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Estimates on the tail probabilities of subordinators and applications to general time fractional equations[☆]

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Abstract

In this paper, we study estimates on tail probabilities of several classes of subordinators under mild assumptions on the tails of their Lévy measures. As an application of that result, we obtain two-sided estimates for fundamental solutions of general homogeneous time fractional equations including those with Dirichlet boundary conditions.

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1. Introduction

1.1. Motivation

The time fractional diffusion equation $\partial_t^\beta u = \Delta u$ $(0 < \beta < 1)$ has been used in various fields to model the diffusions on sticky and trapping environment. Here, ∂_t^β is the Caputo derivative of order β which is defined as

$$\partial_t^\beta u(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta}(u(s) - u(0))ds,$$

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where Γ is the gamma function defined as $\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}dx$. Motivated by this equation, following [6], we consider the following generalized fractional-time derivatives. Let $w : (0, \infty) \to [0, \infty)$ be a function which satisfies the following condition.

(Ker.) w is a right continuous non-increasing function satisfying $\lim_{s\to 0+} w(s) = \infty$, $\lim_{s\to\infty} w(s) = 0$ and $\int_0^\infty \min\{1, s\}(-dw(s)) < \infty$.

Definition 1.1. For a function $u : [0, \infty) \to \mathbb{R}$, the generalized fractional-time derivative ∂_t^w with respect to the kernel w is given by

$$\partial_t^w u(t) := \frac{d}{dt}\int_0^t w(t-s)(u(s)-u(0))ds,$$

whenever the above integral makes sense.

For example, if $w(t) = t^{-\beta}/\Gamma(1-\beta)$ for some $0 < \beta < 1$, then the fractional-time derivative ∂_t^w is nothing but the Caputo derivative of order β.

In [6], Zhen-Qing Chen established the probabilistic representation for the fundamental solution of time fractional equation $\partial_t^w u(t) = \mathcal{L}u$ where $\mathcal{L}$ is the infinitesimal generator of some uniformly bounded strongly continuous semigroup in a Banach space. This procedure can be described as follows: For a given function w satisfying condition **(Ker.)**, define a Bernstein function ϕ by

$$\phi(\lambda) := \int_0^\infty (1-e^{-\lambda s})(-dw(s)) \qquad \text{for all } \lambda \geq 0. \tag{1.1}$$

Since $|1-e^{-\lambda s}| \leq (1+\lambda)\min\{1, s\}$, we see from **(Ker.)** that ϕ is well-defined. Let $\{S_r, r \geq 0\}$ be a subordinator (non-negative valued Lévy process with $S_0 = 0$) whose Laplace exponent is given by (1.1), that is, $\phi(\lambda) = -\log\mathbb{E}\big[\exp(-\lambda S_1)\big]$ for all $\lambda \geq 0$. Then, define its inverse as $E_t := \inf\{r > 0 : S_r > t\}$ for $t > 0$. Since condition **(Ker.)** holds, we have $\lim_{s\to 0+} w(s) = \infty$ so that S_r is not a compounded Poisson process. Therefore, almost surely, $r \mapsto S_r$ is strictly increasing and hence $t \mapsto E_t$ is continuous. Denote by T_t the semigroup corresponding to the generator $\mathcal{L}$ in a Banach space. Then, for every $f \in \mathcal{D}(\mathcal{L})$, where $\mathcal{D}(\mathcal{L})$ denotes the domain of $\mathcal{L}$, the unique solution (in some suitable sense) to the following general homogeneous time fractional equation

$$\partial_t^w u(t,x) = \mathcal{L}u(t,x) \quad \text{with } u(0,x) = f(x) \tag{1.2}$$

is given by

$$u(t,x) = \mathbb{E}_x\big[T_{E_t} f(x)\big]. \tag{1.3}$$

In [8], the second named author, jointly with Zhen-Qing Chen, Takashi Kumagai and Jian Wang, proved that when T_t is the transition semigroup of a symmetric strong Markov process, (1.3) is the unique weak solution to Eq. (1.2) (see [8, Theorem 2.4] for a precise statement). Moreover, they obtained two-sided estimates for the fundamental solution under the condition that ϕ satisfies $\mathbf{WS}(\alpha_1, \alpha_2)$ for some $0 < \alpha_1 \leq \alpha_2 < 1$ (see Definition 1.2 for the definition of $\mathbf{WS}(\alpha_1, \alpha_2)$). The key ingredients to obtain those estimates were the estimates on tail probabilities $\mathbb{P}(S_r \geq t)$ and $\mathbb{P}(S_r \leq t)$ established in [21,29]. Particularly, the weak scaling conditions for ϕ were needed to get sharp estimates on $\mathbb{P}(S_r \geq t)$.

In this paper, we study estimates on upper tail probabilities $\mathbb{P}(S_r \geq t)$ of a general class of subordinators. Our results cover some cases when the lower scaling index α_1 of ϕ is 0 and the upper scaling index α_2 of ϕ is 1. Indeed, we will see that the lower scaling index has no role in tail probability estimates. On the other hand, when the upper scaling index is 1, various phenomena can arise in the asymptotic behaviors of $\mathbb{P}(S_r \geq t)$ as $t \to \infty$. To assort those phenomena, we impose conditions on the tail measure w instead of the Laplace exponent ϕ and then obtain estimates on $\mathbb{P}(S_r \geq t)$ under each condition. More precisely, we will consider the three cases: (i) w is a polynomial decaying function; (ii) w decreases subexponentially or exponentially; (iii) w is finitely supported. (See, Section 2 for details.)

As applications to these tail probability estimates, we then establish two-sided estimates for fundamental solutions of time fractional equations including the ones with the Dirichlet boundary condition, given by (1.5).

1.2. Settings

In this subsection, we introduce the notions of the fundamental solution for a time fractional equation and the weak scaling properties for non-negative function. Then, we list our main assumptions in this paper.

Let (M, ρ, m) be a separable locally compact Hausdorff metric measure space and $D \subset M$ be an open subset. Let $\{T_t^D, t \geq 0\}$ be a uniformly bounded strongly continuous semigroup with infinitesimal generator $(\mathcal{L}^D, \mathcal{D}(\mathcal{L}^D))$ in some Banach space $(\mathbb{B}, \|\cdot\|)$. Let w be a function satisfying condition **(Ker.)**. Then, we consider the following time fractional equation with Dirichlet boundary condition.

$$\begin{cases} \partial_t^w u(t,x) = \mathcal{L}^D u(t,x), & x \in D, \quad t > 0, \\ u(0,x) = f(x), & x \in D, \\ u(t,x) = 0, & \text{vanishes continuously on } \partial D \ \text{ for all } \ t > 0. \end{cases} \tag{1.4}$$

Examples and topics related to the problem (1.4) can be found in [3,16,24–27,31]. See also [19,20] for examples of time fractional equations with non-linear noises.

If we overlook the boundary condition, then it is established in [6, Theorem 2.3] that for all $f \in \mathcal{D}(\mathcal{L}^D)$, $u(t,x) := \mathbb{E}[T_{E_t}^D f(x)]$ is a unique solution to (1.4) in the following sense:
(i) $\sup_{t>0} \|u(t,\cdot)\| < \infty$, $x \mapsto u(t,x)$ is in $\mathcal{D}(\mathcal{L}^D)$ for each $t \geq 0$ with $\sup_{t\geq 0} \|\mathcal{L}^D u(t,\cdot)\| < \infty$, and both $t \mapsto u(t,\cdot)$ and $t \mapsto \mathcal{L}^D u(t,\cdot)$ are continuous in $(\mathbb{B}, \|\cdot\|)$;
(ii) for every $t > 0$, $I_t^w[u] := \int_0^t w(t-s)(u(s,x) - f(x))ds$ is absolutely convergent in $(\mathbb{B}, \|\cdot\|)$ and

$$\lim_{\delta \to 0} \frac{1}{\delta}(I_{t+\delta}^w[u] - I_t^w[u]) = \mathcal{L}^D u(t,x) \quad \text{in } (\mathbb{B}, \|\cdot\|).$$

Indeed, we will see that if $\{T_t^D, t \geq 0\}$ admits a transition density enjoying certain types of estimates, then the solution $u(t,x)$ satisfies the following boundary condition (see Corollary 1.21 for a precise statement):
(iii) if f is bounded, then for all $t > 0$, $x \mapsto u(t,x)$ vanishes continuously on ∂D.

As discussed in [8], if the semigroup $\{T_t^D, t \geq 0\}$ has a transition density $q(t,x,y)$ with respect to m on M, for any function $f \in \mathcal{D}(\mathcal{L}^D)$,

$$\begin{aligned} u(t,x) = \mathbb{E}_x[T_{E_t}^D f(x)] &= \int_0^\infty T_r^D f(x) d_r \mathbb{P}(E_t \leq r) = \int_0^\infty T_r^D f(x) d_r \mathbb{P}(S_r \geq t) \\ &= \int_0^\infty \int_M f(y) q(r,x,y) m(dy) d_r \mathbb{P}(S_r \geq t) \\ &= \int_M f(y) \left(\int_0^\infty q(r,x,y) d_r \mathbb{P}(S_r \geq t) \right) m(dy). \end{aligned}$$

Therefore, it is natural to say that

$$p(t,x,y) := \int_0^\infty q(r,x,y) d_r \mathbb{P}(S_r \geq t) \tag{1.5}$$

is the fundamental solution to Eq. (1.4).

Next, we introduce the weak scaling properties for non-negative functions.

Definition 1.2. Let $f : (0,\infty) \to [0,\infty)$ be a given function and $\alpha_1, \alpha_2 \in \mathbb{R}$ and $c_0 > 0$ be given constants.
(1) We say that f satisfies $\mathbf{LS^0}(\alpha_1, c_0)$ (resp. $\mathbf{LS^\infty}(\alpha_1, c_0)$) if there exists a constant $c_1 > 0$ such that

$$\frac{f(R)}{f(r)} \geq c_1 \left(\frac{R}{r}\right)^{\alpha_1} \quad \text{for all } r \leq R \leq c_0 \text{ (resp. for all } c_0 \leq r \leq R).$$

(2) We say that f satisfies $\mathbf{US^0}(\alpha_2, c_0)$ (resp. $\mathbf{US^\infty}(\alpha_2, c_0)$) if there exists a constant $c_2 > 0$ such that

$$\frac{f(R)}{f(r)} \leq c_2 \left(\frac{R}{r}\right)^{\alpha_2} \quad \text{for all } r \leq R \leq c_0 \text{ (resp. for all } c_0 \leq r \leq R).$$

(3) If f satisfies both $\mathbf{LS^0}(\alpha_1, c_0)$ and $\mathbf{US^0}(\alpha_2, c_0)$ (resp. $\mathbf{LS^\infty}(\alpha_1, c_0)$ and $\mathbf{US^\infty}(\alpha_2, c_0)$), we say that f satisfies $\mathbf{WS^0}(\alpha_1, \alpha_2, c_0)$ (resp. $\mathbf{WS^\infty}(\alpha_1, \alpha_2, c_0)$). Moreover, if f satisfies both $\mathbf{WS^0}(\alpha_1, \alpha_2, c_0)$ and $\mathbf{WS^\infty}(\alpha_1, \alpha_2, c_0)$, then we say that f satisfies $\mathbf{WS}(\alpha_1, \alpha_2)$.

Throughout this paper, we always assume that the kernel w satisfies condition **(Ker.)**. Here, we enumerate our main assumptions for w.

(S.Poly.)(t_s) There exist constants $t_s > 0$ and $\delta_1 > 0$ such that w satisfies $\mathbf{LS^0}(-\delta_1, t_s)$;

(L.Poly.) There exists a constant $\delta_2 > 0$ such that w satisfies $\mathbf{LS^\infty}(-\delta_2, 1)$;

(Sub.)$(\boldsymbol{\beta}, \boldsymbol{\theta})$ There exist constants $c_0, \theta > 0$ and $\beta \in (0, 1]$ such that

$$w(t) \leq c_0 \exp(-\theta t^\beta) \quad \text{for all } t \geq 1.$$

(Trunc.)(t_f) There exists a constant $t_f > 0$ such that
(i) $w(t) > 0$ for $0 < t < t_f$ and $w(t_f) = 0$;
(ii) w is bi-Lipschitz continuous on $[t_f/4, t_f]$, i.e. there exists a constant $K \geq 1$ such that

$$K^{-1}|t-s| \leq |w(t) - w(s)| \leq K|t-s|, \quad \text{for all } t_f/4 \leq s \leq t \leq t_f;$$

(iii) there exists a constant $\delta_3 > 0$ such that w satisfies $\mathbf{LS^0}(-\delta_3, t_f/2)$.

Remark 1.3. (1) Condition **(S.Poly.)**(t_s) implies that the corresponding Laplace exponent ϕ satisfies $\mathbf{US^\infty}(\min\{\delta_1, 1\}, 1)$. Conversely, if ϕ satisfies $\mathbf{US^\infty}(\delta_1, 1)$ for some $\delta_1 < 1$, then there exists a constant $t_s > 0$ such that condition **(S.Poly.)**(t_s) holds with constant δ_1. Analogously, condition **(L.Poly.)** implies that ϕ satisfies $\mathbf{US^0}(\min\{\delta_2, 1\}, 1)$ and if ϕ satisfies $\mathbf{US^0}(\delta_2, 1)$ with $\delta_2 < 1$, then condition **(L.Poly.)** holds. (See, Lemma 2.1.)
(2) If condition **(L.Poly.)** or **(Sub.)**$(\boldsymbol{\beta}, \boldsymbol{\theta})$ holds, then we can replace the constant 1 with arbitrary positive constant since w is a monotone function. However, we cannot replace the constant t_s in condition **(S.Poly.)**(t_s) with other positive constants in general. For instance, if

$w(t) = (t^{-1/2} - 1)\mathbf{1}_{(0,1]}(t)$, then we can only take t_s strictly smaller than 1. Moreover, the constant t_f in condition **(Trunc.)**(t_f) is uniquely determined by its first condition.

Notations: In this paper, we use the symbol ":=" to denote a definition, which is read as "is defined to be". For $a, b \in \mathbb{R}$, we use the notations $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For $x \in \mathbb{R}$, we define $\log^+ x := 0 \vee \log x$ and $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : x \ge n\}$. We denote by ∂_t the partial derivative with respect to the variable t.

The notation $f(x) \asymp g(x)$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 g(x) \le f(x) \le c_2 g(x)$ for the specified range of the variable x. The notation $f(x) \lesssim g_1(x)+g_2(x)h(cx)$ (resp. $f(x) \gtrsim g_1(x) + g_2(x)h(cx)$) means that there exist constants $c_1, c_2 > 0$ such that

$$f(x) \le c_1\big(g_1(x) + g_2(x)h(c_2x)\big) \quad (\text{resp. } \ f(x) \ge c_1\big(g_1(x) + g_2(x)h(c_2x)\big)),$$

for the specified range of x. Then, the notation $f(x) \simeq g_1(x) + g_2(x)h(cx)$ means that both $f(x) \lesssim g_1(x) + g_2(x)h(cx)$ and $f(x) \gtrsim g_1(x) + g_2(x)h(cx)$ hold for the specified range of x.

For a subset D of some metric space (M, ρ), we let $\mathrm{diam}(D) := \sup_{u,v\in D} \rho(u, v)$ and $\delta_D(x) := \sup_{z\in D} \rho(x, z)$ for $x \in D$. Then, for $x, y \in D$, we define

$$\delta_*(x, y) := \delta_D(x)\delta_D(y), \quad \delta_\wedge(x, y) := \delta_D(x) \wedge \delta_D(y) \quad \text{and} \quad \delta_\vee(x, y) := \delta_D(x) \vee \delta_D(y). \tag{1.6}$$

Lower case letters c's without subscripts denote strictly positive constants whose values are unimportant and which may change even within a line, while values of lower case letters with subscripts $c_i, i = 0, 1, 2, \dots$, are fixed in each statement and proof, and the labeling of these constants starts anew in each proof.

1.3. Some toy models with explicit Dirichlet estimates

Our general estimates on the fundamental solution include a term which is described in an integral form. (See, (1.13).) However, in many applications, we can obtain explicit forms of them. We first represent some special versions of our results which can be described explicitly.

Suppose that the operator $(\mathcal{L}^D, \mathcal{D}(\mathcal{L}^D))$ on (D, ρ, m) admits a heat kernel $q(t, x, y)$ with respect to the measure m. We further assume that one of the following assumptions holds for all $(t, x, y) \in (0, \infty) \times D \times D$.

(J1) $\mathrm{diam}(D) < \infty$ and there exist constants $\alpha, d > 0$ and $\lambda > 0$ such that

$$q(t,x,y) \asymp \begin{cases} \left(1 \wedge \dfrac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \dfrac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \left(t^{-d/\alpha} \wedge \dfrac{t}{\rho(x,y)^{d+\alpha}}\right), & \text{if } 0 < t \le 1; \\ e^{-\lambda t}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}, & \text{if } t \ge 1; \end{cases}$$

(J2) There exist constants $\alpha > 0$ and $d > 0$ such that for all $t > 0$,

$$q(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \left(t^{-d/\alpha} \wedge \frac{t}{\rho(x,y)^{d+\alpha}}\right);$$

(J3) There exist constants $\alpha > 0$ and $d > 0$ such that for all $t > 0$,

$$q(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha} \wedge 1}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha} \wedge 1}\right)^{\alpha/2} \left(t^{-d/\alpha} \wedge \frac{t}{\rho(x,y)^{d+\alpha}}\right);$$

(J4) $\mathrm{diam}(D) < \infty$ and there exist constants $\alpha > 1, d > 0$ and $\lambda > 0$ such that

$$q(t,x,y) \asymp \begin{cases} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} \left(t^{-d/\alpha} \wedge \frac{t}{\rho(x,y)^{d+\alpha}}\right), & \text{if } 0 < t \le 1; \\ e^{-\lambda t}\delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1}, & \text{if } t \ge 1; \end{cases}$$

(D1) $\mathrm{diam}(D) < \infty$ and there exist constants $\alpha > 1, d > 0$ and $\lambda > 0$ such that

$$q(t,x,y) \simeq \begin{cases} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} t^{-d/\alpha} \exp\left(-c\frac{\rho(x,y)^{\alpha/(\alpha-1)}}{t^{1/(\alpha-1)}}\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 < t \le 1; \\ e^{-\lambda t}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}, \qquad\qquad\qquad\qquad\qquad\quad\ \text{if } t \ge 1; \end{cases}$$

(D2) There exist constants $\alpha > 1$ and $d > 0$ such that for all $t > 0$,

$$q(t,x,y) \simeq \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} t^{-d/\alpha} \exp\left(-c\frac{\rho(x,y)^{\alpha/(\alpha-1)}}{t^{1/(\alpha-1)}}\right);$$

(D3) There exist constants $\alpha > 1$ and $d > 0$ such that for all $t > 0$,

$$q(t,x,y) \simeq \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha} \wedge 1}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha} \wedge 1}\right)^{\alpha/2} t^{-d/\alpha} \exp\left(-c\frac{\rho(x,y)^{\alpha/(\alpha-1)}}{t^{1/(\alpha-1)}}\right).$$

An open subset $D \subset \mathbb{R}^d$ ($d \ge 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there is a $C^{1,1}$ function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\Gamma(0) = 0, \nabla\Gamma(0) = (0,\dots,0), \|\Gamma\|_\infty \le \Lambda, |\nabla\Gamma(y) - \nabla\Gamma(z)| \le \Lambda|y-z|$ and an orthonormal coordinate system $CS_z : x = (\tilde{x}, x_d) := (x_1,\dots,x_{d-1},x_d)$ with origin at z such that

$$D \cap B(z,R_0) = \{x \in B(0,R_0) \ \text{ in } \ CS_z : x_d > \Gamma(\tilde{x})\}.$$

A $C^{1,1}$ open set in $\mathbb{R}$ is the union of disjoint intervals such that the minimum of their lengths and the distances between them is positive.

Remark 1.4. When M is $\mathbb{R}^d$, ρ is the usual metric on $\mathbb{R}^d$ and m is the Lebesgue measure, there are many examples of generators $(\mathcal{L}^D, \mathcal{D}(\mathcal{L}^D))$ on (D,ρ,m) which admit a transition density satisfying one of the estimates among **(J1)**, **(J2)**, **(J3)**, **(J4)**, **(D1)**, **(D2)** and **(D3)**. For instance, if $\mathcal{L}^D$ is a generator of a killed symmetric α-stable process with $0 < \alpha < 2$ or a censored α-stable process with $1 < \alpha < 2$, and $D \subset \mathbb{R}^d$ is a bounded $C^{1,1}$ open set, then estimate **(J1)** or **(J4)** holds, respectively. (See, [9,10,14].) Else if $\mathcal{L}^D$ is a generator of a killed symmetric α-stable process with $0 < \alpha < 2 \wedge d$, and D is a half space-like $C^{1,1}$ open set or exterior of a bounded $C^{1,1}$ open set, then estimate **(J2)** or **(J3)** holds, respectively. (See, [5,7].) Moreover, when $d \ge 3$, $\mathcal{L}$ is the Dirichlet laplacian on D, and $D \subset \mathbb{R}^d$ is a bounded connected $C^{1,1}$ open set or half space-like $C^{1,1}$ open set or exterior of a bounded $C^{1,1}$ open set, then estimate **(D1)** or **(D2)** or **(D3)** holds with $\alpha = 2$, respectively. (See, [30,33,34].)

Recall that δ_*, $\delta_\wedge$ and $\delta_\vee$ are defined in (1.6). For $\alpha > 0$, we define two auxiliary functions $F_k^\alpha, F_c^\alpha : \mathbb{R} \times (0,\infty) \times D \times D \to [0,\infty)$ as follows.

$F_k^\alpha(s,t,x,y) :=$

$$
\begin{cases}
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\big(\rho(x,y)^{\alpha}\vee\delta_*(x,y)^{\alpha/2}\big)\phi(t^{-1})^{-s/\alpha}, & \text{if } s<0;\\
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\big(\rho(x,y)^{\alpha}\vee\delta_*(x,y)^{\alpha/2}\big)\log^+\left(\dfrac{2\phi(t^{-1})^{-1}}{\rho(x,y)^{\alpha}\vee\delta_\vee(x,y)^{\alpha}}\right), & \text{if } s=0;\\
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\big(\rho(x,y)^{\alpha-s}\vee\delta_*(x,y)^{\alpha/2}\delta_\vee(x,y)^{-s}\big), & \text{if } s<\dfrac{\alpha}{2};\\
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\left(\rho(x,y)^{\alpha/2}+\delta_\wedge(x,y)^{\alpha/2}\log\left(\dfrac{\rho(x,y)\vee 2\delta_\vee(x,y)}{\rho(x,y)\vee\delta_\wedge(x,y)}\right)\right), & \text{if } s=\dfrac{\alpha}{2};\\
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\big(\rho(x,y)^{\alpha-s}\vee\delta_\wedge(x,y)^{\alpha-s}\big), & \text{if } \dfrac{\alpha}{2}<s<\alpha;\\
1+\log^+\left(\dfrac{2\phi(t^{-1})^{-1}\wedge 2\delta_\wedge(x,y)^{\alpha}}{\rho(x,y)^{\alpha}}\right), & \text{if } s=\alpha;\\
\rho(x,y)^{\alpha-s}, & \text{if } s>\alpha.
\end{cases}
$$

$F_c^{\alpha}(s,t,x,y):=$

$$
\begin{cases}
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\big(\rho(x,y)^{2\alpha-2}\vee\delta_*(x,y)^{\alpha-1}\big)\phi(t^{-1})^{-(2-\alpha-s)/\alpha}, & \text{if } s<2-\alpha;\\
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\big(\rho(x,y)^{2\alpha-2}\vee\delta_*(x,y)^{\alpha-1}\big)\log^+\left(\dfrac{2\phi(t^{-1})^{-1}}{\rho(x,y)^{\alpha}\vee\delta_\vee(x,y)^{\alpha}}\right), & \text{if } s=2-\alpha;\\
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\big(\rho(x,y)^{\alpha-s}\vee\delta(x,y)^{\alpha-1}\delta_\vee(x,y)^{2-\alpha-s}\big), & \text{if } 2-\alpha<s<1;\\
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\left(\rho(x,y)^{\alpha-1}+\delta_\wedge(x,y)^{\alpha-1}\log\left(\dfrac{\rho(x,y)\vee 2\delta_\vee(x,y)}{\rho(x,y)\vee\delta_\wedge(x,y)}\right)\right), & \text{if } s=1;\\
\mathbf{1}_{\{\delta_*(x,y)^{\alpha/2}\le\phi(t^{-1})^{-1}\}}\big(\rho(x,y)^{\alpha-s}\vee\delta_\wedge(x,y)^{\alpha-s}\big), & \text{if } 1<s<\alpha;\\
1+\log^+\left(\dfrac{2\phi(t^{-1})^{-1}\wedge 2\delta_\wedge(x,y)^{\alpha}}{\rho(x,y)^{\alpha}}\right), & \text{if } s=\alpha;\\
\rho(x,y)^{\alpha-s}, & \text{if } s>\alpha.
\end{cases}
$$

We also define

$$\overline{\phi}_{\alpha}(\lambda):=\inf\{s>0: s^{\alpha}\phi(s)^{-1}\ge\lambda\}\qquad \text{for }\ \lambda\ge 0. \tag{1.7}$$

Recall that for an integral kernel w satisfying condition **(Ker.)**, the fundamental solution $p(t,x,y)$ of the time fractional equation (1.4) is given by (1.5). We first give the small time estimates for $p(t,x,y)$ under condition **(S.Poly.)**(t_s).

Theorem 1.5. *Assume that w satisfies conditions* **(Ker.)** *and* **(S.Poly.)**(t_s)*. Then, the following estimates for $p(t,x,y)$ hold for all $(t,x,y)\in(0,t_s]\times D\times D$.*
(i) (Near diagonal estimates) Suppose that $\phi(t^{-1})\rho(x,y)^{\alpha}\le 1/(4e^2)$.
(a) If one of the estimates among **(J1)**, **(J2)**, **(J3)**, **(D1)**, **(D2)** *and* **(D3)** *holds, then we have*

$$p(t,x,y)\asymp\left(1\wedge\frac{\delta_*(x,y)}{\phi(t^{-1})^{-2/\alpha}}\right)^{\alpha/2}\phi(t^{-1})^{d/\alpha}+w(t)\left(1\wedge\frac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha/2}F_k^{\alpha}(d,t,x,y). \tag{1.8}$$

(b) Otherwise, if **(J4)** *holds, then we have*

$$p(t,x,y) \asymp \left(1 \wedge \frac{\delta_*(x,y)}{\phi(t^{-1})^{-2/\alpha}}\right)^{\alpha-1} \phi(t^{-1})^{d/\alpha} + w(t)\left(1 \wedge \frac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha-1} F_c^\alpha(d,t,x,y).$$

(ii) (Off diagonal estimates) Suppose that $\phi(t^{-1})\rho(x,y)^\alpha > 1/(4e^2)$.

(a) If **(J1)** *or* **(J2)** *or* **(J3)** *holds, then we have*

$$p(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)}{\phi(t^{-1})^{-1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{\phi(t^{-1})^{-1/\alpha}}\right)^{\alpha/2} \frac{\phi(t^{-1})^{-1}}{\rho(x,y)^{d+\alpha}}. \tag{1.9}$$

(b) If **(J4)** *holds, then we have*

$$p(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)}{\phi(t^{-1})^{-1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{\phi(t^{-1})^{-1/\alpha}}\right)^{\alpha-1} \frac{\phi(t^{-1})^{-1}}{\rho(x,y)^{d+\alpha}}.$$

(c) Otherwise, if **(D1)** *or* **(D2)** *or* **(D3)** *holds, then we have*

$$p(t,x,y) \simeq \left(1 \wedge \frac{\delta_D(x)}{\phi(t^{-1})^{-1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{\phi(t^{-1})^{-1/\alpha}}\right)^{\alpha/2} \phi(t^{-1})^{d/\alpha} \exp\left(-ct\overline{\phi}_\alpha\left(\left(\frac{\rho(x,y)}{t}\right)^\alpha\right)\right), \tag{1.10}$$

where the function $\overline{\phi}_\alpha$ *is defined as* (1.7).

Next, under condition **(L.Poly.)**, we get the large time estimates for $p(t,x,y)$. Hereinafter, we let $R_D := \mathrm{diam}(D)$ and $T_D := [\phi^{-1}(4^{-1}e^{-2}R_D^{-\alpha})]^{-1}$.

Theorem 1.6. *Assume that* w *satisfies conditions* **(Ker.)** *and* **(L.Poly.)**. *Then, for every fixed* $T > 0$, *the following estimates hold for all* $(t,x,y) \in [T,\infty) \times D \times D$.

(i) If **(J1)** *or* **(D1)** *holds and* $R_D < \infty$, *then we have*

$$p(t,x,y) \asymp w(t)\left(1 \wedge \frac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha/2} \left(\delta_*(x,y)^{\alpha/2} + F_k^\alpha(d,T_D,x,y)\right).$$

(ii) If **(J4)** *holds and* $R_D < \infty$, *then we have*

$$p(t,x,y) \asymp w(t)\left(1 \wedge \frac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha-1} \left(\delta_*(x,y)^{\alpha-1} + F_c^\alpha(d,T_D,x,y)\right).$$

(iii) If **(J2)** *holds, then* (1.8) and (1.9) *hold for all* $(t,x,y) \in [T,\infty) \times D \times D$ *satisfying* $\phi(t^{-1})\rho(x,y)^\alpha \le 1/(4e^2)$ *and* $\phi(t^{-1})\rho(x,y)^\alpha > 1/(4e^2)$, *respectively.*

(iv) If **(D2)** *holds, then* (1.8) and (1.10) *hold for all* $(t,x,y) \in [T,\infty) \times D \times D$ *satisfying* $\phi(t^{-1})\rho(x,y)^\alpha \le 1/(4e^2)$ *and* $\phi(t^{-1})\rho(x,y)^\alpha > 1/(4e^2)$, *respectively.*

(v) Assume that either of the estimates **(J3)** *or* **(D3)** *holds.*

(a) If $\phi(t^{-1})\rho(x,y)^\alpha \le 1/(4e^2)$, *then we have*

$$\begin{aligned} p(t,x,y) \asymp{}& \left(1 \wedge \delta_D(x)\right)^{\alpha/2}\left(1 \wedge \delta_D(y)\right)^{\alpha/2}\left(\phi(t^{-1})^{d/\alpha} + w(t)G_d^\alpha(t, 1 \vee \rho(x,y))\right) \\ &+ \mathbf{1}_{\{\rho(x,y)\le 1\}} w(t)\left(1 \wedge \frac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha/2} F_k^\alpha(d, [\phi^{-1}(4^{-1}e^{-2})]^{-1}, x, y), \end{aligned}$$

where the function $G_d^\alpha(t,l)$ is defined as follows:

$$G_d^\alpha(t,l) := \begin{cases} 0, & \text{if } d < \alpha; \\ \log\left(\dfrac{2\phi(t^{-1})^{-1}}{l^\alpha \phi(T^{-1})^{-1}}\right), & \text{if } d = \alpha; \\ l^{\alpha-d}, & \text{if } d > \alpha. \end{cases}$$

(b) If $\phi(t^{-1})\rho(x,y)^\alpha > 1/(4e^2)$, then we have

$$p(t,x,y) \simeq \big(1 \wedge \delta_D(x)\big)^{\alpha/2}\big(1 \wedge \delta_D(y)\big)^{\alpha/2} \times \begin{cases} \dfrac{\phi(t^{-1})^{-1}}{\rho(x,y)^{d+\alpha}}, & \text{if } (\mathbf{J3}) \text{ holds}; \\ \phi(t^{-1})^{d/\alpha} \exp\Big(-ct\overline{\phi}_\alpha\big((\dfrac{\rho(x,y)}{t})^\alpha\big)\Big), & \text{if } (\mathbf{D3}) \text{ holds}, \end{cases}$$

where the function $\overline{\phi}_\alpha$ is defined as (1.7).

Example 1.7. Let $0 < \alpha \le 2$, $d \ge 1$ and $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set. Suppose that D is a bounded set or half space-like set or exterior of a bounded set. If $\alpha = 2$ then we further assume that $d \ge 2$ and D is connected. If D is unbounded, then we also assume that $d > \alpha$. In this example, we consider the following time fractional equation.

$$\begin{aligned} &\frac{d}{dt}\int_0^t w(t-s)\big(u(t,x) - f(x)\big)ds = \Delta^{\alpha/2}u(t,x), \quad x \in D, \quad t > 0, \\ &u(0,x) = f(x), \quad x \in D, \qquad u(t,x) = 0, \quad x \in \mathbb{R}^d \setminus D, \quad t > 0, \end{aligned} \tag{1.11}$$

where $w(s) = s^{-\beta}/\Gamma(1-\beta)$ for some $0 < \beta < 1$. Then, the fractional-time derivative is the Caputo derivative of order β and conditions **(Ker.)**, **(S.Poly.)(1)** and **(L.Poly.)** are satisfied. Thus, by Remark 1.4 and Theorems 1.5 and 1.6, we obtain the global estimates on the fundamental solution $p_\beta(t,x,y)$ of (1.11). Denote by $p_\beta^0(t,x,y)$ the fundamental solution of (1.11) when $D = \mathbb{R}^d$. Two-sided estimates on $p_\beta^0(t,x,y)$ are obtained in [8, Corollary 1.5]. (See also [18, Theorem 2.2] for the exact asymptotic formulas of $p_\beta^0(t,x,y)$.)

1.7.1. Small time estimates. Suppose that $t \in (0,2]$.

(1) Assume that $|x-y| \le t^{\beta/\alpha}$. Then, by calculating the function $F_k^\alpha(d,t,x,y)$, since $d > \alpha/2$ in this example, we have

(i) if $d < \alpha$, then

$$p_\beta(t,x,y) \asymp \begin{cases} t^{-\beta d/\alpha}, & \text{if } \delta_D(x)\delta_D(y) > t^{2\beta/\alpha}; \\ \left(1 \wedge \dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} t^{-\beta}\Big(|x-y| \vee \big(\delta_D(x) \wedge \delta_D(y)\big)\Big)^{\alpha-d}, & \text{if } \delta_D(x)\delta_D(y) \le t^{2\beta/\alpha}. \end{cases}$$

(ii) if $d = \alpha$, then

$$p_\beta(t,x,y) \asymp \begin{cases} t^{-\beta}\log\left(\dfrac{2t^{\beta/\alpha}}{|x-y|}\right), & \text{if } \delta_D(x)\delta_D(y) > t^{2\beta/\alpha}; \\ \left(1 \wedge \dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} t^{-\beta}\left(1 + \log^+\left(\dfrac{\delta_D(x) \wedge \delta_D(y)}{|x-y|}\right)\right), & \text{if } \delta_D(x)\delta_D(y) \le t^{2\beta/\alpha}. \end{cases}$$

(iii) if $d > \alpha$, then

$$p_\beta(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} t^{-\beta}|x-y|^{\alpha-d}.$$

Note that interior estimates in (i), (ii) and (iii) coincide with estimates on $p^0_\beta(t,x,y)$. We also note that in any cases, for $x, y \in D$ satisfying $2\delta_D(x) \le |x-y| \le t^{\beta/\alpha}$,

$$p_\beta(t,x,y) \asymp \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} t^{-\beta}|x-y|^{-d},$$

while for $x, y \in D$ satisfying $|x-y| \le 2\delta_D(x) \le t^{\beta/\alpha}$,

$$p_\beta(t,x,y) \asymp \begin{cases} \delta_D(x)^{\alpha-d}t^{-\beta}, & \text{if } d<\alpha; \\ \log\big(\delta_D(x)/|x-y|\big)t^{-\beta}, & \text{if } d=\alpha; \\ t^{-\beta}|x-y|^{\alpha-d}, & \text{if } d>\alpha. \end{cases}$$

Hence, the decay rate of the boundary term in $p_\beta(t,x,y)$ depends on whether $|x-y| \le 2\delta_D(x)$ or not. Indeed, if $|x-y| \le 2\delta_D(x)$, then the decay rate becomes smaller than $\alpha/2$. We mention that there is no boundary term in the estimate of $p_\beta(t,x,y)$ when $d > \alpha$ and $|x-y| \le 2\delta_D(x)$.

(2) Next, assume that $|x-y| > t^{\beta/\alpha}$. Then, we have

$$p_\beta(t,x,y) \simeq \left(1 \wedge \frac{\delta_D(x)}{t^{\beta/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{t^{\beta/\alpha}}\right)^{\alpha/2} \times \begin{cases} \dfrac{t^\beta}{|x-y|^{d+\alpha}}, & \text{if } 0<\alpha<2; \\ t^{-\beta d/2}\exp\left(-c\dfrac{|x-y|^{2/(2-\beta)}}{t^{\beta/(2-\beta)}}\right), & \text{if } \alpha=2. \end{cases}$$

In particular, if $|x-y| > t^{\beta/\alpha}$, then by combining with the results in [8, Corollary 1.5], we get

$$p_\beta(t,x,y) \simeq \left(1 \wedge \frac{\delta_D(x)}{t^{\beta/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{t^{\beta/\alpha}}\right)^{\alpha/2} p^0_\beta(t,x,y).$$

1.7.2. Large time estimates. Suppose that $t \in [2,\infty)$.

(1) Suppose that D is bounded. Then, by Theorem 1.6(i), we get

$$p_\beta(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} t^{-\beta} \times \begin{cases} \Big(|x-y| \vee \big(\delta_D(x)\wedge\delta_D(y)\big)\Big)^{\alpha-d}, & \text{if } d<\alpha; \\ \left(1+\log^+\left(\dfrac{\delta_D(x)\wedge\delta_D(y)}{|x-y|}\right)\right), & \text{if } d=\alpha; \\ |x-y|^{\alpha-d}, & \text{if } d>\alpha. \end{cases}$$

These estimates are exactly the same as the ones given in *1.7.1. Small time estimates* when $|x-y| \le t^{\beta/\alpha}$ and $\delta_D(x)\delta_D(y) \le t^{2\beta/\alpha}$. We can see that since D is bounded so that $\delta_D(x)$, $\delta_D(y)$ and $|x-y|$ are small, those two inequalities always hold (up to constant) in this case.

(2) Suppose that D is a half space-like set and $d > \alpha$. Then, by Theorem 1.6(iii) and (iv), we see that the estimates given in *1.7.1. Small time estimates* hold not only on $t \in (0, 2]$ but also on $t \in (0, \infty)$.
(3) Suppose that D is exterior of a bounded set and $d > \alpha$. Then, by Theorem 1.6(v), we obtain the following:
If $|x - y| \le t^{\beta/\alpha}$, then

$$p_\beta(t, x, y) \asymp \left(1 \wedge \frac{\delta_D(x)}{|x - y| \wedge 1}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x - y| \wedge 1}\right)^{\alpha/2} t^{-\beta}|x - y|^{\alpha-d},$$

Otherwise, if $|x - y| > t^{\beta/\alpha}$, then

$$p_\beta(t, x, y) \simeq (1 \wedge \delta_D(x))^{\alpha/2}(1 \wedge \delta_D(y))^{\alpha/2} \times \begin{cases} \dfrac{t^\beta}{|x - y|^{d+\alpha}}, & \text{if } 0 < \alpha < 2; \\ t^{-\beta d/2} \exp\left(-c\dfrac{|x - y|^{2/(2-\beta)}}{t^{\beta/(2-\beta)}}\right), & \text{if } \alpha = 2. \end{cases}$$

In both cases, according to [8, Corollary 1.5], we get

$$p_\beta(t, x, y) \simeq \left(1 \wedge \frac{\delta_D(x)}{|x - y| \wedge 1}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x - y| \wedge 1}\right)^{\alpha/2} p^0_\beta(t, x, y). \quad \square$$

Example 1.8. Under the settings of Example 1.7, let $p_{\beta,\eta}(t, x, y)$ be the fundamental solution of (1.11) with $w(s) = s^{-\beta}\mathbf{1}_{(0,1]}(s) + s^{-\eta}\mathbf{1}_{(1,\infty)}(s)$ for some $\eta > 1$. Note that still conditions **(Ker.)**, **(S.Poly.)**(1) and **(L.Poly.)** are satisfied. Thus, by Remark 1.4 and Theorems 1.5 and 1.6, we obtain the global estimates on $p_{\beta,\eta}(t, x, y)$.

1.8.1. Small time estimates. For all $t \in (0, 2]$ and $x, y \in D$, it holds that

$$p_{\beta,\eta}(t, x, y) \simeq p_\beta(t, x, y).$$

1.8.2. Large time estimates. Suppose that $t \in [2, \infty)$ and $x, y \in D$. Since $\phi'(0) < \infty$ in this example, we see that $\phi(t^{-1}) \asymp t^{-1}$ for all $t \ge 2$. Define

$$p^{bdd}_\eta(t, x, y) := \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x - y|^2}\right)^{\alpha/2} t^{-\eta} \times \begin{cases} \Big(|x - y| \vee \big(\delta_D(x) \wedge \delta_D(y)\big)\Big)^{\alpha-d}, & \text{if } d < \alpha; \\ \left(1 + \log^+\left(\dfrac{\delta_D(x) \wedge \delta_D(y)}{|x - y|}\right)\right), & \text{if } d = \alpha; \\ |x - y|^{\alpha-d}, & \text{if } d > \alpha. \end{cases}$$

(1) Suppose that D is bounded. Then, by Theorem 1.6(i), we get

$$p_{\beta,\eta}(t, x, y) \asymp p^{bdd}_\eta(t, x, y).$$

(2) Suppose that D is a half space-like set and $d > \alpha$. Then, by Theorem 1.6(iii) and (iv), we obtain the following:

If $|x-y| \le t^{1/\alpha}$, then

$$p_{\beta,\eta}(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{t^{2/\alpha}}\right)^{\alpha/2} t^{-d/\alpha} + \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} t^{-\eta}|x-y|^{\alpha-d}.$$

In particular, if $\left(\delta_D(x)\delta_D(y)\right)^{1/2} \vee |x-y| \le t^{1/\alpha} t^{-(\eta-1)/d}$, then $p_{\beta,\eta}(t,x,y) \asymp p_\eta^{bdd}(t,x,y)$.

Otherwise, if $|x-y| > t^{1/\alpha}$, then

$$p_{\beta,\eta}(t,x,y) \simeq \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \times \begin{cases} \dfrac{t}{|x-y|^{d+\alpha}}, & \text{if } \ 0 < \alpha < 2; \\ t^{-d/2} \exp\left(-c\dfrac{|x-y|^2}{t}\right), & \text{if } \ \alpha = 2. \end{cases}$$

(3) Suppose that D is exterior of a bounded set and $d > \alpha$. Then, by Theorem 1.6(v), we obtain the following:

If $|x-y| \le t^{1/\alpha}$, then

$$p_{\beta,\eta}(t,x,y) \asymp \left(1 \wedge \delta_D(x)\right)^{\alpha/2} \left(1 \wedge \delta_D(x)\right)^{\alpha/2} t^{-d/\alpha} + \left(1 \wedge \frac{\delta_D(x)}{|x-y| \wedge 1}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y| \wedge 1}\right)^{\alpha/2} t^{-\eta}|x-y|^{\alpha-d}.$$

Otherwise, if $|x-y| > t^{1/\alpha}$, then

$$p_{\beta,\eta}(t,x,y) \simeq \left(1 \wedge \delta_D(x)\right)^{\alpha/2} \left(1 \wedge \delta_D(y)\right)^{\alpha/2} \times \begin{cases} \dfrac{t}{|x-y|^{d+\alpha}}, & \text{if } \ 0 < \alpha < 2; \\ t^{-d/2} \exp\left(-c\dfrac{|x-y|^2}{t}\right), & \text{if } \ \alpha = 2. \quad \square \end{cases}$$

Under condition **(L.Poly.)**, even if D is bounded so that $q(t,x,y)$ decreases exponentially as $t \to \infty$, the fundamental solution $p(t,x,y)$ decreases polynomially. (See, Theorem 1.6(i) and (ii).) We introduce a condition which makes $p(t,x,y)$ decrease subexponentially.

(Sub*.)($\boldsymbol{\beta}, \boldsymbol{\theta}$) There exist constants $c_0 > 1, \theta > 0$ and $\beta \in (0,1)$ such that

$$c_0^{-1} \exp(-\theta t^\beta) \le w(t) \le c_0 \exp(-\theta t^\beta) \quad \text{for all } \ t \ge 1.$$

Under condition **(Sub*.)($\boldsymbol{\beta}, \boldsymbol{\theta}$)**, we obtain estimates for $p(t,x,y)$ which have exactly the same exponential terms as w.

Theorem 1.9. *Assume that w satisfies conditions* **(Ker.)** *and* **(Sub*.)($\boldsymbol{\beta}, \boldsymbol{\theta}$)**. *We further assume that* **(J1)** *or* **(J4)** *or* **(D1)** *holds. Then, for every fixed $T > 0$, the following estimates hold for all $(t,x,y) \in [T,\infty) \times D \times D$.*

(i) If **(J1)** *or* **(D1)** *holds and $R_D < \infty$, then we have*

$$p(t,x,y) \asymp \exp(-\theta t^\beta) \left(1 \wedge \frac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha/2} \left(\delta_*(x,y)^{\alpha/2} + F_k^\alpha(d, T_R, x, y)\right).$$

(ii) If **(J4)** *holds and* $R_D < \infty$*, then we have*

$$p(t,x,y) \asymp \exp(-\theta t^{\beta}) \left(1 \wedge \frac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha-1} \left(\delta_*(x,y)^{\alpha-1} + F_c^{\alpha}(d, T_R, x, y)\right).$$

Notice that condition **(Trunc.)**(t_f) implies condition **(S.Poly.)**(t_s) with $t_s = t_f/2$. Hence, we obtain the small time estimates $(0 < t \le t_f/2)$ under condition **(Trunc.)**(t_f) from Theorem 1.5. Here, we give the large time behaviors of $p(t,x,y)$ under condition **(Trunc.)**(t_f).

Theorem 1.10. *Assume that* w *satisfies conditions* **(Ker.)** *and* **(Trunc.)**(t_f)*. Then, the following estimates hold for all* $(t,x,y) \in [t_f/2, \infty) \times D \times D$*. Let* $n_t := \lfloor t/t_f \rfloor + 1 \in \mathbb{N}$*.*
(i) If **(J1)** *or* **(D1)** *holds and* $R_D < \infty$*, then we have*

$$p(t,x,y) \simeq \begin{cases} \left(1 \wedge \dfrac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha/2} \Big[\delta_*(x,y)^{\alpha/2} + F_k^{\alpha}(d-\alpha n_t, T_D, x, y) \\ \qquad + \big(n_t t_f - t\big)^{n_t} F_k^{\alpha}(d-\alpha(n_t-1), T_D, x, y)\Big], & \text{if } t < \lfloor \dfrac{d+\alpha}{\alpha} \rfloor t_f; \\ \delta_*(x,y)^{\alpha/2} e^{-ct}, & \text{if } t \ge \lfloor \dfrac{d+\alpha}{\alpha} \rfloor t_f. \end{cases}$$

(ii) If **(J4)** *holds and* $R_D < \infty$*, then we have*

$$p(t,x,y) \simeq \begin{cases} \left(1 \wedge \dfrac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha-1} \Big[\delta_*(x,y)^{\alpha/2} + F_c^{\alpha}(d-\alpha n_t, T_D, x, y) \\ \qquad + \big(n_t t_f - t\big)^{n_t} F_c^{\alpha}(d-\alpha(n_t-1), T_D, x, y)\Big], & \text{if } t < \lfloor \dfrac{d+2\alpha-2}{\alpha} \rfloor t_f; \\ \delta_*(x,y)^{\alpha-1} e^{-ct}, & \text{if } t \ge \lfloor \dfrac{d+2\alpha-2}{\alpha} \rfloor t_f. \end{cases}$$

(iii) If **(J2)** *or* **(J3)** *or* **(D2)** *or* **(D3)** *holds, then we have*

$$p(t,x,y) \simeq \begin{cases} \left(1 \wedge \dfrac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha/2} \Big[\delta_*(x,y)^{\alpha/2} \wedge \phi(t^{-1})^{-1} + F_k^{\alpha}(d-\alpha n_t, t, x, y) \\ \qquad + \big(n_t t_f - t\big)^{n_t} F_k^{\alpha}(d-\alpha(n_t-1), t, x, y)\Big], \\ \qquad\qquad \text{if } \rho(x,y)^{\alpha} \le \phi(t^{-1})^{-1} \text{ and } t < \lfloor (d+\alpha)/\alpha \rfloor t_f; \\ q(ct,x,y), \qquad \text{if } \rho(x,y)^{\alpha} > \phi(t^{-1})^{-1} \text{ or } t \ge \lfloor (d+\alpha)/\alpha \rfloor t_f. \end{cases}$$

Remark 1.11. When $d > \alpha$, we have that $F_k^{\alpha}(d,t,x,y) = F_c^{\alpha}(d,t,x,y) = \rho(x,y)^{\alpha-d}$. Thus, by Theorems 1.6 and 1.9, under either of the conditions **(L.Poly.)** or **(Sub*.)**$(\boldsymbol{\beta}, \boldsymbol{\theta})$, $\lim_{y\to x} p(t,x,y) = \infty$ for all large t even if D is bounded. However, under condition **(Trunc.)**(t_f), by Theorem 1.10, $p(t,x,x) < \infty$ for all t large enough. Indeed, we see that when the kernel w is finitely supported, the singularity of $p(t,x,y)$ at $x = y$ recedes as the number $\lfloor t/t_f \rfloor$ increases.

1.4. General results

In this subsection, we present our estimates for the fundamental solution in full generality.

Throughout the remainder of this paper, we always assume that $\{V(x,\cdot) : x \in D\}$ is a family of strictly positive functions satisfying the condition $\mathbf{WS}(d_1, d_2)$ for some $d_2 \geq d_1 > 0$ uniformly, that is, there exist constants $c_1, c_2 > 0$ such that

$$c_1 \left(\frac{l_2}{l_1}\right)^{d_1} \leq \frac{V(x,l_2)}{V(x,l_1)} \leq c_2 \left(\frac{l_2}{l_1}\right)^{d_2} \quad \text{for all } x \in D, \ \ 0 < l_1 \leq l_2 < \infty.$$

We also always assume that $\Phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function such that $\Phi(0) = 0$ and satisfies $\mathbf{WS}(\alpha_1, \alpha_2)$ for some $\alpha_2 \geq \alpha_1 > 0$.

For a given non-decreasing function $\Psi : (0,\infty) \to [0,\infty)$ such that $\Phi(l) \leq \Psi(l)$ for all $l > 0$ and satisfies $\mathbf{WS}(\gamma_1, \gamma_2)$ for some $\gamma_2 \geq \gamma_1 > 0$, we define

$$q^j(t,x,l;\, \Phi,\, \Psi) := \frac{t}{tV(x,\, \Phi^{-1}(t)) + \Psi(l)V(x,l)}.$$

Besides, for a given function $\mathcal{M} : (0,\infty) \times (0,\infty) \to [0,\infty)$ and a constant $a > 0$, we define

$$q^d(a,t,x,l;\, \Phi, \mathcal{M}) := \frac{\exp\big(-a\mathcal{M}(t,l)\big)}{V(x,\, \Phi^{-1}(t))}.$$

We will use the functions q^j and q^d to describe interior estimates for $q(t,x,y)$.

On the other hand, for $\gamma \in [0,1)$ and $(t,x,y) \in (0,\infty) \times D \times D$, we define

$$\begin{aligned} a_1^\gamma(t,x,y) &:= \left(\frac{\Phi(\delta_D(x))}{\Phi(\delta_D(x)) + t}\right)^\gamma \left(\frac{\Phi(\delta_D(y))}{\Phi(\delta_D(y)) + t}\right)^\gamma, \\ a_2^\gamma(t,x,y) &:= a_1^\gamma(t/(t+1), x, y). \end{aligned}$$

These functions will be used to describe boundary behaviors of $q(t,x,y)$.

Remark 1.12. Observe that for any positive constants a, b and c, it holds that $a/(b+c) \leq (a/b) \wedge (a/c) \leq 2a/(b+c)$. Hence, we have that

$$\begin{aligned} q^j(t,x,l;\, \Phi,\, \Psi) &\asymp \frac{1}{V(x,\, \Phi^{-1}(t))} \wedge \frac{t}{\Psi(l)V(x,l)}, \\ a_1^\gamma(t,x,y) &\asymp \left(1 \wedge \frac{\Phi(\delta_D(x))}{t}\right)^\gamma \left(1 \wedge \frac{\Phi(\delta_D(y))}{t}\right)^\gamma, \\ a_2^\gamma(t,x,y) &\asymp \left(1 \wedge \frac{\Phi(\delta_D(x))}{t \wedge 1}\right)^\gamma \left(1 \wedge \frac{\Phi(\delta_D(y))}{t \wedge 1}\right)^\gamma. \end{aligned}$$

We list our candidates for the estimates of the transition density $q(t,x,y)$.

Definition 1.13. Let $\gamma \in [0,1)$, $\lambda \in [0,\infty)$ and $k \in \{1,2\}$.
(1) We say that $q(t,x,y)$ enjoys the estimate $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\, \Psi)$ if

$$q(t,x,y) \asymp a_1^\gamma(t,x,y) q^j(t,x,\rho(x,y);\, \Phi,\, \Psi) \qquad \text{for all } (t,x,y) \in (0,1] \times D \times D,$$

and for all $(t,x,y) \in [1,\infty) \times D \times D$,

$$q(t,x,y) \asymp \begin{cases} a_k^\gamma(t,x,y) q^j(t,x,\rho(x,y);\, \Phi,\, \Psi), & \text{if} \quad \lambda = 0, \\ a_1^\gamma(1,x,y) e^{-\lambda t}, & \text{if} \quad \lambda > 0. \end{cases}$$

(2) We say that $q(t,x,y)$ enjoys the estimate $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$ if $\alpha_1 > 1$ where α_1 is the lower scaling index of Φ, and

$$q(t,x,y) \simeq a_1^{\gamma}(t,x,y) q^d(c,t,x,\rho(x,y);\Phi,\mathcal{M}) \qquad \text{for all } (t,x,y) \in (0,1] \times D \times D,$$

and for all $(t,x,y) \in [1,\infty) \times D \times D$,

$$q(t,x,y) \simeq \begin{cases} a_k^{\gamma}(t,x,y) q^d(c,t,x,\rho(x,y);\Phi,\mathcal{M}), & \text{if} \quad \lambda = 0, \\ a_1^{\gamma}(1,x,y) e^{-\lambda t}, & \text{if} \quad \lambda > 0, \end{cases}$$

where the function $\mathcal{M}(t,l)$ is a strictly positive for all $t,l > 0$, non-increasing on $(0,\infty)$ for each fixed $l > 0$ and determined by the following relation:

$$\frac{t}{\mathcal{M}(t,l)} \asymp \Phi\left(\frac{l}{\mathcal{M}(t,l)}\right) \qquad \text{for all } t,l > 0. \tag{1.12}$$

(3) We say that $q(t,x,y)$ enjoys the estimate $\mathbf{HK}_M^{\gamma,\lambda,k}(\Phi,\Psi)$ if $\alpha_1 > 1$ where α_1 is the lower scaling index of Φ, and there are functions q^j, q^d such that

$$q(t,x,y) = q^j(t,x,y) + q^d(t,x,y) \qquad \text{for all} \quad (t,x,y) \in (0,\infty) \times D \times D,$$

and q^j and q^d enjoy the estimate $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Psi)$ and $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$, respectively.

In the rest of this subsection, we always assume that $q(t,x,y)$ enjoys one of the estimates $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Phi)$, $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$ and $\mathbf{HK}_M^{\gamma,\lambda,k}(\Phi,\Psi)$ for some $\gamma \in [0,1)$, $\lambda \geq 0$ and $k \in \{1,2\}$. If $\lambda > 0$, then we further assume that D is bounded so that $R_D = \mathrm{diam}(D) < \infty$.

Example 1.14. (1) Examples of estimates $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Psi)$, $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$ and $\mathbf{HK}_M^{\gamma,\lambda,k}(\Phi,\Psi)$ include all estimates given in Section 1.3. For example, we see that estimate **(J1)** is nothing but estimate $\mathbf{HK}_J^{1/2,\lambda,1}(\Phi_\alpha,\Phi_\alpha)$ for $\lambda > 0$ where $\Phi_\alpha(x) := x^\alpha$.
(2) The factor $e^{-\lambda t} a_1^{\gamma}(1,x,y)$ usually appears in the global estimates of the Dirichlet heat kernel when D is a $C^{1,1}$ bounded open set, $a_1^{\gamma}(t,x,y)$ appears when D is a half space-like $C^{1,1}$ open set and $a_2^{\gamma}(t,x,y)$ appears when D is an exterior of a bounded $C^{1,1}$ open set. Various examples are given in [4,7,13,15,23,30,33].
(3) Recently, in [17], we, jointly with Renming Song and Zoran Vondraček give examples of generators whose transition density satisfies estimate $\mathbf{HK}_J^{\gamma,\lambda,1}(\Phi_\alpha,\Phi_\alpha)$ for each $0 < \alpha < 2$ and $\gamma \in [0 \vee (\alpha-1)/\alpha, 1)$.
(4) Examples of symmetric Markov processes (including non Lévy processes) satisfying the mixed heat kernel estimates $\mathbf{HK}_M^{\gamma,\lambda,k}(\Phi,\Psi)$ can be found in [1,2,23,29]. We will show that one of the explicit expressions of the function $\mathcal{M}$ is given by

$$\mathcal{M}(t,l) := \sup_{s>0} \left\{ \frac{l}{s} - \frac{t}{\Phi(s)} \right\},$$

which appears in the exponential terms in [1]. (See, Lemma 3.2(i).)

We introduce some functions which will be used in near diagonal estimates for the fundamental solution. Define for $(t,x,y) \in (0,\infty) \times D \times D$, $\gamma \in [0,1)$ and $k \in \{1,2\}$,

$$\mathcal{I}_k^{\gamma}(t,x,y) := \int_{\Phi(\rho(x,y))}^{1/(2e^2\phi(t^{-1}))} \frac{a_k^{\gamma}(r,x,y)}{V(x,\Phi^{-1}(r))} dr,$$

$$\mathcal{J}_k^{\gamma}(t,x,y) := \frac{a_k^{\gamma}(1/\phi(t^{-1}),x,y)}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)} + w(t)\mathcal{I}_k^{\gamma}(t,x,y). \tag{1.13}$$

Under certain weak scaling conditions for V and Φ, we can calculate the integral term $\mathcal{I}_k^\gamma$ explicitly. (See, Proposition 1.22.) Now, we are ready to state the main results.

Theorem 1.15. *Let $p(t,x,y)$ be given by (1.5). Assume that w satisfies conditions* **(Ker.)** *and* **(S.Poly.)(t_s)**. *Then the following estimates hold for all $(t,x,y) \in (0,t_s] \times D \times D$.*
(i) (Near diagonal estimates) If $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/(4e^2)$, then we have

$$p(t,x,y) \asymp \mathcal{J}_k^\gamma(t,x,y).$$

(ii) (Off diagonal estimates) Suppose that $\Phi(\rho(x,y))\phi(t^{-1}) > 1/(4e^2)$.
(a) If $q(t,x,y)$ enjoys the estimate $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Phi)$, then we have

$$p(t,x,y) \asymp \frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})\Phi(\rho(x,y))V(x,\rho(x,y))}.$$

(b) If $q(t,x,y)$ enjoys the estimate $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$, then we have

$$p(t,x,y) \simeq a_k^\gamma(1/\phi(t^{-1}),x,y)\frac{\exp\big(-c\mathcal{N}(t,\rho(x,y))\big)}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)},$$

where $\mathcal{N}(\cdot,l)$ is a strictly positive function which is determined by the following relation

$$\frac{1}{\phi\big(\mathcal{N}(t,l)/t\big)} \asymp \Phi\left(\frac{l}{\mathcal{N}(t,l)}\right), \qquad t,l>0. \tag{1.14}$$

(c) If $q(t,x,y)$ enjoys the estimate $\mathbf{HK}_M^{\gamma,\lambda,k}(\Phi,\Psi)$, then we have

$$p(t,x,y) \simeq a_k^\gamma(1/\phi(t^{-1}),x,y)\left(\frac{1}{\phi(t^{-1})\Psi(\rho(x,y))V(x,\rho(x,y))} + \frac{\exp\big(-c\mathcal{N}(t,\rho(x,y))\big)}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)}\right).$$

Recall that $R_D = \mathrm{diam}(D)$ and $T_D = [\phi^{-1}(4^{-1}e^{-2}R_D^{-\alpha})]^{-1}$.

Theorem 1.16. *Let $p(t,x,y)$ be given by (1.5). Assume that w satisfies conditions* **(Ker.)** *and* **(L.Poly.)**. *Then for every fixed $T>0$, the following estimates hold for all $(t,x,y) \in [T,\infty) \times D \times D$.*
(i) If $\lambda = 0$, then estimates given in Theorem 1.15 hold for all $(t,x,y) \in [T,\infty)\times D\times D$.
(ii) If $\lambda > 0$ and $R_D < \infty$, then we have

$$p(t,x,y) \asymp w(t)F_1^\gamma(T_D,x,y) = w(t)\int_{\Phi(\rho(x,y))}^{2\Phi(R_D)} \frac{a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr.$$

Remark 1.17. (1) By Lemma 3.2(i), one of the explicit expressions of the function $\mathcal{N}$ satisfying (1.14) is given by

$$\mathcal{N}(t,l) := \sup_{s>0}\left\{\frac{l}{s} - t\phi^{-1}(1/\Phi(s))\right\}.$$

(2) Theorems 1.15 and 1.16 recover [8, Theorems 1.6 and 1.8]. Indeed, the assumptions in [8] can be interpreted as the kernel w satisfies conditions **(Ker.)**, **(S.Poly.)(t_s)** and (**L.Poly**.) for some $0<\delta_1,\delta_2<1$ and $q(t,x,y)$ enjoys either of the estimates $\mathbf{HK}_J^{0,0,1}(\Phi,\Phi)$ or $\mathbf{HK}_D^{0,0,1}(\Phi)$.
(3) In off diagonal situations, that is, when $\Phi(\rho(x,y)) \ge \phi(t^{-1})^{-1}$, estimates for $p(t,x,y)$ can be factorized into the boundary factors and the rest. However, there is no such factorization

on near diagonal situation in general since $\mathcal{J}_k^\gamma(t,x,y)$ cannot be factorized commonly. (cf. Theorem 1.5.)

When condition **(Sub.)($\boldsymbol{\beta},\boldsymbol{\theta}$)** holds, the bounds for fundamental solution decrease subexponentially as $t\to\infty$. Moreover, when $0<\beta<1$ and D is bounded, we obtain the sharp upper bounds that decrease with exactly the same rate as the upper bound for w as $t\to\infty$.

Theorem 1.18. *Let $p(t,x,y)$ be given by (1.5). Assume that w satisfies conditions* **(Ker.)** *and* **(Sub.)($\boldsymbol{\beta},\boldsymbol{\theta}$)**. *Then for every fixed $T>0$, the following estimates hold for all $(t,x,y)\in[T,\infty)\times D\times D$.*
(i) Suppose that $\lambda=0$.
(a) If $\Phi(\rho(x,y))\phi(t^{-1})\le 1/(4e^2)$, then there exists a constant $c>1$ such that

$$c^{-1}\left(\frac{a_k^\gamma(t,x,y)}{V\big(x,\Phi^{-1}(t)\big)}+w(t)\int_{\Phi(\rho(x,y))}^{1/(2e^2\phi(t^{-1}))}\frac{a_k^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr\right)$$
$$\le p(t,x,y)\le c\left(\frac{a_k^\gamma(t,x,y)}{V\big(x,\Phi^{-1}(t)\big)}+\exp\big(-\frac{\theta}{2}t^\beta\big)\int_{\Phi(\rho(x,y))}^{1/(2e^2\phi(t^{-1}))}\frac{a_k^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr\right),$$

where $\theta>0$ is the constant in condition **(Sub.)($\boldsymbol{\beta},\boldsymbol{\theta}$)**.
(b) If $\Phi(\rho(x,y))\phi(t^{-1})>1/(4e^2)$, then we have

$$p(t,x,y)\simeq q(ct,x,y).$$

(ii) Suppose that $\lambda>0$ and $R_D<\infty$. Then, there exist constants $L_1,L_2>0$ independent of λ and $c>1$ such that in the case when $\beta\in(0,1)$, we have

$$c^{-1}w(t)\int_{\Phi(\rho(x,y))}^{2\Phi(R_D)}\frac{a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr$$
$$\le p(t,x,y)\le c\exp\big(-\theta t^\beta\big)\int_{\Phi(\rho(x,y))}^{2\Phi(R_D)}\frac{a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr,$$

and in the case when $\beta=1$, we have

$$c^{-1}\left(w(t)\int_{\Phi(\rho(x,y))}^{2\Phi(R_D)}\frac{a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr+e^{-\lambda L_1 t}\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma\right)$$
$$\le p(t,x,y)$$
$$\le c\left(\exp\big(-\frac{\theta}{2}t\big)\int_{\Phi(\rho(x,y))}^{2\Phi(R_D)}\frac{a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr+e^{-\lambda L_2 t}\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma\right),$$

where $\theta>0$ is the constant in condition **(Sub.)($\boldsymbol{\beta},\boldsymbol{\theta}$)**.

Our last theorem gives the estimates for $p(t,x,y)$ when w is finitely supported.

Theorem 1.19. *Let $p(t,x,y)$ be given by (1.5). Assume that w satisfies conditions* **(Ker.)** *and* **(Trunc.)**(t_f). *Then the following estimates hold for all $(t,x,y)\in[t_f/2,\infty)\times D\times D$. Let $n_t:=\lfloor t/t_f\rfloor+1\in\mathbb{N}$.*
(i) Suppose that $\lambda=0$.
(a) If $\Phi(\rho(x,y))\le t\le\lfloor d_2/\alpha_1+2\gamma\rfloor t_f$, then

$$p(t,x,y)\asymp\int_{\Phi(\rho(x,y))}^{2t}\frac{r^{n_t}a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr+(n_t t_f-t)^{n_t}\int_{\Phi(\rho(x,y))}^{2t}\frac{r^{n_t-1}a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr,$$

(b) If $\Phi(\rho(x,y)) \le t$ and $t > \lfloor d_2/\alpha_1 + 2\gamma \rfloor t_f$, then

$$p(t,x,y) \asymp \frac{a_k^\gamma(t,x,y)}{V\big(x,\Phi^{-1}(t)\big)} \asymp q(t,x,y).$$

(c) If $\Phi(\rho(x,y)) > t$, then

$$p(t,x,y) \simeq q(ct,x,y).$$

(ii) Suppose that $\lambda > 0$ and $R_D < \infty$.

(a) If $t \le \lfloor d_2/\alpha_1 + 2\gamma \rfloor t_f$, then

$$p(t,x,y) \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(R_D)} \frac{r^{n_t} a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))} dr + (n_t t_f - t)^{n_t} \int_{\Phi(\rho(x,y))}^{2\Phi(R_D)} \frac{r^{n_t-1} a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))} dr,$$

(b) If $t \ge \lfloor d_2/\alpha_1 + 2\gamma \rfloor t_f$, then

$$p(t,x,y) \simeq e^{-ct}\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma \simeq q(t,x,y).$$

Remark 1.20. Note that under settings of Theorem 1.19, we can apply Theorem 1.15 to obtain the estimates of $p(t,x,y)$ for all $(t,x,y) \in (0, t_f/2] \times D \times D$. Hence, we have the global estimates for $p(t,x,y)$ under those settings.

As a consequence of the estimates for the fundamental solution, we have that the solution to the Dirichlet problem (1.4) vanishes continuously on the boundary of D. Indeed, under mild conditions, the solution $u(t,x)$ vanishes exactly the same rate as a transition density $q(t,x,y)$.

(V.) There exists a constant $c_V > 1$ such that for all $x \in D$ and $0 < l \le R_D = \mathrm{diam}(D)$,

$$c_V^{-1} V(x,l) \le m\big(\{y \in D : \rho(x,y) \le l\}\big) \le c_V V(x,l).$$

Corollary 1.21. *Suppose that (D,ρ,m) satisfies* **(V.)***, and w satisfies conditions* **(Ker.)***,* **(S.Poly.)**(t_s) *and one among* **(L.Poly.)***,* **(Sub.)**$(\boldsymbol{\beta},\boldsymbol{\theta})$ *and* **(Trunc.)**(t_f)*. We also assume that $q(t,x,y)$ enjoys one of the estimates $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Phi)$, $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$ and $\mathbf{HK}_M^{\gamma,\lambda,k}(\Phi,\Psi)$ for some $0 < \gamma < 1$, $\lambda \ge 0$ and $k \in \{1,2\}$. When $\lambda > 0$, we further assume that D is bounded. Then, for all bounded measurable function f on D, $u(t,x) := \mathbb{E}[T^D_{E_t} f(x)]$ satisfies the following boundary condition:*

For any fixed $t > 0$, there exists a constant $c_1 > 0$ such that for every $x \in D$,

$$|u(t,x)| \le c_1 \|f\|_\infty \Phi(\delta_D(x))^\gamma.$$

Proof. Since the ideas are similar, we only give the proof for the case when w satisfies **(Ker.)**, **(S.Poly.)**(t_s) and **(L.Poly.)** and $q(t,x,y)$ enjoys estimate $\mathbf{HK}_J^{\gamma,0,2}(\Phi,\Phi)$ for some $\gamma \in (0,1)$. Fix $t > 0$ and we let $A_t := \Phi^{-1}\big(1/(4e^2\phi(t^{-1}))\big)$. By Theorems 1.15 and 1.16, for every $x \in D$,

$$\begin{aligned}
|u(t,x)| &= \left| \int_D p(t,x,y) f(y) m(dy) \right| \\
&\le c\|f\|_\infty \Phi(\delta_D(x))^\gamma \\
&\quad \times \left(\int_{\{y\in D:\rho(x,y)\le A_t\}} \frac{\mathcal{J}_k^\gamma(t,x,y)}{\Phi(\delta_D(x))^\gamma} m(dy) + \int_{\{y\in D:\rho(x,y)>A_t\}} \frac{m(dy)}{\Phi(\rho(x,y))V(x,\rho(x,y))} \right) \\
&=: c\|f\|_\infty \Phi(\delta_D(x))^\gamma (I_1 + I_2).
\end{aligned}$$

Set $\eta = (d_1/(2\alpha_2)) \wedge ((1-\gamma)/2)$. Since $\eta < d_1/\alpha_2$, by [4, Theorem 2.2.2], we have that for all $x \in D$ and $0 < s < t$,

$$\inf_{r \in (s,t]} r^{-\eta} V(x, \Phi^{-1}(r)) \asymp s^{-\eta} V(x, \Phi^{-1}(s)). \tag{1.15}$$

Then, by Fubini's theorem, (1.15), condition **(V.)** and the weak scaling properties of V and Φ, since $\gamma + \eta < 1$,

$$\begin{aligned} I_1 &\le c \sum_{k=1}^{\infty} \int_{\{y \in D: 2^{-k} A_t < \rho(x,y) \le 2^{-(k-1)} A_t\}} \int_{\Phi(\rho(x,y))}^{1/(2e^2\phi(t^{-1}))} \frac{dr}{r^{\gamma} V(x, \Phi^{-1}(r))} m(dy) \\ &\le c \sum_{k=1}^{\infty} \int_{\{y \in D: 2^{-k} A_t < \rho(x,y) \le 2^{-(k-1)} A_t\}} m(dy) \int_{\Phi(2^{-k} A_t)}^{1/(2e^2\phi(t^{-1}))} \frac{dr}{r^{\gamma+\eta} r^{-\eta} V(x, \Phi^{-1}(r))} \\ &\le c \sum_{k=1}^{\infty} \frac{V(x, 2^{-(k-1)} A_t)}{\Phi(2^{-k} A_t)^{-\eta} V(x, 2^{-k} A_t)} \int_{\Phi(2^{-k} A_t)}^{1/(2e^2\phi(t^{-1}))} \frac{dr}{r^{\gamma+\eta}} \\ &\le c\, \Phi(A_t)^{\eta} \left(\int_0^{1/(2e^2\phi(t^{-1}))} \frac{dr}{r^{\gamma+\eta}} \right) \sum_{k=1}^{\infty} 2^{-k\eta\alpha_1} \le c. \end{aligned}$$

Moreover, we also have that by condition **(V.)** and the weak scaling properties of V and Φ,

$$\begin{aligned} I_2 &\le c \sum_{k=1}^{\infty} \int_{\{y \in D: 2^{k-1} A_t < \rho(x,y) \le 2^{k} A_t\}} \frac{1}{\Phi(\rho(x,y)) V(x, \rho(x,y))} m(dy) \\ &\le c \sum_{k=1}^{\infty} \frac{V(x, 2^{k} A_t)}{\Phi(2^{k-1} A_t) V(x, 2^{k-1} A_t)} \le c \sum_{k=1}^{\infty} \frac{2^{-k\alpha_1}}{\Phi(A_t)} \le c. \end{aligned}$$

Therefore, we get the result. $\square$

In the end of this section, we study explicit forms of $\mathcal{J}_k^{\gamma}(t,x,y)$ $(0 \le \gamma < 1)$ under some weak scaling conditions for V and Φ. Recall that $\Phi(\cdot)$ satisfies $\mathbf{WS}(\alpha_1, \alpha_2)$ and $V(x,\cdot)$ satisfies $\mathbf{WS}(d_1, d_2)$ uniformly. We define $\delta_*^{\Phi}(x,y) := \Phi(\delta_D(x))\Phi(\delta_D(y))$.

Proposition 1.22. *Let $\gamma \in [0,1)$. If $\gamma = 0$, then we redefine $\delta_D(x) = \infty$ for all $x \in D$. Then, the following estimates hold for all $(t,x,y) \in (0,\infty) \times D \times D$ satisfying $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/(4e^2)$.*

(a) If $d_2/\alpha_1 < 1 - 2\gamma$, then

$$\mathcal{J}_1^{\gamma}(t,x,y) \asymp \left(1 \wedge \frac{\delta_*^{\Phi}(x,y)}{\phi(t^{-1})^{-2}}\right)^{\gamma} \frac{1}{V\big(x, \Phi^{-1}(1/\phi(t^{-1}))\big)}.$$

(b) If $\alpha_1 = \alpha_2$, $d_1 = d_2 = (1-2\gamma)\alpha_1$ and $\gamma > 0$, then

$$\begin{aligned} \mathcal{J}_1^{\gamma}(t,x,y) \asymp &\left(1 \wedge \frac{\delta_*(x,y)^{\alpha_1}}{\phi(t^{-1})^{-2}}\right)^{\gamma} \phi(t^{-1})^{1-2\gamma} + \mathbf{1}_{\{\delta_*(x,y)^{\alpha_1/2} \le \phi(t^{-1})^{-1}\}} w(t) \\ &\times \delta_*(x,y)^{\alpha_1\gamma} \log^{+}\left(\frac{2\phi(t^{-1})^{-1}}{\big(\rho(x,y) \vee \delta_\vee(x,y)\big)^{\alpha_1}} \right). \end{aligned}$$

(c) If $1 - 2\gamma < d_1/\alpha_2 \le d_2/\alpha_1 < 1 - \gamma$, then

$$\mathcal{J}_1^{\gamma}(t,x,y) \asymp \left(1 \wedge \frac{\delta_*^{\Phi}(x,y)}{\phi(t^{-1})^{-2}}\right)^{\gamma} \frac{1}{V\big(x, \Phi^{-1}(1/\phi(t^{-1}))\big)} + \mathbf{1}_{\{\delta_*^{\Phi}(x,y)^{1/2} \le \phi(t^{-1})^{-1}\}} w(t)$$

$$\times \left(1 \wedge \frac{\delta_*^\Phi(x,y)}{\Phi(\rho(x,y))^2}\right)^\gamma \left(\frac{\Phi(\rho(x,y))}{V(x,\rho(x,y))} \vee \frac{\delta_*^\Phi(x,y)^\gamma\,\Phi(\delta_\vee(x,y))^{1-2\gamma}}{V(x,\delta_\vee(x,y))}\right).$$

(d) If $\alpha_1 = \alpha_2$, $d_1 = d_2 = (1-\gamma)\alpha_1$ and $\gamma > 0$, then

$$\mathcal{J}_1^\gamma(t,x,y) \asymp \left(1 \wedge \frac{\delta_*(x,y)^{\alpha_1}}{\phi(t^{-1})^{-2}}\right)^\gamma \phi(t^{-1})^{1-\gamma}$$
$$+ \mathbf{1}_{\{\delta_*(x,y)^{\alpha_1/2} \le \phi(t^{-1})^{-1}\}} w(t) \left(1 \wedge \frac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha_1\gamma}$$
$$\times \left(\rho(x,y)^{\alpha_1\gamma} + \delta_\wedge(x,y)^{\alpha_1\gamma} \log\left(\frac{\rho(x,y) \vee 2\delta_\vee(x,y)}{\rho(x,y) \vee \delta_\wedge(x,y)}\right)\right).$$

(e) If $1-\gamma < d_1/\alpha_2 \le d_2/\alpha_1 < 1$, then

$$\mathcal{J}_1^\gamma(t,x,y) \asymp \left(1 \wedge \frac{\delta_*^\Phi(x,y)}{\phi(t^{-1})^{-2}}\right)^\gamma \frac{1}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)} + \mathbf{1}_{\{\delta_*^\Phi(x,y)^{1/2} \le \phi(t^{-1})^{-1}\}} w(t)$$
$$\times \left(1 \wedge \frac{\delta_*^\Phi(x,y)}{\Phi(\rho(x,y))^2}\right)^\gamma \left(\frac{\Phi(\rho(x,y))}{V(x,\rho(x,y))} \vee \frac{\Phi(\delta_\wedge(x,y))}{V(x,\delta_\wedge(x,y))}\right).$$

(f) If $\alpha_1 = \alpha_2 = d_1 = d_2$, then

$$\mathcal{J}_1^\gamma(t,x,y) \asymp \left(1 \wedge \frac{\delta_*(x,y)^{\alpha_1}}{\phi(t^{-1})^{-2}}\right)^\gamma \phi(t^{-1})$$
$$+ w(t)\left(1 \wedge \frac{\delta_*(x,y)}{\rho(x,y)^2}\right)^{\alpha_1\gamma} \left(1 + \log^+\left(\frac{2\phi(t^{-1})^{-1} \wedge 2\delta_\wedge(x,y)^{\alpha_1}}{\rho(x,y)^{\alpha_1}}\right)\right).$$

(g) If $1 < d_1/\alpha_2$, then

$$\mathcal{J}_1^\gamma(t,x,y) \asymp \left(1 \wedge \frac{\delta_*^\Phi(x,y)}{\phi(t^{-1})^{-2}}\right)^\gamma \frac{1}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)}$$
$$+ w(t)\left(1 \wedge \frac{\delta_*^\Phi(x,y)}{\Phi(\rho(x,y))^2}\right)^\gamma \frac{\Phi(\rho(x,y))}{V(x,\rho(x,y))}.$$

Proof. See Appendix. □

Remark 1.23. We can obtain closed forms of $\mathcal{J}_2^\gamma$ from closed forms of $\mathcal{J}_1^\gamma$ and $\mathcal{J}_1^0$. Indeed, for every fixed $T > 0$, we can see that $\mathcal{J}_2^\gamma(t,x,y) \asymp \mathcal{J}_1^\gamma(t,x,y)$ for all $\gamma \in [0,1)$ and $(t,x,y) \in (0,T] \times D \times D$. Moreover, for all large t such that $\Phi(1)\phi(t^{-1}) \le 1/(8e^2)$,

$$\int_{\Phi(\rho(x,y))}^{1/(2e^2\phi(t^{-1}))} \frac{a_2^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))} dr$$
$$\asymp a_1^\gamma(1,x,y) \int_{2\Phi(1)\vee\Phi(\rho(x,y))}^{1/(2e^2\phi(t^{-1}))} \frac{1}{V(x,\Phi^{-1}(r))} dr + \mathbf{1}_{\{\rho(x,y)\le 1\}} \int_{\Phi(\rho(x,y))}^{2\Phi(1)} \frac{a_1^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))} dr.$$

Add an isolated point y_0 to D and define $\rho(x,y_0) = 1$ for all $x \in D$. By the above observation, we have that for any fixed $T > 0$, the following comparison holds for all $\gamma \in [0,1)$ and $(t,x,y) \in [T,\infty) \times D \times D$:

$$\mathcal{J}_2^\gamma(t,x,y)$$
$$\asymp \big(1 \wedge \Phi(\delta_D(x))\big)^\gamma \big(1 \wedge \Phi(\delta_D(y))\big)^\gamma \mathcal{J}_1^0(t,x,y') + \mathbf{1}_{\{\rho(x,y)\le 1\}} \mathcal{J}_1^\gamma([\phi^{-1}(4^{-1}e^{-2})]^{-1}, x, y),$$

where $y' = y$ if $\rho(x,y) \ge 1$ and $y' = y_0$ if $\rho(x,y) < 1$. (cf. Theorem 1.6(v)(a).)

2. Estimates for subordinator

Throughout this section, we always assume that S be the subordinator whose Laplace exponent has the following representation with a function w satisfying condition **(Ker.)**:

$$\phi(\lambda) = -\log \mathbb{E}\big[\exp(-\lambda S_1)\big] = \int_0^\infty (1 - e^{-\lambda s})(-dw(s)) \quad \text{for all } \lambda \geq 0.$$

Following [21], we let

$$H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda) \quad \text{for all } \lambda \geq 0.$$

In [21], Naresh C. Jain and William E. Pruitt studied asymptotic properties of lower tail probabilities of subordinators, $\mathbb{P}(S_r \leq t)$, in terms of the function H. Then, in [29], Ante Mimica obtained estimates for upper tail probabilities, $\mathbb{P}(S_r \geq t)$, in terms of the function H as well. Those estimates were crucial ingredients in [8] to establish the estimates for the fundamental solution $p(t, x, y)$.

In this section, we will improve the results in [29] and obtain tail probability estimates in terms of the tail measure w instead of the function H. This allows us to get estimates for the fundamental solution in more general situations.

2.1. General estimates for subordinator

Lemma 2.1. *(i) For every $\lambda > 0$, we have*

$$\phi(\lambda) \asymp \lambda \int_0^{1/\lambda} w(s)ds \quad \text{and} \quad H(\lambda) \asymp \lambda^2 \int_0^{1/\lambda} sw(s)ds.$$

(ii) If w satisfies $\mathbf{LS^0}(-\alpha_1, c_0)$ (resp. $\mathbf{LS^\infty}(-\alpha_1, c_0)$) for some $\alpha_1 \geq 0$ and $c_0 > 0$, then ϕ satisfies $\mathbf{US^\infty}(\alpha_1 \wedge 1, 1/c_0)$ (resp. $\mathbf{US^0}(\alpha_1 \wedge 1, 1/(2c_0))$) and H satisfies $\mathbf{US^\infty}(\alpha_1 \wedge 2, 1/c_0)$ (resp. $\mathbf{US^0}(\alpha_1 \wedge 2, 1/(2c_0))$).
(iii) If w satisfies $\mathbf{LS^0}(-\alpha_1, c_0)$ (resp. $\mathbf{LS^\infty}(-\alpha_1, c_0)$) for some $\alpha_1 < 2$ and $c_0 > 0$, then

$$w(s) \asymp H(s^{-1}), \qquad \text{for all } 0 < s \leq c_0. \quad (\text{resp. for all } s \geq 2c_0.)$$

(iv) If ϕ satisfies $\mathbf{WS^\infty}(\alpha_2, \alpha_1, c_0)$ (resp. $\mathbf{WS^0}(\alpha_2, \alpha_1, c_0)$) for some $0 \leq \alpha_2 \leq \alpha_1 < 1$, $c_0 > 0$, or H satisfies $\mathbf{WS^\infty}(\alpha_2, \alpha_1, c_0)$ (resp. $\mathbf{WS^0}(\alpha_2, \alpha_1, c_0)$) for some $0 \leq \alpha_1 \leq \alpha_2 < 2$, $c_0 > 0$, then there exists $c_1 > 0$ such that w satisfies $\mathbf{WS^0}(-\alpha_1, -\alpha_2, c_1)$. (resp. $\mathbf{WS^\infty}(-\alpha_1, -\alpha_2, c_1)$.)

Proof. (i) By the integration by parts and Fubini's theorem,

$$\frac{\phi(\lambda)}{\lambda} = \int_0^\infty \int_0^s e^{-\lambda u} du(-dw(s)) = \int_0^{1/\lambda} e^{-\lambda u} w(u)du + \int_{1/\lambda}^\infty e^{-\lambda u} w(u)du =: I_1 + I_2.$$

First, we see that $I_1 \asymp \int_0^{1/\lambda} w(s)ds$. Moreover, since w is non-increasing,

$$I_2 \leq w(1/\lambda) \int_{1/\lambda}^\infty e^{-\lambda u} du = \frac{w(1/\lambda)}{e\lambda} \leq \int_{1/(2\lambda)}^{1/\lambda} w(s)ds \leq \int_0^{1/\lambda} w(s)ds.$$

Hence, the first claim holds. On the other hand, note that by the definition of H,

$$\frac{H(\lambda)}{\lambda^2} = -(\lambda^{-1}\phi(\lambda))' = \int_0^\infty ue^{-\lambda u} w(u)du.$$

Then, we can deduce that $H(\lambda) \asymp \lambda^2 \int_0^{1/\lambda} sw(s)ds$ by a similar argument.

(ii) First, assume that w satisfies $\mathbf{LS^0}(-\alpha_1, c_0)$. By (i),

$$\phi(\kappa\lambda) \asymp \kappa\lambda \int_0^{1/(\kappa\lambda)} w(s)ds = \lambda \int_0^{1/\lambda} \frac{w(s/\kappa)}{w(s)} w(s)ds \quad \text{for all } \kappa \geq 1,\ \lambda \geq 1/c_0. \tag{2.1}$$

Besides, by the assumption, there exists $c_2 > 0$ such that $w(s/\kappa)/w(s) \leq c_2\kappa^{\alpha_1}$ for all $\kappa \geq 1,\ \lambda \geq 1/c_0$. Thus, we deduce that ϕ satisfies $\mathbf{US^\infty}(\alpha_1, 1/c_0)$ from (2.1) and (i). Since ϕ always satisfy $\mathbf{WS}(0, 1)$, we get the result for ϕ. Then, by a similar argument and the fact that H always satisfy $\mathbf{WS}(0, 2)$, we can also deduce that H satisfies $\mathbf{US^\infty}(\alpha_1 \wedge 2, 1/c_0)$.

Now, assume that w satisfies $\mathbf{LS^\infty}(-\alpha_1, c_0)$. Then, by (i),

$$\phi(\lambda) \asymp \lambda \int_0^{1/\lambda} w(s)ds \asymp \lambda \int_{c_0}^{1/\lambda} w(s)ds \quad \text{for all } 0 < \lambda \leq 1/(2c_0). \tag{2.2}$$

The second comparison holds since $\int_0^{c_0} w(s)ds \leq c_3 \leq c_4 \int_{c_0}^{2c_0} w(s)ds \leq c_4 \int_{c_0}^{1/\lambda} w(s)ds$ for all $\lambda \leq 1/(2c_0)$. Then, we get that for all $\kappa \geq 1$ and $0 < \lambda \leq 1/(2\kappa c_0)$,

$$\frac{\phi(\kappa\lambda)}{\phi(\lambda)} \leq c_5\kappa \frac{\int_{c_0}^{1/(\kappa\lambda)} w(s)ds}{\int_{c_0}^{1/\lambda} w(s)ds} = c_5 \frac{\int_{c_0\kappa}^{1/\lambda} w(s/\kappa)ds}{\int_{c_0}^{1/\lambda} w(s)ds} \leq c_5 \frac{\int_{c_0\kappa}^{1/\lambda} w(s/\kappa)ds}{\int_{c_0\kappa}^{1/\lambda} w(s)ds} \leq c_6\kappa^{\alpha_1},$$

which proves that ϕ satisfies $\mathbf{US^0}(\alpha_1 \wedge 1, 1/(2c_0))$. The proof for the assertion on H is similar.

(iii) We first assume that w satisfies $\mathbf{LS^0}(-\alpha_1, c_0)$ for $\alpha_1 < 2$ and $c_0 > 0$. Then, by (i),

$$H(s^{-1}) \asymp s^{-2}w(s) \int_0^s u\frac{w(u)}{w(s)}du \asymp s^{-2}w(s) \int_0^s s^{\alpha_1}u^{1-\alpha_1}du \asymp w(s) \quad \text{for all } 0 < s \leq c_0.$$

Next, assume that w satisfies $\mathbf{LS^\infty}(-\alpha_1, c_0)$ for $\alpha_1 < 2$ and $c_0 > 0$. Then, by the same arguments as the ones given in the proof for (ii) and (i), we get

$$H(s^{-1}) \asymp s^{-2}w(s) \int_{c_0}^s u\frac{w(u)}{w(s)}du \asymp s^{-2}w(s) \int_{c_0}^s s^{\alpha_1}u^{1-\alpha_1}du \asymp w(s) \quad \text{for all } s \geq 2c_0.$$

(iv) Suppose that there is $c_0 > 0$ such that ϕ satisfies $\mathbf{WS^\infty}(\alpha_2, \alpha_1, c_0)$ for some constants $0 \leq \alpha_2 \leq \alpha_1 < 1$ or H satisfies $\mathbf{WS^\infty}(\alpha_2, \alpha_1, c_0)$ for some constants $0 \leq \alpha_2 \leq \alpha_1 < 2$. In either case, by [29, Lemma 2.6 and Proposition 2.9], H satisfies $\mathbf{WS^\infty}(\alpha_2, \alpha_1, c_0)$ and there exists a constant $c_1 > 0$ such that $w(s) \asymp H(s^{-1})$ for $0 < s < c_1$. Then, the result follows.

The cases when either of ϕ and H satisfies the weak scaling properties at the origin can be proved by similar arguments. □

Lemma 2.2. *Suppose that there exist $\delta > 0$ and $t_0 > 0$ such that w satisfies $\mathbf{LS^0}(-\delta, t_0)$. Then, there exists a constant $c_1 > 0$ such that for every $t \in (0, t_0]$,*

$$H(t^{-1})^{\delta+1} \leq c_1\phi(t^{-1})^\delta w(t).$$

Similarly, if there exist $\delta' > 0$ and $t_0' > 0$ such that w satisfies $\mathbf{LS^\infty}(-\delta', t_0')$, then there exists a constant $c_2 > 0$ such that for every $t \in [t_0', \infty)$,

$$H(t^{-1})^{\delta+1} \leq c_2\phi(t^{-1})^{\delta'} w(t).$$

Proof. Since the proofs are similar, we only give the proof for the first assertion. If $\delta < 2$, then by Lemma 2.1(iii), we have that for all $t \in (0, t_0]$,

$$H(t^{-1})^{\delta+1} \leq cH(t^{-1})^\delta w(t) \leq c\phi(t^{-1})^\delta w(t).$$

Now, assume that $\delta \geq 2$. By Lemma 2.1(i) and Hölder's inequality, for every $t \in (0, t_0]$,

$$H(t^{-1}) \leq ct^{-2} \int_0^t sw(s)ds \leq ct^{-2} \left(\int_0^t w(s)ds \right)^{1-1/(\delta+1)} \left(\int_0^t s^{\delta+1} w(s)ds \right)^{1/(\delta+1)}$$
$$\leq ct^{-2} \big(t\phi(t^{-1})\big)^{1-1/(\delta+1)} \big(t^{\delta+2} w(t)\big)^{1/(\delta+1)} = c\phi(t^{-1})^{1-1/(\delta+1)} w(t)^{1/(\delta+1)}.$$

We used Lemma 2.1(i) and [4, 2.12.16] in the third inequality. □

Lemma 2.3. *Suppose that there exist $\delta > 0$ and $t_0 > 0$ such that w satisfies $\mathbf{LS}^\infty(-\delta, t_0)$. Then, there exists a constant $c_1 > 0$ such that for every $t \in [t_0, \infty)$,*

$$\phi(t^{-1})^{\delta+1} \leq c_1 w(t).$$

Proof. We first assume that $\int_{1/(2t_0)}^\infty w(s)ds < \infty$. By Lemma 2.1(i), we have that $\phi(t^{-1}) \asymp t^{-1}$ for all $t \geq t_0$. Then, by Potter's theorem, (see, [4, Theorem 1.5.6],) for all $t \geq t_0$,

$$\phi(t^{-1})^{\delta+1} \leq ct^{-\delta-1} \leq cw(t).$$

Now, assume that $\int_{t_0/2}^\infty w(s)ds = \infty$. In this case, by Lemma 2.1(i), $\phi(t^{-1}) \asymp t^{-1} \int_{t_0/2}^t w(s)ds$ for all $t \geq t_0$. We also have that by [4, 2.12.16], $w(t) \asymp t^{-\delta-1} \int_{t_0/2}^t s^\delta w(s)ds$ for all $t \geq t_0$. Then, by l'Hospital's rule and the fact that w is non-increasing, we get

$$\limsup_{t\to\infty} \frac{w(t)}{\phi(t^{-1})^{\delta+1}} \leq c \limsup_{t\to\infty} \frac{t^{-\delta-1} \int_{t_0/2}^t s^\delta w(s)ds}{\left(t^{-1} \int_{t_0/2}^t w(s)ds\right)^{\delta+1}} = c \limsup_{t\to\infty} \frac{\int_{t_0/2}^t s^\delta w(s)ds}{\left(\int_{t_0/2}^t w(s)ds\right)^{\delta+1}}$$
$$\leq c \limsup_{t\to\infty} \frac{t^\delta w(t)}{w(t)\left(\int_{t_0/2}^t w(s)ds\right)^\delta} \leq c \limsup_{t\to\infty} \frac{t^\delta}{\big(tw(t_0/2)\big)^\delta} = c. \quad \square$$

For $s > 0$, we define

$$b(s) := s\phi'(H^{-1}(1/s)).$$

Lemma 2.4. *(i) b is strictly increasing on $(0, \infty)$, $\lim_{s\to0} b(s) = 0$ and $\lim_{s\to\infty} b(s) = \infty$.*
(ii) For every $s > 0$, we have that

$$\phi(s^{-1})^{-1} \leq b^{-1}(s) \leq \frac{e^2 - e}{e - 2} \phi(s^{-1})^{-1}.$$

Proof. (i) Since H is strictly increasing on $(0, \infty)$ and ϕ' is strictly decreasing on $(0, \infty)$, b is strictly increasing on $(0, \infty)$. Moreover, we have that $\lim_{s\to0} b(s) \leq \phi'(H^{-1}(1)) \lim_{s\to0} s = 0$ and $\lim_{s\to\infty} b(s) \geq \phi'(H^{-1}(1)) \lim_{s\to\infty} s = \infty$.

(ii) From the concavity of ϕ, since $\phi^{-1}(\lambda) \leq H^{-1}(\lambda)$, we have that for all $s > 0$,

$$b(s) \leq \frac{\phi^{-1}(s^{-1})}{\phi(\phi^{-1}(s^{-1}))} \frac{\phi(H^{-1}(s^{-1}))}{H^{-1}(s^{-1})} \frac{1}{\phi^{-1}(s^{-1})} \leq \frac{1}{\phi^{-1}(s^{-1})}.$$

Therefore, we get $b^{-1}(s) \geq \phi(s^{-1})^{-1}$ since both ϕ and b are strictly increasing.

On the other hand, we note that from the definition of ϕ and H, for every $\lambda > 0$,

$$\phi(\lambda) \leq \lambda \int_{(0,1/\lambda]} u(-dw(u)) + w(1/\lambda),$$

$$\phi'(\lambda) \geq e^{-1} \int_{(0,1/\lambda]} u(-dw(u)), \qquad H(\lambda) \geq e^{-1}(e-2)w(1/\lambda).$$

Let $a := (e^2 - e)/(e-2)$. Then, for all $s > 0$,

$$\begin{aligned}
b\big(a\phi(s^{-1})^{-1}\big) &= a\phi(s^{-1})^{-1}\phi'\big(H^{-1}\big(\phi(s^{-1})/a\big)\big) \\
&\geq ae^{-1}\phi(s^{-1})^{-1} \int_{(0,[H^{-1}(\phi(s^{-1})/a)]^{-1}]} u(-dw(u)) \\
&\geq ae^{-1}\phi(s^{-1})^{-1}\left[\int_{(0,s]} u(-dw(u)) + s\int_{(s,[H^{-1}(\phi(s^{-1})/a)]^{-1}]} (-dw(u))\right] \\
&= ae^{-1}\phi(s^{-1})^{-1}\left[\int_{(0,s]} u(-dw(u)) + sw(s) - sw\big([H^{-1}(\phi(s^{-1})/a)]^{-1}\big)\right] \\
&\geq ae^{-1}\phi(s^{-1})^{-1}\left[s\phi(s^{-1}) - e(e-2)^{-1}sH\big(H^{-1}\big(\phi(s^{-1})/a\big)\big)\right] \\
&= ae^{-1}(1 - e(e-2)^{-1}a^{-1})s = s.
\end{aligned}$$

Again, since b is strictly increasing, we conclude that $b^{-1}(s) \leq a\phi(s^{-1})^{-1}$. $\square$

We will use Chebyshev's inequality in tail probability estimates several times. To applying Chebyshev's inequality for subordinators, we need the following lemma.

Lemma 2.5. *Assume that w is finitely supported, that is, there exists a constant $T > 0$ such that $w(T) = 0$. Then, for every $\lambda \in \mathbb{R}$, $r > 0$ and $n \in \{0\} \cup \mathbb{N}$, we have that*

$$\mathbb{E}[(S_r)^n e^{\lambda S_r}] = \frac{d^n}{d\lambda^n} \exp\left(r\int_{(0,T]} (e^{\lambda s} - 1)(-dw(s))\right).$$

Proof. Fix $r > 0$ and let $\xi(dt) := \mathbb{P}(S_r \in dt)$. For $z \in \mathbb{C}$, define

$$f(z) = \int_{[0,\infty)} e^{-zt}\xi(dt).$$

Then, it is well known that there exists the abscissa of convergence $\sigma_0 \in [-\infty, \infty]$ such that $f(z)$ converges for Re $z > \sigma_0$, diverges for Re $z < \sigma_0$ and has a singularity at σ_0. Moreover, $f(z)$ is analytic in the half-plane Re $z > \sigma_0$ so that for every $n \in \mathbb{N}$ and $x > \sigma_0$, it holds that

$$\frac{d^n}{dx^n} f(x) = (-1)^n \int_{[0,\infty)} t^n e^{-xt}\xi(dt). \tag{2.3}$$

(See, [32, p.37 and p.58] and [28].) On the other hand, we also have that for $\lambda > 0$,

$$f(\lambda) = \mathbb{E}\big[\exp(-\lambda S_r)\big] = \exp\big(-r\phi(\lambda)\big) = \exp\left(r\int_{(0,T]} (e^{\lambda s} - 1)(-dw(s))\right) =: g(\lambda).$$

Since w is finitely supported, the function $\lambda \mapsto g(\lambda)$ is a well-defined differentiable function on $\mathbb{R}$. If $\sigma_0 > -\infty$, then from the uniqueness of the analytic continuation, the function $g(\lambda)$ should have a singularity at $\lambda = \sigma_0$. Since there is no such singularity, we get $\sigma_0 = -\infty$. Then, the result follows from the definition of f and (2.3). $\square$

2.2. Tail probability estimates for subordinator

In this section, we study two tail probabilities $\mathbb{P}(S_r \geq t)$ and $\mathbb{P}(S_r \leq t)$ under mild assumption for w. We first give the general lower bounds for upper tail probability $\mathbb{P}(S_r \geq t)$ which are established in [29]. Note that these bounds hold for every subordinator.

Lemma 2.6. *For every $L > 0$, it holds that for all $r, t > 0$ satisfying $r\phi(t^{-1}) \leq L$,*

$$\mathbb{P}(S_r \geq t) \geq e^{-eL} r w(t).$$

Proof. Note that $r\phi(t^{-1}) \leq L$ implies that $rw(t) \leq er\phi(t^{-1}) \leq eL$. Thus, by [29, Proposition 2.5], for all $r, t > 0$ satisfying $r\phi(t^{-1}) \leq L$, we have that

$$\mathbb{P}(S_r \geq t) \geq 1 - e^{-rw(t)} \geq rw(t)e^{-rw(t)} \geq e^{-eL} rw(t). \quad \square$$

Now, we study the upper bounds for $\mathbb{P}(S_r \geq t)$.

Proposition 2.7. *Assume that condition* **(S.Poly.)**(t_s) *holds. Then, there exists a constant $c_1 > 0$ such that for all $r, t > 0$ satisfying $0 < t \leq t_s$ and $r\phi(t^{-1}) \leq 1/(4e^2)$,*

$$\mathbb{P}(S_r \geq t) \leq c_1 r w(t).$$

Proof. Fix $r, t > 0$ satisfying $0 < t \leq t_s$ and $r\phi(t^{-1}) \leq 1/(4e^2)$. Set

$$\mu^1 := \mathbf{1}_{(0,1/H^{-1}(1/r)]} \cdot (-dw), \quad \mu^2 := \mathbf{1}_{(1/H^{-1}(1/r),t]} \cdot (-dw), \quad \mu^3 := \mathbf{1}_{(t,\infty)} \cdot (-dw).$$

Let S^1, S^2 and S^3 be independent subordinators without drift and having Lévy measure μ^1, μ^2 and μ^3, respectively. Then, we have $S_r \leq S_r^1 + S_r^2 + S_r^3$ and hence

$$\mathbb{P}(S_r \geq t) \leq \mathbb{P}(S_r^1 + S_r^2 + S_r^3 \geq t) \leq \mathbb{P}(S_r^1 \geq t/2) + \mathbb{P}(S_r^2 \geq t/2) + \mathbb{P}(S_r^3 > 0). \tag{2.4}$$

First, since S^3 is a compounded Poisson process, $\mathbb{P}(S_r^3 > 0) = 1 - e^{-rw(t)} \leq rw(t)$.

Next, we note that by Lemma 2.4(ii), $t = b(b^{-1}(t)) \geq b(\phi(t^{-1})^{-1}) \geq b(4e^2 r) \geq 4e^2 b(r)$. By Chebyshev's inequality and Lemma 2.5, we have that for every $\lambda > 0$,

$$\begin{aligned}
\mathbb{P}\big(S_r^1 \geq t/2\big) &\leq \mathbb{E}\big[\exp\big(-\lambda t/2 + \lambda S_r^1\big)\big] \\
&= \exp\bigg(-\frac{\lambda t}{2} + r\int_{(0,1/H^{-1}(1/r)]} (e^{\lambda s} - 1)(-dw(s))\bigg) \\
&\leq \exp\bigg(-\frac{\lambda t}{2} + \lambda r e^{\lambda/H^{-1}(1/r)} \int_{(0,1/H^{-1}(1/r)]} s(-dw(s))\bigg) \\
&\leq \exp\bigg(-\frac{\lambda t}{2} + e\lambda b(r) e^{\lambda/H^{-1}(1/r)}\bigg).
\end{aligned}$$

Thus, by letting $\lambda = H^{-1}(1/r)$, we get

$$\mathbb{P}\big(S_r^1 \geq t/2\big) \leq \exp\big(-2^{-1} t H^{-1}(1/r) + e^2 b(r) H^{-1}(1/r)\big) \leq \exp\big(-4^{-1} t H^{-1}(1/r)\big).$$

Thirdly, let $f_0(s) := w(s)\mathbf{1}_{(0,t]}(s) + w(t)t^2 s^{-2}\mathbf{1}_{(t,\infty)}(s)$ for $s > 0$. Then, we see that f_0 is non-increasing and for every Borel set $A \subset \mathbb{R}$, it holds that

$$\mu^2(A) \leq w(\mathrm{dist}(0, A))\mathbf{1}_{(0,t]}(\mathrm{dist}(0, A)) \leq f_0(\mathrm{dist}(0, A)),$$

where $\mathrm{dist}(0, A) := \inf\{|y| : y \in A\}$. Moreover, since w satisfies $\mathbf{LS^0}(-\delta, t_s)$, for all $u, v > 0$,

$$\int_u^\infty f_0\big(v \vee y - \frac{y}{2}\big)\mu^2(dy) \le f_0(v/2)w(u) \le c_1 f_0(v)H(1/u).$$

Therefore, by [22, Proposition 1 and Lemma 9], we have that for every $x > 0$ and $\rho \in (0, x/3]$,

$$\mathbb{P}(S_r^2 \in [x-\rho, x+\rho]) \le c_2 r f_0(x/3).$$

It follows that

$$\begin{aligned}\mathbb{P}(S_r^2 \ge t/2) &\le \sum_{i=0}^\infty \mathbb{P}\big(S_r^2 \in [2^{i-1}t, 2\cdot 2^{i-1}t]\big)\\ &\le cr\sum_{i=0}^\infty f_0(2^{i-2}t) \le crw(t)\sum_{i=0}^\infty 2^{-2i} = crw(t).\end{aligned}$$

Combining the above inequalities, by (2.4) and Lemma 2.2, we deduce that

$$\begin{aligned}\mathbb{P}(S_r \ge t) &\le crw(t) + \exp\Big(-2(\delta+1)\cdot\frac{tH^{-1}(1/r)}{8(\delta+1)}\Big) \le crw(t) + \Big(\frac{1 \vee 8(\delta+1)}{1 \vee tH^{-1}(1/r)}\Big)^{2\delta+2}\\ &\le crw(t) + c\Big(\frac{H(t^{-1})}{H(H^{-1}(1/r))}\Big)^{\delta+1}\\ &\le crw(t) + cr\phi(t^{-1})^{-\delta}H(t^{-1})^{\delta+1} \le crw(t).\end{aligned}$$

In the second inequality, we used the fact that $e^x \ge x$ for all $x > 0$ and in the third inequality, we used the fact that $H(\lambda x) \le (1 \vee \lambda^2)H(x)$ for all $\lambda, x > 0$. Also, the fourth inequality holds since $r \le L\phi(t^{-1})^{-1}$. $\Box$

By the same argument, we also get analogous estimates for large time t.

Proposition 2.8. *Assume that condition* **(L.Poly.)** *holds. Then, for every $T > 0$, there exists a constant $c_1 > 0$ such that for all $r, t > 0$ satisfying $t \ge T$ and $r\phi(t^{-1}) \le 1/(4e^2)$,*

$$\mathbb{P}(S_r \ge t) \le c_1 r w(t).$$

Proof. Follow the proof of Proposition 2.7. The only difference occurs in the definition of f_0. In this case, we use $f_1(s) := \frac{e}{e-2}H(s^{-1})\mathbf{1}_{(0,T/2]}(s) + w(s)\mathbf{1}_{(T/2,\infty)}(s)$ instead of $f_0(s)$. $\Box$

Proposition 2.9. *Assume that condition* **(Sub.)($\boldsymbol{\beta}, \boldsymbol{\theta}$)** *holds. Then, for every $T > 0$, there exist constants $c_2 > 0$ and $L \in (0, 1]$ such that for all $r, t > 0$ satisfying $t \ge T$ and $rt^{-1} \le L$,*

$$\mathbb{P}(S_r \ge t) \le c_2 r \exp\big(-\frac{\theta}{2}t^\beta\big).$$

Proof. Fix $t \ge T$ and $r \in (0, Lt)$ where the constant $L \in (0, 1]$ will be chosen later. Let $\widehat{S}^1$ and $\widehat{S}^2$ be independent subordinators without drift and having Lévy measures

$$\widehat{\mu}^1 := \mathbf{1}_{(0,t]}\cdot(-dw) \quad \text{and} \quad \widehat{\mu}^2 := \mathbf{1}_{(t,\infty)}\cdot(-dw), \qquad \text{respectively.}$$

Then, since $S_r = \widehat{S}^1_r + \widehat{S}^2_r$, by condition **(Sub.)($\boldsymbol{\beta}, \boldsymbol{\theta}$)**, we have

$$\mathbb{P}(S_r \ge t) \le \mathbb{P}(\widehat{S}^1_r \ge t) + \mathbb{P}(\widehat{S}^2_r > 0) \le \mathbb{P}(\widehat{S}^1_r \ge t) + rw(t) \le \mathbb{P}(\widehat{S}^1_r \ge t) + cr\exp\big(-\theta t^\beta\big).$$

It remains to bound $\mathbb{P}(\widehat{S}^1_r \geq t)$. By Chebyshev's inequality and Lemma 2.5, for all $\lambda > 0$,

$$\mathbb{P}(\widehat{S}^1_r \geq t) \leq \mathbb{E}\big[t^{-1}\widehat{S}^1_r \exp\big(\lambda \widehat{S}^1_r - \lambda t\big)\big]$$
$$\leq t^{-1}e^{-\lambda t} r \Big(\int_{(0,t]} s e^{\lambda s}(-dw(s))\Big) \exp\Big(r\int_{(0,t]}(e^{\lambda s}-1)(-dw(s))\Big). \tag{2.5}$$

Note that by the integration by parts and condition **(Sub.)($\boldsymbol{\beta}, \boldsymbol{\theta}$)**, we get

$$\int_{(0,t]} s e^{\lambda s}(-dw(s)) \leq \int_{(0,t]} w(s)e^{\lambda s}ds + \lambda\int_{(0,t]} s w(s) e^{\lambda s} ds$$
$$\leq 2\lambda e^{\lambda}\int_{(0,1]} w(s)ds + c_0\int_{(1,t]}\exp\big(-\theta s^{\beta} + \lambda s\big)ds + c_0\lambda\int_{(1,t]} s\exp\big(-\theta s^{\beta}+\lambda s\big)ds,$$

and

$$\int_{(0,t]}(e^{\lambda s}-1)(-dw(s)) \leq \lambda e^{\lambda}\int_{(0,1]} w(s)ds + c_0\lambda\int_{(1,t]}\exp\big(-\theta s^{\beta}+\lambda s\big)ds.$$

Take $\lambda = 2\theta t^{\beta-1}/3 \in (0, 2\theta T^{\beta-1}/3]$. Then, since $s \mapsto -2\theta s^{\beta}/3 + \lambda s$ is a convex function,

$$\int_{(1,t]} s\exp\big(-\theta s^{\beta}+\lambda s\big)ds \leq \sup_{s\in(1,t]}\Big[-\frac{2\theta s^{\beta}}{3} + \lambda s\Big]\cdot\int_{(1,t]} s\exp\big(-\frac{\theta s^{\beta}}{3}\big)ds$$
$$\leq \big(-\frac{2\theta}{3} + \lambda - \frac{2\theta t^{\beta}}{3} + \lambda t\big)\int_{(1,t]} s\exp\big(-\frac{\theta s^{\beta}}{3}\big)ds \leq c.$$

Using this observation and the fact that $\int_{(0,1]} w(s)ds < \infty$, (2.5) implies that

$$\mathbb{P}(\widehat{S}^1_r \geq t) \leq c_3 t^{-1} r\exp\big(-\frac{2\theta}{3}t^{\beta} + c_4 r t^{\beta-1}\big),$$

for some constants $c_3, c_4 > 0$. Now, we choose $L = 1 \wedge (\theta/(6c_4))$. Then, we get

$$\mathbb{P}(\widehat{S}^1_r \geq t) \leq c_3 T^{-1} r\exp\big(-\frac{2\theta}{3}t^{\beta} + c_4 L t^{\beta}\big) \leq c_2 r\exp\big(-\frac{\theta}{2}t^{\beta}\big). \quad \square$$

When w decreases subexponentially ($0 < \beta < 1$), we obtain small time sharp upper bounds for $\mathbb{P}(S_r \geq t)$ which decrease with exactly the same rate as the bounds for w as $t \to \infty$.

Proposition 2.10. *Assume that condition* **(Sub.)($\boldsymbol{\beta}, \boldsymbol{\theta}$)** *holds with constant* $0 < \beta < 1$. *Then, for every fixed* $k > 0$ *and* $T > 0$, *there exist constants* $c_2 > 0$ *and* $L \in (0, 1]$ *such that for all* $r, t > 0$ *satisfying* $t \geq T$ *and* $rt^{-1} \leq L$,

$$\mathbb{P}(S_r \geq t) \leq c_2 r\exp\big(-\theta t^{\beta} + kr\big).$$

Proof. Let $\widetilde{S}^1$ and $\widetilde{S}^2$ be independent subordinators without drift and having Lévy measures

$$\widetilde{\mu}^1 := \mathbf{1}_{(0,t/2]}\cdot(-dw) \quad \text{and} \quad \widetilde{\mu}^2 := \mathbf{1}_{(t/2,\infty)}\cdot(-dw), \qquad \text{respectively.}$$

Then, since $S_r = \widetilde{S}^1_r + \widetilde{S}^2_r$, we get

$$\mathbb{P}(S_r \geq t) = \int_0^{\infty}\mathbb{P}(\widetilde{S}^2_r \geq t-u)\mathbb{P}(\widetilde{S}^1_r \in du)$$
$$\leq \mathbb{P}(\widetilde{S}^2_r \geq t - T/2) + \int_{T/2}^{t-T/2}\mathbb{P}(\widetilde{S}^2_r \geq t-u)\mathbb{P}(\widetilde{S}^1_r \in du) + \mathbb{P}(\widetilde{S}^1_r \geq t - T/2). \tag{2.6}$$

By Chebyshev's inequality, Lemma 2.5 and the integration by parts, for $u > 0$ and $\lambda > 0$,

$$\begin{aligned}\mathbb{P}(\widetilde{S}_r^1 \ge u) &\le \mathbb{E}\big[u^{-2}(\widetilde{S}_r^1)^2 \exp\big(-\lambda u + \lambda \widetilde{S}_r^1\big)\big]\\ &= u^{-2}\left[r\int_{(0,t/2]} s^2 e^{\lambda s}(-dw(s)) + \left(r\int_{(0,t/2]} s e^{\lambda s}(-dw(s))\right)^2\right]\\ &\quad \times \exp\left(-\lambda u + r\int_{(0,t/2]}(e^{\lambda s}-1)(-dw(s))\right)\\ &\le u^{-2}\left[r\int_0^{t/2}(2+\lambda s)s e^{\lambda s} w(s)ds + \left(r\int_0^{t/2}(1+\lambda s)e^{\lambda s} w(s)ds\right)^2\right]\\ &\quad \times \exp\left(-\lambda u + \lambda r\int_0^{t/2} e^{\lambda s} w(s)ds\right).\end{aligned}$$

Take $\lambda = \theta t^{\beta-1} \in (0, \theta T^{\beta-1}]$. Then, for all $1 \le s \le t/2$, we have that $\lambda s \le \theta t^{\beta-1}(t/2)^{1-\beta}s^\beta \le 2^{\beta-1}\theta s^\beta$. It follows that

$$\begin{aligned}&\int_0^{t/2}(2+\lambda s)s e^{\lambda s} w(s)ds\\ \le&(2+\lambda)e^{\lambda}\int_0^1 w(s)ds + c_0(2+\lambda)\int_1^{t/2} s^2\exp\big(2^{\beta-1}\theta s^\beta - \theta s^\beta\big)ds\\ \le& c + c\int_1^\infty s^2 \exp\big(-\theta(1-2^{\beta-1})s^\beta\big)ds \le c_4,\end{aligned}$$

where the constant $c_4 > 0$ is independent of $t \in [T, \infty)$. By similar calculations, by taking c_4 larger, we may assume that

$$\int_0^{t/2}(1+\lambda s)e^{\lambda s} w(s)ds \le c_4 \quad \text{and} \quad \int_0^{t/2} e^{\lambda s} w(s)ds \le c_4.$$

Therefore, we have that for every $u > 0$,

$$\mathbb{P}(\widetilde{S}_r^1 \ge u) \le (c_4 + c_4^2)u^{-2}(r + r^2)\exp\big(-\theta t^{\beta-1}u + \theta c_4 t^{\beta-1} r\big).$$

In particular,

$$\begin{aligned}\mathbb{P}(\widetilde{S}_r^1 \ge t - T/2) &\le c t^{-2}(r+r^2)\exp\big(-\theta t^\beta + \theta t^{\beta-1}T/2 + \theta c_4 t^{\beta-1} r\big)\\ &\le cT^{-2}\exp\big(\theta T^\beta/2\big) r \exp\big(-\theta t^\beta + (\theta c_4 t^{\beta-1} + k/2)r\big).\end{aligned}$$

On the other hand, note that $\widetilde{S}_r^2 = \sum_{i=1}^{N(r)} D_i$ where $N(r)$ is a Poisson process with rate $w(t/2)$ and D_i are i.i.d. random variables with distribution $\mathbb{P}(D_i > u) = w\big(u \vee (t/2)\big)/w\big(t/2\big)$. Thus, for every $0 < u < t$,

$$\begin{aligned}\mathbb{P}(\widetilde{S}_r^2 \ge u) &\le \mathbb{P}(N(r) = 1,\ D_1 \ge u) + \mathbb{P}(N(r) \ge 2)\\ &\le r w(u \vee (t/2)) + 1 - e^{-rw(t/2)} - rw(t/2)e^{-rw(t/2)}\\ &\le cr\exp\big(-\theta u^\beta\big) + r^2 w(t/2)^2 \le cr\exp\big(-\theta u^\beta\big) + cLrt\exp\big(-\theta 2^{1-\beta}t^\beta\big)\\ &\le cr\exp\big(-\theta u^\beta\big) + cLr\exp\big(-\theta t^\beta\big).\end{aligned}$$

It follows that

$$\mathbb{P}(\widetilde{S}_r^2 \ge t - T/2) \le cr\exp\big(-c_1(t-T/2)^\beta\big) \le cr\exp\big(-c_1 t^\beta\big).$$

The second inequality holds since $t^\beta - (t - T/2)^\beta \le (T/2)^\beta$.

Using the above inequalities, by (2.6) and the integration by parts, we obtain

$$\begin{aligned}\mathbb{P}(S_r \ge t) &\le cr\exp\big(-\theta t^\beta + (\theta c_4 t^{\beta-1} + \frac{k}{2})r\big) + cr\int_{T/2}^t \exp\big(-\theta(t-u)^\beta\big)\mathbb{P}(\widetilde{S}^1_r \in du)\\ &\le cr\exp\big(-\theta t^\beta + (\theta c_4 t^{\beta-1} + \frac{k}{2})r\big) + cr\int_{T/2}^t \mathbb{P}(\widetilde{S}^1_r \ge u)(t-u)^{\beta-1}\exp\big(-\theta(t-u)^\beta\big)du\\ &\le cr\exp\big(-\theta t^\beta + (\theta c_4 t^{\beta-1} + \frac{k}{2})r\big)\Big(1 + c(T/2)^{\beta-1}\int_{T/2}^t u^{-2}\exp\big(-f(u)\big)du\Big),\end{aligned}$$

where $f(u) := \theta(t-u)^\beta + \theta t^{\beta-1}u - \theta t^\beta$. Observe that

$$f'(u) = -\beta\theta(t-u)^{\beta-1} + \theta t^{\beta-1} = -\theta t^{\beta-1}(t-u)^{\beta-1}(\beta t^{1-\beta} - (t-u)^{1-\beta}).$$

Hence, f is decreasing on $(0, (1-\beta^{1/(1-\beta)})t)$ and increasing on $((1-\beta^{1/(1-\beta)})t, t)$. Since $f(0) = f(t) = 0$, we deduce that $f(u) \le 0$ for $u \in (0,t)$ and hence $\int_{T/2}^t u^{-2}\exp\big(-f(u)\big)du \le \int_{T/2}^\infty u^{-2}du \le c$. It follows that

$$\mathbb{P}(S_r \ge t) \le cr\exp\big(-\theta t^\beta + (\theta c_4 t^{\beta-1} + \frac{k}{2})r\big).$$

Hence, if $t \ge (k/(2\theta c_4))^{-1/(1-\beta)} =: c_5$, we are done. Moreover, if $t < c_5$, then we get

$$\exp\big((\theta c_4 t^{\beta-1} + \frac{k}{2})r\big) \le \exp\big(\theta c_4 t^\beta + \frac{k}{2}t\big) \le c,$$

since $r \le Lt \le t$. This completes the proof. □

Here, we state the estimates on lower tail probabilities $\mathbb{P}(S_r \le t)$ when r is large enough compared to $b^{-1}(t)$, which are established in [21].

Lemma 2.11 (*[21, Lemma 5.2]*). *For every $N > 0$, there exist constants $c_1, c_2 > 0$ such that*

$$c_1\exp\big(-c_2 rH((\phi')^{-1}(t/r))\big) \le \mathbb{P}(S_r \le t) \le \exp\big(-rH((\phi')^{-1}(t/r))\big),$$

for all $r, t > 0$ satisfying $r \ge Nb^{-1}(t)$.

Proof. If $N \ge 1$, then $r \ge Nb^{-1}(t)$ implies that $rH((\phi')^{-1}(t/r)) \ge rH((\phi')^{-1}(b(r)/r)) = 1$ and hence the result follows from [21, Lemma 5.2]. Suppose that $N \in (0,1)$. Since $r \mapsto S_r$ is strictly increasing almost surely, we deduce that for all $r \in (Nb^{-1}(t), b^{-1}(t)]$,

$$\mathbb{P}(S_r \le t) \ge \mathbb{P}(S_{b^{-1}(t)} \le t) \ge c \ge c\exp\big(-c_2 rH((\phi')^{-1}(t/r))\big). \quad \square$$

Corollary 2.12. *If condition* **(S.Poly.)**(t_s) *holds, then there exist constants $N > \varepsilon_1 > 0$ such that for all $t \in (0, t_s]$, it holds that*

$$\mathbb{P}(S_{N/\phi(t^{-1})} \ge t) - \mathbb{P}(S_{\varepsilon_1/\phi(t^{-1})} \ge t) \ge 1/4. \tag{2.7}$$

On the other hand, if either of the conditions **(L.Poly.)** *or* **(Sub.)**$(\boldsymbol{\beta}, \boldsymbol{\theta})$ *holds, then for every fixed $T > 0$, there exist constants $N > \varepsilon_1 > 0$ such that (2.7) holds for all $t \in [T, \infty)$.*

Proof. By Lemmas 2.11 and 2.4(ii), there exists a constant $N > 0$ such that for all $t > 0$, $\mathbb{P}(S_{N/\phi(t^{-1})} < t) \le 1/4$ and hence $\mathbb{P}(S_{N/\phi(t^{-1})} \ge t) \ge 3/4$. On the other hand, by Proposition 2.7 (resp. Proposition 2.8 or Proposition 2.9) and the facts that $\phi(t^{-1}) \asymp t^{-1}$ for all $t \ge T$ under condition **(Sub.)**$(\boldsymbol{\beta}, \boldsymbol{\theta})$ and $\phi(t^{-1}) \ge e^{-1}w(t)$ for all $t > 0$, we can find a constant $\varepsilon_1 > 0$ such that $\mathbb{P}(S_{\varepsilon_1/\phi(t^{-1})} \ge t) \le 1/2$ for all $t \in (0, t_s]$ (resp. for all $t \in [T, \infty)$). □

By Corollary 2.12, we get a priori estimates for the fundamental solution $p(t,x,y)$.

Corollary 2.13. *Assume that condition* **(S.Poly.)**(t_s) *holds. Let* $p(t,x,y)$ *be given by* (1.5). *Then, there exist constants* $N > \varepsilon_1 > 0$ *and* $c > 0$ *such that for all* $t \in (0, t_s]$,

$$p(t,x,y) \geq c \inf_{r \in (\varepsilon_1/\phi(t^{-1}), N/\phi(t^{-1}))} q(r,x,y). \tag{2.8}$$

On the other hand, if either of the conditions **(L.Poly.)** *or* **(Sub.)**$(\boldsymbol{\beta},\boldsymbol{\theta})$ *holds, then for every fixed* $T > 0$, *there exist constants* $N > \varepsilon_1 > 0$ *such that* (2.8) *holds for all* $t \in [T, \infty)$.

2.3. Estimates for truncated subordinator

In this subsection, we obtain tail probability estimates when the kernel w is finitely supported. Throughout this subsection, we always assume that condition **(Trunc.)**(t_f) holds. An example of such kernel is given by $w(t) := \frac{1}{\Gamma(1-\beta)}(t^{-\beta} - 1)\mathbf{1}_{(0,1]}(t)$ $(0 < \beta < 1)$. Those integral kernels are used in the fractional-time derivative whose value at time t depends only on the finite range of the past. (See, [6, Example 2.5].)

Proposition 2.14. *There exists a constant* $r_0 > 0$ *such that for all* $r \in (0, r_0]$ *and* $t \geq t_f/2$,

$$\mathbb{P}(S_r \geq t) \simeq \big[r + (nt_s - t)^n\big] r^n \exp\big(-ct\log t\big),$$

where $n := \lfloor t/t_f \rfloor + 1$.

Proof. Take r_0 small enough so that $r\phi(r^{-1}) \leq 1/(4e^2)$ and $r \leq t_f/6$ for all $r \in (0, r_0]$. Since $\lim_{r\to 0} r\phi(r^{-1}) = 0$, we can always find such constant r_0. Then, fix $r \in (0, r_0]$ and $t \geq t_f/2$. Note that since $n = \lfloor t/t_f \rfloor + 1$, we have $\big((n-1) \vee 1/2\big)t_f \leq t < nt_f$.

(Lower bound) Let U^1 and U^2 be the driftless subordinators with Lévy measures

$$\nu_1 := \mathbf{1}_{(t/(n+1),\infty)} \cdot (-dw) \quad \text{and} \quad \nu_2 := \mathbf{1}_{(t/n,\infty)} \cdot (-dw), \qquad \text{respectively.}$$

Observe that both U^1 and U^2 are compounded Poisson processes and their jump sizes are at least bigger than $t/(n+1)$ and t/n, respectively. Since $S_r \geq U_r^1 \geq U_r^2$, it follows that

$$\begin{aligned}
&2\mathbb{P}(S_r \geq t) \geq \mathbb{P}(U_r^1 \geq t) + \mathbb{P}(U_r^2 \geq t) \\
&\geq \mathbb{P}\big(U^1 \text{ jumps } (n+1) \text{ times before time } r\big) + \mathbb{P}\big(U^2 \text{ jumps } n \text{ times before time } r\big) \\
&\geq \exp\big(-rw(t/(n+1))\big)\frac{\big(rw(t/(n+1))\big)^{n+1}}{(n+1)!} + \exp\big(-rw(t/n)\big)\frac{\big(rw(t/n)\big)^n}{n!}.
\end{aligned} \tag{2.9}$$

Since $s \mapsto w(s)$ is non-increasing, we have $w(t/(n+1)) \leq w(t_f/4)$ and $w(t/n) \leq w(t_f/2)$. Moreover, by condition **(Trunc.)**(t_f)(i) and (ii),

$$\begin{aligned}
w(t/(n+1)) &\geq K^{-1}(t_f - t/(n+1)) \geq K^{-1}(n+1)^{-1}t_f, \\
w(t/n) &\geq K^{-1}(t_f - t/n) \geq K^{-1}n^{-1}(nt_f - t).
\end{aligned}$$

Using these observations, Stirling's formula and the fact that $n \asymp t$, by (2.9), we obtain

$$\begin{aligned}
\mathbb{P}(S_r \geq t) &\geq e^{-rw(t_f/4)}\frac{t_f^{n+1}r^{n+1}}{2K^{n+1}(n+1)^{n+1}(n+1)!} + e^{-rw(t_f/2)}\frac{(nt_f - t)^n r^n}{2K^n n^n n!} \\
&\gtrsim r^{n+1}\exp\big(-ct - 2n\log n\big) + (nt_f - t)^n r^n \exp\big(-ct - 2n\log n\big)
\end{aligned}$$

$$\gtrsim \big[r+(nt_f-t)^n\big]r^n\exp\big(-ct\log t\big).$$

(Upper bound) Let U^3 and U^4 be the driftless subordinators with Lévy measures

$$\nu_3 := \mathbf{1}_{(0,t/9]}\cdot(-dw) \quad \text{and} \quad \nu_4 := \mathbf{1}_{(t/9,\infty)}\cdot(-dw), \qquad \text{respectively.}$$

Then, we have that $S_r = U^3_r + U^4_r$ and $U^4_r = \sum_{i=1}^{P(r)} J_i$ where $P(r)$ is a Poisson process with rate $w(t_f/9)$ and J_i are i.i.d. random variables with distribution

$$F(u) := \mathbb{P}(J_i \ge u) = w(t_f/9)^{-1}w\big(u\vee(t_f/9)\big).$$

Hence, we get

$$\begin{aligned}\mathbb{P}(S_r\ge t) &= \sum_{j=0}^{\infty}\mathbb{P}\big(U^3_r+U^4_r\ge t,\, P(r)=j\big)\\ &\le \mathbb{P}\big(U^3_r\ge t\big)+\sum_{j=1}^{n}\mathbb{P}\big(U^3_r+U^4_r\ge t\,|\,P(r)=j\big)\mathbb{P}\big(P(r)=j\big)+\mathbb{P}\big(P(r)>n\big).\end{aligned}$$

First, by Stirling's formula, the definition of Poisson process and the fact that $n\asymp t$,

$$\mathbb{P}(P(r)>n)\le\frac{er^{n+1}}{(n+1)!}\simeq r^{n+1}\exp\big(-ct\log t\big).$$

Secondly, by Chebyshev's inequality and Lemma 2.5, for all $u>0$ and $\lambda>0$,

$$\begin{aligned}\mathbb{P}(U^3_r\ge u)&\le\mathbb{E}\big[\exp\big(-\lambda u+\lambda U^3_r\big)\big]=\exp\Big(-\lambda u+r\int_{(0,t_s/9]}(e^{\lambda s}-1)(-dw(s))\Big)\\ &\le\exp\Big(-\lambda u+\lambda e^{\lambda t_f/9}r\int_{(0,t_f/9]}s(-dw(s))\Big)\le\exp\big(-\lambda u+c_1\lambda e^{\lambda t_f/9}r\big).\end{aligned}$$

Hence, by taking $\lambda=9t_f^{-1}\log\big(u/(9c_1r)\big)$, we have that for every $u>0$,

$$\mathbb{P}(U^3_r\ge u)\le\exp\big(-8\lambda u/9\big)=\big(9c_1r/u\big)^{8u/t_f}. \tag{2.10}$$

In particular, since $t\ge\big((n-1)\vee 1/2\big)t_f$, we have that

$$\mathbb{P}(U^3_r\ge t)\le\big(9c_1r/t\big)^{8t/t_f}\lesssim r^{8t/t_f}\exp\big(-ct\log t\big)\le cr^{n+1}\exp\big(-ct\log t\big).$$

Moreover, we also have that

$$\begin{aligned}\sum_{j=1}^{n-2}&\mathbb{P}\big(U^3_r+U^4_r\ge t\,|\,P(r)=j\big)\mathbb{P}\big(P(r)=j\big)\le\sum_{j=1}^{n-2}\frac{r^jw(t_f/9)^j}{j!}\mathbb{P}\big(U^3_r\ge(n-1-j)t_f\big)\\ &\le\sum_{j=1}^{n-2}\frac{r^jw(t_f/9)^j}{j!}\left(\frac{9c_1r}{(n-j-1)t_f}\right)^{8(n-1-j)}\\ &\lesssim e^{ct}\sum_{j=1}^{n-2}r^{8(n-1)-7j}\frac{1}{j!(n-j-1)^{8(n-j-1)}}\\ &\lesssim r^{n+1}\sum_{j=1}^{n-2}\exp\big(ct-cj\log j-c(n-j-1)\log(n-j-1)\big)\\ &\lesssim r^{n+1}\exp\big(ct-c(n-1)\log(n-1)\big)\simeq r^{n+1}\exp\big(-ct\log t\big).\end{aligned}$$

The first inequality holds since the jump sizes of U_r^4 are at most t_f and the third line follows form Stirling's formula. Lastly, the fourth line holds by the facts that $4(a\log a + b\log b) \geq 2(a\vee b)\log(2(a\vee b)) \geq (a+b)\log(a+b)$ for all $a, b \geq 1$ satisfying $a \vee b \geq 2$ and that $n \asymp t$.

It remains to bound probabilities $\mathbb{P}\big(U_r^3 + U_r^4 \geq t,\, P(r) = j\big)$ for $j = n-1$ (when $n \geq 2$) and $j = n$. Observe that by Stirling's formula, we have

$$
\begin{aligned}
&\mathbb{P}\big(U_r^3 + U_r^4 \geq t | P(r) = n-1\big)\mathbb{P}\big(P(r) = n-1\big)\\
&\leq \frac{r^{n-1}w(t_f/9)^{n-1}}{(n-1)!}\int_0^{(n-1)t_f} \mathbb{P}\big(U_r^3 \geq t-(n-1)t_f + u\big) d_u \mathbb{P}\Big(\sum_{i=1}^{n-1} J_i \geq (n-1)t_f - u\Big)\\
&\lesssim r^{n-1}\exp\big(-ct\log t\big)\\
&\quad \times \Bigg[\bigg(\int_0^r + \int_{t_f/4}^{(n-1)t_f} + \int_r^{t_f/4}\bigg)\mathbb{P}\big(U_r^3 \geq t-(n-1)t_f + u\big) d_u \mathbb{P}\Big(\sum_{i=1}^{n-1} J_i \geq (n-1)t_f - u\Big)\Bigg]\\
&\leq r^{n-1}\exp\big(-ct\log t\big)\Bigg[\mathbb{P}(U_r^3 \geq t_s/4) + \mathbb{P}\big(U_r^3 \geq t-(n-1)t_f\big)\mathbb{P}\Big(\sum_{i=1}^{n-1} J_i \geq (n-1)t_f - r\Big)\\
&\qquad + \int_r^{t_f/4} \mathbb{P}\big(U_r^3 \geq t-(n-1)t_f + u\big) d_u \mathbb{P}\Big(\sum_{i=1}^{n-1} J_i \geq (n-1)t_f - u\Big)\Bigg]\\
&=: r^{n-1}\exp\big(-ct\log t\big)\big[A_1 + A_2 + A_3\big]
\end{aligned}
$$

and by the same way, we also have that

$$
\begin{aligned}
&\mathbb{P}\big(U_r^3 + U_r^4 \geq t | P(r) = n\big)\mathbb{P}\big(P(r) = n\big)\\
&\quad \lesssim r^n \exp\big(-ct\log t\big)\Bigg[\mathbb{P}(U_r^3 \geq t_f/4) + \mathbb{P}\Big(\sum_{i=1}^{n} J_i \geq nt_f - (nt_f - t + r)\Big)\\
&\qquad + \int_{nt_f - t + r}^{t_f/4} \mathbb{P}\big(U_r^3 \geq t - nt_f + u\big) d_u \mathbb{P}\Big(\sum_{i=1}^{n} J_i \geq nt_f - u\Big)\Bigg]\\
&\quad =: r^n \exp\big(-ct\log t\big)\big[B_1 + B_2 + B_3\big].
\end{aligned}
$$

To bound A_i and B_i, we claim that for every $k \in \mathbb{N}$ and $u \in (0, t_f/4]$, it holds that

$$
\mathbb{P}\Big(\sum_{i=1}^{k} J_i \geq kt_f - u\Big) \leq \big(Kw(t_f/9)^{-1}\big)^k u^k, \tag{2.11}
$$

where $K \geq 1$ is the constant in **(Trunc.)**(t_f)(ii). Indeed, if $k = 1$, then by **(Trunc.)**(t_f)(i) and (ii), we get $\mathbb{P}(J_1 \geq t_f - u) = F(t_f - u) = w(t_f/9)^{-1}w(t_f - u) \leq Kw(t_f/9)^{-1}u$. Suppose that the claim holds for k. Then, by **(Trunc.)**(t_f)(i) and (ii), for all $u \in (0, t_f/4]$,

$$
\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{k+1} J_i \geq (k+1)t_f - u\Big)\\
&= \int_{\{\sum_{i=1}^k u_i \leq u\}} F\Big(t_f - u + \sum_{i=1}^{k} u_i\Big) d_{u_k} F(t_f - u_k)...d_{u_1}F(t_f - u_1)\\
&\leq Kw(t_f/9)^{-1}\int_{\{\sum_{i=1}^k u_i \leq u\}} \Big(u - \sum_{i=1}^{k} u_i\Big) d_{u_k} F(t_f - u_k)...d_{u_1}F(t_f - u_1)\\
&\leq Kw(t_f/9)^{-1}u\int_{\{\sum_{i=1}^k u_i \leq u\}} d_{u_k} F(t_f - u_k)...d_{u_1}F(t_f - u_1)
\end{aligned}
$$

$$\le Kw(t_f/9)^{-1}u\mathbb{P}\big(\sum_{i=1}^{k} J_i \ge kt_f - u\big) \le \big(Kw(t_f/9)^{-1}u\big)^{k+1}.$$

Therefore, the claim holds by induction.

We consider the following two cases that when t is very close to nt_f and not.

Case 1. $(n-1/12)t_f \le t < nt_f$;

At first, by (2.10), we obtain $A_1 + A_2 + A_3 \le 3\mathbb{P}(U_r^3 \ge t_f/4) \le cr^2$. On the other hand, by (2.10), (2.11), Proposition 2.7, the change of the variables and the integration by parts,

$$\begin{aligned} B_1 + B_2 + B_3 &\le cr^2 + c^n(nt_f - t + r)^n + cr\int_r^{t_f/4+t-nt_f} w(u)d_u\mathbb{P}\big(\sum_{i=1}^{n} J_i \ge t-u\big) \\ &\le c^n r + c^n(nt_f - t)^n + c^n r\int_r^{t_f/4+t-nt_f} (nt_f - t + u)^n(-dw(u)) \\ &\le c^n r + c^n(nt_f - t)^n + c^n(nt_f - t)^n rw(r) + c^n r\int_r^{t_f/4+t-nt_f} u^n(-dw(u)) \\ &\le c^n\big(r + (nt_f - t)^n\big). \end{aligned}$$

In the third inequality, we used the fact that $(a+b)^k \le 2^k(a^k + b^k)$ for all $a, b > 0$ and $k \in \mathbb{N}$ and in the fourth inequality, we used the assumption that $rw(r) \le er\phi(r^{-1}) \le 1/(4e)$. Therefore, since $n \asymp t$ so that $c^n \le ce^{ct}$, we get the result in this case.

Case 2. $(n-1)t_f \le t < (n-1/12)t_f$;

By (2.10), (2.11), Proposition 2.7 and the integration by parts, we obtain

$$\begin{aligned} A_1 + A_2 + A_3 &\le (36c_1r/t_f)^2 + c^n r^{n-1} + cr\int_r^{t_f/4} w(u)d_u\mathbb{P}\big(\sum_{i=1}^{n-1} J_i \ge (n-1)t_f - u\big) \\ &\le c^n r + c^n r\int_r^{t_f/4} u(-dw(u)) \le c^n r. \end{aligned}$$

Since $B_1 + B_2 + B_3 \le 3$, $n \asymp t$ and $(nt_f - t) \asymp 1$ in this case, we finish the proof. $\square$

Lemma 2.15. *There exists a constant $L \in (0,1)$ such that for all $t, r > 0$ satisfying $t \ge t_f/2$ and $rt^{-1} \le L$,*

$$\mathbb{P}(S_r \ge t) \simeq \left(\frac{r}{t}\right)^{ct} \simeq \exp\big(-ct\log\frac{t}{r}\big).$$

Proof. Fix $r, t > 0$ satisfying $t \ge t_f/2$ and $rt^{-1} \le L$ where the constant L will be chosen later. Pick any $t_e \in (0, t_f)$ such that $w(t_e) \ge 1$ and let S^* be the driftless subordinator with Lévy measure $\mathbf{1}_{(t_e,\infty)} \cdot (-dw)$. By condition **(Ker.)**, we can always find such constant t_e. Since $S_r \ge S_r^*$ and jump sizes of S^* are at least bigger than t_e, by Stirling's formula, we get

$$\begin{aligned} \mathbb{P}(S_r \ge t) \ge \mathbb{P}(S_r^* \ge t) &\ge \mathbb{P}\big(S^* \text{ jumps } (\lfloor t/t_e\rfloor + 1) \text{ times before time } r\big) \\ &= \exp\big(-rw(t_e)\big)\frac{\big(rw(t_e)\big)^{(\lfloor t/t_e\rfloor+1)}}{(\lfloor t/t_e\rfloor + 1)!} \\ &\ge \exp\big(-rw(t_e) - (\lfloor t/t_e\rfloor + 3/2)\log(\lfloor t/t_e\rfloor + 1) + \lfloor t/t_e\rfloor + (\lfloor t/t_e\rfloor + 1)\log r\big) \\ &\ge \exp\big(-ct\log\frac{t}{r} + t/(2t_e) - rw(t_e)\big) \ge \exp\big(-ct\log\frac{t}{r} + t/(2t_e) - Ltw(t_e)\big). \end{aligned}$$

Hence, by taking L sufficiently small so that $Lw(t_e) \le 1/(2t_e)$, we get the lower bound.

On the other hand, by Chebyshev's inequality and Lemma 2.5, for all $\lambda > 0$,

$$\mathbb{P}(S_r \ge t) \le e^{-\lambda t}\mathbb{E}\big[e^{\lambda S_r}\big] = \exp\Big(-\lambda t + r\int_0^{t_f}(e^{\lambda u}-1)(-dw(u))\Big) \le \exp\big(-\lambda t + c_0\lambda r e^{\lambda t_f}\big),$$

where $c_0 := \int_0^{t_f} u(-dw(u)) \in (0,\infty)$. Then, by taking $\lambda = t_f^{-1}\log\big(t/(2c_0 r)\big)$, we obtain

$$\mathbb{P}(S_r \ge t) \le \exp\Big(-\frac{\lambda t}{2}\Big) \lesssim \exp\Big(-ct\log\frac{t}{r}\Big). \quad \square$$

3. Properties of the estimates $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Psi)$, $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$ and $\mathbf{HK}_M^{\gamma,\lambda,k}(\Phi,\Psi)$

A function $f:(0,\infty)\to\mathbb{R}$ is called a completely monotone function if f is infinitely differentiable and $(-1)^n f^{(n)}(\lambda) \ge 0$ for all $n \in \mathbb{N}$ and $\lambda > 0$. A Bernstein function is said to be a complete Bernstein function if its Lévy measure has a completely monotone density with respect to Lebesgue measure.

Lemma 3.1 (*[8, Lemmas 3.1 and 3.2]*)**.** *Assume that a family of non-negative functions $\{f(x,\cdot)\}_{x\in M}$ satisfies the weak scaling property uniformly with (α_1,α_2) for some $0<\alpha_1\le\alpha_2<\infty$, that is, there are constants $c_1, c_2 > 0$ such that for all $x \in M$,*

$$c_1(R/r)^{\alpha_1} \le f(x,R)/f(x,r) \le c_2(R/r)^{\alpha_2}, \quad 0<r\le R<\infty.$$

Then for any $\alpha_3 > \alpha_2$, there is a family of complete Bernstein functions $\{\varphi(x,\cdot)\}_{x\in M}$ such that for all $x\in M$ and $r>0$, we have that

$$f(x,r) \asymp \varphi(x,r^{-\alpha_3})^{-1} \quad \textit{and} \quad \partial_r\varphi(x,r) \asymp r^{-1}\varphi(x,r).$$

By Lemma 3.1, we can assume that all functions $\Phi(r)$, $\Psi(r)$ and $V(x,r)$ are differentiable in variable r and their derivatives are comparable to the function obtained by dividing r, i.e., $\Phi'(r) \asymp r^{-1}\Phi(r)$, $\Psi'(r)\asymp r^{-1}\Psi(r)$ and $\partial_r V(x,r) \asymp r^{-1}V(x,r)$ for all $r>0$ and $x\in M$. Indeed, for example, by Lemma 3.1, we have $V(x,r) \asymp \widetilde{V}(x,r) := \varphi(x,r^{-d_3})^{-1}$ for some complete Bernstein functions $\{\varphi(x,\cdot)\}_{x\in M}$ and $d_3 > d_2$. Then, for all $r>0$ and $x\in M$,

$$r\partial_r\widetilde{V}(x,r) \asymp \frac{r^{-d_3}\varphi'(x,r^{-d_3})}{\varphi(x,r^{-d_3})^2} \asymp \widetilde{V}(x,r).$$

Therefore, by using $\widetilde{V}$ instead of V, we get the desired properties.

Recall that for a strictly increasing function $\Phi:[0,\infty)\mapsto[0,\infty)$ which satisfies $\mathbf{WS}(\alpha_1,\alpha_2)$ for some $\alpha_2\ge\alpha_1>1$ and $\Phi(0)=0$, a function $\mathcal{M}$ is determined by the relation (1.12),

$$\frac{t}{\mathcal{M}(t,l)} \asymp \Phi\left(\frac{l}{\mathcal{M}(t,l)}\right) \qquad \text{for all } \ t,l>0.$$

For example, if $\Phi(l)=l^\alpha$ for some $\alpha>1$, then we have $\mathcal{M}(t,l) = l^{\alpha/(\alpha-1)}t^{-1/(\alpha-1)}$.

Lemma 3.2. *(i) For $t,l>0$, define*

$$\Phi_1(t,l) := \sup_{s>0}\left\{\frac{l}{s} - \frac{t}{\Phi(s)}\right\}.$$

Then, $\Phi_1(t,l)$ is strictly positive for all $t,l>0$, non-increasing on $(0,\infty)$ for fixed $l>0$ and satisfies (1.12). In other words, $\Phi_1(t,l)$ is one of the explicit forms of the function $\mathcal{M}(t,l)$.

(ii) $\mathcal{M}(\Phi(l), l) \asymp 1$ *for all* $l > 0$.
(iii) There are constants $c_3, c_4 > 0$ *such that for all* $l > 0$ *and* $0 < t \le T$,

$$c_3 \left(\frac{T}{t}\right)^{-1/(\alpha_1-1)} \le \frac{\mathcal{M}(T,l)}{\mathcal{M}(t,l)} \le c_4 \left(\frac{T}{t}\right)^{-1/(\alpha_2-1)}.$$

Proof. (i) Fix $t, l > 0$ and define for $s > 0$,

$$g(s) := \frac{l\Phi(s) - ts}{s\Phi(s)}, \quad k(s) := \frac{\Phi(s)}{s}.$$

We also define $k^{-1}(x) := \inf\{s : k(s) \ge x\}$ for $x > 0$. Since $\Phi(s) \asymp s\Phi'(s)$ for all $s > 0$, there exists a constant $c_1 > 0$ such that

$$(s\Phi(s))^2 g'(s) = s(ts\Phi'(s) - lk(s)\Phi(s)) \ge s\Phi(s)(c_1 t - lk(s)).$$

It follows that for $s_* := k^{-1}(c_1 t/l)$, we have $\Phi_1(t,l) = \sup_{s>0} g(s) = \sup_{s \ge s_*} g(s) \le l/s_*$.

On the other hand, for any $a > 1$, we have

$$\Phi_1(t,l) \ge \frac{l}{as_*} - \frac{t}{\Phi(as_*)} \ge \frac{l}{as_*} - c_2 \frac{t}{a^{\alpha_1}\Phi(s_*)} = \frac{l}{as_*}\left(1 - \frac{c_1^{-1}c_2}{a^{\alpha_1-1}}\right).$$

Hence, by choosing $a = 2 \vee (2c_1^{-1}c_2)^{1/(\alpha_1-1)}$, we get $\Phi_1(t,l) \asymp l/s_*$. Then, we conclude that

$$\Phi\left(\frac{l}{\Phi_1(t,l)}\right) \asymp \Phi(s_*) = s_* k(s_*) \asymp \frac{t}{\Phi_1(t,l)}.$$

(ii), (iii) These are consequences of the relation (1.12). □

By Lemma 3.2(iii) and Lemma 3.1, we can assume that $\mathcal{M}(t,l)$ is differentiable in variable t for every fixed $l > 0$ and there exists a constant $c_1 > 1$ such that for all $t, l > 0$,

$$c_1^{-1} t^{-1}\mathcal{M}(t,l) \le -\partial_t \mathcal{M}(t,l) \le c_1 t^{-1}\mathcal{M}(t,l). \tag{3.1}$$

From [8, Lemma 5.1], we get the following time derivative estimates for $q(a,t,x,l;\, \Phi, \mathcal{M})$.

Lemma 3.3. *For every* $a > 0$, *there are constants* $c_1, c_2 > 0$ *such that*

$$\left|\partial_t q^d(a,t,x,l;\, \Phi, \mathcal{M})\right| \le c_1 t^{-1} q^d(c_2,t,x,l;\, \Phi, \mathcal{M}), \quad t, l > 0, \quad x \in D,$$

Moreover, there are constants $c_3 > 0$ *and* $c_u \in (1,\infty)$ *such that for all* $x \in D$,

$$\partial_t q^d(a,t,x,l;\, \Phi, \mathcal{M}) \ge c_3 t^{-1} q^d(a,t,x,l;\, \Phi, \mathcal{M}) \quad \text{if } \Phi(l) \ge c_u t.$$

We obtain the upper time derivative estimates for $q^j(t,x,l;\, \Phi,\, \Psi)$ and $a_k^\gamma(t,x,y)$.

Lemma 3.4. *(i) There is a constant* $c_1 > 0$ *such that for all* $t, l > 0$ *and* $x \in D$,

$$\left|\partial_t q^j(t,x,l;\, \Phi,\, \Psi)\right| \le c_1 t^{-1} q^j(t,x,l;\, \Phi,\, \Psi).$$

(ii) For all $\gamma \in [0,1)$, $t > 0$, $x, y \in D$ *and* $j \in \{1,2\}$,

$$\left|\partial_t a_k^\gamma(t,x,y)\right| \le 2t^{-1} a_k^\gamma(t,x,y).$$

Proof. (i) Observe that

$$\partial_t q^j(t,x,l;\, \Phi,\, \Psi) = \frac{\Psi(l)V(x,l) - t^2 \partial_r V(x, \Phi^{-1}(t))\partial_t \Phi^{-1}(t)}{(tV(x,\Phi^{-1}(t)) + \Psi(l)V(x,l))^2},$$

By using the comparisons $\partial_r V(x,r) \asymp r^{-1}V(x,r)$ and $\partial_t \Phi^{-1}(t) \asymp t^{-1}\Phi^{-1}(t)$, we get

$$|\partial_t q^j(t,x,l;\,\Phi,\,\Psi)| \le \frac{ctV(x,\,\Phi^{-1}(t)) + \Psi(l)V(x,l)}{(tV(x,\,\Phi^{-1}(t)) + \Psi(l)V(x,l))^2} \le ct^{-1}q^j(t,x,l;\,\Phi,\,\Psi)$$

(ii) From the definition of a_j^γ, we get

$$|\partial_t a_1^\gamma(t,x,y)| = \left(\frac{\gamma}{t+\Phi(\delta_D(x))} + \frac{\gamma}{t+\Phi(\delta_D(y))}\right) a_1^\gamma(t,x,y) \le 2t^{-1}a_1^\gamma(t,x,y),$$

$$\begin{aligned}|\partial_t a_2^\gamma(t,x,y)| &= \left(\frac{\gamma}{t+(1+t)\Phi(\delta_D(x))} + \frac{\gamma}{t+(1+t)\Phi(\delta_D(y))}\right)\frac{a_2^\gamma(t,x,y)}{1+t} \\ &\le 2t^{-1}a_2^\gamma(t,x,y). \quad \square\end{aligned}$$

4. Proof of main theorems

In this section, we give the proof for Theorems 1.15, 1.16, 1.18 and 1.19. Throughout this section, we assume that there exist $\gamma \in [0,1)$, $\lambda \ge 0$ and $k \in \{1,2\}$ such that $q(t,x,y)$ enjoys the one of the estimates $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\,\Phi)$, $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$ and $\mathbf{HK}_M^{\gamma,\lambda,k}(\Phi,\,\Psi)$. Let $p(t,x,y)$ be given by (1.5).

Proposition 4.1. (On-diagonal lower bounds) *If condition* **(S.Poly.)**(t_s) *holds, then there exists* $c>0$ *such that for all* $(t,x,y) \in (0,t_s] \times D \times D$ *satisfying* $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/(4e^2)$,

$$p(t,x,y) \ge c\mathcal{J}_k^\gamma(t,x,y). \tag{4.1}$$

On the other hand, if condition **(L.Poly.)** *holds and* $\lambda = 0$, *then for every fixed* $T>0$, (4.1) *holds for all* $(t,x,y) \in [T,\infty) \times D \times D$ *satisfying* $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/(4e^2)$.

Proof. Since the proofs are similar, we only give the proof when condition **(S.Poly.)**(t_s) holds. Fix $(t,x,y) \in (0,t_s] \times D \times D$ satisfying $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/(4e^2)$ and set $l := \rho(x,y)$. By Proposition 2.7, there is a constant $\varepsilon_2 \in (0,1/2]$ such that for all $t \in (0,t_s]$, we have that $\mathbb{P}(S_{\varepsilon_2\Phi(l)} \ge t) \le 1/2$. Then, by the Markov property, we get

$$\begin{aligned}\mathbb{P}(S_{2\varepsilon_2\Phi(l)} \ge t) &\ge \mathbb{P}(S_{2\varepsilon_2\Phi(l)} - S_{\varepsilon_2\Phi(l)} \ge t \quad \text{or} \quad S_{\varepsilon_2\Phi(l)} \ge t) \\ &\ge 1-(1-\mathbb{P}(S_{\varepsilon_2\Phi(l)} \ge t))^2 \ge \frac{3}{2}\mathbb{P}(S_{\varepsilon_2\Phi(l)} \ge t).\end{aligned}$$

We used the inequality that $1-(1-x)^2 \ge 3x/2$ for $x \in (0,1/2]$. It follows that

$$\mathbb{P}(S_{2\varepsilon_2\Phi(l)} \ge t) - \mathbb{P}(S_{\varepsilon_2\Phi(l)} \ge t) \ge \frac{1}{2}\mathbb{P}(S_{\varepsilon_2\Phi(l)} \ge t).$$

and hence by the scaling properties of V and Φ and the monotonicity of $r \mapsto a_k^\gamma(r,x,y)$,

$$p(t,x,y) \ge c\int_{\varepsilon_2\Phi(l)}^{2\varepsilon_2\Phi(l)} \frac{a_k^\gamma(r,x,y)}{V(x,\,\Phi^{-1}(r))}d_r\mathbb{P}(S_r \ge t) \ge c_2\frac{a_k^\gamma(\Phi(l),x,y)}{V(x,l)}\mathbb{P}(S_{\varepsilon_2\Phi(l)} \ge t). \tag{4.2}$$

Besides, by the integration by parts and Lemma 2.6,

$$p(t,x,y) \ge c\int_{\varepsilon_2\Phi(l)}^{1/(2e^2\phi(t^{-1}))} \frac{a_k^\gamma(r,x,y)}{V(x,\,\Phi^{-1}(r))}d_r\mathbb{P}(S_r \ge t)$$

$$
\begin{aligned}
&\geq -cw(t)\int_{\varepsilon_2\Phi(l)}^{1/(2e^2\phi(t^{-1}))} r d_r\left(\frac{a_k^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}\right) - c_3\frac{a_k^\gamma(\Phi(l),x,y)}{V(x,l)}\mathbb{P}(S_{\varepsilon_2\Phi(l)}\geq t)\\
&\geq c_4 w(t)\int_{\Phi(l)}^{1/(2e^2\phi(t^{-1}))}\frac{a_k^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr - c_3\frac{a_k^\gamma(\Phi(l),x,y)}{V(x,l)}\mathbb{P}(S_{\varepsilon_2\Phi(l)}\geq t). \qquad (4.3)
\end{aligned}
$$

Finally, by Corollary 2.13, (4.2) and (4.3), we deduce that

$$
\begin{aligned}
&(1+c_3+c_2)p(t,x,y)\\
&\geq c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)} + c_2c_4w(t)\int_{\Phi(l)}^{1/(2e^2\phi(t^{-1}))}\frac{a_k^\gamma(r,x,y)}{V(x,\Phi^{-1}(r))}dr. \qquad \Box
\end{aligned}
$$

In the rest of this section, we fix $(x,y)\in D\times D$ and then define $l := \rho(x,y)$ and $V(r) := V(x,r)$.

4.1. Pure jump case

In this subsection, we give the proofs when $q(t,x,y)$ enjoys the estimate $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Phi)$.

Proof of Theorem 1.15. Fix $t\in(0,t_s]$. Since we only deal with small time t, we can assume that $\lambda=0$. By (1.5) and the integration by parts, we have that for $L := 1/(4e^2)$,

$$
\begin{aligned}
p(t,x,y) &\asymp \int_0^\infty q(r,x,y)d_r\mathbb{P}(S_r\geq t)\\
&= \int_0^{L/\phi(t^{-1})} q(r,x,y)d_r\mathbb{P}(S_r\geq t) - \int_{L/\phi(t^{-1})}^\infty q(r,x,y)d_r\mathbb{P}(S_r\leq t)\\
&= q\big(L/\phi(t^{-1}),x,y\big) - \int_0^{L/\phi(t^{-1})}\mathbb{P}(S_r\geq t)d_r q(r,x,y) + \int_{L/\phi(t^{-1})}^\infty \mathbb{P}(S_r\leq t)d_r q(r,x,y)\\
&=: q\big(L/\phi(t^{-1}),x,y\big) - I_1 + I_2. \qquad (4.4)
\end{aligned}
$$

Case 1. $\Phi(l)\phi(t^{-1})\leq 1/(4e^2)$;

By Proposition 4.1, it remains to prove the upper bound. We first note that

$$
q\big(L/\phi(t^{-1}),x,y\big)\leq c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}.
$$

Next, by Proposition 2.7, Lemma 3.4 and the definition of $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Phi)$,

$$
\begin{aligned}
|I_1| &\leq cw(t)\int_0^{L/\phi(t^{-1})} q(r,x,y)dr\\
&\leq cw(t)\int_0^{\Phi(l)/2}\frac{ra_k^\gamma(r,x,y)}{\Phi(l)V(l)}dr + cw(t)\int_{\Phi(l)/2}^{L/\phi(t^{-1})}\frac{a_k^\gamma(r,x,y)}{V(\Phi^{-1}(r))}dr =: I_{1,1}+I_{1,2}.
\end{aligned}
$$

Observe that since $\gamma<1$ and $r\mapsto r^{2\gamma}a_k^\gamma(r,x,y)$ is increasing, we have that

$$
\begin{aligned}
I_{1,1} &\leq cw(t)\int_0^{\Phi(l)/2}\frac{r^{2\gamma}a_k^\gamma(r,x,y)r^{1-2\gamma}}{\Phi(l)V(l)}dr\\
&\leq cw(t)\frac{\Phi(l)^{2\gamma}a_k^\gamma(\Phi(l),x,y)}{\Phi(l)V(l)}\int_0^{\Phi(l)/2} r^{1-2\gamma}dr
\end{aligned}
$$

$$\le cw(t)a_k^\gamma(\Phi(l),x,y)\frac{\Phi(l)}{V(l)} \le cw(t)\int_{\Phi(l)/2}^{\Phi(l)} \frac{a_k^\gamma(r,x,y)}{V(\Phi^{-1}(r))}dr \le I_{1,2}.$$

Therefore, we have that $|I_1| \le cI_{1,2} \le c\mathcal{J}_k^\gamma(t,x,y)$.

Lastly, by Lemma 3.4 and the change of variables, we get

$$\begin{aligned}|I_2| &\le c\int_{L/\phi(t^{-1})}^{\infty} \frac{a_k^\gamma(r,x,y)}{rV(\Phi^{-1}(r))}dr \le ca_k^\gamma(1/\phi(t^{-1}),x,y)\int_L^\infty \frac{1}{sV\big(\Phi^{-1}(s/\phi(t^{-1}))\big)}ds \\ &\le c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}\int_L^\infty s^{-1-d_1/\alpha_2}ds = c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}.\end{aligned}$$

Therefore, we obtain the upper bound from (4.4).

Case 2. $\Phi(l)\phi(t^{-1}) > 1/(4e^2)$;

In this case, we have

$$q\big(L/\phi(t^{-1}),x,y\big) \le c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})\Phi(l)V(l)}.$$

By Proposition 2.7, Lemma 3.4 and the fact that $\phi(t^{-1}) \ge e^{-1}w(t)$,

$$\begin{aligned}|I_1| &\le cw(t)\int_0^{L/\phi(t^{-1})} \frac{ra_k^\gamma(r,x,y)}{\Phi(l)V(l)}dr \le cw(t)\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})^{2\gamma}}\int_0^{L/\phi(t^{-1})}\frac{r^{1-2\gamma}}{\Phi(l)V(l)}dr \\ &\le cw(t)a_k^\gamma(1/\phi(t^{-1}),x,y)\frac{\phi(t^{-1})^{-2}}{\Phi(l)V(l)} \le c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})\Phi(l)V(l)}.\end{aligned}$$

Moreover, by Lemmas 3.4, 2.11, 2.4(ii) and the change of variables,

$$\begin{aligned}|I_2| &\le c\int_{L/\phi(t^{-1})}^{b^{-1}(t)} \frac{a_k^\gamma(r,x,y)}{V(l)\Phi(l)}dr + c\int_{b^{-1}(t)}^{\infty}\frac{a_k^\gamma(r,x,y)\exp\big(-rH(\phi'^{-1}(t/r))\big)}{V(l)\Phi(l)}dr \\ &\le c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})V(l)\Phi(l)}\left[1+\int_1^\infty \exp\Big(-b^{-1}(t)sH\big(\phi'^{-1}(t/(b^{-1}(t)s))\big)\Big)ds\right] \\ &\le c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})V(l)\Phi(l)}\left[1+\int_1^\infty \exp(-s)ds\right] \le c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})V(l)\Phi(l)}.\end{aligned}$$

In the third inequality, we used the fact that $s \mapsto H\big(\phi'^{-1}(t/(b^{-1}(t)s))\big)$ is increasing and $b^{-1}(t)H\big(\phi'^{-1}(t/(b^{-1}(t)))\big) = 1$. This proves the upper bound.

On the other hand, by Corollary 2.13 and the definition of $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Phi)$, we obtain

$$p(t,x,y) \ge c\frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})V(l)\Phi(l)}. \quad \square$$

Proof of Theorem 1.16. If $\lambda = 0$, then by using Proposition 2.8 instead of Proposition 2.7, the proof is essentially the same as the one for Theorem 1.15. Hence, we omit it in here. Now, assume that $\lambda > 0$ and $R_D = \mathrm{diam}(D) < \infty$. Let $T_* := 1/(4e^2\phi(T^{-1}))$. Then, by Proposition 2.8, Lemma 3.4 and the integration by parts,

$$\begin{aligned}p(t,x,y) &\asymp \int_0^{T_*} q(r,x,y)d_r\mathbb{P}(S_r \ge t) + \Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma\int_{T_*}^\infty e^{-\lambda r}d_r\mathbb{P}(S_r \ge t) \\ &\le q(T_*,x,y)\mathbb{P}(S_{T_*} \ge t)\end{aligned}$$

$$
\begin{aligned}
&+ c w(t) \int_0^{T_*} q(r,x,y) dr + \lambda \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma \int_{T_*}^\infty e^{-\lambda r} \mathbb{P}(S_r \ge t) dr \\
&=: q(T_*,x,y)\mathbb{P}(S_{T_*} \ge t) + J_1 + J_2 \le c w(t) \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma + J_1 + J_2.
\end{aligned}
$$

By Proposition 2.8 and Lemma 2.3, we obtain

$$
\begin{aligned}
J_2 &\le c \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma \left(w(t) \int_{T_*}^{1/(4e^2\phi(t^{-1}))} r e^{-\lambda r} dr + \int_{1/(4e^2\phi(t^{-1}))}^\infty e^{-\lambda r} dr \right) \\
&\le \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma \big(w(t) + \exp\big(-c/\phi(t^{-1})\big)\big) \\
&\le c \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma \big(w(t) + \phi(t^{-1})^{\delta_2+1}\big) \le c w(t) \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma .
\end{aligned}
$$

In the third inequality, we used the fact that for every $\delta > 0$, $e^{-1/x} \le \delta^\delta e^{-\delta} x^\delta$ for all $x > 0$.

On the other hand, we note that

$$
\begin{aligned}
2 \int_{T_* \Phi(l)/(2\Phi(R_D))}^{T_*} \frac{a_1^\gamma(r,x,y)}{V(\Phi^{-1}(r))} dr &\ge \int_{T_* \Phi(l)/(2\Phi(R_D))}^{T_* \Phi(l)/\Phi(R_D)} \frac{r a_1^\gamma(r,x,y)}{V(l)\Phi(l)} dr + \int_{T_*/2}^{T_*} \frac{a_1^\gamma(r,x,y)}{V(\Phi^{-1}(r))} dr \\
&\ge c \frac{a_1^\gamma(\Phi(l),x,y)\Phi(l)}{V(l)} + c \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma \\
&\ge c \int_0^{T_* \Phi(l)/(2\Phi(R_D))} \frac{r a_1^\gamma(r,x,y)}{V(l)\Phi(l)} dr + c \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma .
\end{aligned}
$$

Thus, by the scaling properties of a_1^γ, V and Φ, we get

$$
\begin{aligned}
J_1 &\asymp w(t) \int_{T_* \Phi(l)/(2\Phi(R_D))}^{T_*} \frac{a_1^\gamma(r,x,y)}{V(\Phi^{-1}(r))} dr \asymp w(t) \int_{\Phi(l)}^{2\Phi(R_D)} \frac{a_1^\gamma(r,x,y)}{V(\Phi^{-1}(r))} dr \\
&\ge c w(t) \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma \ge c J_2 .
\end{aligned}
$$

This proves the upper bound.

On the other hand, by essentially the same proof as the one for Proposition 4.1, we get the lower bound. We omit the details in here. □

Proof of Theorem 1.18. If $\lambda = 0$, then by using Proposition 2.9 instead of Proposition 2.7 and the fact that $\phi(t^{-1}) \asymp t^{-1}$ for all $t \ge T$ which follows from Lemma 2.1(i), we get the desired results. Hence, we assume that $\lambda > 0$ and $R_D = \mathrm{diam}(D) < \infty$. Let $L > 0$ be the minimum of the constants in Propositions 2.9 and 2.10. By the integration by parts, Proposition 2.10 with $k = \lambda/2$ and the argument given in the proof of Theorem 1.16,

$$
\begin{aligned}
p(t,x,y) &\le c \int_0^{LT} q(r,x,y) d_r \mathbb{P}(S_r \ge t) + c \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma \int_{LT}^\infty e^{-\lambda r} d_r \mathbb{P}(S_r \ge t) \\
&\le c \Big[\mathbf{1}_{\{0<\beta<1\}} \exp\big(-c_1 t^\beta\big) + \mathbf{1}_{\{\beta=1\}} \exp\big(-\frac{c_1}{2} t\big) \Big] \left(q(LT,x,y) + \int_{\Phi(l)}^{2\Phi(R_D)} \frac{a_1^\gamma(r,x,y)}{V(\Phi^{-1}(r))} dr \right) \\
&\quad + c\lambda \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma \Big[\mathbf{1}_{\{0<\beta<1\}} \exp\big(-c_1 t^\beta\big) + \mathbf{1}_{\{\beta=1\}} \exp\big(-\frac{c_1}{2} t\big) \Big] \int_{LT}^{Lt} r e^{-\lambda r/2} dr \\
&\quad + c\lambda \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma \int_{Lt}^\infty e^{-\lambda r} \mathbb{P}(S_r \ge t) dr \\
&\le c \Big[\mathbf{1}_{\{0<\beta<1\}} \exp\big(-c_1 t^\beta\big) + \mathbf{1}_{\{\beta=1\}} \exp\big(-\frac{c_1}{2} t\big) \Big] \int_{\Phi(l)}^{2\Phi(R_D)} \frac{a_1^\gamma(r,x,y)}{V(\Phi^{-1}(r))} dr \\
&\quad + c\lambda e^{-\lambda L t} \Phi(\delta_D(x))^\gamma \, \Phi(\delta_D(y))^\gamma .
\end{aligned}
$$

This proves the upper bound.

On the other hand, by the proof for Proposition 4.1, we can obtain that

$$p(t,x,y) \geq cw(t)\int_{\Phi(l)}^{2\Phi(R_D)} \frac{a_1^\gamma(r,x,y)}{V(\Phi^{-1}(r))}dr.$$

Furthermore, by Corollary 2.12 and the fact that $\phi(t^{-1})^{-1} \asymp t$ for all $t \geq T$, there exists a constant $L_1 > 0$ such that

$$p(t,x,y) \geq c \inf_{r\in(0,L_1 t)} q(r,x,y) \geq ce^{-\lambda L_1 t}\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma.$$

Hence, we get the lower bound. □

4.2. *Diffusion case*

In this subsection, we provide the proof when $q(t,x,y)$ enjoys the estimate $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$. Set $k(c_0,r) := a_k^\gamma(r,x,y)q(c_0,r,x,l;\,\Phi,\mathcal{M})$ for $c_0 > 0$ and $r > 0$ where the function $\mathcal{M}$ is determined by the relation (1.12).

Proof of Theorem 1.15. Since we only consider small time t, we can assume that $\lambda = 0$. For every fixed $t \in (0, t_s]$, by the integration by parts, we have that for $L := 1/(4e^2)$,

$$\begin{aligned}
p(t,x,y) &\simeq \int_0^\infty k(c,r)d_r\mathbb{P}(S_r \geq t)\\
&= k(c, L/\phi(t^{-1})) - \int_0^{L/\phi(t^{-1})} \mathbb{P}(S_r \geq t)d_r k(c,r) + \int_{L/\phi(t^{-1})}^\infty \mathbb{P}(S_r \leq t)d_r k(c,r)\\
&=: k(c, L/\phi(t^{-1})) - I_1 + I_2.
\end{aligned}$$

Case 1. $\Phi(l)\phi(t^{-1}) \leq 1/(4e^2)$;

Note that by a similar proof as the one given in Section 4.1, we obtain

$$I_2 \leq ck\big(c_1, 1/(4e^2\phi(t^{-1}))\big) \leq \frac{a_k^\gamma(1/\phi(t^{-1}),x,y)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}.$$

Hence, by Proposition 4.1, it remains to get upper bound for I_1.

By Lemma 3.3, Proposition 2.7, the change of variables and Lemma 3.2(iii),

$$\begin{aligned}
|I_1| &\leq cw(t)\int_0^{\Phi(l)/2} \frac{a_k^\gamma(r,x,y)}{V(\Phi^{-1}(r))}e^{-c_2\mathcal{M}(r,l)}dr + cw(t)\int_{\Phi(l)/2}^{L/\phi(t^{-1})} \frac{a_k^\gamma(r,x,y)}{V(\Phi^{-1}(r))}dr\\
&\leq cw(t)\Phi(l)^{1-2\gamma}\int_0^{1/2} \frac{(\Phi(l)s)^{2\gamma}a_1^\gamma(\Phi(l)s,x,y)}{s^{2\gamma}V(\Phi^{-1}(\Phi(l)s))}e^{-c_2\mathcal{M}(\Phi(l)s,l)}ds + c\mathcal{J}_k^\gamma(t,x,y)\\
&\leq cw(t)\frac{\Phi(l)a_1^\gamma(\Phi(l)/2,x,y)}{V(l)}\int_0^{1/2} s^{-d_2/\alpha_1-2\gamma}\exp\big(-c_3 s^{-1/(\alpha_2-1)}\big)ds + c\mathcal{J}_k^\gamma(t,x,y)\\
&\leq cw(t)a_k^\gamma(\Phi(l),x,y)\frac{\Phi(l)}{V(l)} + c\mathcal{J}_k^\gamma(t,x,y)\\
&\leq cw(t)\int_{\Phi(l)/2}^{\Phi(l)} \frac{a_k^\gamma(r,x,y)}{V(\Phi^{-1}(r))}dr + c\mathcal{J}_k^\gamma(t,x,y) \leq c\mathcal{J}_k^\gamma(t,x,y).
\end{aligned}$$

Case 2. $\Phi(l)\phi(t^{-1}) > 1/(4e^2)$;

Define for every $a > 0$ and $r > 0$,

$$g(a,r) := \frac{\exp\big(-a\mathcal{M}(r,l)\big)}{r^{1+2\gamma}V(\Phi^{-1}(r))}.$$

Then, we see that

$$\begin{aligned}\frac{dg(a,r)}{dr} &= \big(-ar\partial_r\mathcal{M}(r,l) - (1+2\gamma) - r\partial_r V(\Phi^{-1}(r))\cdot V(\Phi^{-1}(r))^{-1}\big)\frac{\exp\big(-a\mathcal{M}(r,l)\big)}{r^{2+2\gamma}V(\Phi^{-1}(r))}\\ &\geq \big(ac_4\mathcal{M}(r,l) - c_5\big)\frac{\exp\big(-a\mathcal{M}(r,l)\big)}{r^{2+2\gamma}V(\Phi^{-1}(r))},\end{aligned}$$

for some positive constants c_4 and c_5 independent of a and r. By Lemma 3.2(ii) and (iii), for each fixed $a > 0$, there exists a constant $\delta > 0$ such that $g(a,r)$ is increasing on $0 < r < \delta\Phi(l)$. By Lemma 3.3 and the fact that $r \mapsto r^{2\gamma}a_k^\gamma(r,x,y)$ is increasing on $r > 0$, we get

$$\begin{aligned}|I_1| &\leq c\int_0^{L/\phi(t^{-1})} r^{-1}k(c_6,r)dr = c\int_0^{L/\phi(t^{-1})} r^{2\gamma}a_k^\gamma(r,x,y)g(c_6,r)dr\\ &\leq c\phi(t^{-1})^{-(1+2\gamma)}a_k^\gamma(1/\phi(t^{-1}),x,y)\sup_{0<r<\phi(t^{-1})^{-1}} g(c_6,r).\end{aligned}$$

Therefore, if $\phi(t^{-1})^{-1} < \delta(c_6)\Phi(l)$, then we get

$$|I_1| \leq ca_k^\gamma(1/\phi(t^{-1}),x,y)\frac{\exp\big(-c_7\mathcal{M}(1/\phi(t^{-1}),l)\big)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}.$$

Otherwise, if $\phi(t^{-1})^{-1} \geq \delta(c_6)\Phi(l)$, then $\phi(t^{-1})^{-1} \asymp \Phi(l)$ and hence by Lemma 3.2(iii),

$$\begin{aligned}|I_1| &\leq c\phi(t^{-1})^{-(1+2\gamma)}a_k^\gamma(1/\phi(t^{-1}),x,y)\sup_{\delta(c_6)\Phi(l)<r<\phi(t^{-1})^{-1}} g(c_6,r)\\ &\leq ca_k^\gamma(1/\phi(t^{-1}),x,y)\frac{\exp\big(-c_9\mathcal{M}(1/\phi(t^{-1}),l)\big)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}.\end{aligned}$$

Next, by Lemmas 3.3, 2.11 and 2.4(ii), we have

$$\begin{aligned}|I_2| &\leq c\int_{L/\phi(t^{-1})}^{b^{-1}(t)} r^{-1}k(c_6,r)dr + c\int_{b^{-1}(t)}^\infty r^{-1}k(c_6,r)\exp\big(-rH(\phi'^{-1}(t/r))\big)dr\\ &=: I_{2,1} + I_{2,2}.\end{aligned}$$

By Lemmas 2.4(ii) and 3.2(iii), we have

$$I_{2,1} \leq ca_k^\gamma(1/\phi(t^{-1}),x,y)\frac{\exp\big(-c_{10}\mathcal{M}(1/\phi(t^{-1}),l)\big)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}.$$

To control the exponential terms in $I_{2,2}$, we consider the following two functions that $e_1(r) := rH(\phi'^{-1}(t/r))$ and $e_2(r) := \mathcal{M}(r,l)$. (cf. [8].) Note that e_1 is non-decreasing and e_2 is non-increasing. Moreover, by the definition of the function b, $e_1(b^{-1}(t)) = 1$ for all $t > 0$ and $e_1(\infty) = \infty$ and by Lemma 3.2(ii) and (iii), $e_2(\Phi(l)) \asymp 1$ for all $l > 0$ and $e_2(\infty) = 0$. Thus, by the intermediate value theorem, there are constants $a_1 > 0$ and $a_2 > 0$ independent of t and l such that for all $t, l > 0$ with $\Phi(l)\phi(t^{-1}) > 1/(4e^2)$, there exists a unique $r^* = r^*(t,l) \in (b^{-1}(t), a_1\Phi(l))$ such that $e_1(r^*) = a_2e_2(r^*)$. Now, we have

$$a_k^\gamma(1/\phi(t^{-1}),x,y)^{-1}I_{2,2}$$

$$\leq c\int_{b^{-1}(t)}^{r^*}\frac{\exp\big(-c_6\mathcal{M}(r,l)\big)}{rV(\Phi^{-1}(r))}dr + c\int_{r^*}^{\infty}\frac{\exp\big(-rH(\phi'^{-1}(t/r))\big)}{rV(\Phi^{-1}(r))}dr =: I_{2,2,1}+I_{2,2,2}.$$

By the change of variables and Lemma 3.2(ii) and (iii), we get

$$\begin{aligned} I_{2,2,1} &= c\int_{b^{-1}(t)/r^*}^{1}\frac{\exp\big(-c_6\mathcal{M}(r^*s,l)\big)}{sV(\Phi^{-1}(r^*s))}ds \\ &\leq c\frac{\exp\big(-\frac{c_6}{2}\mathcal{M}(r^*,l)\big)}{V(\Phi^{-1}(r^*))}\int_0^1 s^{-1-d_2/\alpha_1}\exp\big(-cs^{-1/(\alpha_2-1)}\big)ds \leq c\frac{\exp\big(-\frac{c_6}{2}\mathcal{M}(r^*,l)\big)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}. \end{aligned}$$

Also, by the change of variables, we have

$$\begin{aligned} I_{2,2,2} &= c\int_{r^*/b^{-1}(t)}^{\infty}\frac{\exp\big(-b^{-1}(t)sH\big(\phi'^{-1}(t/(b^{-1}(t)s))\big)\big)}{sV(\Phi^{-1}(b^{-1}(t)s))}ds \\ &\leq c\frac{\exp\big(-e_1(r^*)\big)}{V(\Phi^{-1}(b^{-1}(t)))}\int_1^{\infty}s^{-1-d_1/\alpha_2}ds \leq c\frac{\exp\big(-a_2\mathcal{M}(r^*,l)\big)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}. \end{aligned}$$

To determine the function $\mathcal{M}(r^*,l)$, we note that by (1.12), $e_1(r^*) \asymp e_2(r^*)$ implies that

$$\frac{r^*}{r^*H(\phi'^{-1}(t/r^*))} \asymp \Phi\left(\frac{l}{r^*H(\phi'^{-1}(t/r^*))}\right).$$

Let $s^* = 1/H(\phi'^{-1}(t/r^*))$. Then, $b(s^*)/s^* = \phi'(H^{-1}(1/s^*)) = t/r^*$. Therefore, by Lemma 2.4(ii), the function $\mathcal{N}(t,l) := a_2\mathcal{M}(r^*,l) = e_1(r^*) = r^*/s^*$ is determined by the relation

$$\frac{1}{\phi\big(\mathcal{N}(t,l)/t\big)} \asymp b^{-1}\left(\frac{t}{\mathcal{N}(t,l)}\right) = s^* \asymp \Phi\left(\frac{l}{\mathcal{N}(t,l)}\right).$$

Since $\mathcal{M}(b^{-1}(t),l) \geq ce_1(r^*)$, we finish the proof for the upper bound.

Now, we prove the lower bound. By Lemma 3.3 and the integration by parts, we have

$$\begin{aligned} p(t,x,y) &\geq -c\int_{Nb^{-1}(t)}^{\infty}k(c_8,r)d_r\mathbb{P}(S_r\leq t) \\ &\geq c\int_{Nb^{-1}(t)}^{\Phi(l)/c_u}r^{-1}\mathbb{P}(S_r\leq t)k(c_8,r)dr - c\int_{\Phi(l)/c_u}^{\infty}r^{-1}\mathbb{P}(S_r\leq t)k(c_9,r)dr \\ &\geq c_{10}\int_{Nb^{-1}(t)}^{\infty}r^{-1}\mathbb{P}(S_r\leq t)k(c_8,r)dr - c_{11}\int_{\Phi(l)/c_u}^{\infty}r^{-1}\mathbb{P}(S_r\leq t)k(c_9,r)dr \\ &:= J_1 - J_2, \end{aligned} \tag{4.5}$$

where $N := (e-2)/(8c_ue^2(e^2-e))$. Note that by Lemma 2.4(ii), we have that $Nb^{-1}(t) \leq 1/(8e^2c_u\phi(t^{-1})) \leq \Phi(l)/(2c_u)$. By taking c_u large enough, we may assume that $N \in (0,1/2)$. Then, by Lemmas 2.11 and 2.4(ii),

$$J_1 \geq ca_k^{\gamma}(1/\phi(t^{-1}),x,y)\phi(t^{-1})^{-2\gamma}\int_{Nb^{-1}(t)}^{\infty}\frac{\exp\big(-c_{12}rH(\phi'^{-1}(t/r)) - c_8\mathcal{M}(r,l)\big)}{r^{1+2\gamma}V(\Phi^{-1}(r))}dr.$$

Let $e_3(r) = c_{12}rH(\phi'^{-1}(t/r))$ and $e_4(r) = c_8\mathcal{M}(r,l)$ for $r > 0$. By the same argument as in the proof for the upper bounds, there are constants $a_3, a_4 > 0$ independent of t and l such that for all $t, l > 0$ with $\Phi(l)\phi(t^{-1}) > 1/(4e^2)$, there exists a unique $r_* = r_*(t,l) \in (b^{-1}(t), a_3\Phi(l))$ such that $e_3(r_*) = a_4e_4(r_*)$. Moreover, from the monotonicity,

$$e_3(r) < a_4e_4(r) \quad \text{for } r \in (b^{-1}(t), r_*) \quad \text{and} \quad e_3(r) > a_4e_4(r) \quad \text{for } r > r_*.$$

Therefore, by the change of variables, Lemma 2.4(ii) and the weak scaling properties,

$$\phi(t^{-1})^{-2\gamma}\int_{Nb^{-1}(t)}^{\infty}\frac{\exp\big(-e_3(r)-e_4(r)\big)}{r^{1+2\gamma}V(\Phi^{-1}(r))}dr=\int_N^{\infty}\frac{\exp\big(-e_3(b^{-1}(t)s)-e_4(b^{-1}(t)s)\big)}{s^{1+2\gamma}V\big(\Phi^{-1}(b^{-1}(t)s)\big)}ds$$

$$\geq\frac{c}{V\big(\Phi^{-1}(\phi(t^{-1})^{-1})\big)}\int_{r_*/(2b^{-1}(t))}^{r_*/b^{-1}(t)}s^{-1-d_2/\alpha_1-2\gamma}\exp\big(-(1+a_4)e_4(b^{-1}(t)s)\big)ds$$

$$\geq\frac{c}{V\big(\Phi^{-1}(\phi(t^{-1})^{-1})\big)}(r_*/b^{-1}(t))^{-d_2/\alpha_1-2\gamma}\exp\big(-c_{12}e_4(r_*)\big)$$

$$\geq c\frac{\exp\big(-2c_{12}e_4(r_*)\big)}{V\big(\Phi^{-1}(\phi(t^{-1})^{-1})\big)}(r_*/b^{-1}(t))^{-d_2/\alpha_1-2\gamma}\exp\Big(\frac{c_{12}}{a_4}e_3(r_*)\Big)$$

$$\geq c\frac{\exp\big(-2c_{12}e_4(r_*)\big)}{V\big(\Phi^{-1}(\phi(t^{-1})^{-1})\big)}(r_*/b^{-1}(t))^{-d_2/\alpha_1-2\gamma}\exp\Big(\frac{c_{12}r_*}{a_4b^{-1}(t)}e_3(b^{-1}(t))\Big)$$

$$\geq c\frac{\exp\big(-2c_{12}e_4(r_*)\big)}{V\big(\Phi^{-1}(\phi(t^{-1})^{-1})\big)}.$$

In the last inequality, we used the fact that $e_3(b^{-1}(t))=c_{12}$ and that for every $p>0$, there exists a constant $c(p)>0$ such that $e^x\geq c(p)x^p$ for all $x>0$. It follows that

$$J_1\geq c_{13}a_k^{\gamma}(1/\phi(t^{-1}),x,y)\frac{\exp\big(-c_{14}e_3(r_*)\big)}{V\big(\Phi^{-1}(\phi(t^{-1})^{-1})\big)}.$$

On the other hand, by Lemma 2.11, we have that

$$J_2\leq ca_k^{\gamma}(\Phi(l),x,y)\int_{\Phi(l)}^{\infty}\frac{\exp\big(-rH(\phi'^{-1}(t/r))\big)}{rV(\Phi^{-1}(r))}dr$$

$$\leq ca_k^{\gamma}(1/\phi(t^{-1}),x,y)\frac{\exp\big(-c_{15}e_3(\Phi(l))\big)}{V(l)}.$$

Since $e_3(Ar)\geq Ae_3(r)$ for all $r>0$ and $A\geq1$, from (4.5), we deduce that there exists a constant $A>0$ such that $\Phi(l)>Ar_*$ implies that

$$p(t,x,y)\geq ca_k^{\gamma}(1/\phi(t^{-1}),x,y)\frac{\exp\big(-c_{14}e_3(r_*)\big)}{V\big(\Phi^{-1}(\phi(t^{-1})^{-1})\big)},$$

which yields the result. Otherwise, if $\Phi(l)\leq Ar_*$, then by Lemma 3.2(ii) and (iii),

$$e_3(b^{-1}(t))=c_{12}\geq ce_4(\Phi(l))\geq ce_4(r_*)=ca_4^{-1}e_3(r_*)\geq ca_4^{-1}(r_*/b^{-1}(t))e_3(b^{-1}(t)).$$

It follows that $b^{-1}(t)\asymp r_*\asymp\Phi(l)$ in this case. Since by Corollary 2.13, we have that

$$p(t,x,y)\geq ca_k^{\gamma}(1/\phi(t^{-1}),x,y)\frac{1}{V\big(\Phi^{-1}(\phi(t^{-1})^{-1})\big)},$$

we still get the result in this case. □

Proof of Theorems 1.16 and 1.18. Observe that both $\mathbf{HK}_J^{\gamma,\lambda,k}(\Phi,\Phi)$ and $\mathbf{HK}_D^{\gamma,\lambda,k}(\Phi)$ give the same estimates for $q(t,x,y)$ on near diagonal situation, that is, when $t\geq c\rho(x,y)$ for some constant $c>0$. Using this fact, we deduce the result by the same argument given in Section 4.1. □

4.3. Mixed type case

In this subsection, we give the proof when $q(t, x, y)$ enjoys the estimate $\mathbf{HK}_M^{\gamma,\lambda,k}(\Phi, \Psi)$. Since the ideas for proofs are similar, we only provide the proof of Theorem 1.15. This completes the proof for Theorems 1.15, 1.16 and 1.18.

Proof of Theorem 1.15. Define for $r > 0$ and $c_0 > 0$,

$$m_1(r) := a_k^\gamma(r, x, y)q^j(r, x, l; \Phi, \Psi), \qquad m_2(c_0, r) := a_k^\gamma(r, x, y)q^d(c_0, r, x, l; \Phi, \mathcal{M}).$$

We also define for $t > 0$ and $c_0 > 0$,

$$p_1(t) := \int_0^\infty m_1(r)d_r\mathbb{P}(S_r \geq t), \quad p_2(c_0, t) := \int_0^\infty m_2(c_0, r)d_r\mathbb{P}(S_r \geq t).$$

Then, from the definition, we get

$$p(t, x, y) \simeq p_1(t) + p_2(c, t). \tag{4.6}$$

Case 1. $\Phi(l)\phi(t^{-1}) \leq 1/(4e^2)$;

By the proof given in Section 4.2, for each fixed $c_0 > 0$, $p_2(c_0, t) \asymp \mathcal{J}_1^\gamma(t, x, y)$. On the other hand, since $\Psi(l) \geq \Phi(l)$ for all $l > 0$, by the proof given in Section 4.1, $p_1(t) \leq c\int_0^\infty a_1^\gamma(r, x, y)q^j(r, x, l; \Phi, \Phi)d_r\mathbb{P}(S_r \geq t) \leq c\mathcal{J}_1^\gamma(t, x, y)$. Therefore, (4.6) yields the result.

Case 2. $\Phi(l)\phi(t^{-1}) > 1/(4e^2)$;

By the proof given in Section 4.2, we get

$$p_2(c, t) \simeq a_k^\gamma(1/\phi(t^{-1}), x, y)\frac{\exp\big(-c\mathcal{N}(t, \rho(x, y))\big)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}.$$

On the other hand, by Lemma 3.4, the integration by parts, Proposition 2.7 and Lemma 2.11,

$$\begin{aligned}
p_1(t) &= m_1\big(1/(4e^2\phi(t^{-1}))\big) - \int_0^{1/(4e^2\phi(t^{-1}))}\mathbb{P}(S_r \geq t)d_r m_1(r) \\
&\quad + \int_{1/(4e^2\phi(t^{-1}))}^\infty \mathbb{P}(S_r \leq t)d_r m_1(r) \\
&\leq c\frac{a_k^\gamma(1/\phi(t^{-1}), x, y)}{\phi(t^{-1})V(l)\Psi(l)} + cw(t)\int_0^{1/(4e^2\phi(t^{-1}))}\frac{r^{2\gamma}a_k^\gamma(r, x, y)}{V(l)\Psi(l)}r^{1-2\gamma}dr \\
&\quad + ca_k^\gamma(1/\phi(t^{-1}), x, y)\int_{1/(4e^2\phi(t^{-1}))}^\infty \frac{\exp\big(-rH(\phi'^{-1}(t/r))\big)}{V(l)\Psi(l)}dr \\
&\leq c\frac{a_k^\gamma(1/\phi(t^{-1}), x, y)}{\phi(t^{-1})V(l)\Psi(l)}.
\end{aligned}$$

We also have that by Corollary 2.13,

$$p_1(t, x, y) \geq c\frac{a_k^\gamma(1/\phi(t^{-1}), x, y)}{\phi(t^{-1})V(l)\Psi(l)}.$$

Hence, we get the result from (4.6). $\square$

4.4. Truncated kernel

In this subsection, we give the proof for Theorem 1.19. Throughout this subsection, we further assume that condition **(Trunc.)**(t_f) holds.

Proposition 4.2. *There are comparison constants independent of x and y such that for all $t \geq (\lfloor d_2/\alpha_1 + 2\gamma \rfloor \vee 1/2)t_f$, it holds that*

$$p(t,x,y) \simeq q(ct,x,y).$$

Proof. Note that by Lemma 2.1(i), $\phi(t^{-1}) \asymp t^{-1}$ for all $t \geq t_f$. Thus, by Corollary 2.13, we obtain the lower bound. Since condition **(Trunc.)**(t_f) implies condition **(Sub.)(1,1)**, by Theorem 1.18, there exists a constant $a > 0$ such that if $\lambda = 0$ and $a\Phi(\rho(x,y)) \geq t$, then $p(t,x,y) \simeq q(ct,x,y)$. Moreover, if $\lambda > 0$, then since D is bounded, by taking a small enough, we can assume that there is no $x, y \in D$ such that $a\Phi(\rho(x,y)) \geq t$. Hence, it remains to prove the upper bound when $a\Phi(\rho(x,y)) < t$. Assume that $a\Phi(\rho(x,y)) < t$.

Let r_0 and L be the constants in Proposition 2.14 and Lemma 2.15, respectively. Using the same arguments as in the ones given in the proof of Theorem 1.15,

$$\begin{aligned} p(t,x,y) &\asymp \int_0^{Lt} q(r,x,y)d_r\mathbb{P}(S_r \geq t) - \int_{Lt}^{\infty} q(r,x,y)d_r\mathbb{P}(S_r \leq t) \\ &\leq cq(Lt,x,y) + c\int_{a L\Phi(l)/2}^{Lt} r^{-1}q(r,x,y)\mathbb{P}(S_r \geq t)dr. \end{aligned}$$

Case 1. $\lambda = 0$;

If $aL\Phi(l)/2 \geq r_0$, then by Lemma 2.15 and the fact that $r \mapsto r^{2\gamma}a_k^{\gamma}(r,x,y)$ is increasing,

$$\begin{aligned} \int_{aL\Phi(l)/2}^{Lt} r^{-1}q(r,x,y)\mathbb{P}(S_r \geq t)dr &\leq c\int_{r_0}^{Lt} \frac{r^{2\gamma}a_k^{\gamma}(r,x,y)}{r^{1+2\gamma}V(\Phi^{-1}(r))}\left(\frac{r}{t}\right)^{ct} dr \\ &\leq ct^{2\gamma}a_k^{\gamma}(t,x,y)L^{ct}\int_{r_0}^{Lt} dr \leq ca_k^{\gamma}(t,x,y)e^{-ct} \leq c\frac{a_k^{\gamma}(t,x,y)}{V(\Phi^{-1}(t))} \asymp q(Lt,x,y). \end{aligned}$$

Otherwise, if $aL\Phi(l)/2 < r_0$, then by Proposition 2.14 and Lemma 2.15 and the weak scaling properties of V and Φ,

$$\begin{aligned} &\int_{aL\Phi(l)/2}^{Lt} r^{-1}q(r,x,y)\mathbb{P}(S_r \geq t)dr \\ &\leq c\exp\big(-ct\log t\big)\int_{aL\Phi(l)/2}^{r_0} \frac{r^{\lfloor t/t_f \rfloor + 2\gamma}a_k^{\gamma}(r,x,y)}{r^{2\gamma}V(\Phi^{-1}(r))}dr + c\int_{r_0}^{Lt}\frac{r^{2\gamma}a_k^{\gamma}(r,x,y)}{r^{1+2\gamma}V(\Phi^{-1}(r))}\left(\frac{r}{t}\right)^{ct} dr \\ &\leq c\frac{a_k^{\gamma}(t,x,y)}{V(\Phi^{-1}(t))}\left(1 + \int_{aL\Phi(l)/2}^{r_0} r^{\lfloor t/t_f \rfloor - 2\gamma - d_2/\alpha_1}dr\right) \leq c\frac{a_k^{\gamma}(t,x,y)}{V(\Phi^{-1}(t))} \asymp q(Lt,x,y). \end{aligned}$$

In the last inequality, we used the assumption that $t/t_f \geq \lfloor d_2/\alpha_1 + 2\gamma \rfloor$.

Case 2. $\lambda > 0$;

If $aL\Phi(l)/2 \geq r_0$, then by Lemma 2.15,

$$\begin{aligned}\int_{aL\Phi(l)/2}^{Lt} r^{-1}q(r,x,y)\mathbb{P}(S_r \geq t)dr &\leq c\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma \int_{r_0}^{Lt} r^{-1}e^{-\lambda r}\left(\frac{r}{t}\right)^{ct} dr \\ &\leq c\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma L^{ct}\int_{r_0}^{Lt} dr \leq c\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma e^{-ct} \simeq q(ct,x,y).\end{aligned}$$

Otherwise, if $aL\Phi(l)/2 < r_0$, then by Proposition 2.14 and Lemma 2.15 and the above calculation,

$$\begin{aligned}&\int_{aL\Phi(l)/2}^{Lt} r^{-1}q(r,x,y)\mathbb{P}(S_r \geq t)dr \\ &\leq ce^{-ct\log t}\int_{aL\Phi(l)/2}^{r_0} \frac{r^{\lfloor t/t_f\rfloor+2\gamma}a_k^\gamma(r,x,y)}{r^{2\gamma}V(\Phi^{-1}(r))}dr + c\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma e^{-ct} \\ &\leq c\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma\left(e^{-ct\log t}\int_{aL\Phi(l)/2}^{r_0} r^{\lfloor t/t_f\rfloor-2\gamma-d_2/\alpha_1}dr + ce^{-ct}\right) \\ &\leq c\,\Phi(\delta_D(x))^\gamma\,\Phi(\delta_D(y))^\gamma e^{-ct} \simeq q(ct,x,y). \quad \square\end{aligned}$$

Proof of Theorem 1.19. By Proposition 4.2 and the second paragraph in its proof, it remains to consider the case when $\Phi(l) \leq t \leq \lfloor d_2/\alpha_1 + 2\gamma\rfloor t_f$. Then, by using Proposition 2.14 instead of Proposition 2.7, we get the result by the same argument as in the proof for Theorem 1.15. We omit in here. $\square$

Proof of Theorems 1.5, 1.6, 1.9 and 1.10. Let $\Phi_\alpha(x) := x^\alpha$. Then, we can check that **(J1)** equals $\mathbf{HK}_J^{1/2,\lambda,1}(\Phi_\alpha,\Phi_\alpha)$, **(J2)** equals $\mathbf{HK}_J^{1/2,0,1}(\Phi_\alpha,\Phi_\alpha)$, **(J3)** equals $\mathbf{HK}_J^{1/2,0,2}(\Phi_\alpha,\Phi_\alpha)$, **(J4)** equals $\mathbf{HK}_J^{(\alpha-1)/\alpha,\lambda,1}(\Phi_\alpha,\Phi_\alpha)$, **(D1)** equals $\mathbf{HK}_D^{1/2,\lambda,1}(\Phi_\alpha)$, **(D2)** equals $\mathbf{HK}_D^{1/2,0,1}(\Phi_\alpha)$ and **(D3)** equals $\mathbf{HK}_D^{1/2,\lambda,2}(\Phi_\alpha)$ where the underlying function $V(x,r) := r^d$ for all $x \in D$ and $r > 0$. Hence, we can apply Theorems 1.15, 1.16, 1.18 and 1.19. Combining these results with Proposition 1.22 and Remark 1.23, we get the result. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

In this section, we give the sketch of proof of Proposition 1.22. Fix $t > 0$ and $x, y \in D$ satisfying $\Phi(\rho(x,y))\phi(t^{-1}) \leq 1/(4e^2)$ and set $V(r) := V(x,r)$ and $l := \rho(x,y)$ as before.

Lemma A.1. *Fix $p \in \mathbb{R}$. For $0 < A < B/2$, define*

$$S_p(A,B) := \int_A^B \frac{1}{r^pV(\Phi^{-1}(r))}dr.$$

Then, the followings are true.

(i) There exists a constant $c > 0$ independent of A and B such that

$$S_p(A,B) \geq c\big(A^{1-p}V(\Phi^{-1}(A))^{-1} + B^{1-p}V(\Phi^{-1}(B))^{-1}\big).$$

(ii) If $d_1 > \alpha_2(1-p)$, *then* $S_p(A,B) \asymp A^{1-p}V(\Phi^{-1}(A))^{-1}$.
(iii) If $d_2 < \alpha_1(1-p)$, *then* $S_p(A,B) \asymp B^{1-p}V(\Phi^{-1}(B))^{-1}$.
(iv) If $d_1 = d_2 = (1-p)\alpha_1 = (1-p)\alpha_2$, *then* $S_p(A,B) \asymp \log(B/A)$.

Proof. (i) By the monotonicities and the weak scaling properties of V and Φ,

$$\begin{aligned} 2S_p(A,B) &\geq \int_A^{2A} \frac{1}{r^pV(\Phi^{-1}(r))}dr + \int_{B/2}^{B} \frac{1}{r^pV(\Phi^{-1}(r))}dr \\ &\geq \frac{A^{1-p}}{2^pV(\Phi^{-1}(2A))} + \frac{B^{1-p}}{2V(\Phi^{-1}(B))} \\ &\geq c\big(A^{1-p}V(\Phi^{-1}(A))^{-1} + B^{1-p}V(\Phi^{-1}(B))^{-1}\big). \end{aligned}$$

(ii), (iii) See [4, 2.12.16].

(iv) In this case, since the assumptions imply that $V(r) \asymp r^{d_1}$ and $\Phi^{-1}(r) \asymp r^{1/\alpha_1}$ for all $r > 0$, we get $S_p(A,B) \asymp \int_A^B r^{-p-d_1/\alpha_1}dr = \int_A^B r^{-1}dr = \log(B/A)$. $\square$

Recall that $\delta_*^\Phi(x,y) = \Phi(\delta_D(x))\Phi(\delta_D(y))$. Without loss of generality, by symmetry, we can assume that $\delta_D(x) \leq \delta_D(y)$. We first claim that if $\Phi(l)\phi(t^{-1}) \leq 1/(4e^2)$, then

$$\big(\phi(t^{-1})^{-1} + \Phi(\delta_D(x))\big)\big(\phi(t^{-1})^{-1} + \Phi(\delta_D(y))\big) \asymp \phi(t^{-1})^{-2} + \delta_*^\Phi(x,y).$$

Indeed, it is clear that $(RHS) \leq (LHS)$ and we also have that

$$\begin{aligned} (LHS) &\leq \phi(t^{-1})^{-2} + \delta_*^\Phi(x,y) + 2\phi(t^{-1})^{-1}\Phi(\delta_D(x)+l) \\ &\leq \phi(t^{-1})^{-2} + \delta_*^\Phi(x,y) + 2\phi(t^{-1})^{-1}\big(\Phi(2\delta_D(x)) + \Phi(2l)\big) \\ &\leq c\phi(t^{-1})^{-2} + \delta_*^\Phi(x,y) + c\Phi(\delta_D(x))^2 + c\Phi(l)^2 \leq c(RHS). \end{aligned}$$

In the third line, we used the fact that $2ab \leq a^2+b^2$ for $a,b \in \mathbb{R}$, the weak scaling properties of Φ and the assumption that $\phi(t^{-1})^{-1} \geq 4e^2\Phi(l)$. Thus, if $\Phi(l)\phi(t^{-1}) \leq 1/(4e^2)$, then

$$\begin{aligned} a_1^\gamma(1/\phi(t^{-1}),x,y) &= \left(\frac{\delta_*^\Phi(x,y)}{\big(\phi(t^{-1})^{-1} + \Phi(\delta_D(x))\big)\big(\phi(t^{-1})^{-1} + \Phi(\delta_D(y))\big)}\right)^\gamma \\ &\asymp \left(\frac{\delta_*^\Phi(x,y)}{\phi(t^{-1})^{-2} + \delta_*^\Phi(x,y)}\right)^\gamma \asymp \left(1 \wedge \frac{\delta_*^\Phi(x,y)}{\phi(t^{-1})^{-2}}\right)^\gamma \asymp \left(1 \wedge \frac{\delta_*^\Phi(x,y)^\gamma}{\phi(t^{-1})^{-2\gamma}}\right). \end{aligned}$$

Now, We consider the following three scenarios.
(Sc.1) $\Phi(\delta_D(x)) \leq 4\Phi(l)$.
(Sc.2) $4\Phi(l) < \Phi(\delta_D(x))$ and $\Phi(\delta_D(y)) \leq 1/(4e^2\phi(t^{-1}))$.
(Sc.3) $4\Phi(l) < \Phi(\delta_D(x))$ and $\Phi(\delta_D(y)) > 1/(4e^2\phi(t^{-1}))$.
If (Sc.1) is true, then we have

$$\mathcal{I}_1^\gamma(t,x,y) \asymp \delta_*^\Phi(x,y)^\gamma S_{2\gamma}\big(\Phi(l), 1/(2e^2\phi(t^{-1}))\big).$$

Else if (Sc.2) is true, then we have

$$\begin{aligned} \mathcal{I}_1^\gamma(t,x,y) \asymp\ & S_0\big(\Phi(l), \Phi(\delta_D(x))/2\big) + \Phi(\delta_D(x))^\gamma S_\gamma\big(\Phi(\delta_D(x))/2, \Phi(\delta_D(y))\big) \\ &+ \delta_*^\Phi(x,y)^\gamma S_{2\gamma}\big(\Phi(\delta_D(y)), 1/(2e^2\phi(t^{-1}))\big). \end{aligned}$$

Otherwise, if (Sc.3) is true, then we get

$$F_1^\gamma(t,x,y) \asymp F_1^0(t,x,y) \asymp S_0\big(\Phi(l), 1/(2e^2\phi(t^{-1}))\big).$$

Hence, by applying Lemma A.1 with $p = 0, \gamma$ and 2γ, we obtain the following estimates.

(a) Suppose that $d_2/\alpha_1 < 1-2\gamma$. Then,

$$\mathcal{I}_1^\gamma(t,x,y) \asymp \begin{cases} \delta_*^\Phi(x,y)^\gamma \phi(t^{-1})^{2\gamma-1} V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)^{-1}, & \text{if (Sc.1) is true;} \\ \delta_*^\Phi(x,y)^\gamma \phi(t^{-1})^{2\gamma-1} V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)^{-1}, & \text{if (Sc.2) is true;} \\ \phi(t^{-1})^{-1} V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)^{-1}, & \text{if (Sc.3) is true.} \end{cases}$$

(b) Suppose that $\alpha_1 = \alpha_2$, $d_1 = d_2 = (1-2\gamma)\alpha_1$ and $\gamma > 0$. Then, $V(r) \asymp r^{d_1}$, $\Phi(r) \asymp r^{\alpha_1}$ and

$$\mathcal{I}_1^\gamma(t,x,y) \asymp \begin{cases} \delta_*^\Phi(x,y)^\gamma \log\left(\dfrac{1}{\Phi(l)\phi(t^{-1})}\right), & \text{if (Sc.1) is true;} \\ \delta_*^\Phi(x,y)^\gamma \log\left(\dfrac{1}{\Phi(\delta_D(y))\phi(t^{-1})}\right), & \text{if (Sc.2) is true;} \\ \phi(t^{-1})^{-1} V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)^{-1}, & \text{if (Sc.3) is true.} \end{cases}$$

(c) Suppose that $1-2\gamma < d_1/\alpha_2 \le d_2/\alpha_1 < 1-\gamma$. Then,

$$\mathcal{I}_1^\gamma(t,x,y) \asymp \begin{cases} \delta_*^\Phi(x,y)^\gamma\, \Phi(l)^{1-2\gamma} V(l)^{-1}, & \text{if (Sc.1) is true;} \\ \delta_*^\Phi(x,y)^\gamma\, \Phi(\delta_D(y))^{1-2\gamma} V(\delta_D(y))^{-1}, & \text{if (Sc.2) is true;} \\ \phi(t^{-1})^{-1} V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)^{-1}, & \text{if (Sc.3) is true.} \end{cases}$$

(d) Suppose that $\alpha_1 = \alpha_2$, $d_1 = d_2 = (1-\gamma)\alpha_1$ and $\gamma > 0$. Then, $V(r) \asymp r^{d_1}$, $\Phi(r) \asymp r^{\alpha_1}$ and

$$\mathcal{I}_1^\gamma(t,x,y) \asymp \begin{cases} \delta_*^\Phi(x,y)^\gamma\, \Phi(l)^{1-2\gamma} V(l)^{-1}, & \text{if (Sc.1) is true;} \\ \Phi(\delta_D(x))^\gamma \log\left(\dfrac{2\Phi(\delta_D(y))}{\Phi(\delta_D(x))}\right), & \text{if (Sc.2) is true;} \\ \phi(t^{-1})^{-1} V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)^{-1}, & \text{if (Sc.3) is true.} \end{cases}$$

(e) Suppose that $1-\gamma < d_1/\alpha_2 \le d_2/\alpha_1 < 1$. Then,

$$\mathcal{I}_1^\gamma(t,x,y) \asymp \begin{cases} \delta_*^\Phi(x,y)^\gamma\, \Phi(l)^{1-2\gamma} V(l)^{-1}, & \text{if (Sc.1) is true;} \\ \Phi(\delta_D(x)) V(\delta_D(x))^{-1}, & \text{if (Sc.2) is true;} \\ \phi(t^{-1})^{-1} V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)^{-1}, & \text{if (Sc.3) is true.} \end{cases}$$

(f) Suppose that $d_1 = d_2 = \alpha_1 = \alpha_2$. Then, $V(r) \asymp r^{d_1}$, $\Phi(r) \asymp r^{\alpha_1}$ and

$$\mathcal{I}_1^\gamma(t,x,y) \asymp \begin{cases} \delta_*^\Phi(x,y)^\gamma\, \Phi(l)^{-2\gamma}, & \text{if (Sc.1) is true;} \\ \log\left(\dfrac{\Phi(\delta_D(x))}{\Phi(l)}\right), & \text{if (Sc.2) is true;} \\ \log\left(\dfrac{1}{\Phi(l)\phi(t^{-1})}\right), & \text{if (Sc.3) is true.} \end{cases}$$

(g) Suppose that $1 < d_1/\alpha_2$. Then,

$$\mathcal{I}_1^\gamma(t,x,y) \asymp \begin{cases} \delta_*^\Phi(x,y)^\gamma\, \Phi(l)^{1-2\gamma} V(l)^{-1}, & \text{if (Sc.1) is true;} \\ \Phi(l) V(l)^{-1}, & \text{if (Sc.2) is true;} \\ \Phi(l) V(l)^{-1}, & \text{if (Sc.3) is true.} \end{cases}$$

Together with the fact that $\phi(t^{-1}) \geq t^{-1}\int_0^t e^{-s/t} w(s)ds \geq e^{-1}w(t)$, we get the result.

Appendix B. Further examples

Example B.1 (*Cf. [6, Example 2.5(ii)]*)**.** Let $0 < \alpha \leq 2$, $0 < \beta < 1$ and $\delta > 0$. Then, we consider the fundamental solution of the following Cauchy problem.

$$\begin{aligned} \frac{d}{dt}\int_{(t-\delta)\vee 0}^{t} \big[(t-s)^{-\beta} - \delta^{-\beta}\big]\big(u(s,x) - f(x)\big)ds &= \Delta^{\alpha/2}u(t,x), \qquad x \in \mathbb{R}^d, \quad t > 0, \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}^d. \end{aligned} \tag{B.1}$$

In this case, we see that $w(s) = w_\delta(s) = (s^{-\beta} - \delta^{-\beta})\mathbf{1}_{(0,\delta]}(s)$ and hence conditions **(Ker.)** and **(Trunc.)**(δ) hold. Moreover, it is well known that for the function $\Phi_\alpha(x) = x^\alpha$, the heat kernel $q(t,x,y)$ corresponding to the generator $\Delta^{\alpha/2}$ enjoys estimate $\mathbf{HK}_J^{0,0,0}(\Phi_\alpha, \Phi_\alpha)$ if $0 < \alpha < 2$ and estimate $\mathbf{HK}_D^{0,0,0}(\Phi_\alpha)$ if $\alpha = 2$. By Theorems 1.15 and 1.19, we obtain the global estimates for the fundamental solution $p(t,x,y)$ of Eq. (B.1).

(i) For every $t \in (0, \delta/2]$ and $x, y \in \mathbb{R}^d$, we have

$$p(t,x,y) \simeq \begin{cases} t^{-\beta d/\alpha}, & \text{if } |x-y| \leq t^{\beta/\alpha} \text{ and } d < \alpha; \\ t^{-\beta}\log\left(\dfrac{2t^{\beta/\alpha}}{|x-y|}\right), & \text{if } |x-y| \leq t^{\beta/\alpha} \text{ and } d = \alpha; \\ t^{-\beta}|x-y|^{\alpha-d}, & \text{if } |x-y| \leq t^{\beta/\alpha} \text{ and } d > \alpha; \\ \dfrac{t^\beta}{|x-y|^{d+\alpha}}, & \text{if } |x-y| > t^{\beta/\alpha} \text{ and } 0 < \alpha < 2; \\ t^{-\beta d/\alpha}\exp\big(-c|x-y|^{2/(2-\beta)}t^{-\beta/(2-\beta)}\big), & \text{if } |x-y| > t^{\beta/\alpha} \text{ and } \alpha = 2. \end{cases}$$

(ii) Fix any $t \in [\delta/2, \infty)$ and $x, y \in \mathbb{R}^d$. Let $n_t = \lfloor t/\delta \rfloor + 1$. Then, we have

$$p(t,x,y) \simeq \begin{cases} \Big[|x-y|^\alpha t^{-1} + (n_t\delta - t)^{n_t}\Big]t^{-n_t}|x-y|^{\alpha n_t - d}, \\ \qquad \text{if } |x-y|^\alpha \leq t \text{ and } \delta/2 \leq t < \lfloor (d-\alpha)/\alpha \rfloor\delta; \\ t^{-d/\alpha} + (n_t\delta - t)^{n_t}t^{-n_t}|x-y|^{\alpha n_t - d}, \\ \qquad \text{if } d/\alpha \notin \mathbb{N}, \ |x-y|^\alpha \leq t \text{ and } \lfloor (d-\alpha)/\alpha \rfloor\delta \leq t < \lfloor d/\alpha \rfloor\delta; \\ t^{-d/\alpha} + (\dfrac{d\delta}{\alpha t} - 1)^{d/\alpha}\log\left(\dfrac{2t}{|x-y|^\alpha}\right), \\ \qquad \text{if } d/\alpha \in \mathbb{N}, \ |x-y|^\alpha \leq t \text{ and } (d-\alpha)\delta/\alpha \leq t < d\delta/\alpha; \\ t^{-d/\alpha}, \qquad \text{if } |x-y|^\alpha \leq t \text{ and } \lfloor d/\alpha \rfloor\delta \leq t; \\ \dfrac{t}{|x-y|^{d+\alpha}}, \qquad \text{if } |x-y|^\alpha > t \text{ and } 0 < \alpha < 2, \\ t^{-d/\alpha}\exp\big(-c|x-y|^2t^{-1}\big), \qquad \text{if } |x-y|^\alpha > t \text{ and } \alpha = 2. \end{cases}$$

In particular, for every $t > 0$ and $x \in \mathbb{R}^d$, $p(t,x,x) < \infty$ if and only if $t \geq \lfloor d/\alpha \rfloor\delta$. $\square$

Recall that in Example 1.7, we obtain the global two-sided estimates on the fundamental solution $p_\beta(t,x,y)$ of the following time fractional equation:

$$\begin{aligned} &\frac{d}{dt}\int_0^t w(t-s)\big(u(t,x)-f(x)\big)ds = \Delta^{\alpha/2}u(t,x), \quad x\in D, \quad t>0, \\ &u(0,x)=f(x), \quad x\in D, \qquad u(t,x)=0, \quad x\in\mathbb{R}^d\setminus D, \quad t>0, \end{aligned} \tag{B.2}$$

with $w(s)=s^{-\beta}/\Gamma(1-\beta)$.

Example B.2. Let $0<\alpha\le 2, d\ge 1$ and $D\subset\mathbb{R}^d$ be a bounded $C^{1,1}$ open set. We further assume that if $\alpha=2$, then $d\ge 2$ and D is connected. Let $p^{tem}(t,x,y)$ be the fundamental solution of (B.2) with

$$w(s)=\int_s^\infty \frac{\beta}{\Gamma(1-\beta)}\frac{e^{-\theta y}}{y^{1+\beta}}dy, \quad (0<\beta<1, \ \ \theta>0).$$

The corresponding subordinator is called a tempered stable subordinator in the literature. (See, e.g. [27, Section 3].) By Theorems 1.15 and 1.18, we obtain the global two-sided estimates on $p^{tem}(t,x,y)$.

6.2.1. Small time estimates. For all $t\in(0,2]$ and $x,y\in D$, $p^{tem}(t,x,y)\simeq p_\beta(t,x,y)$.

6.2.2. Large time estimates. For all $t\in[2,\infty)$ and $x,y\in D$, there are constants $c_1,c_2,c_3,c_4>0$ and $L_1\ge L_2>0$ such that

$$\begin{aligned} p^{tem}(t,x,y) \ge\ & c_1e^{-L_1t}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}+c_2\exp\big(-2\theta t\big)\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} \\ &\times\begin{cases}\Big(|x-y|\vee\big(\delta_D(x)\wedge\delta_D(y)\big)\Big)^{\alpha-d}, & \text{if } d<\alpha; \\ \left(1+\log^+\left(\dfrac{\delta_D(x)\wedge\delta_D(y)}{|x-y|}\right)\right), & \text{if } d=\alpha; \\ |x-y|^{\alpha-d}, & \text{if } d>\alpha,\end{cases} \end{aligned}$$

and

$$\begin{aligned} p^{tem}(t,x,y) \le\ & c_3e^{-L_2t}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}+c_4\exp\big(-\frac{\theta}{2}t\big)\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} \\ &\times\begin{cases}\Big(|x-y|\vee\big(\delta_D(x)\wedge\delta_D(y)\big)\Big)^{\alpha-d}, & \text{if } d<\alpha; \\ \left(1+\log^+\left(\dfrac{\delta_D(x)\wedge\delta_D(y)}{|x-y|}\right)\right), & \text{if } d=\alpha; \\ |x-y|^{\alpha-d}, & \text{if } d>\alpha. \quad \square\end{cases} \end{aligned}$$

Following [12], for a function f on $\mathbb{R}^d$, we define for $1<\alpha<2$ and $r>0$,

$$M_f^\alpha := \sup_{x\in\mathbb{R}^d}\int_{|y-x|<r}\frac{|f(y)|}{|x-y|^{d+1-\alpha}}dy.$$

Then, a function f on $\mathbb{R}^d$ is said to belong to the Kato class $\mathbb{K}^{\alpha-1}$ if $\lim_{r\to 0+}M_f^\alpha(r)=0$.

Example B.3. Let $1<\alpha<2$, $d\ge 1$ and $D\subset\mathbb{R}^d$ be a bounded $C^{1,1}$ open set. In [12], the authors studied the stability of Dirichlet heat kernel estimates under gradient perturbation.

More precisely, for every $b \in \mathbb{K}^{\alpha-1}$, an operator $\big(\Delta^{\alpha/2} + b \cdot \nabla\big)|_D$ enjoys the estimates $\mathbf{HK}_J^{1/2,\lambda^b,1}(\Phi_\alpha, \Phi_\alpha)$ for some constant $\lambda^b > 0$.

Let $p^{per}(t,x,y)$ be the fundamental solution of (B.2) replacing the operator $\Delta^{\alpha/2}$ with $\Delta^{\alpha/2}+b\cdot\nabla$ for some $b \in \mathbb{K}^{\alpha-1}$ and $w(s) = s^{-\beta}/\Gamma(1-\beta)$ $(0 < \beta < 1)$. Then, since the operators $\Delta^{\alpha/2} + b\cdot\nabla$ and $\Delta^{\alpha/2}$ admit the same form of heat kernel estimates $\mathbf{HK}_J^{1/2,\lambda,1}(\Phi_\alpha, \Phi_\alpha)$ (with possibly different λ), we see that $p^{per}(t,x,y) \simeq p_\beta(t,x,y)$ for all $t \in (0,\infty)$ and $x, y \in D$.

Example B.4. Let $0 < \alpha' < 2$, $d \geq 2$ and $D \subset \mathbb{R}^d$ be a bounded connected $C^{1,1}$ open set. Let $p^{mix}(t,x,y)$ be the fundamental solution of (B.2) replacing the operator $\Delta^{\alpha/2}$ with $\Delta + \Delta^{\alpha'/2}$ and $w(s) = s^{-\beta}/\Gamma(1-\beta)$ $(0 < \beta < 1)$. According to [11, Theorem 1.3], the heat kernel corresponding to the operator $\Delta + \Delta^{\alpha'/2}$ enjoys the estimate $\mathbf{HK}_M^{1/2,\lambda',1}(\Phi_2 \wedge \Phi_{\alpha'}, \Phi_{\alpha'})$ for some $\lambda' > 0$. Hence, by Theorems 1.15 and 1.16, the fundamental solution $p^{mix}(t,x,y)$ admits the same estimates as $p_\beta(t,x,y)$ with $\alpha = 2$, unless $t \in (0,2]$ and $|x-y| > t^{\beta/2}$. For those values of t and $x, y \in D$, by Theorem 1.15(ii)(c), we get

$$\begin{aligned} &p^{mix}(t,x,y) \\ &\simeq \left(1 \wedge \frac{\delta_D(x)}{t^{\beta/2}}\right)\left(1 \wedge \frac{\delta_D(y)}{t^{\beta/2}}\right)\left(\frac{t^\beta}{|x-y|^{d+\alpha'}} + t^{-\beta d/2}\exp\left(-c|x-y|^{2/(2-\beta)}/t^{\beta/(2-\beta)}\right)\right). \end{aligned}$$

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