &= \mathbf{1}_{\{0\le z-x<v\le z\}} e^{1+z-x-v}(1 - x/z)^{1/2} \ln[(1 - x/z)^{-1}]/2,\\ h_2(x, v) &= \mathbf{1}_{\{0\le z-x<z<v\}} z e^{-x}/v,\end{aligned}$$
and do the following steps.
   (a) Sample $\iota \in \{0, 1\}$, such that $\Pr\{\iota = i\} = w_i/(w_0 + w_1 + w_2)$. If $\iota = 0$, then set $x = z$, $v = 0$, $\eta = 1$; if $\iota = 1$, then sample $\beta \sim \mathrm{Beta}(t, 1/2)$, $\xi \sim U(0, 1)$, and set $x = z\beta$, $v = z(1 - \beta)^\xi$, $\eta = h_1(x, v)$; if $\iota = 2$, then sample $\beta \sim \mathrm{Beta}(t, 1)$, $\xi \sim \mathrm{Exp}(1)$, and set $x = z\beta$, $v = z + \xi$, $\eta = h_2(x, v)$.
   (b) Sample $U \sim U(0, 1)$. If $U > \eta$, then go back to step 3(a).
4. If $t = D < t_1$, then sample $\gamma \sim \mathrm{Gamma}(D, 1)$ conditional on $\gamma \le z$. Set $x = \gamma$, $v = 0$.
5. The rest is the same as steps 6–8 in Table 1.

To verify the algorithm, from (3.1) and $S(t) \sim \mathrm{Gamma}(t, 1)$,

$$\Pr\{\tau_b^S \le t\} = \Pr\{S(t) \ge b(t)\} = \frac{1}{\Gamma(t)} \int_{b(t)}^\infty x^{t-1} e^{-x}\,\mathrm{d}x,$$

which is continuous and strictly increasing in t. Thus step 2 uses the inversion method to sample τ_b^S and the (t_1, t, z) it generates is a sample of $(\tau_b^S, \tau^*, b(\tau^*))$, where $\tau^* := \tau_b^S \wedge D$. Given $\tau_b^S = t_1$ and $\tau^* = t$, by step 3 in Table 1, if $t = t_1 < D$, then we need to sample $(S(t-), \Delta_S(t))$

conditional on $\tau_b^S = t$. Let g_t denote the pdf of Gamma$(t, 1)$. From Theorem 4.4, for $x, v > 0$,

$$\begin{aligned}\Pr\{S(t-) \in \mathrm{d}x,\ \Delta_S(t) \in \mathrm{d}v \mid \tau_b^S = t\} &\propto |b'(t)| g_t(z)\delta(\mathrm{d}x - z)\delta(\mathrm{d}v) \\ &\quad + \mathbf{1}_{\{0\le z-x<v\}} g_t(x) v^{-1} e^{-v}\,\mathrm{d}x\,\mathrm{d}v \\ &\propto |b'(t)|\delta(\mathrm{d}x - z)\delta(\mathrm{d}v) \\ &\quad + q_1(x, v)\,\mathrm{d}x\,\mathrm{d}v + q_2(x, v)\,\mathrm{d}x\,\mathrm{d}v,\end{aligned}$$

where, letting $q(x, v) = g_t(x)e^{-v}/[v g_t(z)]$,

$$q_1(x, v) = \mathbf{1}_{\{0\le z-x<v\le z\}} q(x, v), \qquad q_2(x, v) = \mathbf{1}_{\{0\le z-x<z<v\}} q(x, v).$$

Now $q(x, v) = (x/z)^{t-1} e^{z-x-v}/v$. Let

$$\begin{aligned}\rho_1(x, v) &= \mathbf{1}_{\{0\le z-x<v\le z\}} \frac{(x/z)^{t-1}(1 - x/z)^{-1/2}}{B(t, 1/2) z}\, \frac{1}{v \ln[(1 - x/z)^{-1}]}, \\ \rho_2(x, v) &= \mathbf{1}_{\{0\le z-x<z<v\}} t (x/z)^{t-1} e^{z-v}/z.\end{aligned}$$

For $i = 1, 2$, $\rho_i(x, v)$ is a pdf and $q_i(x, v) = w_i h_i(x, v)\rho_i(x, v)$, where w_i and h_i are defined in the algorithm. It is easy to check $h_i(x, v) \le 1$ for $i = 1, 2$. Define $h_0(x, v) \equiv 1$. Define (ι, ζ, V) such that $\iota \in \{0, 1, 2\}$ with $\Pr\{\iota = i\} = w_i/(w_0 + w_1 + w_2)$, conditional on $\iota = 0$, $\zeta = z$ and $V = 0$, and conditional on $\iota = i \in \{1, 2\}$, (ζ, V) has pdf ρ_i. Then

$$\Pr\{S(t-) \in \mathrm{d}x,\ \Delta_S(t) \in \mathrm{d}v \mid \tau_b^S = t\} = \int h_i(x, v) \Pr\{\iota \in \mathrm{d}i,\ \zeta \in \mathrm{d}x,\ V \in \mathrm{d}v\},$$

with only i being integrated. It is easy to check that $(z\beta_t, z(1 - \beta_t)^U)$ has pdf ρ_1, with $\beta_t \sim$ Beta$(t, 1/2)$ and $U \sim U(0, 1)$ independent, and $(z\beta_t', z+\xi)$ has pdf ρ_2, with $\beta_t' \sim$ Beta$(t, 1)$ and $\xi \sim$ Exp(1) independent. Then step 3 in the algorithm is rejection sampling of $(S(t-), \Delta_S(t))$ conditional on $\tau_b^S = t$. Since $S(t-)$ sampled by step 3 or 4 is exactly $X_1(t-)$, there is no need for a step like step 5 in Table 1. We can directly proceed to steps 6–8 in Table 1.

4. Distributions of the FPE

4.1. Main results and the proof of Theorem 2.1

Consider the following general setting. Let $X \sim \mathrm{BV}(\Pi)$ take values in $[0, \infty)^k$ and Δ_X be its jump process. Then $S = \|X\| \sim \mathrm{BV}(\Pi_S)$ is a subordinator with $\Pi_S(\mathrm{d}s) = \mathbf{1}_{\{s>0\}} \int \mathbf{1}_{\{\|x\|\in \mathrm{d}s\}} \Pi(\mathrm{d}x)$ and $\Delta_S = \|\Delta_X\|$. Denote $\overline{\Pi}_S(s) = \Pi_S(s, \infty)$.

Theorem 4.1. *Suppose $|\Pi| = \infty$. Fix a nonincreasing function $c \in C(0, \infty)$ with $c(0+) > 0$ and put $\tau = \tau_c^S$. Then* (1) *for $t > 0$, $u \in [0, \infty)^k$, and $0 \ne v \in [0, \infty)^k$,*

$$\Pr\{\tau \in \mathrm{d}t,\ X(\tau-) \in \mathrm{d}u,\ \Delta_X(\tau) \in \mathrm{d}v\} = \mathbf{1}_{\{0\le c(t)-\|u\|<\|v\|\}} \mathrm{d}t\ \Pr\{X(t) \in \mathrm{d}u\}\, \Pi(\mathrm{d}v) \tag{4.1}$$

and

$$\begin{aligned}&\Pr\{\tau \in \mathrm{d}t,\ X(\tau-) \in \mathrm{d}u,\ \Delta_X(\tau) = 0\} \\ &\quad = \Pr\{\tau \in \mathrm{d}t,\ S(\tau) = c(\tau)\} \Pr\{X(t) \in \mathrm{d}u \mid S(t) = c(t)\};\end{aligned} \tag{4.2}$$

and (2) *for $t > 0$, $s \in [0, c(t)]$, $z > c(t) - s$, $u \in [0, \infty)^k$, and $0 \ne v \in [0, \infty)^k$,*

$$\begin{aligned}&\Pr\{X(\tau-) \in \mathrm{d}u,\ \Delta_X(\tau) \in \mathrm{d}v \mid \tau = t,\ S(\tau-) = s,\ \Delta_S(\tau) = z\} \\ &\quad = \Pr\{X(t) \in \mathrm{d}u \mid S(t) = s\}\, \Pi_z(\mathrm{d}v)\end{aligned} \tag{4.3}$$

and

$$\Pr\{X(\tau-) \in \mathrm{d}u \mid \tau = t,\ \Delta_S(\tau) = 0\} = \Pr\{X(t) \in \mathrm{d}u \mid S(t) = c(t)\}, \tag{4.4}$$

where $\Pi_z(\mathrm{d}v) = \Pr\{V \in \mathrm{d}v \mid \|V\| = z\}$ *with* V *following the normalization of* $(\|v\| \wedge 1)\Pi(\mathrm{d}v)$.

We next obtain the pdf of τ_c^S at the event that S creeps across a differentiable segment of c. For linear c, the pdf is already obtained in [20]. We need the following condition on the distribution of S, which is satisfied by both stable and Gamma processes.

Definition 4.2. S is said to satisfy the *continuous density condition*, if $S(t)$ has a pdf g_t on $(0, \infty)$ for each $t > 0$ and the mapping $(t, x) \to g_t(x)$ is continuous on $(0, \infty) \times (0, \infty)$.

Proposition 4.3. *Let* $c \in C(0, \infty)$ *be nonincreasing with* $c(0+) > 0$ *and differentiable on an open nonempty* $G \subset (0, \infty)$. *Put* $\tau = \tau_c^S$. *If* S *satisfies the continuous density condition, then for* $t \in G$

$$\Pr\{\tau \in \mathrm{d}t,\ S(\tau) = c(\tau)\} = -c'(t) g_t(c(t))\, \mathrm{d}t. \tag{4.5}$$

Theorem 4.4. *Let* c *be regular. Fix a closed set* F *with* $\ell(F) = 0$ *such that* c *is differentiable on* $(0, \infty) \setminus F$. *Put* $\tau = \tau_c^S$. *Then under the continuous density condition on* S, *w.p.* 1, $\tau \in (0, \infty) \setminus F$ *and for* $u \in [0, \infty)^k$ *and* $0 \neq v \in [0, \infty)^k$,

$$\Pr\{X(\tau-) \in \mathrm{d}u,\ \Delta_X(\tau) \in \mathrm{d}v \mid \tau\} = Z(\tau)^{-1} \mu_\tau(\mathrm{d}u, \mathrm{d}v), \tag{4.6}$$

$$\Pr\{X(\tau-) \in \mathrm{d}u,\ \Delta_X(\tau) = 0 \mid \tau\} = Z(\tau)^{-1} \nu_\tau(\mathrm{d}u), \tag{4.7}$$

where for $t \in (0, \infty) \setminus F$, *letting* $\tilde{c}(t) = -c'(t) g_t(c(t))$,

$$Z(t) = \tilde{c}(t) + \int_0^{c(t)} \overline{\Pi}_S(c(t) - s) \Pr\{S(t) \in \mathrm{d}s\},$$

$\mu_t(\mathrm{d}u, \mathrm{d}v) = \mathbf{1}_{\{0 \le c(t) - \|u\| < \|v\|\}} \Pr\{X(t) \in \mathrm{d}u\}\, \Pi(\mathrm{d}v)$, *and* $\nu_t(\mathrm{d}u) = \tilde{c}(t) \Pr\{X(t) \in \mathrm{d}u \mid S(t) = c(t)\}$.

Based on the above results, we are ready to give

Proof of Theorem 2.1. We only deal with the case where c is a regular function. The case where $c \equiv \infty$ and $K < \infty$ is similar. Let Z, S, X_1, X_2, X_3, and Q be defined as in Section 2.1. Put $X = (X_1, X_2, X_3)$. Then $S = \|X\|$.

Consider the first iteration in Table 1. As $A = K$, $D = \tilde{\tau} \wedge K$. For $t < D$, $Z(t) = X_1(t)$. Note that $b = c \wedge r$ is regular. In step 2, t_1 is a sample of τ_b^S and t that of $\tau^* := \tau_b^S \wedge \tilde{\tau} \wedge K$. By independence of S and Q, $t_1 \neq D$ w.p. 1, so either $t_1 < D$ or $t_1 > D$. Then the (s, v) in steps 3–4 is a sample of $(S(\tau^*-), \Delta_S(\tau^*))$ conditional on $\tau^* = t$. For step 3, this is clear. As for step 4, notice that if $t_1 > D$, then w.p. 1, $S(D-) = S(D) < b(D)$.

Given $(\tau^*, S(\tau^*-), \Delta_S(\tau^*)) = (t, s, v)$, steps 5–6 sample $X_1(\tau^*-)$ and $\Delta_1(\tau^*) := X_1(\tau^*) - X_1(\tau^*-)$ from their joint conditional distribution. Indeed, if $t = t_1 < D$, then by Theorem 4.1(2), $X_1(\tau^*-)$ and $\Delta_1(\tau^*)$ are independent under the conditional distribution, following the distribution of $X_1(t)$ conditional on $S(t) = s$ and that of $\Delta_1(t)$ conditional on $\Delta_S(t) = v$, respectively. This is still true if $t = D < t_1$, as $X(D-) = X(D)$ and $\Delta_1(D) = \Delta_S(D) = 0$ w.p. 1. By $s \le b(t) \le r$, $\Pr\{X_1(t) \in \mathrm{d}x \mid S(t) = s\} = \Pr\{X_1(t) \in \mathrm{d}x \mid X_1(t) + X_2(t) = s\}$, hence the sampling of x in step 5. Clearly, $\Delta_S(t) = 0$ implies

$\Delta_1(t) = 0$. Suppose $\Delta_S(t) = v > 0$. The support of Π_X is within $\{(x_1, x_2, x_3) : x_i \ge 0$, at most one is nonzero$\}$, such that for $y > 0$, $\Pi_X(\mathrm{d}y \times \{0\} \times \{0\}) = e^{-qy}\mathbf{1}_{\{y \le r\}}\Lambda(\mathrm{d}y)$, $\Pi_X(\{0\} \times \mathrm{d}y \times \{0\}) = (1 - e^{-qy})\mathbf{1}_{\{y\le r\}}\Lambda(\mathrm{d}y)$, and $\Pi_X(\{0\} \times \{0\} \times \mathrm{d}y) = \mathbf{1}_{\{y>r\}}\Lambda(\mathrm{d}y)$. Then by Theorem 4.1(2), $\Pr\{\Delta_1(t) \in \mathrm{d}y \mid \Delta_S(t) = v\} = \Pi_v(\mathrm{d}y \times \{0\} \times \{0\}) = \mathbf{1}_{\{y=v\le r\}}e^{-qv}$, hence the updating of v in step 6.

Put together, the (t, x, v) generated by the end of step 6 is a sample of $(\tau^*, X_1(\tau^*-), \Delta_1(\tau^*))$. It follows that the Δ and z in step 7 are samples of $\Delta_Z(\tau^*) = \Delta_1(\tau^*) + \Delta_Q(\tau^*)$ and $Z(\tau^*) = X_1(\tau^*-) + \Delta_Z(\tau^*)$, respectively. If the condition of termination is not satisfied, the updating of A and b in step 8 renews the sampling. Note that D is the distance in time to the current jump of Q. Once D becomes 0, the next jump of Q has to be sampled.

Let $T_0 = 0$, and for $n \ge 1$, (T_n, H_n, Δ_n) the value of (T, H, Δ) obtained by the end of the nth iteration. By induction, we can make the following conclusion. For $n \ge 1$, if $Z(T_{n-1}) < c(T_{n-1})$ and $T_{n-1} < K$, then

$$T_n = \inf\{t > T_{n-1} : S(t) - S(T_{n-1}) > [c(t) - Z(T_{n-1})] \wedge r \text{ or } \Delta_Q(t) > 0\} \wedge K, \tag{4.8}$$

$H_n = Z(T_n)$ and $\Delta_n = \Delta_Z(T_n)$. To show that the procedure stops w.p. 1 and returns a sample of $(\tau, Z(\tau-), \Delta_Z(\tau))$, it suffices to show $\Pr\{T_n = \tau \text{ eventually }\} = 1$. Clearly, $T_0 < \tau$. For $n \ge 1$, if $T_{n-1} < \tau$, then, since Z is strictly increasing w.p. 1, $Z(T_{n-1}) < Z(\tau-) \le c(\tau) \le c(T_{n-1})$. Then by (4.8), $T_n > T_{n-1}$. For $t \in (T_{n-1}, T_n)$, since there are no jumps of Q in the interval,

$$Z(t) - Z(T_{n-1}) = X_1(t) - X_1(T_{n-1}) \le S(t) - S(T_{n-1}) \le c(t) - Z(T_{n-1}),$$

with the last inequality due to (4.8). Then $Z(t) \le c(t)$ and hence $T_n \le \tau$. Assume that $T_n \ne \tau$ for all $n \ge 1$. Then $T_1 < T_2 < \cdots < \tau \le K$. Let $\theta = \lim T_n$. Then $\theta \le \tau < \infty$. By quasi-left-continuity of Lévy processes [3, Proposition I.7], $(X(T_n), Q(T_n)) \to (X(\theta), Q(\theta))$. Then $Z(T_n) - Z(T_{n-1}) \to 0$. Meanwhile, since the CP processes X_2, X_3 and Q only have a finite number of jumps in $(0, \theta)$, eventually they have no jumps in (T_n, θ). It follows that for $n \gg 1$,

$$r > Z(T_{n+1}) - Z(T_n) = S(T_{n+1}) - S(T_n) \ge [c(T_{n+1}) - Z(T_n)] \wedge r.$$

It is easy to see that the inequalities imply $Z(T_{n+1}) \ge c(T_{n+1})$ and hence $T_{n+1} \ge \tau$. The contradiction shows that w.p. 1, $T_n = \tau$ for some n. □

4.2. *Proofs of Theorems 2.2 and 2.3*

Proposition 4.5. *Let $X \in \mathrm{BV}(\Pi, d)$ take values in $\mathbb{R}$ with $|\Pi| = \infty$ and $d \le 0$. Then*

$$\Pr\{\exists t > 0 \text{ s.t. } X(s) < X(t) = a \text{ for all } s < t\} = 0, \quad a > 0.$$

Proof. Given $a > 0$, let $\tau^* = \inf\{t : X(s) < X(t) = a \text{ for all } s < t\}$. It suffices to show $\Pr\{\tau^* < \infty\} = 0$. By $\tau^* \le \tau := \tau_a^X$, $\Pr\{\tau^* < \infty\} = \Pr\{\tau^* = \tau < \infty\} + \Pr\{\tau^* < \tau\} \le \Pr\{X(\tau) = a\} + \Pr\{\tau^* < \tau\}$. Since X is not CP, by the argument for Proposition III.2 in [3], $\Pr\{X(\tau) = a, \Delta_X(\tau) \ne 0\} = 0$. Meanwhile, since X has bounded variation and nonpositive drift, $\Pr\{X(\tau) = a, \Delta_X(\tau) = 0\} = 0$ [3, Exercise VI.9]. Then $\Pr\{X(\tau) = a\} = 0$. Assume $\Pr\{\tau^* < \tau\} > 0$. Let $\eta \sim \mathrm{Exp}(1)$ be independent of X. Then $\Pr\{\tau^* < \eta < \tau\} > 0$ and hence $\Pr\{\overline{X}(\eta) = a\} > 0$, where $\overline{X}(t) = \sup\{X(s) : 0 \le s \le t\}$. From the fluctuation identity [3, Theorem VI.5], $\overline{X}(\eta)$ is either constant 0 or ID with Lévy measure $\nu(\mathrm{d}x) = \mathbf{1}_{\{x>0\}} \int_0^\infty t^{-1}e^{-t} \Pr\{X(t) \in \mathrm{d}x\}\,\mathrm{d}t$. In the latter case, the potential measure $U(\mathrm{d}x) = \int_0^\infty e^{-t} \Pr\{X(t) \in \mathrm{d}x\}\,\mathrm{d}t$ is diffuse [3, Proposition I.15], so ν is also diffuse, implying the distribution of $\overline{X}(\eta)$ is continuous on $(0, \infty)$ [34, Remark 27.3 and Theorem 27.4]. As a result, $\Pr\{\overline{X}(\eta) = a\} = 0$. The contradiction completes the proof. $\square$

Applying the result to X and $-X$ respectively and using union-sum inequality, we get

Corollary 4.6. *Let $X \in \mathrm{BV}(\Pi)$ take values in $\mathbb{R}$ with $|\Pi| = \infty$. Then*

$$\Pr\{\exists t > 0 \text{ s.t. } -b < X(s) < a \text{ for all } s < t,\ X(t) = -b \text{ or } a\} = 0, \quad a, b > 0.$$

Proof of Theorem 2.2. Let $T_0 = 0$, $H_0^+ = H^- = 0$, and for $n \ge 1$, let (T_n, H_n^+, H_n^-, v_n) be the value of (T, H^+, H^-, v) at the end of the nth iteration. By induction, for $n \ge 1$, the procedure continues to the nth iteration if and only if $Z(T_k) < a$ and $T_k < K$ for $0 \le k < n$, in which case $T_n = \inf\{t > T_{n-1} : Z^+(t) - Z^+(T_{n-1}) > a - Z(T_{n-1})\} \wedge K > T_{n-1}$, $H_n^+ = Z^+(T_n)$, $H_n^- = Z^-(T_n)$, and $v_n = \Delta_Z(T_n)$. Note that for $n \ge 1$, if the procedure continues to the nth iteration, then

$$Z(t) < a \quad \text{for all } t < T_n. \tag{4.9}$$

Indeed, for $n = 1$, since at least one of $Z^\pm$ is strictly increasing, for $0 \le t < T_1$, $Z(t) = Z^+(t) - Z^-(t) < Z^+(T_1-) \le a$. On the other hand, for $n \ge 2$, by renewal argument, $Z(t) - Z(T_{n-1}) < a - Z(T_{n-1})$ for all $T_{n-1} \le t < T_n$. Then by induction, (4.9) follows.

By assumption, $\tau < \infty$ w.p. 1. To finish the proof, it suffices to show w.p. 1, the procedure stops eventually at the end of an iteration with $T_n = \tau$. The compliment of the event consists of two cases. The first one is that the procedure stops at the end of an iteration with $T_n \ne \tau$. In this case, $T_n < \tau \le K$, otherwise there would be $\tau < T_n \le K$, which leads a contradiction to (4.9). On the one hand, $T_n < \tau$ implies $Z(T_n) \le a$, on the other, $T_n < K$ together with the stopping rule of the procedure implies $Z(T_n) \ge a$. Then $Z(T_n) = a$. By (4.9) and Proposition 4.5, the chance of this is 0. The second case is that the procedure goes on forever. In this case, as pointed out at the beginning, T_n is strictly increasing, $T_n < K$, and $Z(T_n) < a$. Then by (4.9), $T_n < \tau$, giving $\theta = \lim T_n < \infty$. For any $t < \theta$, by (4.9), $Z(t) < a$. Meanwhile, by $Z^+(T_{n+1}) - Z^+(T_n) \ge a - Z(T_n) > 0$ and quasi-left-continuity of Lévy processes [3], letting $n \to \infty$ yields $Z(\theta) = a$. By Proposition 4.5, the chance for such θ to exist is also 0. $\square$

Proof of Theorem 2.3. First, $\tau < \infty$ w.p. 1 [3, Theorem VI.12]. Let $T_0 = 0$, $H_0^+ = H_0^- = 0$, and for $n \ge 1$, let $(T_n, H_n^+, H_n^-, \Delta_n^+, \Delta_n^-)$ be the value of $(T, H^+, H^-, \Delta^+, \Delta^-)$ obtained by the end of the nth iteration. By induction and the same argument as in the proof of Theorem 2.1, for $n \ge 1$, the procedure continues to the nth iteration if and only if $Z(T_k) \in (-a^-, a^+)$ and $T_k < K$ for $0 \le k < n$, and in this case,

$$\begin{aligned} T_n = \inf\{t > T_{n-1} : {}& S^+(t) - S^+(T_{n-1}) > a^+ - Z(T_{n-1}), \\ & S^-(t) - S^-(T_{n-1}) > a^- + Z(T_{n-1}), \text{ or } \Delta_Q(t) > 0\} \wedge K > T_{n-1}, \end{aligned}$$
$$H_n^+ = Z^+(T_n), \qquad H_n^- = Z^-(T_n), \qquad \Delta_n^+ = \Delta_{Z^+}(T_n), \qquad \Delta_n^- = \Delta_{Z^-}(T_n),$$

and, similar to (4.9), $Z(t) \in (-a^-, a^+)$ for all $t < T_n$. The rest of the proof follows that of Theorem 2.2, except Corollary 4.6 is used. $\square$

4.3. Proof of Theorem 4.1

(1) It is clear that $0 < \tau < \infty$ w.p. 1. To show (4.1), following the proof for Proposition III.2 in [3], let $f \ge 0$ be an arbitrary Borel function on $\Omega := (0, \infty) \times [0, \infty)^k \times ([0, \infty)^k \setminus \{0\})$ such

that $f(t, u, v) = 0$ when $\|v\| = c(t) - \|u\|$. Then

$$f(\tau, X(\tau-), \Delta_X(\tau)) = \sum_t f(t, X(t-), \Delta_X(t)) \mathbf{1}_{\{0 \le c(t) - S(t-) < \|\Delta_X(t)\|\}}. \tag{4.10}$$

For each $t > 0$, define function $H_t(v) = f(t, X(t-), v)\mathbf{1}_{\{0 \le c(t) - S(t-) < \|v\|\}}$ on $[0, \infty)^k$. Since $H = (H_t)$ is a predictable process with respect to the filtration generated by Δ_X, by the compensation formula [3, p. 7]

$$\begin{aligned} \mathrm{E}[f(\tau, X(\tau-), \Delta_X(\tau))] &= \int_0^\infty \mathrm{d}t\, \mathrm{E}\left[\int f(t, X(t-), v) \mathbf{1}_{\{0 \le c(t) - S(t-) < \|v\|\}} \Pi(\mathrm{d}v)\right] \\ &\overset{\text{(a)}}{=} \int_0^\infty \mathrm{d}t \int f(t, u, v) \mathbf{1}_{\{0 \le c(t) - \|u\| < \|v\|\}} \Pr\{X(t) \in \mathrm{d}u\}\, \Pi(\mathrm{d}v) \\ &= \int_\Omega \mathbf{1}_{\{0 \le c(t) - \|u\| < \|v\|\}} f(t, u, v)\, \mathrm{d}t\, \Pr\{X(t) \in \mathrm{d}u\}\, \Pi(\mathrm{d}v), \end{aligned}$$

with (a) due to $X(t-) \sim X(t)$. This shows (4.1) for $(t, u, v) \in \Omega$ with $\|v\| \ne c(t) - \|u\|$. Now let $(t, u, v) \in \Omega$ with $\|v\| = c(t) - \|u\|$. Then the RHS of (4.1) is 0. Letting $f(t, u, v) = \mathbf{1}_{\{v = c(t) - u > 0\}}$, by similar derivation as in the above display, but applied to S instead of X,

$$\Pr\{S(\tau-) < S(\tau) = c(\tau)\} = \int_0^\infty \mathrm{d}t \int \Pr\{S(t) \in \mathrm{d}u\} \Pi_S(\{c(t) - u\}).$$

For each t, there is only a countable set of u with $\Pi_S(\{c(t) - u\}) > 0$. On the other hand, by $|\Pi_S| = |\Pi| = \infty$, the distribution of $S(t)$ is continuous, i.e., $\Pr\{S(t) = u\} = 0$ for all u [34, Theorem 27.4]. As a result, $\int \Pr\{S(t) \in \mathrm{d}u\} \Pi_S(\{c(t) - u\}) = 0$ for all $t > 0$, and so the multiple integral is 0. Finally, the proof of (4.1) is complete by

$$\Pr\{\Delta_X(\tau) \ne 0,\ S(\tau) = c(\tau)\} = \Pr\{S(\tau-) < S(\tau) = c(\tau)\} = 0. \tag{4.11}$$

Now consider (4.2). By $|\Pi_S| = \infty$, S is strictly increasing w.p. 1. Clearly, $\Delta_X(\tau) = 0$ implies $S(\tau) = c(\tau)$. Conversely, from (4.11), on the event $S(\tau) = c(\tau)$, $\Delta_X(\tau) = 0$ w.p. 1. Define $\tau^* = \inf\{t \ge 0 : S(t) = c(t)\}$. Then w.p. 1,

$$\{\tau^* < \infty\} = \{\tau = \tau^*\} = \{S(\tau) = c(\tau)\}. \tag{4.12}$$

Let $f \ge 0$ be a Borel function on $(0, \infty) \times [0, \infty)^k$ with bounded support. Then there are two ways to express $\mathrm{E}[f(\tau, X(\tau-))\mathbf{1}_{\{S(\tau) = c(\tau)\}}]$. First, it equals

$$\int f(t, u) \mathbf{1}_{\{\|u\| = c(t)\}} \Pr\{\tau \in \mathrm{d}t,\ X(\tau-) \in \mathrm{d}u,\ \Delta_X(\tau) = 0\}. \tag{4.13}$$

Second, from (4.12), it also equals

$$\begin{aligned} \mathrm{E}[f(\tau, X(\tau))\mathbf{1}_{\{S(\tau) = c(\tau)\}}] &= \mathrm{E}[f(\tau^*, X(\tau^*))\mathbf{1}_{\{\tau^* < \infty\}}] \\ &= \int \mathrm{E}[f(t, X(t)) \mid \tau^* = t] \Pr\{\tau^* \in \mathrm{d}t\} \\ &= \int f(t, u) \Pr\{X(t) \in \mathrm{d}u \mid \tau^* = t\} \Pr\{\tau^* \in \mathrm{d}t\}. \end{aligned}$$

From the definition of τ^* and (4.12), the last integral is equal to

$$\int f(t, u) \Pr\{\tau \in \mathrm{d}t,\ S(\tau) = c(\tau)\} \Pr\{X(t) \in \mathrm{d}u \mid S(t) = c(t)\}. \tag{4.14}$$

Since f is arbitrary, comparing the integrals in (4.13) and (4.14) then yields

$$\mathbf{1}_{\{\|u\|=c(t)\}} \Pr\{\tau \in \mathrm{d}t,\ X(\tau-) \in \mathrm{d}u,\ \Delta_X(\tau) = 0\}$$
$$= \Pr\{\tau \in \mathrm{d}t,\ S(\tau) = c(\tau)\} \Pr\{X(t) \in \mathrm{d}u \mid S(t) = c(t)\}.$$

The qualifier $\mathbf{1}_{\{\|u\|=c(t)\}}$ is redundant in the identity. Then (4.2) follows.

(2) Since $(X, S) \sim \mathrm{BV}(\tilde{\Pi})$, with $\tilde{\Pi}(\mathrm{d}x, \mathrm{d}s) = \Pi(\mathrm{d}x)\,\delta(\mathrm{d}s - \|x\|)$. By similar argument as (1), for $t > 0$, $s \ge 0$, $z > 0$, $u \in [0,\infty)^k$, and $0 \ne v \in [0,\infty)^k$,

$$\Pr\{\tau \in \mathrm{d}t,\ X(\tau-) \in \mathrm{d}u,\ S(\tau-) \in \mathrm{d}s,\ \Delta_X(\tau) \in \mathrm{d}v,\ \Delta_S(\tau) \in \mathrm{d}z\}$$
$$= \mathbf{1}_{\{0 \le c(t)-s<z\}}\mathrm{d}t\ \Pr\{X(t) \in \mathrm{d}u,\ S(t) \in \mathrm{d}s\}\,\Pi(\mathrm{d}v)\,\delta(\mathrm{d}z - \|v\|).$$

On the other hand, applying (4.1) directly to S,

$$\Pr\{\tau \in \mathrm{d}t,\ S(\tau-) \in \mathrm{d}s,\ \Delta_S(\tau) \in \mathrm{d}z\} = \mathbf{1}_{\{0 \le c(t)-s<z\}}\mathrm{d}t\,\Pr\{S(t) \in \mathrm{d}s\}\,\Pi_S(\mathrm{d}z).$$

In order to get (4.3), it then suffices to show $\Pi(\mathrm{d}v)\delta(\mathrm{d}z - \|v\|) = \Pi_z(\mathrm{d}v)\Pi_S(\mathrm{d}z)$ for $0 \ne v \in [0,\infty)^k$ and $z \in (0,\infty)$. Put $C = \int (\|u\| \wedge 1)\Pi(\mathrm{d}u)$, which is positive and finite. Then $\Pr\{V \in \mathrm{d}v\} = (\|v\| \wedge 1)\Pi(\mathrm{d}v)/C$ and $\Pr\{\|V\| \in \mathrm{d}z\} = (z \wedge 1)\Pi_S(\mathrm{d}z)/C$. It follows that

$$\begin{aligned}\Pi_z(\mathrm{d}v)\Pi_S(\mathrm{d}z) &= C\Pr\{V \in \mathrm{d}v \mid \|V\| = z\}\Pr\{\|V\| \in \mathrm{d}z\}/(z \wedge 1)\\ &= C\Pr\{V \in \mathrm{d}v,\ \|V\| \in \mathrm{d}z\}/(z \wedge 1) = C\delta(\mathrm{d}z - \|v\|)\Pr\{V \in \mathrm{d}v\}/(z \wedge 1),\end{aligned}$$

which yields the desired equality.

4.4. *Proof of Theorem 4.4*

We need a few auxiliary results. First, from Theorem 4.1 it is easy to get the following.

Corollary 4.7. *For $a > 0$ and $t > 0$, define*

$$\psi_a(t) = \int_0^a \overline{\Pi}_S(a-u)\Pr\{S(t) \in \mathrm{d}u\}. \tag{4.15}$$

Then, under the same assumption as Theorem 4.1,

$$\Pr\{\tau \in \mathrm{d}t,\ S(\tau) > c(\tau)\} = \psi_{c(t)}(t)\,\mathrm{d}t. \tag{4.16}$$

In particular, if c is constant $a \in (0,\infty)$, then τ has pdf $\psi_a(t)$.

Proof. Apply (4.1) in Theorem 4.1 directly to S to get

$$\Pr\{\tau \in \mathrm{d}t,\ S(\tau) > c(\tau)\} = \mathrm{d}t \int \mathbf{1}_{\{0 \le c(t)-u<v\}} \Pr\{S(t) \in \mathrm{d}u\}\Pi_S(\mathrm{d}v),$$

which is (4.16). If $c \equiv a$, then by $\Pr\{S(\tau) > a\} = 1$ [3, Theorem III.4], $\Pr\{\tau \in \mathrm{d}t\} = \psi_a(t)\,\mathrm{d}t$. $\square$

The following two lemmas will be proved in Section 4.5.

Lemma 4.8. *Under the continuous density condition on S, the mapping $(a,t) \to \psi_a(t)$ is continuous on $(0,\infty) \times (0,\infty)$.*

Lemma 4.9. *Let c be continuous and nonincreasing on $(0,\infty)$ with $c(0+) > 0$. Put $\tau = \tau_c^S$. If S satisfies the continuous density condition, then $\Pr\{\tau \in A\} = 0$ for any $A \subset (0,\infty)$*

with $\ell(A) = c(A) = 0$, where $c(A)$ is the absolute value of the Riemann–Stieltjes integral $\int \mathbf{1}_{\{x\in A\}}\,\mathrm{d}c(x)$.

Proof of Proposition 4.3. It suffices to consider $t \in G$ with $c(t) > 0$. Given such t, put $a = c(t)$. For $\varepsilon > 0$, let $q(\varepsilon) = \Pr\{t-\varepsilon < \tau \le t\}$. Then $q(\varepsilon) = \Pr\{S(t-\varepsilon) < c(t-\varepsilon),\ S(t) \ge a\} = q_1(\varepsilon) + q_2(\varepsilon)$, where $q_1(\varepsilon) = \Pr\{S(t-\varepsilon) < a \le S(t)\}$, $q_2(\varepsilon) = \Pr\{a \le S(t-\varepsilon) < c(t-\varepsilon)\}$. Then $q_1(\varepsilon) = \Pr\left\{t-\varepsilon < \tau_a^S \le t\right\}$. By Corollary 4.7 and Lemma 4.8, the distribution function of τ_a^S is differentiable with derivative $\psi_a(t)$ at t. Then $q_1(\varepsilon)/\varepsilon \to \psi_a(t) = \psi_{c(t)}(t)$ as $\varepsilon \downarrow 0$. On the other hand, $q_2(\varepsilon) = \int_0^{c(t-\varepsilon)-c(t)} g_{t-\varepsilon}(a+x)\,\mathrm{d}x$. Since $(t,x) \to g_t(x)$ is continuous on $(0,\infty)\times(0,\infty)$ and c is differentiable at t, $q_2(\varepsilon)/\varepsilon \to -c'(t)g_t(c(t))$ as $\varepsilon \downarrow 0$. We thus get

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}[\Pr\{\tau \le t\} - \Pr\{\tau \le t-\varepsilon\}] = -c'(t)g_t(c(t)) + \psi_{c(t)}(t).$$

Likewise, as $\varepsilon \downarrow 0$, $\varepsilon^{-1}[\Pr\{\tau \le t+\varepsilon\} - \Pr\{\tau \le t\}]$ converges to the same limit. Thus, the distribution function of τ is differentiable everywhere in the open set $\{t \in G : c(t) > 0\}$, and so $-c'(t)g_t(c(t)) + \psi_{c(t)}(t)$ is the pdf of τ on the set [33, Theorem 7.21]. Then by Corollary 4.7, $\Pr\{\tau \in \mathrm{d}t\} = -c'(t)g_t(c(t))\,\mathrm{d}t + \Pr\{\tau \in \mathrm{d}t,\ S(\tau) > c(\tau)\}$, which yields (4.5). $\square$

We are ready to complete the proof of Theorem 4.4. Since c is absolutely continuous, $c(F) = 0$, so by Lemma 4.9, $\tau \in (0,\infty)\setminus F$ w.p. 1. By Theorem 4.1 and Proposition 4.3, for $t \in (0,\infty)\setminus F$, $u \in [0,\infty)^k$, $0 \ne v \in [0,\infty)^k$,

$$\Pr\{\tau \in \mathrm{d}t,\ X(\tau-) \in \mathrm{d}u,\ \Delta_X(\tau) \in \mathrm{d}v\} = \mathbf{1}_{\{0\le c(t)-\|u\|<\|v\|\}}\mathrm{d}t\,\Pr\{X(t) \in \mathrm{d}u\}\,\Pi(\mathrm{d}v)$$

and

$$\Pr\{\tau \in \mathrm{d}t,\ X(\tau-) \in \mathrm{d}u,\ \Delta_X(\tau) = 0\} = -c'(t)g_t(c(t))\,\mathrm{d}t\,\Pr\{X(t) \in \mathrm{d}u \mid S(t) = c(t)\},$$

which, by definition, are $\mathrm{d}t\,\mu_t(\mathrm{d}u,\mathrm{d}v)$ and $\mathrm{d}t\,\nu_t(\mathrm{d}u)$, respectively. Integrate over u and v to get $\Pr\{\tau \in \mathrm{d}t\} = Z(t)\,\mathrm{d}t$. Then (4.6) and (4.7) follow.

4.5. Additional technical details

To prove Lemmas 4.8 and 4.9, we start with two more lemmas.

Lemma 4.10. *For $t > 0$ and $0 < a < b < \infty$, let*

$$L_1(t,a,b) = \int_0^a [\overline{\Pi}_S(a-u) - \overline{\Pi}_S(b-u)]g_t(u)\,\mathrm{d}u,$$

$$L_2(t,a,b) = \int_a^b \overline{\Pi}_S(b-u)g_t(u)\,\mathrm{d}u.$$

Then for any $E = [t_0,t_1] \subset (0,\infty)$ and $I = [\alpha,\beta] \subset (0,\infty)$,

$$\lim_{r\downarrow 0} \sup\{L_i(t,a,b) : t \in E,\ a \in I,\ a \le b \le a+r\} = 0, \quad i = 1,2. \tag{4.17}$$

Proof. For $a \le b \le a+r$, $L_1(t,a,b) \le L_1(t,a,a+r)$. Given $\varepsilon \in (0,\alpha/2)$, $L_1(t,a,a+r) = J_1 + J_2$ with $J_1 = \int_0^\varepsilon h$ and $J_2 = \int_\varepsilon^a h$, where $h(u) = [\overline{\Pi}_S(a-u) - \overline{\Pi}_S(a+r-u)]g_t(u)$. Then

$$J_1 \le \int_0^\varepsilon \overline{\Pi}_S(a-u)g_t(u)\,\mathrm{d}u \le \overline{\Pi}_S(a-\varepsilon)\int_0^\varepsilon g_t(u)\,\mathrm{d}u$$
$$= \overline{\Pi}_S(\alpha-\varepsilon)\Pr\{S(t) \le \varepsilon\} \le \overline{\Pi}_S(\alpha-\varepsilon)\Pr\{S(t_0) \le \varepsilon\}$$

and letting $M = \sup\{g_t(u) : t \in E,\ \varepsilon \le u \le \beta\}$,

$$J_2 \le M \int_\varepsilon^a [\overline{\Pi}_S(a-u) - \overline{\Pi}_S(a-u+r)]\,\mathrm{d}u \le M \int_0^\beta [\overline{\Pi}_S(u) - \overline{\Pi}_S(u+r)]\,\mathrm{d}u.$$

By assumption on $g_t(x)$, $M < \infty$. Also, $\int_0^\beta \overline{\Pi}_S(u)\,\mathrm{d}u = \int_0^\infty (v \wedge \beta)\Pi_S(\mathrm{d}v) < \infty$. Then by monotone convergence, as $r \downarrow 0$, $J_2 \to 0$ uniformly for $(t, a) \in E \times I$, and so for L_1, the limit in (4.17) is at most $\overline{\Pi}_S(\alpha - \varepsilon)\Pr\{S(t_0) \le \varepsilon\}$. Since $\Pr\{S(t_0) > 0\} = 1$ and ε is arbitrary, the limit 0. Thus (4.17) holds for L_1. Next, for $t \in E$, $a \in I$, and $b \in [a, a+r]$,

$$\begin{aligned} L_2(t,a,b) = \int_0^\infty \Pi_S(\mathrm{d}x) \int_{a\vee(b-x)}^b g_t &\le \int_0^\varepsilon \Pi_S(\mathrm{d}x) \int_{b-x}^b g_t + \int_\varepsilon^\infty \Pi_S(\mathrm{d}x) \int_a^b g_t \\ &\le M' \left[\int_0^\varepsilon x\,\Pi_S(\mathrm{d}x) + r\overline{\Pi}_S(\varepsilon)\right], \end{aligned}$$

where $M' = \sup\{g_t(u) : t \in E, \alpha - \varepsilon \le u \le \beta\}$. Then, as $r \downarrow 0$, the limit for L_2 in (4.17) is at most $M' \int_0^\varepsilon x\,\Pi_S(\mathrm{d}x)$. Since ε is arbitrary, the limit is 0. □

Lemma 4.11. *Let h be a bounded function on $(0,\infty) \times (0,\infty)$. For $a, t \in (0,\infty)$, define*

$$H(a,t) = \int \mathbf{1}_{\{u \le a < x\}} h(u,x) \Pr\{S(t) \in \mathrm{d}u\}\, \Pi_S(\mathrm{d}x - u).$$

Then under the continuous density condition, H is continuous on $(0,\infty) \times (0,\infty)$.

Proof. Let $|h(u,x)| \le 1$. It suffices to show $H \in C(R)$ for any $R = [\alpha,\beta] \times [t_0,t_1] \subset (0,\infty) \times (0,\infty)$. Let $(a,s), (b,t) \in R$. Then

$$|H(b,t) - H(a,s)| \le |H(b,t) - H(a,t)| + |H(a,t) - H(a,s)|.$$

Let L_1 and L_2 be as in Lemma 4.10. Let $a' = a \wedge b$ and $b' = a \vee b$. Then

$$\begin{aligned} |H(b,t) - H(a,t)| &\le \int |\mathbf{1}_{\{u \le b < x\}} - \mathbf{1}_{\{u \le a < x\}}| \Pr\{S(t) \in \mathrm{d}u\}\, \Pi_S(\mathrm{d}x - u) \\ &\le \int \left(\mathbf{1}_{\{u \le a' < x \le b'\}} + \mathbf{1}_{\{a' < u \le b' < x\}}\right) \Pr\{S(t) \in \mathrm{d}u\}\, \Pi_S(\mathrm{d}x - u) \\ &= L_1(t,a',b') + L_2(t,a',b'). \end{aligned}$$

Then by Lemma 4.10, as $(b,t) \to (a,s)$, $H(b,t) - H(a,t) \to 0$. On the other hand, given $\varepsilon \in (0,\alpha)$, let $M = \sup\{g_t(u) : u \in [\alpha - \varepsilon, \beta], t \in [t_0,t_1]\}$. Then

$$\begin{aligned} |H(a,t) - H(a,s)| &\le \int \mathbf{1}_{\{u \le a < x\}} |g_t(u) - g_s(u)| \Pi_S(\mathrm{d}x - u)\,\mathrm{d}u \\ &= \int \mathbf{1}_{\{u \le a\}} |g_t(u) - g_s(u)| \overline{\Pi}_S(a-u)\,\mathrm{d}u. \end{aligned}$$

Bounding the integral on $[a-\varepsilon, a]$ and $[0, a-\varepsilon]$ separately, we obtain

$$|H(a,t) - H(a,s)| \le 2M \int_0^\varepsilon \overline{\Pi}_S(u)\,\mathrm{d}u + \overline{\Pi}_S(\varepsilon) \int |g_t(u) - g_s(u)|\,\mathrm{d}u.$$

Let $t \to s$. Since point-wise convergence of g_t to g_s implies convergence in total variation, $\overline{\lim}\,|H(a,t) - H(a,s)| \le 2M \int_0^\varepsilon \overline{\Pi}_S(u)\,\mathrm{d}u < \infty$. Letting $\varepsilon \to 0$ gets $H(a,t) - H(a,s) \to 0$. □

Proof of Lemma 4.8. Apply (4.15) and Lemma 4.11, with $h(a, t) \equiv 1$ therein. □

Proof of Lemma 4.9. $G := \{t > 0 : c(t) > 0\}$ is an open interval and $\Pr\{\tau \in G\} = 1$. To prove the lemma, it suffices to show that for any $I = [t_0, t_1] \subset G$, $\Pr\{\tau \in A \cap I\} = 0$. Let $\alpha = c(t_1)$ and $\beta = c(t_0)$. Given $\varepsilon > 0$, $A \cap I$ can be covered by at most countably many disjoint intervals $(a_i, b_i) \subset (t_0, t_1)$ such that $\sum(b_i - a_i) < \varepsilon$ and $\sum[c(a_i) - c(b_i)] < \varepsilon$. For each i,

$$\begin{aligned}\Pr\{\tau \in (a_i, b_i)\} &\le \Pr\{S(a_i) \le c(a_i),\ S(b_i) > c(b_i)\}\\ &\le \Pr\{c(b_i) < S(a_i) \le c(a_i)\} + \Pr\{S(a_i) \le c(b_i) < S(b_i)\}\\ &= \Pr\{c(b_i) < S(a_i) \le c(a_i)\} + \Pr\{\tau^S_{c(b_i)} \in (a_i, b_i)\}.\end{aligned}$$

By the continuous density condition and Lemma 4.8, the RHS is at most $M_1[c(a_i) - c(b_i)] + M_2(b_i - a_i)$, where $M_1 = \sup g_t(x)$ and $M_2 = \sup \psi_x(t)$ over $(t, x) \in [t_0, t_1] \times [\alpha, \beta]$. Therefore, $\Pr\{\tau \in A \cap I\} \le \sum \Pr\{\tau \in (a_i, b_i)\} \le (M_1 + M_2)\varepsilon$. Since ε is arbitrary, this yields the proof. □

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