Markov branching processes with disasters: Extinction, survival and duality to p-jump processes

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Abstract

A p-jump process is a piecewise deterministic Markov process with multiplicative jumps by a factor of p. We prove a limit theorem for such processes on the unit interval. Via duality with respect to probability generating functions, we deduce limiting results for the survival probabilities of time-homogeneous branching processes with arbitrary offspring distributions, underlying binomial disasters. Extending this method, we obtain corresponding results for time-inhomogeneous birth–death processes underlying time-dependent binomial disasters and continuous state branching processes with p-jumps.
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1. Introduction

Consider a population evolving according to a branching process $\mathcal{Z}'$. In addition to reproduction events, global events called disasters occur at some random times (independent of $\mathcal{Z}'$) that kill off every individual alive with probability $1 - p \in (0, 1)$, independently of each other. The resulting process of population sizes $\mathcal{Z}$ will be called a branching process subject to binomial disasters with survival probability p. Provided no information regarding fitness of the individuals in terms of resistance against disasters, this binomial approach appears intuitive, since the survival events of single individuals are iid. Applications span from natural disasters

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as floods and droughts to effects of radiation treatment or chemotherapy on cancer cells as well as antibiotics on populations of bacteria. Also, Bernoulli sampling comes into mind as in the Lenski's long-term evolutionary experiment (LTEE; cf. [21], and [14]).

In the general setting of Bellman–Harris processes with non-lattice lifetime-distribution subject to binomial disasters, Kaplan et al. [17] and Athreya and Kaplan [1] have studied the almost sure asymptotic behaviour as well as asymptotics of the expectation of the population size and showed that such processes almost surely either go extinct or explode, giving necessary and sufficient conditions for extinction. They also computed the limit of the age-distribution on the set of explosion. In the special case of homogeneous birth–death processes with binomial disasters, Bühler and Puri [8] obtained more explicit results regarding asymptotics and normalised limit distributions as well as the distribution of the extinction probability conditioned on the disaster times. Bartoszynski et al. [4] added the analysis of extinction probabilities in a multi-type setting. Furthermore, Peng et al. [25], Thilaka et al. [28] and Kumar et al. [19] studied single-type and multi-type population models underlying binomial disasters with survival probabilities depending on the time the last disaster occurred. These models reflect disasters like earthquakes where pressure builds up over time and increases severity. A more general disaster mechanism in a birth–death-scenario has been discussed by Brockwell et al. [7], Brockwell [6], Pakes [22] and Pakes and Pollett [23], where the absolute population decline after a catastrophe follows geometric, uniform or even an arbitrary distribution independent of the population size. Additionally, the rate of catastrophes is linear in the population size. For continuous state branching processes with disasters according to some intensity measure ν, Bansaye et al. [3] have studied the probability of extinction.

We will add to this literature of branching processes with disasters precise results for the asymptotic extinction probability at late times. As Theorem 2 shows, if extinction occurs, the survival probability decays exponentially at a rate which has a phase transition. We will also be dealing with the time-inhomogeneous case (see Theorem 3), and extinction probabilities for continuous state branching processes (CSBP) with binomial disasters (see Theorem 4).

The main technique we are going to use in our study is duality. Recall that duality of branching systems to a solution of a differential equation has particularly proven useful for continuous state branching processes and measure-valued processes. (See [11] for an overview and the beginning of Section 4 for a brief introduction to this notion of duality.) Bringing this notion back to a branching process in continuous time (without disasters) $\mathcal{Z}$, where every individual branches at rate λ and has offspring distribution with probability generating function (pgf) h, the distribution at time t can be computed via the duality relation

$$\mathbb{E}[x^{Z_t}|Z_0 = z] = X_t^z,$$

where $X_0 = x$ and X_t solves

$$\dot{X} = -\lambda(X - h(X)) \tag{1.1}$$

(cf. [2], Chapter III.3). We will generalise this equation in order to include binomial disasters. Here, the dual process will be a piecewise deterministic Markov process (PDMP) $\mathcal{X}$ on $[0, 1]$, where $1 - \mathcal{X}$ evolves according to (1.1) and jumps by a factor of p at the rate of the disasters. Such processes will be called p-jump processes below. In Theorem 1 we will give general limit results for these processes, which become more precise and concise under concavity conditions shown in Corollary 2.4. These findings will then translate into limit and asymptotics results for survival and extinction probabilities of branching processes with binomial disasters in Theorem 2.

The results of Theorem 2 will be expanded in two directions. First, we are dealing with the time-inhomogeneous case in Theorem 3, i.e. branching rate, pgf and disaster rate may depend on time. Here – using results of Kendall [18] for the case without disasters – we are able to derive the pgf of a binary branching process subject to binomial disasters, conditioned on the times of disasters – similar to the approach of Bühler and Puri [8] for the homogeneous case. In this case, we also give in Proposition 4.11 some limits of pgfs. Second, we apply the duality technique to continuous state branching processes (CSBP) with binomial disasters. Here, we derive in Theorem 4 limit results for the extinction probabilities.

The manuscript is organised as follows. In Section 2, we give our main results on p-jump processes with Theorem 1 and Corollary 2.4. The results on (time-homogeneous) branching processes with disasters are collected in Theorem 2. The case of a birth–death process, i.e. a binary branching process, is given in Corollary 2.7 and extended further to the time-inhomogeneous case in Theorem 3. The CSBP with disasters is treated in Theorem 4. In Section 3, we will prove Theorem 1 and Corollary 2.4. The duality of branching processes with disasters and p-jump PDMP will be established in Section 4, where we will also prove Theorem 2. For the time-inhomogeneous case, we first need in Section 4.2 some results on regularly varying functions, as collected in Theorem 4.6, which might be of interest in their own right. The proof of Theorem 3 is then given in Section 4.3.

2. Results

2.1. p-jump processes

Let us begin by clarifying the notion of p-jump processes.

Definition 2.1. Let $I = [0, \upsilon]$ for $\upsilon > 0$ or $I = \mathbb{R}^+ := [0, \infty)$. Then, let $\alpha : I \to \mathbb{R}$, $p \in [0, 1]$ and $\mathcal{X}$ be a right-continuous Markov process on I, that performs unit-rate jumps from a state x to px and between jumps fulfils $\dot{X}_t = \alpha(X_t)$. Such a process has generator

$$\mathcal{G}_{\mathcal{X}} f(x) = f(px) - f(x) + \alpha(x)f'(x) \tag{2.1}$$

for $f \in \mathcal{C}^1(I)$ and is called a *p-jump process on I with drift* α.

The drift α is called *proper*, if either $\alpha(0) > 0$ and the initial value problem $f' = \alpha(f)$, $f(0) = X_0$ has a unique solution for every $X_0 \in I$, or $\alpha(0) = 0$ and $f' = \alpha(f)$, $f(0) = X_0$ has a unique solution for each $X_0 \in I \setminus \{0\}$. A p-jump process is called *proper* if its drift is proper and if either $\alpha(0) > 0$ or 0 is an absorbing state.

Remark 2.2.

1. Due to its multiplicative jumps, if $p \in (0, 1)$ a p-jump process repeatedly contracts by a factor of p and can thus only have 0 as an absorbing state.
2. 1-jump processes are deterministic, since their jumps have no effect, while 0-jump processes always jump to 0. These two special cases will be left aside in Theorem 1 but considered in Corollary 2.4 for concave α, where a more concise conclusion is possible.
3. Note that properness of α in the above definition implies existence and uniqueness of a proper p-jump process with drift α in the following sense: If $\alpha : I \to \mathbb{R}$ with $\alpha(0) > 0$ is proper, there exists a unique p-jump process $\mathcal{X}$ with drift α. On the other hand, if $\alpha : I \to \mathbb{R}$ with $\alpha(0) = 0$ is proper, there exists a unique p-jump process $\mathcal{X}$ with drift α, which has 0 as an absorbing state. The requirement that 0 is an absorbing state

is important since initial value problems may have infinitely many solutions, such as $f' = 2\sqrt{f}$ with $f(0) = 0$ solved by $f(x) = 0$ for $x \leq x_0$ and $f(x) = (x - x_0)^2$ for $x > x_0$, for any $x_0 > 0$. However, $f = 0$ is the only solution which has 0 as an absorbing state. This will be discussed in more detail in Lemma 3.1.*2*.

4. The Picard–Lindelöf Theorem implies that α is proper whenever it is Lipschitz-continuous. In the later applications however, we will need a less restrictive criterion, allowing $\alpha'(0) = \infty$. Lemma 3.1.*1* provides sufficient conditions for properness of α in that case.

At first, we present our most general limit results for p-jump processes.

Theorem 1 (*Convergence of p-jump-processes*)**.** *Let $p \in (0, 1)$ and $\mathcal{X}$ be a proper p-jump process with drift $\alpha : I \to \mathbb{R}$ and denote $\alpha'_0 := \lim_{x \to 0} \frac{1}{x}\alpha(x)$ and $\hat{\alpha} = \sup_{x \in I \setminus \{0\}} \frac{1}{x}\alpha(x)$. Furthermore, if $I = \mathbb{R}^+$, assume that $s_\alpha := \sup\{x : \alpha(x) > 0\} < \infty$. Then, the following statements hold:*

1. *If $\hat{\alpha} < \log \frac{1}{p}$, then $\mathbb{P}(X_t \xrightarrow{t \to \infty} 0) = 1$. Additionally, for the kth moment of X_t, $k \geq 1$, the following estimates hold:*

   (U_1) *In general (i.e. even for all $\hat{\alpha} \in \mathbb{R}$),*

   $$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[X_t^k] \leq -(1 - p^k - \hat{\alpha}k).$$

   (U_2) *If $\hat{\alpha} > p^k \log \frac{1}{p}$, letting $\lambda := \frac{1}{\hat{\alpha}} \log \frac{1}{p}$, we obtain the stronger bound*

   $$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[X_t^k] \leq -\left(1 - \frac{1}{\lambda}(1 + \log \lambda)\right).$$

   (L_1) *If there are $\delta \in \mathbb{R}$ and $\vartheta > 0$ such that $\alpha(x) \geq \delta x - \vartheta x^2$ for all $x \in I$, such that $\delta \leq p^k \log \frac{1}{p}$, then*

   $$\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}[X_t^k] \geq -(1 - p^k - \delta k).$$

   (L_2) *If δ in (L_1) can be chosen positive, letting $\gamma := \frac{1}{\delta} \log \frac{1}{p}$, we obtain*

   $$\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}[X_t^k] \geq -\left(1 - \frac{1}{\gamma}(1 + \log \gamma)\right),$$

   which is a stronger bound than (L_1), if $\delta > p^k \log \frac{1}{p}$.

2. *If $\alpha'_0 \in (\log \frac{1}{p}, \infty]$, let $x_\alpha := \min\{x \in I \setminus \{0\} : \alpha(x) = 0\}$. Then, $\mathcal{X}$ converges weakly and its limit X_∞ satisfies $\mathbb{P}(X_\infty \in (0, x_\alpha]) = 1$, $\mathbb{E}[X_\infty^{-1}\alpha(X_\infty)] = \log \frac{1}{p}$ and for every $k \geq 1$*

   $$\mathbb{E}[X_\infty^k] = \frac{k}{1 - p^k} \mathbb{E}[X_\infty^{k-1}\alpha(X_\infty)].$$

   Also, the distribution of X_∞ is the unique stationary distribution and for every bounded and measurable function $f : I \to \mathbb{R}$ almost surely

   $$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s)ds = \mathbb{E}[f(X_\infty)].$$
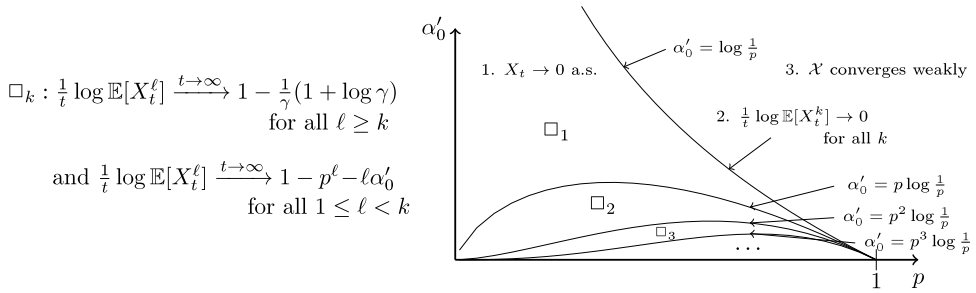
Fig. 1. The different cases in Corollary 2.4. Within $\alpha'_0 > \log \frac{1}{p}$, we are in case 3. and obtain weak convergence. For case 1., different limits arise for different exponents, depending on α'_0 and p.

Remark 2.3.

1. Note that in Theorem 1.*1* α'_0 is finite, since $\alpha'_0 \leq \hat{\alpha}$. Hence, $\alpha(0)$ has to be 0 and it holds $\alpha'_0 = \alpha'(0)$. Therefore the similar notation. Case *2.* accounts for both possibilities $\alpha(0) = 0$ with $\alpha'(0) = \infty$ as well as $\alpha(0) > 0$.
2. Although the theorem shows existence of a *p*-jump process on $\mathbb{R}^+$, it is notable that such a process will only assume values in $[0, \max\{X_0, s_\alpha\}]$, since by definition it can never grow beyond $\sup(\{X_0\} \cup \{x : \alpha(x) > 0\})$.

In the case where α is concave, the bounds (U_i) and (L_i) align and the continuity conditions (C_i) become evident such that we can give a much more concise result. Here, we will also include the cases $p = 0$ and $p = 1$. Fig. 1 illustrates the various parameter regimes. Furthermore, we obtain existence and uniqueness if 0 is absorbing.

Corollary 2.4. *Let* $I \subset \mathbb{R}$ *and* $\alpha : I \to \mathbb{R}$ *concave with* $\alpha(0) = 0$ *such that* $\alpha''(0) \in [-\infty, 0]$ *exists and either*

– $I = [0, \upsilon]$, $\alpha(\upsilon) \leq 0$ *and* $\alpha'(\upsilon) > -\infty$ *or*
– $I = \mathbb{R}^+$ *and there is an* $x > 0$ *such that* $\alpha(x) = 0$.

Then, for all $p \in [0, 1]$ *and* $X_0 \in I$ *there is a unique proper p-jump process* $\mathcal{X}$ *with drift* α *that, letting* $\alpha'_0 = \lim_{x \to 0} \frac{1}{x}\alpha(x)$, *holds:*

1. *If* $\alpha'_0 < \log \frac{1}{p}$ *or* $p = 0$, *then* $X_t \xrightarrow{t \to \infty} 0$ *almost surely. Also, for* $k \geq 1$

$$\lim_{t \to \infty} -\frac{1}{t} \log \mathbb{E}[X_t^k] = \begin{cases} 1 + \max\{0, -k\alpha'_0\} & \text{if } p = 0, \\ 1 - p^k - k\alpha'_0 & \text{if } \alpha'_0 \leq p^k \log \frac{1}{p}, \\ 1 - \frac{1}{\gamma}(1 + \log \gamma) & \text{otherwise,} \end{cases}$$

   where $\gamma = \log(\frac{1}{p})/\alpha'_0$.
2. *If* $\alpha'_0 = \log \frac{1}{p}$, *then* $\frac{1}{t} \log \mathbb{E}[X_t^k] \xrightarrow{t \to \infty} 0$ *and* $\limsup_t X_t \leq m_\alpha := \sup\{x \in [0, x_\alpha] : \alpha(x) = x\alpha'_0\}$ *almost surely. In particular, if* α *is strictly concave on an interval* $(0, \varepsilon)$, *then* $X_t \to m_\alpha = 0$ *almost surely.*

3. *If $\alpha'_0 > \log \frac{1}{p}$, then $x_\alpha := \sup\{x : \alpha(x) > 0\} \in (0, \infty)$, $\mathcal{X}$ converges weakly and its limit X_∞ satisfies $\mathbb{P}(X_\infty \in (0, x_\alpha]) = 1$, $\mathbb{E}[X_\infty^{-1}\alpha(X_\infty)] = \log \frac{1}{p}$ and for every $k \geq 1$*

$$\mathbb{E}[X_\infty^k] = \frac{k}{1 - p^k}\mathbb{E}[X_\infty^{k-1}\alpha(X_\infty)].$$

Also, the distribution of X_∞ is the unique stationary distribution and for every bounded and measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ almost surely

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s)ds = \mathbb{E}[f(X_\infty)].$$

2.2. Branching processes with binomial disasters

Applying Corollary 2.4 to a p-jump process, via duality with respect to probability generating functions in Theorem 2 we obtain immediate limit results for the following class of branching processes with binomial disasters.

Definition 2.5. Let $\lambda > 0$, $q = (q_k)_{k \geq 0}$ a distribution on $\mathbb{N}_0$, $\kappa > 0$ and $p \in [0, 1]$. A Markov process $\mathcal{Z}$ on $\mathbb{N}_0$ with generator

$$\mathcal{G}_{\mathcal{Z}} f(z) = \lambda z \sum_{k \geq 0} q_k \big(f(z - 1 + k) - f(z)\big) + \kappa \sum_{k=0}^{z} \binom{z}{k} p^k (1 - p)^{z-k} \big(f(k) - f(z)\big)$$

for $f \in \mathcal{B}(\mathbb{N}_0)$, the set of real-valued, bounded functions on $\mathbb{N}$, is called a *homogeneous branching process with death-rate λ and offspring distribution (q_k), subject to binomial disasters at rate κ with survival probability p and will be denoted by $\mathcal{Z}_{\lambda,q,\kappa,p}^{hom}$.*

Such a process describes the size of a population that behaves in the following way: Every individual dies with rate λ and leaves behind a random number of offsprings distributed according to (q_k). Independent of this growth mechanism, with rate κ global events occur that kill off every individual alive at that time with probability $1 - p$ independently of each other.

Theorem 2. *Let $\lambda > 0$, $q = (q_k)_{k \geq 0}$ a distribution on $\mathbb{N}_0$ with expectation $\mu := \sum_k kq_k$, $\kappa > 0$, $p \in [0, 1)$ and $Z_0 = z_0 > 0$. Then, if $p = 0$, $\mathcal{Z} := \mathcal{Z}_{\lambda,q,\kappa,p}^{hom}$ goes extinct almost surely with*

$$\lim_{t \to \infty} -\frac{1}{t} \log \mathbb{P}(Z_t > 0) = \kappa + \max\{\lambda(1 - \mu), 0\}.$$

Otherwise, letting $\nu := \lambda(\mu - 1)/(\kappa \log \frac{1}{p})$, $\mathcal{Z}$ satisfies

1. *if $\nu \leq p$, $\mathcal{Z}$ goes extinct almost surely and*

$$\lim_{t \to \infty} -\frac{1}{t} \log \mathbb{P}(Z_t > 0) = (1 - p)\kappa - \lambda(\mu - 1).$$

2. *if $p < \nu \leq 1$, $\mathcal{Z}$ goes extinct almost surely and*

$$\lim_{t \to \infty} -\frac{1}{t} \log \mathbb{P}(Z_t > 0) = \kappa(1 - \nu - \nu \log(\tfrac{1}{\nu})).$$

3. *if $v > 1$, then $\mathcal{Z}$ survives with positive probability, where*

$$0 < \mathbb{P}(\lim_{t \to \infty} Z_t = \infty) = 1 - \mathbb{P}(\lim_{t \to \infty} Z_t = 0) = \sum_{k=1}^{z_0} \binom{z_0}{k}(-1)^{k-1}\mathbb{E}[X^k] < 1 - x_*^{z_0},$$

(2.2)

where $h(x) = \sum_k x^k q_k$ is the pgf of q, x_* is the smallest fixed point of h and X is a random variable on $(0, 1 - x_*]$ satisfying $\mathbb{E}[X^{-1}(1 - h(1 - X))] = 1 + \frac{\kappa}{\lambda}\log\frac{1}{p}$ and

$$\mathbb{E}[X^k] = \frac{\lambda k}{\lambda k + \kappa(1 - p^k)}\mathbb{E}[X^{k-1}(1 - h(1 - X))].$$

(2.3)

Remark 2.6.

1. Rearranging the inequalities in terms of μ, we obtain

   1. if $\mu \leq 1 + \frac{\kappa p}{\lambda}\log\frac{1}{p}$,   2. if $1 + \frac{\kappa p}{\lambda}\log\frac{1}{p} < \mu \leq 1 + \frac{\kappa}{\lambda}\log\frac{1}{p}$,   3. if $\mu > 1 + \frac{\kappa}{\lambda}\log\frac{1}{p}$.

   This gives insight into how supercritical the underlying branching process has to be in order to survive the disasters.

   Also, this formulation illustrates the continuity of the theorem in $p = 1$: Classical results for such processes without disasters (cf. [15, Theorem 11.1, p.109]) show that $\mathcal{Z}$ goes extinct almost surely if $\mu \leq 1$ with $-\frac{1}{t}\log\mathbb{P}(Z_t > 0) \to \lambda(1 - \mu)$ as $t \to \infty$, which aligns with *1.*, while for $\mu > 1$, $\mathbb{P}(Z_t \to \infty) = 1 - \mathbb{P}(Z_t \to 0) = 1 - x_*^{z_0}$, which is the upper bound for the survival probability in (2.2) for $p < 1$.

2. While [17] have already shown the almost sure extinction in *1.* and *2.* as well as the fact in *3.* that $\mathcal{Z}$ almost surely either goes extinct or explodes, we offer an alternative proof via our duality results plus rates of convergence for the survival probability including the case $\mu = \infty$. Also, making use of (2.3), our result offers a way to compute the exact extinction probability in *3.* Since the recursion in (2.3) depends on the offspring pgf h, in general this formula can be difficult to compute. Corollary 2.7, however, shows that in the example of homogeneous birth–death-processes it is feasible.

The following corollary applies Theorem 2 to birth–death-processes with disasters. This does not only provide a nice transition to the next theorem, but offers an example where (using the relation (2.3)) we can explicitly compute the survival probability.

Corollary 2.7. *Let $\mathcal{Z} := (Z_t)_t$ be a homogeneous birth–death-process with respective rates $b > 0$ and $d \geq 0$ that underlies binomial disasters at a rate of $\kappa > 0$ with survival probability $p \in (0, 1)$ and let $Z_0 = z_0 > 0$.*

1. *If $b - d \leq \kappa p \log\frac{1}{p}$, $\mathcal{Z}$ goes extinct almost surely and*

   $$\lim_{t \to \infty} -\frac{1}{t}\log\mathbb{P}(Z_t > 0) = (1 - p)\kappa - (b - d).$$

2. *If $\kappa p \log\frac{1}{p} < b - d \leq \kappa \log\frac{1}{p}$, $\mathcal{Z}$ goes extinct almost surely and*

   $$\lim_{t \to \infty} -\frac{1}{t}\log\mathbb{P}(Z_t > 0) = \kappa - \frac{b - d}{\log\frac{1}{p}}\left(1 + \log\left(\frac{\kappa \log\frac{1}{p}}{b - d}\right)\right).$$

3. If $b - d > \kappa \log \frac{1}{p}$, then $\mathbb{P}(\lim_{t \to \infty} Z_t = 0) + \mathbb{P}(\lim_{t \to \infty} Z_t = \infty) = 1$ and

$$\mathbb{P}(\lim_{t \to \infty} Z_t = \infty) = \Big(1 - \frac{d + \kappa \log \frac{1}{p}}{b}\Big) \sum_{\ell=1}^{z_0} \binom{z_0}{\ell}(-1)^{\ell-1} \prod_{m=1}^{\ell-1} \Big(1 - \frac{dm + (1 - p^m)\kappa}{bm}\Big).$$

Proof.   First note that $\mathcal{Z}$ is a $\mathcal{Z}^{hom}_{\lambda,q,\kappa,p}$-process with $\lambda = b + d$, $q_0 = d/(b + d)$ and $q_2 = b/(b+d) = 1 - q_0$. Thus, $\lambda(\mu - 1) = (b+d)(q_2 - q_0) = b - d$ and $\nu = (b-d)/(\kappa \log \frac{1}{p})$, already providing *1.* and *2.* by insertion in Theorem 2. For *3.* we derive $1 - h(1 - x) = 1 - q_0 - q_2(1 - x)^2 = q_2 x(2 - x)$, yielding a simple recursion in (2.3) concluding the proof. □

In Section 4.3 we will develop tools for the analysis of inhomogeneous birth–death processes with time-dependent disasters, generalising the setting of Corollary 2.7. For this, mind the following definition, where the birth, death and disaster rates b, d, κ as well as the survival probability p are now given as functions of t.

Definition 2.8.   Let $b, d, \kappa : \mathbb{R}^+ \to \mathbb{R}^+$ and $p : \mathbb{R}^+ \to [0, 1]$, where we abbreviate $b_s := b(s), d_s := d(s), \kappa_s := \kappa(s)$ and $p_s := p(s)$. A Markov process $\mathcal{Z}$ on $\mathbb{N}_0$ with time-dependent generator (see Section 4.7A of Ethier and Kurtz [12])

$$\mathcal{G}_{\mathcal{Z},t} f(z) = b_t z \big(f(z + 1) - f(z) \big) + d_t z \big(f(z - 1) - f(z) \big)$$

$$+ \kappa_t \sum_{k=0}^{z} \binom{z}{k} p_t^k (1 - p_t)^{z-k} \big(f(k) - f(z) \big)$$

for $t \geq 0$ and $f \in \mathcal{B}(\mathbb{N}_0)$, is called an *inhomogeneous birth–death-process with birth-rate b and death-rate d, subject to binomial disasters with survival probability p occurring at rate κ* and will be denoted by $\mathcal{Z}^{in}_{b,d,\kappa,p}$.

Key to our approach is Lemma 4.9, which computes the conditioned pgf delivering some kind of *stronger* duality, enabling us with Proposition 4.11 to easily give pgf limit results in terms of that dual process. While these tools offer room for further generalisation (cf. Remark 4.12), we give the following theorem as an example of application, where we also make use of Theorem 4.6.

Theorem 3.   *Let b, d, κ be non-negative right-continuous functions on $\mathbb{R}^+$ with left limits and $p : \mathbb{R}^+ \to [0, 1]$ left-continuous with right limits such that $p_t = 0$ only if $\kappa_t = 0$ and, letting $\Lambda_\kappa^{-1}(t) := \inf\{s > 0 : \int_0^s \kappa_s ds > t\}$, such that the map $-\log(p(\Lambda_\kappa^{-1}(\cdot)))$ is regularly varying. Furthermore, let $j : \mathbb{R}^+ \to \mathbb{R}^+$ continuous and non-decreasing with $\lim_{t \to \infty} t^{-\alpha} j(t) = \infty$ for some $\alpha > 0$, as well as $\iota \in \{-1, 1\}$ such that*

$$\frac{1}{j(t)} \int_0^t \Big(b_s - d_s - \kappa_s \log\Big(\frac{1}{p_s}\Big) \Big) ds \xrightarrow{t \to \infty} \iota.$$

Then, $\mathcal{Z} := \mathcal{Z}^{in}_{b,d,\kappa,p}$ satisfies

1. *if $\iota = 1$ and for some $\varepsilon > 0$ holds $\int_0^\infty e^{-(1-\varepsilon)j(s)} b_s ds < \infty$, then $\mathbb{P}(Z_t \xrightarrow{t \to \infty} 0) < 1$.*
2. *if $\iota = -1$ or for some $\varepsilon > 0$ holds $\int_0^\infty e^{-(1+\varepsilon)j(s)} b_s ds = \infty$, then $\mathbb{P}(Z_t \xrightarrow{t \to \infty} 0) = 1$.*

Remark 2.9.

1. Letting $j(t) = t$ and b, d, κ, p constant functions, this result aligns with the homogeneous case (cf. Corollary 2.7).

2. The regular variation condition on p and κ is equivalent to the existence of $\beta \in \mathbb{R}$ and some slowly varying function ℓ, such that $p(t) = \exp(-\Lambda_\kappa(t)^\beta)\ell(\Lambda_\kappa(t))$. We need $t^{-\alpha} j(t) \to \infty$ for some $\alpha > 0$ to handle the case $\beta = -1$ in which Theorem 4.6 is inconclusive. If $\beta \neq -1$, one may choose $\alpha = 0$.

3. The condition that $\kappa = 0$ whenever $p = 0$ ensures that no *terminal* disasters occur, i.e. disasters that render $\mathcal{Z}$ extinct with probability 1. Dropping this, $\int_0^t \kappa_s \log(1/p_s)ds$ might no longer be finite. However, letting $\kappa_t^- := \kappa_t \cdot \delta_{0,p_t}$ and $\kappa_t^+ := \kappa_t - \kappa_t^-$, it is possible to apply Theorem 3 to the process $\mathcal{Z} = \mathcal{Z}_{b,d,\kappa^+,p}^{in}$ without the terminal disasters and separately compute the probability π_{term} that at least one terminal disaster occurs, which for a unit-rate Poisson process (P_t) satisfies

$$\pi_{term} = \mathbb{P}(P_{\int_0^\infty \kappa_t^- \, dt} > 0) = 1 - \exp\left(-\int_{t:\, p(t)=0} \kappa_t \, dt\right).$$

Since the sets $\{\kappa^+ > 0\}$ and $\{\kappa^- > 0\}$ are disjoint, the respective counts of disasters on these sets are independent. This implies that the terminal disasters only affect the positivity of the survival probability, if $\int_{t:\, p(t)=0} \kappa_t \, dt = \infty$.

4. The cases where a normalisation function j as in Theorem 3 does not exist, are discussed in Remark 4.13.1. Rates of convergence for the survival probability in case *2.* will briefly be discussed in Remark 4.13.2.

2.3. Continuous state branching processes with binomial disasters

The application of p-jump processes is not limited to branching processes with discrete states. We will now discuss survival and extinction for continuous state branching processes (see e.g. [20] for an overview) with binomial disasters. A similar model is studied in [3], where multiplicative jumps occur for any factor according to some intensity measure. Their Theorem 1 shows existence and uniqueness of the process we now define.

Definition 2.10. Let N a measure on $\mathbb{R}_+$ with $\int_0^\infty \min(y, y^2)N(dy) < \infty$, $b \in \mathbb{R}, c, \kappa \in \mathbb{R}^+$ and $p \in (0, 1)$. Then, the $\mathbb{R}^+$-valued Markov process with generator

$$\mathcal{G}_{\mathcal{Z}} f(z) = bzf'(z) + czf''(z) + \kappa(f(pz) - f(z)) + \int_0^\infty \big(f(z+y) - f(z) - yf'(z)\big)zN(dy),$$

for $f \in \mathcal{C}_b^2(\mathbb{R}_+)$ (the space of bounded, twice continuously differentiable functions) is called the *continuous state branching process with p-disasters* and will be denoted by $\mathcal{Z}_{b,c,N,\kappa,p}^{cs}$.

Viewing a continuous state branching process as a scaling limit of a discrete branching process, by the law of large numbers the equivalent of a binomial disaster would be a p-jump of the population size, represented by the term $\kappa(f(pz) - f(z))$ in $\mathcal{G}_{\mathcal{Z}}$. The next theorem is similar to but less precise than Corollary 6 of Bansaye et al. [3], using weaker assumptions. (see Remark 2.11 for a more detailed comparison)

Theorem 4. *Let $\mathcal{Z} = \mathcal{Z}^{cs}_{b,c,N,\kappa,p}$ be a continuous state branching process with p-disasters as in Definition 2.10 and let*

$$\alpha : \mathbb{R}^+ \to \mathbb{R}, x \mapsto bx - cx^2 - \int_0^\infty (e^{-xy} - 1 + xy)N(dy)$$

satisfy $\limsup_{x \to \infty} x^{-(1+\varepsilon)}\alpha(x) < 0$ for some $\varepsilon > 0$.

1. *If $b \le \kappa \log \frac{1}{p}$, $Z_t \xrightarrow{t \to \infty} 0$ almost surely and*

$$\lim_{t \to \infty} -\frac{1}{t} \log \mathbb{P}(Z_t > 0) = \begin{cases} (1 - p)\kappa - b & \text{if } b \le \kappa p \log \frac{1}{p}, \\ \kappa - \frac{\kappa}{\gamma}(1 + \log \gamma) & \text{otherwise,} \end{cases}$$

   where $\gamma := \kappa \log(\frac{1}{p})/b$.

2. *If $b > \kappa \log \frac{1}{p}$, $\lim_{t \to \infty} \mathbb{P}(Z_t = 0 | Z_0 = z) \in (e^{-z\xi}, 1)$, where ξ denotes the largest root of α.*

We will only give an outline of the proof, since its structure is illustrative for the proofs to come and it differs only in details.

Sketch of proof. For $(Z_t) = \mathcal{Z}^{cs}_{b,c,N,\kappa,p}$ it holds $(Z_{\kappa t}) \overset{d}{=} \mathcal{Z}^{cs}_{b/\kappa,c/\kappa,N/\kappa,1,p}$ and $\lim \frac{1}{t} \log \mathbb{P}(Z_{\kappa t} > 0) = \lim \frac{\kappa}{t} \log \mathbb{P}(Z_t > 0)$. This rescaling illustrates that we just need to show the case of $\kappa = 1$. Setting $H(x, z) := e^{-xz}$ and applying the generator to the function $z \mapsto H(x, z)$ for x fixed gives

$$\mathcal{G}_{\mathcal{Z}} H(x, .)(z) = \left(bx - cx^2 - \int_0^\infty (e^{-xy} - 1 + xy)N(dy) \right) \frac{\partial H}{\partial x}(x, z) + H(xp, z) - H(x, z).$$

In other words (cf. (D′) at the beginning of Section 4), $\mathcal{Z}$ is dual to the p-jump process with drift α. Now, note that $\xi = \bar{x}_\alpha < \infty$, since $\limsup_{x \to \infty} x^{-(1+\varepsilon)}\alpha(x) < 0$ and thus, $\alpha(x) \to -\infty$ as $x \to \infty$. Hence, either $c > 0$ or $N((0, \infty)) > 0$. In any case, for all $x > 0$

$$\alpha''(x) = -2c - \int_0^\infty x^2 e^{-xy} N(dy) < 0$$

and α is strictly concave. Hence, Corollary 2.4 applies with $\alpha'_0 = \alpha'(0) = b$.

Since $\limsup_{x \to \infty} x^{-(1+\varepsilon)}\alpha(x) < 0$, $\mathcal{X}$ comes down from infinity in the sense that $\mathcal{X}$ is well-defined in the limit of $X_0 \to \infty$. This can be verified in the following way: By a restart argument it suffices to consider $t > 0$ small enough, such that $\alpha(x) \le -sx^{1+\varepsilon}$ for some $s \in (0, -\limsup_{x \to \infty} x^{-(1+\varepsilon)}\alpha(x))$ and for all $x \ge X_t$. Using that $u(t) = (X_0^{-\varepsilon} + \varepsilon st)^{-1/\varepsilon}$ is the only solution to $u' = -su^{1+\varepsilon}$ with $u(0) = X_0$ and the monotonicity of the jumps of $\mathcal{X}$ we see that $X_t \le u(t) \xrightarrow{X_0 \to \infty} (\varepsilon st)^{-1/\varepsilon} < \infty$. Hence, letting $(\mathcal{X}^x)_{x \in \mathbb{R}}$ be p-jump processes with drift α jumping simultaneously with $X_0^x = x$ for all x, we can define $\mathcal{X}^\infty$ via $X_t^\infty := \sup_x X_t^x$.

It follows

$$\mathbb{P}(Z_t = 0 | Z_0 = z) = \lim_{x \to \infty} \mathbb{E}[e^{-xZ_t} | Z_0 = z] = \mathbb{E}[e^{-zX_t^\infty}]. \tag{2.4}$$

Considering $x - \frac{x^2}{2} \le 1 - e^{-x} \le x$ for all $x \ge 0$ to estimate the rates of convergence, Corollary 2.4 concludes the proof. (The almost sure convergence in *1.* follows from the fact that $e^{-zX_t^\infty}$ is bounded, which shows that $Z_t \to 0$ in probability implying almost sure convergence of a sub-sequence. It remains to note that 0 is absorbing.) □

Remark 2.11 (*Connection to Bansaye et al. [3]*)**.** For the critical and subcritical cases, Corollary 6 in [3] uses the stronger assumptions that $c > 0$ and $\int_0^\infty y^2 N(dy) < \infty$. However, they consider a more general setting, where the jump factor p is not constant but random and possibly even larger than one. Under the second moment condition $\mathbb{E}[\log(p)^2] < \infty$ they obtain more precise asymptotics for the survival probability, not only giving the exponential rates of decay of Theorem 4, but also polynomial factors t^0, $t^{-\frac{1}{2}}$ and $t^{-\frac{3}{2}}$ in the strong, intermediate and weak subcritical cases respectively.

In the supercritical case, i.e. $b > \kappa \log \frac{1}{p}$, Corollary 2.iii) in [3] shows positivity of the survival probability while their Corollary 3 shows a central limit theorem conditional on survival. Our results add to that bounds on the survival probability in Theorem 4.2 as well as an approach to compute the survival probability:

In (2.4) we have seen that

$$\mathbb{P}(Z_t = 0 | Z_0 = z) \xrightarrow{t \to \infty} \mathbb{E}[e^{-z X_\infty}] =: L(z),$$

the Laplace transform of the limit of $\mathcal{X}$. Using stationarity it follows from $\mathbb{E}[\mathcal{G}_{\mathcal{X}} f(X_\infty)] = 0$ for $f(x) = e^{-zx}$, that L has to satisfy the functional equation

$$\kappa(L(pz) - L(z)) - bz L'(z) + cz L''(z) + z \int_0^\infty \big(L(z + y) - L(z) + y L'(z) \big) N(dy) = 0,$$

which for appropriate N might deliver a more precise result than Theorem 4.2.

3. Piecewise deterministic Markov processes

In this section, we start by providing an existence and uniqueness result for proper p-jump processes.

Lemma 3.1 (*Properness, Existence and Uniqueness*)**.**

1. Let $I = [0, v]$ or $I = \mathbb{R}^+$ and $\alpha : I \to \mathbb{R}$ with $\alpha(0) = 0$, $\alpha'(0) \in (-\infty, \infty]$ and $\alpha(v) \leq 0$, if $I = [0, v]$. Also, let one of the following conditions hold:

   (C_1) α is Lipschitz-continuous on $I \cap [\varepsilon, \infty)$ for every $\varepsilon > 0$.
   (C_2) α is Lipschitz-continuous on $I \cap [\varepsilon, \varepsilon^{-1}]$ for every $\varepsilon > 0$ and there is $K > 0$ such that $\alpha(x) \leq 0$ for all $x \geq K$.

   Then, α is proper.
2. Let $(P_t)_{t \geq 0}$ a unit-rate Poisson process and α a proper drift. Then, for $p \in [0, 1]$ and $X_0 \in I$ there is a unique proper p-jump process $\mathcal{X}$ with drift α that jumps simultaneously with (P_t) and starts in X_0.

Proof. If $\alpha'(0) < \infty$, α is Lipschitz-continuous either on I by (C_1) or on $I \cap [0, \max\{K, X_0\}]$ by (C_2), where $\alpha(\max\{K, X_0\}) \leq 0$, and hence is proper by the Picard–Lindelöf Theorem. Otherwise, if $\alpha'(0) = \infty$ and $X_0 > 0$, we find $\varepsilon > 0$ such that $\alpha(x) > 0$ for all $0 < x \leq \varepsilon$. Then, α is Lipschitz-continuous either on $I' := I \cap [\min\{X_0, \varepsilon\}, \infty)$ by (C_1) or on $I' := I \cap [\min\{X_0, \varepsilon\}, \max\{K, X_0\}]$ by (C_2), such that Picard–Lindelöf either way implies existence of a unique solution of $f' = \alpha(f)$ and $f(0) = X_0$ for $\alpha : I' \to \mathbb{R}$. Since $\alpha(\min\{X_0, \varepsilon\}) > 0$, the properness of α follows.

Now, 2. follows by iteratively constructing the path of $\mathcal{X}$, considering the deterministic pieces between jumps as solutions of suitable initial value problems and letting $\mathcal{X}$ be absorbed at 0 if $\alpha(0) = 0$. $\square$

Remark 3.2.

1. Note that (C_1) and (C_2) are equivalent if I is bounded, since then we assume $\alpha(v) \leq 0$.
2. The above lemma does not consider the case $\alpha(0) > 0$. This is due to the fact that then, one needs to make additional arguments if $\alpha'(0) = \infty$ for existence and uniqueness and, that the p-jump processes we will use in our applications all hold $\alpha(0) = 0$. However, Theorem 1 also includes that case.

In the following subsection we prove Theorem 1, mainly by applying large deviation results for Poisson processes, given in the Appendix, and the work of Bladt and Nielsen [5] regarding regenerative processes. Using this, we prove Corollary 2.4 in Section 3.2 for concave α offering a more concise result including continuities for $p \in \{0, 1\}$.

3.1. Proof of Theorem 1

Proof. At the very beginning, suppose that the theorem already holds for p-jump processes on $[0, 1]$, let α be as in the assumptions, $s := \max\{X_0, s_\alpha\}$ and consider $a : [0, 1] \to \mathbb{R}^+, x \mapsto \alpha(sx)/s$. It is straightforward to show that a satisfies the conditions of Definition 2.1 as well as the ones of Theorem 1 such that there is a p-jump process $\overline{\mathcal{X}} = (\overline{X}_t)$ with drift a on $[0, 1]$ starting in sX_0 for which the assertions of the theorem hold, where $a_0' = \alpha_0'$, $\hat{a} = \hat{\alpha}$ and $x_a = sx_\alpha$. Also, $a(x) \geq \delta x - \vartheta x^2$ for all $x \in [0, 1]$ iff $\alpha(x) \geq \delta x - \frac{\vartheta}{s} x^2$ for all $x \in [0, s]$. Considering that $t \mapsto X_t := s_\alpha \overline{X}_t$ also performs p-multiplicative jumps at rate 1 and in between satisfies $\dot{X}_t = s \cdot a(\overline{X}_t) = \alpha(X_t)$, we obtain that the theorem holds. Hence, without loss of generality, we assume for the rest of the proof that $I = [0, 1]$.

Note that by definition $\alpha_0' \leq \hat{\alpha}$, and hence $\alpha'(0) = \infty$ can only hold in case 2. Moreover, $\alpha_0' < \infty$ only if $\alpha(0) = 0$, in which case $\alpha_0' = \alpha'(0)$.

1. Let $\beta(y) := e^y \alpha(e^{-y})$ and $Y_t := -\log X_t \in \mathbb{R}^+$. Then, the process $\mathcal{Y} := (Y_t)_{t \geq 0}$ has the generator

$$
\begin{aligned}
\mathcal{G}_{\mathcal{Y}} g(y) &= \big(\mathcal{G}_{\mathcal{X}}(g \circ (-\log))\big)(e^{-y}) \\
&= g\big(-\log(pe^{-y})\big) - g(y) + \alpha(e^{-y}) \cdot \big(-\tfrac{1}{x} g'(-\log x)\big)\big|_{x=e^{-y}} \\
&= g(y + \log \tfrac{1}{p}) - g(y) - \beta(y) g'(y).
\end{aligned}
$$

The assertion implies that $\beta(y) \leq \hat{\alpha} < \log \frac{1}{p}$ for all $y \in \mathbb{R}^+$. Letting (P_t) be the unit-rate Poisson process jumping simultaneously with $\mathcal{Y}$, it follows that

$$
P_t \cdot \log \tfrac{1}{p} - t\hat{\alpha} \leq Y_t \tag{3.1}
$$

for all t, since the jumps are identical, the left side always starts in 0 and has point-wise inferior drift. The law of large numbers, giving us $\lim_{t \to \infty} P_t/t = 1$ and thus almost surely $\liminf_t(Y_t/t) \geq \log \frac{1}{p} - \hat{\alpha} > 0$, shows that $-\log X_t = Y_t \longrightarrow_{t \to \infty} \infty$ with probability 1.

(U_1) Using the generator of $\mathcal{X}$ on $f(x) = x^k$, we obtain

$$
\tfrac{d}{dt} \mathbb{E}[X_t^k] = \mathbb{E}\big[\big(\mathcal{G}_{\mathcal{X}}(\cdot)^k\big)(X_t)\big] = \mathbb{E}[p^k X_t^k - X_t^k + \alpha(X_t) k X_t^{k-1}] \leq \mathbb{E}[X_t^k](p^k - 1 + k\hat{\alpha}). \tag{3.2}
$$

Considering Gronwall's inequality, (U_1) follows (even for arbitrary $\hat{\alpha}$).

(U_2) For the (stronger) upper bound in the case of $\hat{\alpha} > p^k \log \frac{1}{p}$ we reuse (3.1) to compute

$$
\begin{aligned}
\mathbb{E}[X_t^k] &= \mathbb{E}[e^{-kY_t}] \\
&\leq u(t) := \mathbb{E}[e^{-k(P_t \log(1/p) - t\hat{\alpha})} \wedge 1] \tag{3.3}
\end{aligned}
$$

$$= \mathbb{P}\left(P_t \le \frac{\hat{\alpha}}{\log(1/p)}t\right) + \sum_{\ell \ge \hat{\alpha}t/(\log(1/p))} e^{k(\hat{\alpha}t - \log(1/p)\ell)} e^{-t} \frac{t^\ell}{\ell!}$$

$$= \underbrace{\mathbb{P}\left(P_t \le \frac{t}{\lambda}\right)}_{=:u_1(t)} + \underbrace{e^{-t(1-p^k-\hat{\alpha}k)} \mathbb{P}\left(P_{p^k t} \ge \frac{1}{p^k \lambda} p^k t\right)}_{=:u_2(t)}.$$

First, using (A.2) of Lemma A.1 with $x = \frac{1}{\lambda} < 1$, we compute

$$-\frac{1}{t} \log u_1(t) \xrightarrow{t \to \infty} 1 - \frac{1}{\lambda} + \frac{1}{\lambda} \log \frac{1}{\lambda} = 1 - \tfrac{1}{\lambda}(1 + \log \lambda) =: A_\lambda.$$

For the second term, u_2, we see that $p^k \lambda < 1$ and thus, by (A.1) of Lemma A.1,

$$-\frac{1}{t} \log u_2(t) \xrightarrow{t \to \infty} 1 - p^k - \hat{\alpha}k + p^k\left(1 - \frac{1}{p^k \lambda} + \frac{1}{p^k \lambda} \log \frac{1}{p^k \lambda}\right)$$

$$= 1 - \hat{\alpha}k - \frac{1}{\lambda}\left(1 - \log \frac{1}{\lambda} - k \log \frac{1}{p}\right) = A_\lambda.$$

Combining these two results, we obtain $\lim_{t \to \infty} \frac{1}{t} \log u(t) = -A_\lambda$, which equates to the desired bound.

To recognise that the bound in (U_2) is in fact stronger than the one of (U_1), we need to verify that $1 - p^k - \hat{\alpha}k \le A_\lambda$, if $p^k \lambda < 1$. In this case, the function $h : x \mapsto x + \frac{1}{\lambda} \log \frac{1}{x}$ is strictly decreasing on $[p^k, \lambda^{-1}]$, since $h'(x) = 1 - \frac{1}{\lambda x}$. Thus, inserting $\hat{\alpha} = \frac{1}{\lambda} \log \frac{1}{p}$, the difference satisfies

$$A_\lambda - (1 - p^k - \hat{\alpha}k) = p^k + \frac{1}{\lambda} \log \frac{1}{p^k} - \frac{1}{\lambda}(1 + \log \lambda)$$

$$= h(p^k) - h(\lambda^{-1}) > 0.$$

(L_1) and (L_2): Let $w : [0, 1] \to \mathbb{R}, x \mapsto \delta x - \vartheta x^2$. Then, $\mathcal{X}$ is bounded below by a process $\mathcal{W} = (W_t)_t$ with generator

$$\mathcal{G}_{\mathcal{W}} f(x) = f(px) - f(x) + w(x)f'(x),$$

if $\mathcal{X}$ and $\mathcal{W}$ have equal initial values and are coupled in such a way that they jump simultaneously at the jump times of a Poisson process (P_t). Now, we will show that the moments of $\mathcal{W}$ have the desired asymptotic properties, using that $\mathcal{W}$ can be represented explicitly via

$$W_t = \frac{p^{P_t} e^{\delta t}}{X_0^{-1} + \vartheta \int_0^t p^{P_s} e^{\delta s} ds}. \tag{3.4}$$

This can be verified either from [24, Proposition 4] in a more general setting, or by the following computation: Surely, $\mathcal{W}$ starts in X_0 and has the desired jumps at the times of P, since W_t/p^{P_t} is continuous. To verify the desired deterministic growth, let $\eta(t) := X_0^{-1} + \vartheta \int_0^t p^{P_s} e^{\delta s} ds$, the denominator of W_t, and assume that P does not jump in an interval $(t - \varepsilon, t + \varepsilon)$. Then,

$$\frac{d}{dt} W_t = \frac{p^{P_t} \delta e^{\delta t} \eta(t) - p^{P_t} e^{\delta t} \eta'(t)}{\eta(t)^2} = W_t \cdot \frac{\delta \eta(t) - \vartheta p^{P_t} e^{\delta t}}{\eta(t)} = W_t(\delta - \vartheta W_t) = w(W_t).$$

Next, we consider the process $\overline{\mathcal{W}}$ that arises from $\mathcal{W}$ by exchanging for every t the path $(P_s)_{0 \leq s \leq t}$ with its time-reversal $(\overline{P}^t_s)_{0 \leq s \leq t}$ via $\overline{P}^t_s := P_t - P_{t-s}$, given by

$$
\overline{W}_t = \frac{p^{\overline{P}^t_t} e^{\delta t}}{X_0^{-1} + \vartheta \int_0^t p^{\overline{P}^t_s} e^{\delta s} ds} = \frac{1}{X_0^{-1} p^{-P_t} e^{-\delta t} + \vartheta \int_0^t p^{-P_{t-s}} e^{-\delta(t-s)} ds}
$$
$$
= \left(X_0^{-1} p^{-P_t} e^{-\delta t} + \vartheta \int_0^t p^{-P_s} e^{-\delta s} ds \right)^{-1}.
$$

Now, since $(P_s)_{0 \leq s \leq t}$ is equally distributed as $(\overline{P}_{s+})_{0 \leq s \leq t}$ for every $t > 0$, where $\overline{P}_{s+} := \lim_{r \downarrow s} \overline{P}_r$ denotes the right-side limit, we also have that $W_t \overset{d}{=} \overline{W}_t$ for all t and in particular $\mathbb{E}[W_t^k] = \mathbb{E}[\overline{W}_t^k]$. Also note that, in contrast to W_t, $\overline{W}_t$ decreases if the path of (P_t) increases pointwise. Now, using this and (A.3) of Lemma A.1, we obtain for (L_2) (even for arbitrary $\delta > 0$), considering that $\delta \leq \hat{\alpha}$ and thus $\gamma \geq \frac{1}{\hat{\alpha}} \log \frac{1}{p} > 1$ as well as $\delta = \frac{1}{\gamma} \log \frac{1}{p}$,

$$
\frac{1}{t} \log \mathbb{E}[X_t^k] \geq \frac{1}{t} \log \mathbb{E}[\overline{W}_t^k]
$$
$$
= \frac{1}{t} \log \mathbb{E}\left[\left(X_0^{-1} p^{-P_t} e^{-\delta t} + \vartheta \int_0^t p^{-P_s} e^{-\delta s} ds \right)^{-k} \right]
$$
$$
\geq \frac{1}{t} \log \mathbb{E}\left[\left(X_0^{-1} p^{-P_t} e^{-\delta t} + \vartheta \int_0^t p^{-P_s} e^{-\delta s} ds \right)^{-k}, P_s \leq \frac{s}{\gamma} \text{ for all } s \leq t \right]
$$
$$
\geq \frac{1}{t} \log \left(\left(X_0^{-1} e^{(\frac{1}{\gamma} \log \frac{1}{p} - \delta)t} \right. \right.
$$
$$
\left. \left. + \vartheta \int_0^t e^{(\frac{1}{\gamma} \log \frac{1}{p} - \delta)s} ds \right)^{-k} \mathbb{P}(P_s \leq \frac{s}{\gamma} \text{ for all } s \leq t) \right)
$$
$$
\xrightarrow{t \to \infty} -k \cdot 0 - (1 - \frac{1}{\gamma}(1 + \log \gamma)) = A_\gamma.
$$

For the case $\delta \leq p^k \log \frac{1}{p}$, we deduce analogously

$$
\frac{1}{t} \log \mathbb{E}[X_t^k] \geq \frac{1}{t} \log \mathbb{E}\left[\left(X_0^{-1} p^{-P_t} e^{-\delta t} + \vartheta \int_0^t p^{-P_s} e^{-\delta s} ds \right)^{-k} \right]
$$
$$
\geq \frac{1}{t} \log \left(\left(X_0^{-1} e^{(p^k \log \frac{1}{p} - \delta)t} \right. \right.
$$
$$
\left. \left. + \vartheta \int_0^t e^{(p^k \log \frac{1}{p} - \delta)s} ds \right)^{-k} \mathbb{P}(P_s \leq p^k s \text{ for all } s \leq t) \right)
$$
$$
\xrightarrow{t \to \infty} -k(p^k \log \frac{1}{p} - \delta) - (1 - p^k + p^k \log(p^k)) = \delta k - (1 - p^k).
$$

With the same argument as at the end of the proof of (U_2), one can recognise that the bound in (L_1) is stronger than A_γ if $\delta \leq p^k \log \frac{1}{p}$.

2. Recalling that $Y_t := -\log X_t$ and letting $T_z := \inf\{t > 0 : Y_t = z\}$, we show that (i) there is $z \geq 0$ such that $\mathbb{E}[T_z | Y_0 = z] < \infty$ and (ii) $\mathbb{P}(T_z < \infty | Y_0 = y) = 1$ for all $y \geq 0$:

Since $\alpha_0' > \log \frac{1}{p}$, there have to be $\xi > 0$ and $\varepsilon > 0$ such that $\alpha(x)/x \geq (1 + \varepsilon) \log \frac{1}{p}$ for all $x \in (0, \xi]$. Setting $z = -\log \xi < \infty$, we see that $\beta(y) \geq (1 + \varepsilon) \log \frac{1}{p} =: \zeta$ for all $y \geq z$. We define

$$
S := S_{(z, z+\log(1/p)]} := \inf\{t : z < Y_t \leq z + \log(1/p)\}.
$$

Then, $\mathbb{E}[S|Y_0 = y] < \infty$ for all $y \leq z$. Indeed, the probability for at least $z/\log(1/p)$ jumps in some small time interval of length $\varepsilon' > 0$ is positive. After the first such time interval we can be sure that S has occurred. By finiteness of first moments of geometric distributions, $\mathbb{E}[S|Y_0 = y] < \infty$ follows. By a restart argument, we have to show that $\mathbb{E}[T_z|Y_0 = y] < \infty$ for all $z < y \leq z + \log(1/p)$, which will be done by using a comparison argument. For this, let $\mathcal{R} = (R_t)_{t \geq 0}$ be a process with generator

$$\mathcal{G}_{\mathcal{R}} g(y) = g(y + \log(1/p)) - g(y) - \zeta g'(y).$$

If $z < R_0 = Y_0 \leq z + \log(1/p)$, then – using the same Poisson processes for $\mathcal{Y}$ and $\mathcal{R}$ – we have that $T_z \leq T_z^{\mathcal{R}} := \inf\{t \geq 0 : R_t = z\}$ since $\beta(y) \geq \zeta$ for $y \geq z$. Analogously to the argument following (3.1), we see that $R_t \to -\infty$ as $t \to \infty$ almost surely, which implies that $T_z^{\mathcal{R}} < \infty$. Since $(R_t - R_0 + t(\zeta - \log(1/p)))_{t \geq 0}$ is a martingale, we have by optional stopping that $\mathbb{E}[R_0 - R_{T_z^{\mathcal{R}}}] = R_0 - z = (\zeta - \log(1/p))\mathbb{E}[T_z^{\mathcal{R}}] = \varepsilon \log(1/p)\mathbb{E}[T_z^{\mathcal{R}}]$, hence $\mathbb{E}[T_z^{\mathcal{R}}|Y_0 = R_0] \leq 1/\varepsilon < \infty$. It is now straightforward to obtain the properties (i) and (ii).

Now, by (i) and homogeneity, $\mathcal{Y}$ is a positively recurrent delayed regenerative process in the sense of Definition 7.1.1 in [5, p. 387] with regeneration cycles starting at the state z (and so is $\mathcal{X}$ with cycles starting in ξ). By (ii), the delay is almost surely finite. Hence, Theorem 7.1.4 in [5, p. 388] gives us weak convergence of $\mathcal{Y}$ to a finite random variable Y_∞ and thus, also $\mathcal{X}$ has a weak limit $X_\infty := e^{-Y_\infty} > 0$. Furthermore, recalling that $x_\alpha = \min\{x \in (0,1] : \alpha(x) = 0\}$, this is well-defined since α is positive on $(0, \xi)$ and Lipschitz-continuous on $[\xi, 1]$, $\alpha'_0 > 0$ and $\alpha(1) \leq 0$. This implies that the hitting time of $[0, x_\alpha)$ of $\mathcal{X}$ is almost surely finite. After that, $\mathcal{X}$ will never hit $[x_\alpha, 1]$ again, since it will always jump before it can reach x_α. Thus, $\mathbb{P}(X_\infty \in (0, x_\alpha)) = 1$.

If $f : [0, 1] \to \mathbb{R}$ is measurable and bounded, Theorem 7.1.6 of Bladt and Nielsen [5, p. 391] as well as the weak convergence give us that almost surely

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s) ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}[f(X_s)] ds = \mathbb{E}[f(X_\infty)]. \tag{3.5}$$

Since for $t > 0$ and a continuous and bounded function f the distribution μ of X_∞ satisfies

$$\mathbb{E}_\mu[f(X_t)] = \mathbb{E}\big[\mathbb{E}_{X_\infty}[f(X_t)]\big] = \lim_{s \to \infty} \mathbb{E}\big[\mathbb{E}_{X_s}[f(X_t)]\big] = \lim_{s \to \infty} \mathbb{E}[f(X_{t+s})] = \mathbb{E}[f(X_\infty)],$$

μ is a stationary distribution. Letting ν be a stationary distribution of $\mathcal{X}$ we obtain from (3.5) that for all t

$$\mathbb{E}_\nu[f(X_1)] = \frac{1}{t} \int_0^t \mathbb{E}_\nu[f(X_s)] ds = \mathbb{E}[f(X_\infty)],$$

by which the stationary distribution must be unique. In particular, by stationarity $\mathbb{E}[\mathcal{G}_{\mathcal{X}} f(X_\infty)] = 0$ holds. Hence, choosing $f(x) = \log(\rho + x)$ for $\rho > 0$, we obtain

$$0 = \mathbb{E}\left[\log\left(\frac{\rho + pX_\infty}{\rho + X_\infty}\right) + \frac{\alpha(X_\infty)}{\rho + X_\infty}\right] \xrightarrow{\rho \to 0} -\log(\tfrac{1}{p}) + \mathbb{E}[X_\infty^{-1}\alpha(X_\infty)]$$

by monotone convergence, since $X_\infty > 0$ almost surely. (Note here, that this argument works for both cases (C_1) and (C_2).) On the other hand, choosing $f(x) = x^k$, it follows

$$0 = \mathbb{E}[(pX_\infty)^k - X_\infty^k + \alpha(X_\infty)kX_\infty^{k-1}],$$

which implies $\mathbb{E}[X_\infty^k] = \frac{k}{1-p^k}\mathbb{E}[X_\infty^{k-1}\alpha(X_\infty)]$. $\quad\square$

Remark 3.3.

1. For $\alpha_0' < \log \frac{1}{p} < \hat{\alpha}$, Theorem 1 is inconclusive. The weak estimates using $\hat{\alpha}$ in (3.1) and (3.2) offer room for improvement. Also note here that for the application of (L_1) and (L_2) for $\alpha(0) = 0$, one can often choose $\delta = \alpha'(0) = \alpha_0'$.

2. Using the notion of *strong large deviations* (e.g. [9, Theorem 3.5, p.1868]) one can compute exact asymptotics:

    (a) If $x > 1$, then $\mathbb{P}(P_t \geq xt) \sim \frac{\sqrt{x}}{x-1} \cdot \frac{e^{-t(1-x+x\log x)}}{\sqrt{2\pi t}}$,

    (b) If $0 < x < 1$, then $\mathbb{P}(P_t \leq xt) \sim \frac{1}{(1-x)\sqrt{x}} \cdot \frac{e^{-t(1-x+x\log x)}}{\sqrt{2\pi t}}$,

    where $\sim$ denotes asymptotic equivalence, i.e. $f \sim g \Leftrightarrow f(t)/g(t) \to 1$ as $t \to \infty$. Applying these to u_1 and u_2 in (3.3) would provide more precise upper bounds in (U_1) and (U_2). For similar stronger bounds in (L_1) and (L_2) however, one would need strong large deviation results for Poisson processes.

3. Dropping the boundedness of the state space of $\mathcal{X}$, i.e. considering $s_\alpha = \infty$, (3.1) holds letting $\hat{\alpha} = \sup_{x \in \mathbb{R}^+} \frac{1}{x}\alpha(x)$ and so does *1.* as well as (U_1). Also, (L_1) and (L_2) still apply, starting $\mathcal{W}$ at $W_0 := \min\{1, X_0\}$. But we need boundedness from below of $\mathcal{Y}$ to prove (3.3) and the finiteness of $\mathbb{E}[S|Y_0 = y]$ and thus lose (U_2) as well as the limit results of *2.*

4. Whenever (X_t) is a p-jump process with drift α, the process of the kth powers, (X_t^k), is a p^k-jump process with drift $\alpha_k : x \mapsto kx^{1-\frac{1}{k}}\alpha(x^{\frac{1}{k}})$ holding $\hat{\alpha}_k = k\hat{\alpha}$ as well as $(\alpha_k)_0' = k\alpha_0'$. Hence, it would suffice to prove (U_1) and (U_2) for $k = 1$. For (L_1) and (L_2) however, the lower bound on α only implies that $\alpha_k(x) \geq k\delta x - k\vartheta x^{1+\frac{1}{k}}$, such that for $k > 1$ we cannot use the process $\mathcal{W}$ in (3.4) as a lower bound process and the proof fails.

3.2. Proof of Corollary 2.4

Proof. The concavity of α implies that x_α equates to x_α and s_α from Theorem 1. Noting that, if $I = \mathbb{R}^+$ and $s = \max\{x_\alpha, X_0\}$, $\alpha'(s) > -\infty$, by the same scaling argument as in the beginning of the proof of Theorem 1 we can assume that $I = [0, 1]$.

We start with $p \in (0, 1)$: Since α is concave, $x \mapsto \frac{\alpha(x)-\alpha(y)}{x-y}$ decreases on $(y, 1]$ and $y \mapsto \frac{\alpha(x)-\alpha(y)}{x-y}$ decreases on $(0, x)$. Thus, by the assertions, $|\frac{\alpha(x)-\alpha(y)}{x-y}|$ is bounded on each interval $[\varepsilon, 1]$. Hence, Lemma 3.1.*1* shows properness of α and Lemma 3.1.*2* implies existence and uniqueness of $\mathcal{X}$. Thus, Theorem 1 applies and *3* immediately follows by Theorem 1.*2*.

Recall that $\alpha_0' = \alpha'(0)$, if $\alpha_0' < \infty$. Now, for *1.* we want to apply (L_1) and (L_2) of Theorem 1 using $\delta = \alpha_0' = \alpha'(0)$. Let us first assume that $\alpha''(0) > -\infty$. Then, there is $\vartheta > 0$ such that $\alpha'(0)x - \vartheta x^2 \leq \alpha(x)$ for all $x \in [0, 1]$. (The Taylor expansion delivers the existence of a ϑ_0 such that $\alpha'(0)x - \vartheta_0 x^2 \leq \alpha(x)$ holds for x in some interval $[0, \varepsilon)$. The boundedness of α ensures that we can choose ϑ_1 such that $\alpha'(0)x - \vartheta_1 x^2 \leq \alpha(x)$ holds for $x \geq \varepsilon$. Let $\vartheta = \max\{\vartheta_0, \vartheta_1\}$.)

On the other hand, if $\alpha''(0) = -\infty$, α has no parabolic lower bound as we need for (L_1) and (L_2) (e.g. for $\alpha(x) := ax - bx^{3/2}$ with $b \geq a > 0$). We solve this by a coupling argument: Therefore, let p be fixed and for all n define

$$\alpha_n(x) = \min\{\alpha(x), x\alpha(\tfrac{1}{n})\},$$

the minimum of α and the secant of α intersecting at 0 and $\frac{1}{n}$. Surely, the α_n satisfy the conditions of Theorem 1, $\alpha_n \nearrow \alpha$ point-wise, $\alpha''(0) = 0 > -\infty$ and $\alpha_n(0) = 0$ such that

$(\alpha_n)'_0 = \alpha'_n(0)$. Let $\mathcal{X}^{(n)}$, $n \geq 1$, be p-jump processes with respective drifts α_n, each started in X_0, coupled in such a way that they jump simultaneously with $\mathcal{X}$. Then, $X_t^{(n)} \leq X_t$ for all n and t and thus

$$\liminf_{t \to \infty} \tfrac{1}{t} \log \mathbb{E}[X_t^k] \geq \sup_n \liminf_{t \to \infty} \tfrac{1}{t} \log \mathbb{E}[(X_t^{(n)})^k].$$

From the previous case, we obtain for every n

$$\liminf_{t \to \infty} \tfrac{1}{t} \log \mathbb{E}[(X_t^{(n)})^k] \geq \begin{cases} -(1 - p^k - \alpha'_n(0)k) & \text{if } \alpha'_n(0) \leq p^k \log \tfrac{1}{p} \\ -(1 - \tfrac{1}{\gamma_n}(1 + \log \gamma_n)) & \text{if } p^k \log \tfrac{1}{p} < \alpha'_n(0) < \log \tfrac{1}{p}, \end{cases}$$

where $\gamma_n = \log \tfrac{1}{p}/\alpha'_n(0)$. Since $\alpha'_n(0) = \alpha(\tfrac{1}{n})/(\tfrac{1}{n}) \nearrow \alpha'(0)$ and the bounds in either case increase in $\alpha'_n(0)$, we obtain "$\leq$" in *1*. Since α is concave, it satisfies $\alpha'(0) = \hat{\alpha}$ and thus, (U_1) and (U_2) of Theorem 1 provide "$\geq$" and *1*. follows.

For the case $\alpha'_0 = \log \tfrac{1}{p}$ consider a family $(\mathcal{X}^{(p)})_{p \in (0,1)}$ of processes with respective generators as in (2.1) with α fixed and $p \mapsto X_0^{(p)}$ constant, coupled in such a way that the processes jump simultaneously. Then, for every t the map $p \mapsto X_t^{(p)}$ is increasing. Since $p^k \log \tfrac{1}{p} < \alpha'_0$ for p near $p^* := e^{-\alpha'_0}$, we conclude from *1*. and the boundedness of the $\mathcal{X}^{(p)}$ that

$$0 \geq \lim_{t \to \infty} \tfrac{1}{t} \log \mathbb{E}[(X_t^{(p^*)})^k] \geq \sup_{p < p^*} \lim_{t \to \infty} \tfrac{1}{t} \log \mathbb{E}[(X_t^{(p)})^k]$$

$$= \sup_{p < p^*} -\left(1 - \frac{\alpha'_0}{\log \tfrac{1}{p}} \left(1 - \log\left(\frac{\alpha'_0}{\log \tfrac{1}{p}}\right)\right)\right) = 0.$$

Since $x \mapsto \tfrac{1}{x}\alpha(x) = \tfrac{1}{x}(\alpha(x) - \alpha(0)) + \tfrac{1}{x}\alpha(0)$ is decreasing, Theorem 1.2 implies

$$\alpha'_0 \geq \mathbb{E}\left[\liminf_{t \to \infty} \frac{\alpha(X_t^{(p^*)})}{X_t^{(p^*)}}\right] \geq \sup_{p > p^*} \mathbb{E}\left[\liminf_{t \to \infty} \frac{\alpha(X_t^{(p)})}{X_t^{(p)}}\right] = \sup_{p > p^*} \log \tfrac{1}{p} = \alpha'_0$$

and hence almost surely

$$\liminf_{t \to \infty} \frac{\alpha(X_t^{(p^*)})}{X_t^{(p^*)}} = \lim_{t \to \infty} \frac{\alpha(X_t^{(p^*)})}{X_t^{(p^*)}} = \alpha'_0.$$

Since $x \mapsto \alpha(x)/x$ is decreasing, $\limsup_t X_t^{(p^*)}$ has to be bounded by $m_\alpha := \sup\{x \in [0,1] : \alpha(x) = x\alpha'_0\}$. Finally, if α is strictly concave near 0, $\alpha(x)/x$ strictly decreases near 0 and thus $m_\alpha = 0$, which concludes the proof for $p \in (0,1)$.

Considering $p = 1$, we can ignore the jumps and $\mathcal{X}$ becomes deterministic, i.e. the solution of $\dot{X}_t = \alpha(X_t)$. Then, it holds if $\alpha'_0 < 0 = \log(\tfrac{1}{p})$ that

$$\frac{1}{t}(\log(X_t) - \log(X_0)) = \frac{1}{t} \int_0^t \frac{\dot{X}_s}{X_s} ds = \frac{1}{t} \int_0^t \frac{\alpha(X_s)}{X_s} ds \leq \alpha'_0 < 0.$$

Hence, $X_t \to 0$, $\alpha(X_t)/X_t \to \alpha'_0$ and $\tfrac{1}{t} \log(X_t^k) \to k\alpha'_0$, which aligns with *1*. However, if $\alpha'_0 = 0$, since α is concave it is non-positive. Then, $\mathcal{X}$ is constant, if $\alpha(X_0) = 0$, and it converges monotonically to $m_\alpha = \sup\{x \in [0,1] : \alpha(x) = 0\}$ if $X_0 > m_\alpha$. Thus, $\lim_{t \to \infty} \tfrac{1}{t} \log(X_t) = \lim_{t \to \infty} \alpha(X_t)/X_t = 0$ giving us *2*. Lastly, in the case of $\alpha'_0 > 0$, $\mathcal{X}$ will either grow towards x_α if started below, i.e. $X_0 \in (0, x_\alpha)$, or fall towards it if started above. Either way, $X_t \to x_\alpha$, $\alpha(X_t) \to 0$ and we obtain *3*.

Choosing $p = 0$, $\mathcal{X}$ would jump to 0 after an exponentially distributed time T with mean 1 and stay there indefinitely. Thus, we can write $X_t = Y_t \cdot \mathbb{1}_{\{T>t\}}$, where (Y_t) is the deterministic process arising for $p = 1$ discussed above. Then, $\mathcal{X}$ will always converge to 0 almost surely and

$$-\frac{1}{t} \log(\mathbb{E}[X_t^k]) = -\frac{1}{t} \log \mathbb{P}(T > t) - \frac{k}{t} \log(Y_t) \xrightarrow{t \to \infty} 1 + \max\{0, -k\alpha_0'\}. \quad \square$$

4. Branching processes with binomial disasters

In the following subsections, we borrow ideas from the notion of duality of Markov processes; see Chapter 4.4 in [12].

Recall that two Markov processes $\mathcal{Z} = (Z_t)_{t \geq 0}$ and $\mathcal{X} = (X_t)_{t \geq 0}$ with state spaces E and E' are called dual with respect to the function $H : E \times E' \to \mathbb{R}$ if

$$\mathbb{E}[H(Z_t, x)|Z_0 = z] = \mathbb{E}[H(z, X_t)|X_0 = x] \tag{D}$$

for all $z \in E, x \in E'$. When one is interested in the process $\mathcal{Z}$, this relationship is most helpful if the process $\mathcal{X}$ is easier to analyse than the process $\mathcal{Z}$. Moreover, frequently, the set of functions $\{H(\cdot, x) : x \in E'\}$ is separating on E such that the left hand side of (D) determines the distribution of Z_t. In this case, the distribution of the simpler process $\mathcal{X}$ determines via (D) the distribution of $\mathcal{Z}$, so analysing $\mathcal{Z}$ becomes feasible.

There is no straightforward way how to find dual processes, but they arise frequently in the literature; see [16] for a survey. Examples span reflected and absorbed Brownian motion, interacting particle models such as the voter model and the contact process, as well as branching processes.

A simple way to verify (D) for homogeneous $\mathcal{Z}$ and $\mathcal{X}$, is to show that

$$\frac{\partial}{\partial t} \mathbb{E}[H(Z_t, x)|Z_0 = z]\big|_{t=0} = \frac{\partial}{\partial t} \mathbb{E}[H(z, X_t)|X_0 = x]\big|_{t=0} \tag{D'}$$

for all z and x, since then both sides of (D) follow the same evolution.

4.1. Proof of Theorem 2

In this section we will discuss branching processes of the form of Definition 2.5. Hence, let $\mathcal{Z} := \mathcal{Z}_{\lambda,q,\kappa,p}^{hom}$, where $\lambda \in (0, \infty)$ is the death-rate, $q = (q_k)_{k \geq 0}$ the offspring distribution on $\mathbb{N}_0$, $p \in (0, 1)$ the survival probability of the disasters that occur at the jump times of $(D_t)_{t \geq 0}$, a Poisson process with rate $\kappa > 0$. Moreover, let $h : [0, 1] \to [0, 1], x \mapsto \sum_{k \geq 0} q_k x^k$ be the probability generating function of the offspring distribution. We start with establishing a suitable duality for $\kappa = 1$. The general case will follow by a rescaling argument.

Lemma 4.1. *Let $p \in [0, 1]$, $\kappa = 1$ and $\mathcal{X} = (X_t)$ be a proper p-jump process with drift $x \mapsto \lambda(1 - x - h(1 - x))$. Then, the duality relation*

$$\mathbb{E}[(1 - X_t)^z|X_0 = x] = \mathbb{E}[(1 - x)^{Z_t}|Z_0 = z] \tag{4.1}$$

holds for every $x \in [0, 1], z \in \mathbb{N}_0$ and $t \geq 0$.

Proof. Recalling the generator of $\mathcal{Z}$ from Definition 2.5, we obtain for $x \in [0,1]$, $z \in \mathbb{N}_0$ and $H(x,z) := (1-x)^z$ that, recalling (2.1),

$$\big(\mathcal{G}_{\mathcal{Z}} H(x, \cdot)\big)(z)$$

$$= \lambda z (1-x)^{z-1} \sum_{k \geq 0} q_k \big((1-x)^k - (1-x)\big)$$

$$+ \sum_{\ell=0}^{z} \binom{z}{\ell} \big(p(1-x)\big)^{\ell} (1-p)^{z-\ell} - (1-x)^z$$

$$= -\frac{\partial}{\partial x} (1-x)^z \cdot \lambda \big(h(1-x) - (1-x)\big) + \big(p(1-x) + (1-p)\big)^z - (1-x)^z$$

$$= \big(\mathcal{G}_{\mathcal{X}} H(\cdot, z)\big)(x),$$

which resembles (D′). Hence, (D) gives us the desired relation. $\square$

Now, we apply Corollary 2.4 to the dual process $\mathcal{X}$, followed by the proof of Theorem 2.

Lemma 4.2. *For $p \in [0,1]$, such a process $\mathcal{X}$ in Lemma 4.1 exists and satisfies*

1. if $p = 0$ or $h'(1) < 1 + \frac{1}{\lambda} \log \frac{1}{p}$, $X_t \xrightarrow{t \to \infty} 0$ almost surely. Also, for $k \geq 1$

$$\lim_{t \to \infty} -\frac{1}{t} \log \mathbb{E}[X_t^k] = \begin{cases} 1 + \max\{0, -k(h'(1) - 1)\} & \text{if } p = 0, \\ 1 - p^k - k\lambda(h'(1) - 1) & \text{if } h'(1) \leq 1 + \frac{p^k}{\lambda} \log \frac{1}{p}, \\ 1 - \frac{1}{\gamma}(1 + \log \gamma) & \text{otherwise,} \end{cases}$$

where $\gamma = \frac{1}{\lambda} \log \frac{1}{p} / (h'(1) - 1)$.

2. if $h'(1) = 1 + \frac{1}{\lambda} \log \frac{1}{p}$, $X_t \xrightarrow{t \to \infty} 0$ almost surely, while $\frac{1}{t} \log \mathbb{E}[X_t^k] \xrightarrow{t \to \infty} 0$ for all k.

3. if $h'(1) \in (1 + \frac{1}{\lambda} \log \frac{1}{p}, \infty]$, letting x_* be the smallest fixed point of h, $\mathcal{X}$ converges weakly to a random variable X_∞ on $(0, 1 - x_*]$ that satisfies $\mathbb{E}[X_\infty^{-1}(1 - h(1 - X_\infty))] = 1 + \frac{1}{\lambda} \log \frac{1}{p}$ and for $k \geq 1$

$$\mathbb{E}[X_\infty^k] = \frac{\lambda k}{\lambda k + 1 - p^k} \mathbb{E}[X_\infty^{k-1}(1 - h(1 - X_\infty))]. \tag{4.2}$$

Proof. Since h is a convex function, $\alpha : x \mapsto \lambda(1 - x - h(1 - x))$ is concave. Also, $\alpha(0) = \lambda(1 - h(1)) = 0$, $\alpha(1) = -\lambda q_0 \leq 0$, $\alpha'_0 = \alpha'(0) = \lambda(h'(1) - 1) \in (-\lambda, \infty]$ and $\alpha'(1) = \lambda(h'(0) - 1) = \lambda(q_1 - 1) \geq -\lambda > -\infty$. Hence, considering that $\alpha'(0) \geq p^k \log \frac{1}{p}$ iff $h'(1) \leq 1 + \frac{p^k}{\lambda} \log \frac{1}{p}$ for all $k \geq 1$, Corollary 2.4 implies *1*. For *3.*, noting that $\alpha(x) > 0$ only if $0 < x < 1 - x_* = x_\alpha$, only (4.2) remains to be shown. Here, Corollary 2.4.*3* gives us for $k \geq 1$

$$\mathbb{E}[X_\infty^k] = \frac{\lambda k}{1 - p^k} \big(-\mathbb{E}[X_\infty^k] + \mathbb{E}[X_\infty^{k-1}(1 - h(1 - X_\infty))]\big)$$

and (4.2) follows. Finally, if $h'(1) = 1 + \frac{1}{\lambda} \log \frac{1}{p} > 1$, h is strictly convex and thus, α is strictly concave, which gives us *2*. $\square$

Proof of Theorem 2. First, let the theorem hold for $\kappa = 1$ and for arbitrary $\kappa > 0$ consider the process $\mathcal{Z}^* := \mathcal{Z}^{hom}_{\lambda/\kappa, q, 1, p}$. Then, $(Z_t)_t := (Z^*_{\kappa t})_t$ defines a $\mathcal{Z}^{hom}_{\lambda, q, \kappa, p}$-process and we obtain

$$\lim_{t \to \infty} -\frac{1}{t} \log \mathbb{P}(Z_t > 0) = \kappa \cdot \lim_{s \to \infty} -\frac{1}{s} \log \mathbb{P}(Z^*_s > 0),$$

which shows *1.* and *2.* Since $\lim_t Z_t = \lim_t Z_t^*$ almost surely, *3.* follows, where we obtain (2.3) by substitution.

Hence, without loss of generality let $\kappa = 1$. Then, letting $Z_0 = z$ and $X_0 = 1$, Lemma 4.1 implies,

$$\mathbb{P}(Z_t > 0) = 1 - \mathbb{E}[0^{Z_t}] = 1 - \mathbb{E}[(1 - X_t)^z] = \sum_{k=1}^{z} \binom{z}{k}(-1)^{k+1}\mathbb{E}[X_t^k]. \tag{4.3}$$

Considering that $X_t \in [0, 1]$, we obtain from Bernoulli's inequality

$$\mathbb{E}[X_t] \le 1 - \mathbb{E}[(1 - X_t)^z] = \mathbb{P}(Z_t > 0) \le z\mathbb{E}[X_t]. \tag{4.4}$$

Thus, noting that $h'(1) = \mu$ and $\gamma = 1/\nu$, for $\mu \le 1 + \frac{1}{\lambda}\log\frac{1}{p}$ (i.e. $\nu \le 1$) Lemma 4.2.*1* and 4.2.*2* show that

$$\lim_{t\to\infty} -\frac{1}{t}\log\mathbb{P}(Z_t > 0) = \begin{cases} 1 + \max\{0, 1 - h'(1)\} & \text{if } p = 0, \\ 1 - p - \lambda(\mu - 1) & \text{if } \nu \le p, \\ 1 - \nu(1 + \log\frac{1}{\nu}) & \text{if } p < \nu \le 1. \end{cases}$$

Additionally, considering the boundedness and thus the $\mathcal{L}^1$-convergence of $(X_t)_t$, we get from (4.4) that Z_t converges to 0 in probability. Since this implies almost sure convergence of a subsequence and 0 is an absorbing state, we have $Z_t \to 0$ almost surely.

For *2.*, noting that $\mathbb{P}(Z_t \xrightarrow{t\to\infty} 0 \mid Z_s) \ge \frac{1}{1+\lambda}(1-p)^{Z_s}$, i.e. the probability that the next event after s is a disaster that kills all, we obtain

$$\limsup_{s\to\infty}\mathbb{P}(Z_t \xrightarrow{t\to\infty} 0 \mid \sigma(Z_r; r \le s)) \ge \frac{1}{1+\lambda}\limsup_{s\to\infty}(1-p)^{Z_s} = \frac{1}{1+\lambda}(1-p)^{\liminf_{s\to\infty} Z_s}.$$

Thus, Lemma 3.1 of Kaplan et al. [17, p. 54] concludes

$$\mathbb{P}(Z_t \xrightarrow{t\to\infty} 0) + \mathbb{P}(Z_t \xrightarrow{t\to\infty} \infty) = 1.$$

Furthermore, for $\mu > 1 + \frac{1}{\lambda}\log\frac{1}{p}$ Lemma 4.2.*3* shows stationarity of the distribution of X_∞ and hence independence of X_0. Thus, using that 0 is an absorbing state and $\{Z_s = 0\} \subset \{Z_t = 0\}$ for $s \le t$, we obtain from (4.3) that

$$\mathbb{P}(\lim_{t\to\infty} Z_t = 0) = \mathbb{P}\Big(\bigcup_{t>0}\{Z_t = 0\}\Big) = \lim_{t\to\infty}\mathbb{P}(Z_t = 0) = \mathbb{E}[(1 - X_\infty)^z]. \quad \square$$

4.2. Preparation: Regular variation

In this subsection, using results of chapter VIII.9 of Feller [13] and Seneta [27], we will arrange the tools regarding regularly varying functions needed for the proof of Theorem 3. However, we need to establish some additional notation first:

Remark 4.3.

1. We will make use of the Bachmann–Landau notation: For a function $g : \mathbb{R}^+ \to [0, \infty)$, let

$$o(g) := \{f : \mathbb{R}^+ \to \mathbb{R}^+ \mid \limsup_{t\to\infty}\frac{f(t)}{g(t)} = 0\},$$

$$O(g) := \{f : \mathbb{R}^+ \to \mathbb{R}^+ \mid \limsup_{t\to\infty}\frac{f(t)}{g(t)} < \infty\},$$

$$\Omega(g) := \{f : \mathbb{R}^+ \to \mathbb{R}^+ \mid g \in O(f)\}.$$

2. We define the relation of *asymptotic equivalence* for functions $f, g : \mathbb{R}^+ \to \mathbb{R}$ by

$$f \overset{t \to \infty}{\sim} g \quad \Leftrightarrow \quad \lim_{t \to \infty} \frac{f(t)}{g(t)} = 1.$$

Often, when the running variable is either obvious or t, we will just write $f \sim g$.

Definition 4.4. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called *regularly varying with exponent $\beta \in \mathbb{R}$*, if for every $x > 0$

$$\frac{f(xt)}{f(t)} \xrightarrow{t \to \infty} x^\beta$$

holds. A *slowly varying* function is a regularly varying function with exponent 0.

Lemma 4.5. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ regularly varying with exponent $\beta \in \mathbb{R}$ and $F(t) := \int_0^t f(x)dx$.*

1. *F is regularly varying with exponent $\max\{\beta + 1, 0\}$ and for $t \to \infty$, if*

   (a) *$\beta > -1$, then $F(t) \sim tf(t)(\beta + 1)^{-1}$.*
   (b) *$\beta = -1$, then $F(t) \in \Omega(1) \cap O(t^\varepsilon)$ for all $\varepsilon > 0$.*
   (c) *$\beta < -1$, then $F(t) \to c < \infty$.*

2. *Let $(t_n) \subset \mathbb{R}^+$ such that $t_n \xrightarrow{n \to \infty} \infty$ and $t_{n+1}/t_n \xrightarrow{n \to \infty} 1$. Then,*

   $$\frac{F(t_{n+1})}{F(t_n)} \xrightarrow{n \to \infty} 1.$$

3. *There are functions a and ε such that $a(t) \xrightarrow{t \to \infty} c \in \mathbb{R}^+$, $\varepsilon(t) \xrightarrow{t \to \infty} 0$ and*

   $$f(t) = t^\beta a(t) \exp\left(\int_1^t \frac{\varepsilon(y)}{y} dy\right).$$

4. *For each $\alpha > 0$, $f \in O(t^{\beta+\alpha}) \cap \Omega(t^{\beta-\alpha})$.*

Proof. *3.* and *4.* follow from [27], Theorem 1.1 on page 2 and Proposition 1^0 on page 18 respectively, while *1.* is a consequence of *4.* and exercises 2.1, 2.2 and 2.3 on [27, p. 86]. (A proof of these exercises is given by Theorem 1 in [13, p.281].)

Finally, by *1.* F is regularly varying with exponent $\beta' \geq 0$. Applying *3.* we see that there are functions A and $\mathcal{E}$ with $\lim_{t \to \infty} A(t) = c \in (0, \infty)$ and $\lim_{t \to \infty} \mathcal{E}(t) = 0$ such that

$$\frac{F(t_{n+1})}{F(t_n)} = \left(\frac{t_{n+1}}{t_n}\right)^{\beta'} \cdot \frac{A(t_{n+1})}{A(t_n)} \cdot \exp\left(\int_{t_n}^{t_{n+1}} \frac{\mathcal{E}(y)}{y} dy\right).$$

Now, the first two factors converge to 1, while the integral in the exponent is bounded by $|t_{n+1} - t_n| \cdot \frac{1}{t_n} \sup_{y \in [t_n, t_{n+1}]} |\mathcal{E}(y)| \longrightarrow_{n \to \infty} 0$. $\square$

The following Theorem is needed in the proof of Theorem 3 to build a bridge between the asymptotics of the deterministic rate functions and the almost sure asymptotics of the process (L_t) from Lemma 4.9, which is key to the computation of the survival probability in the inhomogeneous case.

Theorem 4.6. *Let $(D_t)_{t \geq 0}$ be an inhomogeneous Poisson process with right continuous rate-function κ with left limits, $\Lambda(t) := \int_0^t \kappa_s ds$, $\Lambda^{-1}(t) := \inf\{s > 0 : \Lambda(s) > t\}$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(\Lambda^{-1}(\cdot))$ is regularly varying with exponent β.*

1. If $\Lambda(t) \xrightarrow{t\to\infty} \Lambda(\infty) < \infty$ or $\beta < -1$, then $\int_0^t f(s)dD_s$ has an almost surely finite limit.
2. If $\Lambda(t) \xrightarrow{t\to\infty} \infty$ and $\beta > -1$,

$$\int_0^t f(s)dD_s \sim \int_0^t f(s)\kappa_s ds$$

   holds almost surely and in $\mathcal{L}^2$.
3. If $\beta = -1$, for arbitrary $\alpha > 0$ it holds $\frac{\int_0^t f(s)dD_s}{\int_0^t f(s)\kappa_s ds} \in O(t^\alpha) \cap \Omega(t^{-\alpha})$ almost surely.

Proof. First note that there is a unit-rate Poisson process, which we denote by (P_t) and its jump times by $(\sigma_k)_k$, such that $D_t = P_{\Lambda(t)}$ for all $t \geq 0$ and the jump times of (D_t) satisfy $\tau_k = \Lambda^{-1}(\sigma_k)$. Then, supposing that 2. holds for $\kappa \equiv 1$, the general case follows as

$$\int_0^t f(s)dD_s = \sum_{k=1}^{D_t} f(\tau_k) = \sum_{k=1}^{P_{\Lambda(t)}} f(\Lambda^{-1}(\sigma_k)) \sim \int_0^{\Lambda(t)} f(\Lambda^{-1}(s))ds = \int_0^t f(s)\kappa_s ds.$$

Thus, without loss of generality, let $\kappa \equiv 1$, $(D_t) = (P_t)$ and f regularly varying with exponent $\beta > -1$. Letting $F(t) = \int_0^t f(x)dx$, it remains to be shown that

$$Y_t := \frac{1}{F(t)} \sum_{k=1}^{D_t} f(\tau_k) \xrightarrow{t\to\infty} 1$$

almost surely and in $\mathcal{L}^2$. Starting with the $\mathcal{L}^2$-convergence, we recall that on the event $\{D_t = n\}$, the jump times $(\tau_1, \ldots, \tau_n)$ are equal in distribution to $(U_{(1)}^t, U_{(2)}^t, \ldots, U_{(n)}^t)$, the order statistic of n iid variables $(U_1^t, \ldots, U_n^t)$, uniformly distributed on $[0, t]$. We obtain

$$\mathbb{E}\Big[\sum_{k=1}^{D_t} f(\tau_k)\Big] = \mathbb{E}\Big[\sum_{k=1}^{D_t} \mathbb{E}\big[f(\tau_k)\big|D_t\big]\Big] = \mathbb{E}\Big[\sum_{k=1}^{D_t} \mathbb{E}\big[f(U_k^t)\big|D_t\big]\Big]$$

$$= t\mathbb{E}[f(U_1^t)] = t \cdot \frac{1}{t}\int_0^t f(s)ds = F(t). \tag{4.5}$$

Hence, $\mathbb{E}[Y_t] = 1$ and we compute

$$\|Y_t - 1\|_{\mathcal{L}^2} = \text{Var}\Big[\frac{1}{F(t)} \sum_{k=1}^{D_t} f(\tau_k)\Big]$$

$$= \frac{1}{F(t)^2}\Big(\text{Var}\Big[\sum_{k=1}^{D_t} \mathbb{E}\big[f(U_k^t)\big|D_t\big]\Big] + \mathbb{E}\Big[\sum_{k=1}^{D_t} \text{Var}\big[f(U_k^t)\big|D_t\big]\Big]\Big)$$

$$= \frac{1}{F(t)^2}\Big(\text{Var}\Big[D_t\mathbb{E}[f(U_1^t)]\Big] + \mathbb{E}\Big[D_t\text{Var}[f(U_1^t)]\Big]\Big)$$

$$= \frac{1}{F(t)^2}\Big(t\mathbb{E}[f(U_1^t)]^2 + t\text{Var}[f(U_1^t)]\Big)$$

$$= \frac{t\mathbb{E}[f(U_1^t)^2]}{F(t)^2} = \frac{\int_0^t f(x)^2 dx}{\big(\int_0^t f(x)dx\big)^2}.$$

Since f^2 is regularly varying with exponent 2β, we obtain from Lemma 4.5.*1* and 4.5.*4* for

- $\beta > -\frac{1}{2}$, that $\|Y_t - 1\|_{\mathcal{L}^2} \sim \frac{1}{t} \cdot \frac{(\beta+1)^2}{2\beta+1}$.
- $\beta = -\frac{1}{2}$, some slowly varying function ℓ and arbitrary $\varepsilon > 0$ that

$$\|Y_t - 1\|_{\mathcal{L}^2} = \frac{\ell(t)}{t} \in O(t^{-1+\varepsilon}).$$

- $\beta \in (-1, -\frac{1}{2})$, the numerator converges to a constant and the denominator converges to ∞.

Either way, the $\mathcal{L}^2$ convergence follows.

For the almost sure convergence first note that there is a subsequence $(t_n)_n$ with $t_n \nearrow \infty$ as well as $\lim_{n\to\infty} Y_{t_n} = 1$ and hence, $\liminf_t Y_t \leq 1 \leq \limsup_t Y_t$ almost surely. Noting that (Y_t) is a piecewise deterministic process, jumping upwards and between jumps decreasing continuously, the maximum and minimum on the nth deterministic piece of the path respectively are given by

$$Y_n^+ := Y_{\tau_n} = \frac{1}{F(\tau_n)} \sum_{k=1}^n f(\tau_k) \quad \text{and} \quad Y_n^- := Y_{\tau_{n+1}-} = \frac{1}{F(\tau_{n+1})} \sum_{k=1}^n f(\tau_k)$$

and we obtain for every t that $Y_{D_t}^- \leq Y_t \leq Y_{D_t}^+$. Also, we deduce from Lemma 4.5.*2* that

$$\frac{Y_n^+}{Y_n^-} = \frac{F(\tau_{n+1})}{F(\tau_n)} \xrightarrow{n\to\infty} 1$$

almost surely, considering that $\tau_{n+1}/\tau_n \xrightarrow{n\to\infty} 1$. Since the values of the local extrema of $(Y_t)_t$ are given by Y^+ and Y^-, it follows that

$$\liminf_{n\to\infty} Y_n^+ = \liminf_{n\to\infty} Y_n^- = \liminf_{t\to\infty} Y_t \leq 1 \leq \limsup_{t\to\infty} Y_t = \limsup_{n\to\infty} Y_n^+.$$

Hence, it suffices to show that $Y_n^+ \xrightarrow{n\to\infty} 1$ almost surely. For this, let $h(n) := \min\{\sqrt{n}, \sqrt{F(n)}\}$ and decompose Y_n^+ in the following way:

$$Y_n^+ = \frac{1}{F(\tau_n)} \sum_{k \leq h(n)} f(\tau_k) + \frac{1}{F(\tau_n)} \sum_{h(n) < k \leq n} f(\tau_k). \tag{4.6}$$

Considering that $A^- := \inf_{n\geq 1} \frac{\tau_n}{n} > 0$ and $A^+ := \sup_{n\geq 1} \frac{\tau_n}{n} < \infty$ almost surely, by Lemma 4.5 we obtain for the first part

$$\frac{1}{F(\tau_n)} \sum_{k \leq h(n)} f(\tau_k) \sim \frac{\beta+1}{n} \sum_{k \leq h(n)} \left(\frac{\tau_k}{\tau_n}\right)^\beta \cdot \frac{a(\tau_k)}{a(\tau_n)} \cdot \exp\left(\int_{\tau_k}^{\tau_n} \frac{\varepsilon(y)}{y} dy\right)$$

$$\leq \frac{C}{n} \cdot \left(\frac{A^+}{A^-}\right)^{|\beta|} \exp\left(\int_0^{nA^+} \frac{|\varepsilon(y)|}{y} dy\right) \cdot \sum_{k \leq h(n)} \left(\frac{k}{n}\right)^\beta,$$

where the constant C arises from the boundedness of a. Now, for $\beta \geq 0$ the remaining sum is bounded above by $h(n) \leq \sqrt{n}$, while for $-1 < \beta < 0$ it holds for some slowly varying function ℓ that

$$\sum_{k \leq h(n)} \left(\frac{k}{n}\right)^\beta \leq \sum_{k \leq h(n)} n^{|\beta|} \leq h(n) n^{|\beta|} = n^{|\beta| + \frac{1+\beta}{2}} \ell(n) = n^{\frac{1+|\beta|}{2}} \ell(n).$$

Thus, noting that $\frac{1+|\beta|}{2} < 1$ and $\exp(\int_0^\bullet \frac{|\varepsilon(y)|}{y}dy)$ is slowly varying, it follows that

$$\frac{1}{F(\tau_n)} \sum_{k \leq h(n)} f(\tau_k) \xrightarrow{n \to \infty} 0$$

almost surely. Hence, since $\tau_n \sim n$ almost surely and thus $F(n) \sim F(\tau_n)$ and $f(\tau_k) \sim f(k)$ by Lemma 4.5.2, it follows from (4.6) and Lemma 4.5.1(a)

$$Y_n^+ \sim \frac{F(n)}{F(\tau_n)} \cdot \frac{1}{F(n)} \sum_{h(n)<k \leq n} \frac{f(\tau_k)}{f(k)} \cdot f(k) \sim \frac{\beta+1}{n} \sum_{h(n)<k \leq n} \frac{f(k)}{f(n)}$$

$$\sim \frac{\beta+1}{n} \sum_{h(n)<k \leq n} \left(\frac{k}{n}\right)^\beta \sim (\beta+1) \int_{h(n)/n}^1 x^\beta dx \xrightarrow{n \to \infty} 1$$

and the proof of 2. is done.

For 1. if $\Lambda(\infty) < \infty$, also $\lim_{t \to \infty} \int_0^t f(s)dD_s = \sum_{k=1}^{P_{\Lambda(\infty)}} f(\tau_k)$ is almost surely finite. Otherwise, we obtain from (4.5) and Lemma 4.5.1 that

$$\mathbb{E}\left[\int_0^t f(s)dD_s\right] = \int_0^t f(s)\kappa_s ds \leq \int_0^\infty f(s)\kappa_s ds = \int_0^\infty f(\Lambda^{-1}(s))ds < \infty,$$

which also, by monotone convergence, implies the finiteness of $\int_0^\infty f(s)dD_s$ and 1. is done.

Lastly, for 3. we conclude that for $\alpha > 0$, $F(t) := \int_0^t f(s)\kappa_s ds$ and $Y_t := \int_0^t f(s)dD_s/F(t)$

$$0 \leq t^{-\alpha}Y_t \leq \frac{1}{F(t)} \int_0^t s^{-\alpha} f(s)dD_s.$$

Now, since $t \mapsto t^{-\alpha} f(t)$ is regularly varying with exponent $-1 - \alpha < -1$, 1. shows that the integral almost surely converges to some finite limit and hence, considering that F is non-decreasing and non-negative, $\limsup_t t^{-\alpha}Y_t < \infty$ almost surely and $Y_t \in O(t^\alpha)$. Similarly, using 2.

$$t^\alpha Y_t \geq \frac{1}{F(t)} \int_0^t s^\alpha f(s)dD_s,$$

which either converges to a positive constant, if $\Lambda(\infty) < \infty$, or is asymptotically equivalent to

$$\frac{\int_0^t s^\alpha f(s)\kappa_s ds}{\int_0^t f(s)\kappa_s ds} \sim \frac{\int_1^t s^\alpha f(s)\kappa_s ds}{\int_1^t f(s)\kappa_s ds} \geq 1.$$

Either way, it follows that $\limsup_t t^\alpha Y_t > 0$ and thus $Y_t \in \Omega(t^{-\alpha})$. □

Remark 4.7 (*More Precise Asymptotics for $\beta = -1$*). In the case $\beta = -1$ it follows from Lemma 4.5 that F is slowly varying and thus, considering its monotonicity, lies in $O(t^\varepsilon) \cap \Omega(1)$ for all $\varepsilon > 0$. As discussed in [26] however, it is not always the case, that a regularly varying function with exponent -1 is integrable on $\mathbb{R}^+$. Supposing that $F(\infty) = \infty$, we obtain the $\mathcal{L}^2$-convergence in Theorem 4.6 analogously to the case $\beta \in (-1, -\frac{1}{2})$, while the methods we used to obtain almost sure convergence fail for $\beta = -1$. Conversely, if $F(\infty) < \infty$, similarly to the proof of Theorem 4.6.1 it follows that

$$\lim_{t \to \infty} \mathbb{E}\left[\int_0^t f(s)dD_s\right] < \infty$$

and the integral has a finite almost sure limit. Surely, Polfeldt [26] can be used to specify the results for this critical case.

4.3. Proof of *Theorem* 3

In this section we generalise the findings of Corollary 2.7 to the time-inhomogeneous case. Recalling Definition 2.8, let $\mathcal{Z} = \mathcal{Z}^{in}_{b,d,\kappa,p}$ with birth, death and disaster rate functions b, d and κ respectively, and $p : \mathbb{R}^+ \to [0, 1]$ the survival probability function. Furthermore, let $(D_t)_t$ be the inhomogeneous Poisson process with rate κ that counts the disasters up to time t. In what follows we will always assume b, d and κ to be right continuous with left limits and p to be left continuous with right limits.

We start by computing the pgf of $\mathcal{Z}$ for $(1 - p)\kappa \equiv 0$, i.e. without disasters, which will be generalised in Lemma 4.9.

Lemma 4.8. *Let* $v(t) := \int_0^t (b_y - d_y)dy$ *and* $(1 - p)\kappa \equiv 0$. *Then, for* $x \in [0, 1]$, $t \geq t_0 \geq 0$ *and* $k \geq 0$ *it holds that*

$$\mathbb{E}[(1 - x)^{Z_t} | Z_{t_0} = k] = (1 - s(t, x))^k,$$

where $s(t, 0) = 0$ *for all* t *and* $s(t, x)^{-1} = \frac{1}{x} e^{v(t_0) - v(t)} + e^{v(t_0)} \int_{t_0}^t b_y e^{-v(y)} dy$ *for* $x > 0$.

Proof. Given that $Z_{t_0} = k$, Z_t is equal in distribution to a sum of k independent copies started in 1 at time t_0. Thus, $\mathbb{E}[(1 - x)^{Z_t} | Z_{t_0} = k] = \mathbb{E}[(1 - x)^{Z_t} | Z_{t_0} = 1]^k$. Hence, without loss of generality we assume $k = 1$. Now, considering [18], where birth- and death-rate are denoted by λ and μ respectively and v is denoted by $-\rho$ (cf. (11)), by (9), (12) and (10b) we can compute for $z \in [0, 1]$ that

$$\mathbb{E}[z^{Z_t} | Z_0 = 1] =: \varphi(z, t) = \frac{1 + e^{v(t)} \int_0^t b_s e^{-v(s)} ds - e^{v(t)} + \left(e^{v(t)} - e^{v(t)} \int_0^t b_s e^{-v(s)} ds\right) z}{1 + \int_0^t b_s e^{-v(s)} ds - \int_0^t b_s e^{-v(s)} ds \cdot z}$$

$$= 1 - \frac{e^{v(t)} - e^{v(t)} z}{1 + \int_0^t b_s e^{-v(s)} ds \cdot (1 - z)}.$$

Substitution of $z = 1 - x$ and reducing the fraction by $x e^{v(t)}$ concludes the proof for $t_0 = 0$. The general case $t_0 \geq 0$ is obtained considering a process $\mathcal{Z}^*$ with birth and death rates at time s given by $b^*(s) := b_{t_0 + s}$ and $d^* := d_{t_0 + s}$ respectively. Then, for $t \geq t_0$

$$\mathbb{E}[(1 - x)^{Z_t} | Z_{t_0} = 1] = \mathbb{E}[(1 - x)^{Z^*_{t - t_0}} | Z^*_0 = 1] = 1 - \frac{1}{\frac{1}{x} e^{-v^*_{t - t_0}} + \int_0^{t - t_0} b^*(s) e^{-v^*_s} ds}, \quad (4.7)$$

where

$$v^*_s = \int_0^s \left(b^*(y) - d^*(y)\right) dy = \int_{t_0}^{t_0 + s} \left(b_y - d_y\right) dy = v(t_0 + s) - v(t_0). \quad (4.8)$$

Substituting $y := s + t_0$ in (4.7) and using (4.8) concludes the proof. $\square$

The following lemma generalises the result above to processes with disasters, i.e. $(1 - p)\kappa \not\equiv 0$. It delivers a dual process $\mathcal{X}$ with respect to the pgf and thus corresponds to Lemma 4.1 in the proof of Theorem 2.

Lemma 4.9 (*A Stronger Duality*)**.** *Let* $\log \frac{1}{0} = -\infty$, $1/0 = \infty$ *and* $1/\infty = 0$. *Then, for* $x \in [0, 1]$, $k \geq 0$ *and* $\mathcal{D}_\infty := \sigma(D_s; s \geq 0)$, *it holds that*

$$\mathbb{E}[(1 - x)^{Z_t} | \mathcal{D}_\infty, Z_0 = k] = (1 - X_t)^k$$

for a piecewise deterministic process $\mathcal{X} = (X_t)_{t \geq 0}$ given by

$$X_t^{-1} = \tfrac{1}{x} e^{-L_t} + \int_0^t e^{-L_s} b_s \, ds, \tag{4.9}$$

where $L_t = \int_0^t (b_s - d_s) ds - \int_0^t \log\left(\tfrac{1}{p_s}\right) dD_s$.

Proof. Let t be fixed, $G_t(x) := \mathbb{E}_k[(1-x)^{Z_t} | \mathcal{D}_\infty]$, $\tau_0 := 0$ and $\tau_1, \tau_2, \ldots$ be the jump times of (D_t), i.e. the disaster times of $\mathcal{Z}$. Note that the binomial disasters provide (with the left-side limit $Z_{\tau_n -} := \lim_{s \uparrow \tau_n} Z_s$),

$$\mathbb{E}[(1-x)^{Z_{\tau_n}} | Z_{\tau_n -} = z] = \sum_{\ell=0}^{z} \binom{z}{\ell} p_{\tau_n}^\ell (1 - p_{\tau_n})^{z-\ell} (1-x)^\ell$$

$$= \left(p_{\tau_n}(1-x) + (1 - p_{\tau_n})\right)^z = (1 - p_{\tau_n} x)^z.$$

Iterating this and Lemma 4.8, we obtain

$$G_t(x) = \mathbb{E}\left[\mathbb{E}[(1-x)^{Z_t} | Z_{\tau_{D_t}}, \mathcal{D}_\infty] \Big| \mathcal{D}_\infty\right] = \mathbb{E}[(1 - s_{D_t})^{Z_{\tau_{D_t}}} | \mathcal{D}_\infty]$$

$$= G_{\tau_{D_t}}(s_{D_t}) \qquad \text{with } s_{D_t}^{-1} = \frac{e^{v(\tau_{D_t}) - v(t)}}{x} + e^{v(\tau_{D_t})} \int_{\tau_{D_t}}^t b_s e^{-v(s)} ds$$

$$= G_{\tau_{D_t} -}(p_{\tau_{D_t}} s_{D_t}) = \cdots$$

$$= G_{\tau_{D_t} - 1}(s_{D_t - 1}) \qquad \text{with } s_{D_t - 1}^{-1} = \frac{e^{v(\tau_{D_t} - 1) - v(\tau_{D_t})}}{p_{\tau_{D_t}} s_{D_t}} + e^{v(\tau_{D_t} - 1)} \int_{\tau_{D_t} - 1}^{\tau_{D_t}} b_s e^{-v(s)} ds$$

$$= G_{\tau_{D_t} - 2}(s_{D_t - 2}) \qquad \text{with } s_{D_t - 2}^{-1} = \frac{e^{v(\tau_{D_t} - 2) - v(\tau_{D_t} - 1)}}{p_{\tau_{D_t} - 1} s_{D_t - 1}} + e^{v(\tau_{D_t} - 2)} \int_{\tau_{D_t} - 2}^{\tau_{D_t} - 1} b_s e^{-v(s)} ds$$

$$= \cdots = G_{\tau_0}(s_0) \qquad \text{with } s_0 = \frac{e^{v(\tau_0) - v(\tau_1)}}{p_{\tau_1} s_1} + e^{v(\tau_0)} \int_{\tau_0}^{\tau_1} b_s e^{-v(s)} ds$$

$$= (1 - s_0)^k.$$

Now, solving the recursion,

$$s_0^{-1} = \left(\left(\cdots \left(\left(\frac{1}{x} e^{v(\tau_{D_t}) - v(t)} + e^{v(\tau_{D_t})} \int_{\tau_{D_t}}^t b_y e^{-v(y)} dy \right) \right.\right.\right.$$

$$\left. \cdot \frac{1}{p_{\tau_{D_t}}} e^{v(\tau_{D_t} - 1) - v(\tau_{D_t})} + e^{v(\tau_{D_t} - 1)} \int_{\tau_{D_t} - 1}^{\tau_{D_t}} b_y e^{-v(y)} dy \right) \cdots \right)$$

$$\left. \cdot \frac{1}{p_{\tau_1}} e^{v(\tau_0) - v(\tau_1)} + e^{v(\tau_0)} \int_{\tau_0}^{\tau_1} b_y e^{-v(y)} dy \right)$$

$$= \frac{1}{x} e^{-v(t)} \prod_{k=1}^{D_t} p_{\tau_k}^{-1} + \sum_{k=0}^{D_t} \prod_{\ell=1}^{k} p_{\tau_\ell}^{-1} \cdot e^{v(\tau_0)} \int_{\tau_k}^{\tau_{k+1} \wedge t} b_y e^{-v(y)} dy,$$

where the empty product equals 1. (Then, for $D_t = 0$ and thus $t \leq \tau_1$, one obtains the deterministic dual from Lemma 4.8.) Letting $\beta(t) := \int_0^t b_s e^{-v(s)} ds$ and considering that

$v(\tau_0) = v(0) = 0$, the sum equates to

$$\sum_{k=0}^{D_t} \prod_{\ell=1}^{k} p_{\tau_\ell}^{-1}\big(\beta(\tau_{k+1} \wedge t) - \beta(\tau_k)\big) = \int_0^t \prod_{\ell=1}^{D_s} p_{\tau_\ell}^{-1} \cdot b_s e^{-v(s)} ds.$$

With

$$\prod_{k=1}^{D_t} p_{\tau_k}^{-1} = \exp\Big(\sum_{s \le t} \log \tfrac{1}{p_s} \cdot (D_s - D_{s-})\Big) = \exp\Big(\int_0^t \log \tfrac{1}{p_s} d D_s\Big),$$

it follows that s_0 equals X_t from (4.9) and the proof is done. $\quad\square$

Remark 4.10.

1. This Lemma holds for arbitrary counting processes $(D_t)_{t \ge 0}$. One might even consider a process with multiple jumps, e.g. $\mathbb{P}(\tau_k = 0) > 0$ for some k.
2. The process $\mathcal{X}$ here is not of the form required for Corollary 2.4 or Theorem 1, even if we choose constant b, d, κ, p to obtain homogeneity: X_t jumps from a state $1/(\frac{a}{x} + b)$ to $1/(\frac{a}{px} + b)$, which is not a p-jump. However, in the homogeneous case, letting $b \equiv \vartheta > 0$, $d \in \mathbb{R}^+$, $\delta := b - d > 0$ and $\kappa \equiv 1$, we can see that X_t equates to $\overline{W}_t$, the time-reversal of W_t in (3.4) we used in the proof of Theorem 1. Similarly one obtains that the (homogeneous) time-reversal $\overline{X}_t$ has the generator, for $f \in \mathcal{C}^1([0, 1])$,

   $$\mathcal{G}_{\overline{\mathcal{X}}} f(x) = \kappa(f(px) - f(x)) + (bx(1 - x) - dx) f'(x).$$

3. The relationship between $\mathcal{X}$ and $\mathcal{Z}$ can be viewed as a *stronger* duality, since the duality relation (D), from the beginning of Section 4, here does not only hold in expectation, but even in conditional expectation. (Taking expectation, (D) follows.)

Although we are not able to use Corollary 2.4 here, from the previous Lemma we immediately obtain the following

Proposition 4.11. *Let L_t as in Lemma 4.9 and $J_t := \int_0^t e^{-L_s} b_s ds$. Supposing that $L_t \xrightarrow{t \to \infty} L \in [-\infty, \infty]$ almost surely and letting $J := \lim_{t \to \infty} J_t$, there are 3 possible outcomes for the limit of $\mathbb{E}_k[(1 - x)^{Z_t} | \mathcal{D}_\infty]$:*

$$\lim_{t \to \infty} \mathbb{E}_k[(1 - x)^{Z_t} | \mathcal{D}_\infty](\omega) = \begin{cases} 1 & \text{if } \omega \in \{L = -\infty\} \cup \{J = \infty\}. \\ (1 - J(\omega)^{-1})^k & \text{if } \omega \in \{L = \infty\} \cap \{J < \infty\}. \\ \big(1 - \frac{x}{\exp(-L(\omega)) + x J(\omega)}\big)^k & \text{if } \omega \in \{L \in \mathbb{R}\} \cap \{J < \infty\}. \end{cases}$$

The third case occurs if and only if

$$\int_0^\infty (b_s + d_s) ds < \infty \quad \text{and} \quad \prod_{k \ge 1} p(\tau_k(\omega)) > 0. \tag{4.10}$$

Proof. By construction and monotonicity of (J_t), these three cases cover all possible outcomes. The results follow by insertion into Lemma 4.9. In case 3. there is $m(\omega) < \infty$ such that $m(\omega) \ge e^{L_t(\omega)}$ for all t, since $(L_t(\omega))_t$ converges in $\mathbb{R}$. Thus, almost surely

$$\int_0^\infty b_s ds \le \int_0^\infty m e^{-L_s} b_s ds = m J < \infty.$$

Now, the convergence of $(L_t(\omega))$ and the non-negativity of d and $\log \frac{1}{p}$ give us that also $\int_0^\infty d_s ds$ as well as

$$\int_0^\infty \log \frac{1}{p_s} dD_s(\omega) = -\sum_{k \geq 1} \log p_{\tau_k(\omega)}$$

have to be finite, which shows that condition (4.10) is necessary for the third case. To see the sufficiency, from (4.10) the finiteness of $L(\omega)$ immediately follows analogously. Then, $(e^{-L_t(\omega)})_t$ is bounded and thus by finiteness of $\int_0^\infty b_s ds$, also $J(\omega)$ has to be finite. $\quad\square$

Remark 4.12.

1. The first part of condition (4.10) implies that with probability 1 there is only a finite number of birth and death events, while the second part offers either the possibility of $\lim_t D_t < \infty$ or p converging to 1 on the support of κ, fast enough to compensate for (D_t).
2. Only in the third case, the limiting probability generating function depends on x, which implies that, as soon as b is bounded away from 0, $\mathcal{Z}$ either goes extinct or explodes.
3. Since this Proposition provides results depending directly on the paths of (D_t), b, d and p, it can easily be applied to random environments in the sense of choosing b, d and/or p to be stochastic processes.
4. Another possible generalisation could be to drop the assertion that the limit L exists. Then, we see that the first case still only holds if $\limsup_t L_t = -\infty$ or $J = \infty$. Secondly, in the case of $J < \infty$ we still obtain a limit independent of x, only if $\liminf_t L_t = \infty$. Hence, only the third case changes, where we obtain bounds on the limit in terms of $\liminf_t L_t$ and $\limsup_t L_t$.

Proof of Theorem 3. First note that the assertions and Theorem 4.6 imply that almost surely

$$L_t = \int_0^t (b_s - d_s) ds - \int_0^t \log\left(\frac{1}{p_s}\right) dD_s \sim \iota j(t). \tag{4.11}$$

(Since $j(t) = \Omega(t^\alpha)$ for some $\alpha > 0$, in the case where $\beta \leq -1$, $\int_0^t \log(1/p_s) dD_s$ has either a finite limit or it lies in $O(t^{\alpha/2}) \subset o(t^\alpha)$ such that in either case it does not contribute to the asymptotics of L_t. Otherwise, it is asymptotically equivalent to $\int_0^t \log(1/p_s)\kappa_s ds$.)

Now, we can apply Proposition 4.11:

1.: If $\iota = 1$, $L_t \xrightarrow{t\to\infty} \infty$. Also, for almost every ω there is a $T(\omega) \in (0, \infty)$ such that $L_t(\omega) \geq (1 - \varepsilon) j(t)$ for all $t \geq T(\omega)$. Thus,

$$J \leq J_T + \int_T^\infty e^{-(1-\varepsilon)j(s)} b_s ds < \infty.$$

Hence, the second case of Proposition 4.11 concludes that

$$\mathbb{P}(Z_t \xrightarrow{t\to\infty} 0) = \lim_{t\to\infty} \mathbb{E}_k[(1-1)^{Z_t}] = \mathbb{E}[(1 - J^{-1})^k] < 1.$$

2.: If $\iota = -1$, it is clear, that $L_t \xrightarrow{t\to\infty} -\infty$ and independently of the integral condition of *2.* the first part of Proposition 4.11 concludes

$$\mathbb{P}(Z_t \xrightarrow{t\to\infty} 0) = \mathbb{E}_k[\lim_{t\to\infty}(1-1)^{Z_t}] = 1.$$

Otherwise, i.e. if $\iota = 1$ but the integral condition holds, $L_t \xrightarrow{t \to \infty} \infty$ and analogously to *1.* we find a finite random variable T' such that $L_t \leq (1 + \varepsilon)j(t)$ for all $t \geq T'$ almost surely. Thus,

$$J = \int_0^\infty e^{-L_s} b_s ds \geq J_{T'} + \int_{T'}^\infty e^{-(1+\varepsilon)j(s)} b_s ds = \infty.$$

Then again, the first part of Proposition 4.11 concludes the proof. □

Remark 4.13 (*Normalisation Function, Rates of Convergence*)**.**

1. There are two major cases in which a normalisation function j as in Theorem 3 does not exist:

    (a) The integral $\ell(t) := \int_0^t (b_s - d_s - \kappa_s \log(\frac{1}{p_s})) ds$ converges to a constant. Then, $\mathcal{Z}$ will exhibit only a finite number of birth events almost surely and converge to a random variable, where the third part of Proposition 4.11 provides a way to compute the limiting distribution.

    (b) The integral ℓ oscillates too strongly – e.g. $\ell(t) = t(1 + \sin(t))$. This might happen in periodic models, which were briefly discussed in [18]. In this case, Lemma 4.9 still holds, while Proposition 4.11 as well as Theorem 4.6 do not apply.

2. In case *2.* of Theorem 3, for the convergence rates of the survival probability conditioned on $\mathcal{D}_\infty$, the σ-algebra of the disaster times, we can estimate for arbitrary $k \geq 1$, using the processes (X_t) and (L_t) from Lemma 4.9 with $X_0 = 1$ and Bernoulli's inequality

$$-\log \mathbb{P}(Z_t > 0 | \mathcal{D}_\infty, Z_0 = z) = -\log\big(1 - (1 - X_t)^z\big)$$

$$\sim -\log(X_t) = \log\Big(e^{-L_t} + \int_0^t e^{-L_s} b_s ds\Big)$$

$$\sim \max\Big\{-L_t, \ \log\Big(\int_0^t e^{-L_s} b_s ds\Big), \ 0\Big\},$$

where $\sim$ denotes asymptotic equivalence, i.e. $f \sim g \Leftrightarrow f(t)/g(t) \to 1$ as $t \to \infty$. Recalling (4.11) it follows

$$-\frac{1}{j(t)} \log \mathbb{P}(Z_t > 0 | \mathcal{D}_\infty, Z_0 = z) \sim \max\Big\{-\iota, 0, \ \frac{1}{j(t)} \log\Big(\int_0^t e^{-L_s} b_s ds\Big)\Big\}. \quad (4.12)$$

It is possible to use similar integral estimates here as in the proof of Theorem 3, i.e. considering a finite random variable T such that L_t is close to $\iota j(t)$ for all $t \geq T$. With more knowledge on j and b, this approach can be used to compute bounds on the convergence rates.

Declaration of competing interest

Acknowledgements

Appendix. Large deviations

In Theorem 1, we make use of large deviations for Poisson processes. These can be e.g. read from [10], Exercise 5.2.12.

Lemma A.1 (*Large Deviations for a Poisson Process*)**.** *Let $P = (P_t)_{t \geq 0}$ be a unit rate Poisson process.*

$$\lim_{t \to \infty} \tfrac{1}{t} \log \mathbb{P}(P_t \geq xt) = -(1 - x + x \log x) \text{ for } x > 1, \tag{A.1}$$

$$\lim_{t \to \infty} \tfrac{1}{t} \log \mathbb{P}(P_t \leq xt) = -(1 - x + x \log x) \text{ for } x \in (0, 1). \tag{A.2}$$

Moreover, for any $x \in (0, 1)$

$$\lim_{t \to \infty} \tfrac{1}{t} \log \mathbb{P}(P_s \leq xs \text{ for all } s \leq t) = -(1 - x + x \log x). \tag{A.3}$$

Proof. The first two assertions, (A.1) and (A.2) are a consequence of Crámer's Theorem [10, Theorem 2.2.3, p.27]. Moreover, the large deviation result (A.3) is an application of Dembo and Zeitouni [10], Exercise 5.2.12, by rescaling and choosing $\phi(t) = xt$. □

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