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On exponential stability criteria of stochastic partial differential equations

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Abstract

Some criteria for the mean square and almost sure exponential stability of nonlinear stochastic partial differential equations are shown in this paper. In particular, the main results obtained in Caraballo and Real (1994, Stochast. Anal. Appl. 12(5), 517–525) are improved, since the new coercivity condition introduced in this work permits the state independent term γ to be time dependent and nonnegative but of subexponential growth, while in Caraballo and Real (1994) this parameter is required to be constant and nonpositive. Several examples are studied to illustrate the theory.

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1. Introduction and preliminaries

The main aim of this paper is to establish some criteria for the mean square and almost sure exponential stability of a class of nonlinear stochastic partial differential equations of monotone type. In fact, a coercivity condition, extending the one considered by Chow (1982) and Caraballo and Real (1994), is introduced and will play the role of a stability criterion. To be precise, under the coercivity condition (Theorem 1.2 below) from Caraballo and Real (1994), almost sure exponential stability of solutions is obtained, while in Chow (1982) pathwise asymptotic stability is proved. However, as we will explain later, coercivity criteria from Caraballo and Real (1994) are too restrictive to be applied to a number of interesting and, in our opinion, important examples, especially in the nonautonomous case. In this work, we shall improve their results to cover the general nonautonomous stochastic differential equations in Hilbert spaces. For this purpose, let us first state some basic notations and notions (mainly from Caraballo and Real, 1994; Chow, 1982).

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Let V be a Banach space and H, K real, separable Hilbert spaces such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V',$$

where the injections are continuous and dense.

Let $||\cdot||$, $|\cdot|$ and $||\cdot||_*$ denote the norms in V, H and V', respectively, $\langle\cdot,\cdot\rangle$ the duality product between V' and V, $(\cdot,\cdot)$ the inner product in H, and β a constant such that

$$|x| \leqslant \beta||x||, \quad \forall x \in V.$$

Let W_t be a Wiener process defined on some complete probability space $(\Omega, \mathscr{F}, P)$ and taking its values in the separable Hilbert space K, with increment covariance operator Q.

Consider the following nonlinear stochastic diffusion equation:

$$X_t = X_0 + \int_0^t A(s, X_s)\,\mathrm{d}s + \int_0^t B(s, X_s)\,\mathrm{d}W_s, \tag{1.1}$$

where $A(t,\cdot) : V \to V'$ is a family of nonlinear operators defined a.e.t. satisfying $A(t,0) = 0$ for all $t \in \mathbb{R}_+$; and where $B(t,\cdot) : V \to \mathscr{L}(K,H)$, the family of all bounded linear operators from K into H, satisfies

(b.1) $B(t,0) = 0$;

(b.2) There exists $k > 0$ such that

$$||B(t,y) - B(t,x)|| \leqslant k||y-x||, \quad \forall x, y \in V, \text{ a.e.}t,$$

(b.3) $t \in (0,T) \to B(t,x) \in \mathscr{L}(K,H)$ is Lebesgue-measurable $\forall x \in V$, $\forall T > 0$.

Definition 1.1. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ be the stochastic basis and W_t a K-valued Wiener process with covariance operator Q. Suppose that X_0 is an H-valued random variable such that $E|X_0|^2 < \infty$. A stochastic process X_t is said to be a *strong solution* on Ω to the SDE (1.1) for $t \in [0,T]$ if the following conditions are satisfied:

(a) X_t is a V-valued $\mathscr{F}_t$-measurable random variable;

(b) $X_t \in I^p(0,T;V) \cap L^2(\Omega; C(0,T;H))$, $p > 1$, $T > 0$, where $I^p(0,T;V)$ denotes the space of all V-valued processes $(X_t)_{t\in[0,T]}$ (we will write X_t for short) measurable (from $[0,T] \times \Omega$ into V), and satisfying

$$E\int_0^T ||X_t||^p\,\mathrm{d}t < \infty.$$

Here $C(0,T;H)$ denotes the space of all continuous functions from $[0,T]$ to H;

(c) Eq. (1.1) is satisfied for every $t \in [0,T]$ with probability one.

If T is replaced by ∞, X_t is called a global strong solution of (1.1).

As we are mainly interested in stability analysis, one always assumes that for each H-valued random variable X_0 with $E|X_0|^2 < \infty$, there exists a global strong solution to (1.1). In this situation, it is reasonable to assume the following (see Pardoux, 1975)

(a.1) (*Coercivity*). There exist $\alpha > 0$, $p > 1$ and λ, $\gamma \in \mathbb{R}$ such that

$$2\langle A(t,x), x\rangle + ||B(t,x)||_2^2 \leqslant -\alpha||x||^p + \lambda|x|^2 + \gamma, \quad \forall x \in V, \text{ a.e.}t.$$

where $||\cdot||_2$ denotes the Hilbert–Schmidt norm of nuclear operator, i.e.,

$$||B(t,x)||_2^2 = \mathrm{tr}(B(t,x)QB(t,x)^*),$$

(a.2) (*Boundedness*). There exists $c > 0$ such that

$$||A(t,x)||_* \leqslant c||x||^{p-1}, \quad \forall x \in V, \text{ a.e.}t,$$

(a.3) (*Monotonicity*).

$$-2\langle A(t,x) - A(t,y), x - y\rangle + \lambda|x - y|^2 \geqslant ||B(t,x) - B(t,y)||_2^2, \quad \forall x, y \in V, \text{ a.e.}t,$$

(a.4) (*Hemicontinuity*). The map $\theta \in \mathbb{R} \mapsto \langle A(t, x + \theta y), z\rangle \in \mathbb{R}$ is continuous $\forall x, y, z \in V$, a.e.t,

(a.5) (*Measurability*). $t \in (0,T) \mapsto A(t,x) \in V'$ is Lebesgue-measurable $\forall x \in V$, a.e.t., $\forall T > 0$.

The following stability criterion is proved in Caraballo and Real (1994):

Theorem 1.2. *Assume conditions* (b.1)–(b.3) *and* (a.1) *hold. We also suppose that* X_t *is a global strong solution to* (1.1). *Then, there exists* $r > 0$ *such that*

$$E|X_t|^2 \leqslant E|X_0|^2 \mathrm{e}^{-rt}, \quad \forall t \geqslant 0, \tag{1.2}$$

if either one of the following hypotheses holds:

(a) $\lambda < 0$, $\gamma \leqslant 0$, $(\forall p > 1)$;

(b) $\lambda\beta^2 - \alpha < 0$, $\gamma \leqslant 0$, $(p = 2)$.

Furthermore, under the same conditions the solution is almost surely stable. That is, there exist positive constants ξ, η *and a subset* $N_0 \subset \Omega$ *with* $P(N_0) = 0$ *such that, for each* $\omega \notin N_0$, *there exists a positive random number* $T(\omega)$ *such that the following holds*:

$$|X_t(\omega)|^2 \leqslant \eta|X_0|^2 \mathrm{e}^{-\xi t}, \quad \forall t \geqslant T(\omega).$$

However, when the time variable does appear in the operators $A(t,\cdot)$ and $B(t,\cdot)$ in an explicit way or the term γ is finally positive so that neither hypothesis (a) nor (b) holds, this criterion cannot be applied (see the examples in Section 3). In the following section, we shall improve this theorem.

2. The main results

In this section, we shall prove the mean square and almost sure exponential stability of the solutions to (1.1). Before introducing the coercivity condition which will guarantee such results, we are going to exhibit two simple examples of one-dimensional linear Itô equations in order to motivate the subexponential growth imposed on the state independent term appearing in such a condition:

Example 2.1. First, assume X_t satisfies the following:

$$\mathrm{d}X_t = -pX_t\,\mathrm{d}t + (1+t)^{-q}\,\mathrm{d}W_t, \quad t \geqslant 0$$

with initial data $X_0 = 0$, where $p, q > 0$ are two positive constants and W_t is a one-dimensional standard Brownian motion.

Let $\langle\cdot,\cdot\rangle$ denote the standard inner product in $\mathbb{R}$ and we set $A(t,x) = -px$, $B(t,x) = (1+t)^{-q}$. It easily follows that

$$2\langle A(t,x),x\rangle + ||B(t,x)||^2 = -2px^2 + (1+t)^{-2q}, \tag{2.1a}$$

and, consequently, Theorem 1.2 cannot be applied to this example since $(1+t)^{-2q} > 0$, for all $t \geqslant 0$, and so one cannot find a $\gamma \leqslant 0$ which satisfies (a.1). However, it is easy to obtain the explicit solution

$$X_t = \mathrm{e}^{-pt}\int_0^t \mathrm{e}^{ps}\cdot(1+s)^{-q}\,\mathrm{d}W_s \equiv \mathrm{e}^{-pt}M_t, \quad t \geqslant 0.$$

Noticing the law of the iterated logarithm

$$\limsup_{t\to\infty} \frac{M_t}{\sqrt{2\langle M_t\rangle \log\log\langle M_t\rangle}} = 1 \quad \text{a.s.}$$

and

$$\limsup_{t\to\infty} \frac{\log(\int_0^t \mathrm{e}^{2ps}(1+s)^{-2q}\,\mathrm{d}s)}{t} = 2p,$$

we get Lyapunov exponent

$$\limsup_{t\to\infty} \frac{1}{t}\log|X_t| = 0 \quad \text{a.s.}$$

which means that almost all the sample paths of the solution will not tend to zero exponentially.

Next, suppose Y_t satisfies

$$\mathrm{d}Y_t = -pY_t\,\mathrm{d}t + \mathrm{e}^{-qt}\,\mathrm{d}W_t, \quad t \geqslant 0$$

with initial data $Y_0 = 0$, and p, q both are positive constants.

Assume $A(t,x) = -px$ and $B(t,x) = \mathrm{e}^{-qt}$, then it is easy to deduce

$$2\langle A(t,x),x\rangle + ||B(t,x)||^2 = -2px^2 + \mathrm{e}^{-2qt}, \tag{2.1b}$$

and again Theorem 1.2 cannot be applied.

However, the explicit solution is now given by

$$Y_t = \mathrm{e}^{-pt}\int_0^t \mathrm{e}^{(p-q)s}\,\mathrm{d}W_s \equiv \mathrm{e}^{-pt}N_t, \quad t \geqslant 0.$$

Taking into account again the law of the iterated logarithm for the process N_t and

$$\limsup_{t\to\infty} \frac{\log(\int_0^t \mathrm{e}^{2(p-q)s}\,\mathrm{d}s)}{t} = 2(p-q),$$

we can obtain Lyapunov exponent

$$\limsup_{t\to\infty} \frac{1}{t}\log|Y_t| = -q \quad \text{a.s.}$$

That is, the solution is almost surely exponentially stable.

Therefore, if the term γ appearing in condition (a.1) is permitted to be nonnegative and time dependent, a polynomial decay of such a term is not sufficient, in general, to ensure exponential stability of the solutions. However, the solution could be exponentially stable provided the term tends to zero with an exponential decay.

Bearing these examples in mind, we can now formulate our stability hypothesis. Once again, we consider the stochastic diffusion equation (1.1) where $A(t,\cdot): V \to V'$ is supposed to be a measurable family of nonlinear operators defined a.e.t. and $B(t,\cdot): V \to \mathscr{L}(K,H)$ a measurable family of operators. Note that, at the moment, we do not assume $A(t,0)=0$ and $B(t,0)=0$, $t \in \mathbb{R}_+$, as in Caraballo and Real (1994).

The following coercivity condition (CC) will play a key role in our stability result: There exist constants $\alpha > 0$, $\mu > 0$, $\lambda \in \mathbb{R}$, and a nonnegative continuous function $\gamma(t)$, $t \in \mathbb{R}_+$, such that

$$2\langle A(t,v),v\rangle + ||B(t,v)||_2^2 \leqslant -\alpha||v||^p + \lambda|v|^2 + \gamma(t)\mathrm{e}^{-\mu t}, \quad v \in V, \tag{2.2}$$

where $p > 1$ and, for arbitrary $\delta > 0$, $\gamma(t)$ satisfies $\gamma(t) = \mathrm{o}(\mathrm{e}^{\delta t})$, as $t \to \infty$, i.e., $\lim_{t\to\infty} \gamma(t)/\mathrm{e}^{\delta t} = 0$.

Remark 2.1. Observe that, owing to the continuity and subexponential growth of the term $\gamma(t)\mathrm{e}^{-\mu t}$, there exists a positive constant $\tilde{\gamma}$ such that $\gamma(t)\mathrm{e}^{-\mu t} \leqslant \tilde{\gamma}$ for all $t \in \mathbb{R}_+$. As a consequence, (2.2) implies (a.1) (by replacing γ by $\tilde{\gamma}$), i.e., this assumption is compatible with the existence of the strong solutions to (1.1).

Theorem 2.2. *Assume conditions* (CC), (b.2) *and* (b.3) *hold. Then, if X_t is a global strong solution to Eq.* (1.1), *there exist constants $\tau > 0$, $C > 0$ such that*

$$E|X_t|^2 \leqslant C \cdot \mathrm{e}^{-\tau t}, \quad \forall t \geqslant 0, \tag{2.3}$$

if either one of the following hypotheses holds

(i) $\lambda < 0$, $(\forall p > 1)$;

(ii) $\lambda\beta^2 - \alpha < 0$, $(p=2)$.

Proof. We only show case (ii). Case (i) can be proved similarly. Firstly, we can choose $\delta > 0$ small enough such that $\mu - \delta > 0$. Then, Itô's formula implies

$$\begin{aligned}
&\mathrm{e}^{(\mu-\delta)t}|X_t|^2 - |X_0|^2 \\
&\quad = (\mu-\delta)\int_0^t \mathrm{e}^{(\mu-\delta)s}|X_s|^2\,\mathrm{d}s + 2\int_0^t \mathrm{e}^{(\mu-\delta)s}\langle A(s,X_s),X_s\rangle\,\mathrm{d}s \\
&\qquad + 2\int_0^t \mathrm{e}^{(\mu-\delta)s}\langle X_s, B(s,X_s)\,\mathrm{d}W_s\rangle \\
&\qquad + \int_0^t \mathrm{e}^{(\mu-\delta)s} tr(B(s,X_s)QB(s,X_s)^*)\,\mathrm{d}s.
\end{aligned} \tag{2.4}$$

Now, since $\int_0^t \mathrm{e}^{(\mu-\delta)s}\langle X_s, B(s,X_s)\,\mathrm{d}W_s\rangle$, $t \in \mathbb{R}_+$, is a continuous martingale, it follows that

$$E\left(\int_0^t \mathrm{e}^{(\mu-\delta)s}\langle X_s, B(s,X_s)\,\mathrm{d}W_s\rangle\right) = 0, \quad t \in \mathbb{R}_+.$$

Therefore, condition (2.2) and the continuous injection $V \hookrightarrow H$ yield

$$\mathrm{e}^{(\mu-\delta)t}E|X_t|^2 \leqslant E|X_0|^2 + (\mu-\delta-v)\int_0^t \mathrm{e}^{(\mu-\delta)s}E|X_s|^2\,\mathrm{d}s + \int_0^t \gamma(s)\mathrm{e}^{-\delta s}\,\mathrm{d}s, \tag{2.5}$$

where $v = (\alpha - \lambda\beta^2)/\beta^2$.

If $\mu - \nu \leqslant 0$, it follows immediately

$$e^{(\mu-\delta)t}E|X_t|^2 \leqslant E|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s}\,ds,$$

which means that there exists a positive constant $k = k(\delta) > 0$ such that

$$E|X_t|^2 \leqslant (E|X_0|^2 + k(\delta))e^{-(\mu-\delta)t}.$$

On the other hand, if $\mu - \nu > 0$, we can choose $\delta > 0$ small enough such that $\mu - \nu - \delta > 0$. Then, from (2.5) and Gronwall's lemma one can obtain

$$e^{(\mu-\delta)t}E|X_t|^2 \leqslant \left(E|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s}\,ds\right) e^{t(\mu-\delta-\nu)},$$

and, once again, there exists a positive constant $k(\delta) > 0$ such that

$$E|X_t|^2 \leqslant (E|X_0|^2 + k(\delta))e^{-\nu t}. \qquad \square$$

Theorem 2.3. *Assume the hypotheses in Theorem 2.2 hold. Then there exist positive constants M, ε and a subset $N_0 \subset \Omega$ with $P(N_0) = 0$ such that, for each $\omega \notin N_0$, there exists a positive random number $T(\omega)$ such that*

$$|X_t|^2 \leqslant M \cdot e^{-\varepsilon t}, \quad \forall t \geqslant T(\omega). \tag{2.6}$$

Proof. We only prove case (ii) as in the last proof. We shall split our proof into several steps, as follows.

Step 1: We claim that there exists $C > 0$, $\tau > 0$, independent of $t \in \mathbb{R}_+$, such that

$$\int_s^t E||B(u, X_u)||_2^2\,du \leqslant Ce^{-\tau s}, \quad 0 \leqslant s \leqslant t. \tag{2.7}$$

Indeed, applying Itô's formula to (1.1) as in Theorem 2.2, we get that for any $\delta > 0$ with $\mu - \delta > 0$

$$e^{(\mu-\delta)t}E|X_t|^2 \leqslant E|X_0|^2 + (\mu - \delta - \nu)\int_0^t e^{(\mu-\delta)s}E|X_s|^2\,ds + \int_0^t \gamma(s)e^{-\delta s}\,ds \tag{2.8}$$

and

$$\begin{aligned} e^{(\mu-\delta)t}E|X_t|^2 \leqslant{}& E|X_0|^2 + (\mu - \delta + \lambda)\int_0^t e^{(\mu-\delta)s}E|X_s|^2\,ds \\ &+ \int_0^t \gamma(s)e^{-\delta s}\,ds - \alpha\int_0^t e^{(\mu-\delta)s}E||X_s||^2\,ds, \end{aligned} \tag{2.9}$$

where $\nu = (\alpha - \lambda\beta^2)/\beta^2$.

Now, if $\mu - \nu \leqslant 0$, it follows from (2.8) that

$$\int_0^t e^{(\mu-\delta)s}E|X_s|^2\,ds \leqslant \frac{E|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s}\,ds}{\nu + \delta - \mu} \tag{2.10}$$

which, together with (2.9), immediately implies

$$\begin{aligned} \int_0^t e^{(\mu-\delta)s}E||X_s||^2\,ds \leqslant{}& \frac{1}{\alpha}\left[E|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s}\,ds\right] \\ &+ \frac{\mu - \delta + \lambda}{\alpha}\int_0^t e^{(\mu-\delta)s}E|X_s|^2\,ds \end{aligned}$$

$$\leqslant \frac{1}{\alpha}\left[\frac{\mu-\delta+\lambda}{\nu+\delta-\mu}+1\right]\left[E|X_0|^2+\int_0^t \gamma(s)\mathrm{e}^{-\delta s}\,\mathrm{d}s\right]$$

$$\leqslant \frac{1}{\alpha}\left[\frac{\mu-\delta+\lambda}{\nu+\delta-\mu}+1\right][E|X_0|^2+k(\delta)]. \tag{2.11}$$

Consequently, for $0\leqslant s\leqslant t$,

$$\begin{aligned}\int_s^t E||X_u||^2\,\mathrm{d}u &\leqslant \int_s^t \mathrm{e}^{(\mu-\delta)(u-s)}E||X_u||^2\,\mathrm{d}u\\ &\leqslant \mathrm{e}^{-(\mu-\delta)s}\int_0^t \mathrm{e}^{(\mu-\delta)u}E||X_u||^2\,\mathrm{d}u\\ &\leqslant \frac{1}{\alpha}\left[\frac{\mu-\delta+\lambda}{\nu+\delta-\mu}+1\right][E|X_0|^2+k(\delta)]\mathrm{e}^{-(\mu-\delta)s}\end{aligned} \tag{2.12}$$

which, together with (b.2) and (2.2), immediately yields that

$$\begin{aligned}\int_s^t E||B(u,X_u)||_2^2\,\mathrm{d}u &\leqslant 2\int_s^t E||B(u,X_u)-B(u,0)||_2^2\,\mathrm{d}u+2\int_s^t E||B(u,0)||_2^2\,\mathrm{d}u\\ &\leqslant k_1\int_s^t E||X_u||^2\,\mathrm{d}u+k_2\int_s^t \gamma(u)\mathrm{e}^{-\mu u}\,\mathrm{d}u\\ &\leqslant C(\delta)\mathrm{e}^{-(\mu-\delta)s},\end{aligned} \tag{2.13}$$

where k_1, k_2 are two positive constants.

On the other hand, if $\mu-\nu>0$, it is always possible to choose a suitable $\delta>0$ such that $\nu-\delta>0$. Then, by applying Itô's lemma to the strong solution X_t, it is easy to deduce

$$\begin{aligned}\mathrm{e}^{(\nu-\delta)t}E|X_t|^2\leqslant& E|X_0|^2+(\nu-\delta+\lambda)\int_0^t \mathrm{e}^{(\nu-\delta)s}E|X_s|^2\,\mathrm{d}s\\ &+\int_0^t \gamma(s)\mathrm{e}^{-(\mu-\nu+\delta)s}\,\mathrm{d}s-\alpha\int_0^t \mathrm{e}^{(\nu-\delta)s}E||X_s||^2\,\mathrm{d}s\\ \leqslant& E|X_0|^2+(\nu-\delta+\lambda)\int_0^t \mathrm{e}^{(\nu-\delta)s}E|X_s|^2\,\mathrm{d}s\\ &+\int_0^t \gamma(s)\mathrm{e}^{-\delta s}\,\mathrm{d}s-\alpha\int_0^t \mathrm{e}^{(\nu-\delta)s}E||X_s||^2\,\mathrm{d}s.\end{aligned} \tag{2.14}$$

Noticing that, in this case, the parameter τ in Theorem 2.2 turns out to be ν, (2.14) yields

$$\alpha\int_0^t \mathrm{e}^{(\nu-\delta)s}E||X_s||^2\,\mathrm{d}s\leqslant E|X_0|^2+k(\delta)+(\nu-\delta+\lambda)\int_0^t \mathrm{e}^{-\delta s}\,\mathrm{d}s,$$

and we can argue in a similar manner as we did previously. Hence our claim is proved.

Step 2: We claim that there exists a positive constant $M>0$ such that

$$E\Big(\sup_{0\leqslant t<\infty}|X_t|^2\Big)\leqslant M.$$

Indeed, Itô's formula implies

$$|X_t|^2 - |X_0|^2 = 2\int_0^t \langle A(s,X_s),X_s\rangle\,\mathrm{d}s + \int_0^t \mathrm{tr}(B(s,X_s)QB(s,X_s)^*)\,\mathrm{d}s$$
$$+2\int_0^t \langle X_s, B(s,X_s)\,\mathrm{d}W_s\rangle. \tag{2.15}$$

On the other hand, from Burkholder–Davis–Gundy's inequality, we get for any $T\in\mathbb{R}_+$

$$2E\left[\sup_{t\in[0,T]}\left|\int_0^t \langle X_s, B(s,X_s)\,\mathrm{d}W_s\rangle\right|\right]$$
$$\leqslant K_1 E\left[\left(\int_0^T |X_s|^2\|B(s,X_s)\|_2^2\,\mathrm{d}s\right)^{1/2}\right]$$
$$\leqslant K_1 E\left\{\sup_{0\leqslant s\leqslant T}|X_s|\left[\int_0^T \|B(s,X_s)\|_2^2\,\mathrm{d}s\right]^{1/2}\right\}$$
$$\leqslant \frac{1}{2}E\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right] + K_2\int_0^T \|B(s,X_s)\|_2^2\,\mathrm{d}s, \tag{2.16}$$

where K_1, K_2 are two positive constants. Therefore, in addition to condition (CC), (2.15) and (2.16) imply

$$E\left(\sup_{0\leqslant s\leqslant T}|X_s|^2\right) \leqslant E|X_0|^2 + v\int_0^T E|X_s|^2\,\mathrm{d}s + \int_0^T \gamma(s)\mathrm{e}^{-\mu s}\,\mathrm{d}s$$
$$+\frac{1}{2}E\left(\sup_{0\leqslant s\leqslant T}|X_s|^2\right) + K_2\int_0^T E\|B(s,X_s)\|_2^2\,\mathrm{d}s. \tag{2.17}$$

Thus, our claim can be easily obtained owing to (2.3), (2.7) and condition (CC).

Step 3: Now, we can finish our proof. We only sketch it because it is similar to that in Haussmann (1978).

Firstly, the coercivity condition (CC) and (2.15) imply

$$|X_T|^2 \leqslant |X_N|^2 + v\int_N^T |X_s|^2\,\mathrm{d}s + \int_N^T \gamma(s)\mathrm{e}^{-\mu s}\,\mathrm{d}s$$
$$+\left[\sup_{t\in[N,T]}\left|\int_N^t \langle X_s, B(s,X_s)\,\mathrm{d}W_s\rangle\right|\right] \tag{2.18}$$

for $T\geqslant N$, where N is a natural number.

In particular, taking $N\in\mathbb{N}$ large enough, we can easily obtain

$$P\left\{\sup_{t\in[N,N+1]}|X_t|^2\geqslant\varepsilon_N^2\right\}$$
$$\leqslant P\{|X_N|^2\geqslant\varepsilon_N^2/4\} + P\left\{v\int_N^{N+1}|X_s|^2\,\mathrm{d}s\geqslant\varepsilon_N^2/4\right\}$$
$$+P\left\{\left[\sup_{t\in[N,N+1]}\left|\int_N^t \langle X_s, B(s,X_s)\,\mathrm{d}W_s\rangle\right|\right]\geqslant\varepsilon_N^2/4\right\}, \tag{2.19}$$

where $\varepsilon_N^2 = C\mathrm{e}^{-\tau N/4}$.

Now, we can estimate the terms on the right-hand side of (2.19) using Kolmogorov's inequality and (2.3) for the first two terms, and Burkholder–Davis–Gundy's lemma, Hölder's inequality and an argument similar to that used in Steps 1 and 2 for the last one. Consequently, there exists a positive constant $K_3 > 0$ such that

$$P\left[\sup_{t\in[N,N+1]} |X_t|^2 \geqslant \varepsilon_N^2\right] \leqslant K_3 \mathrm{e}^{-\tau N/4}.$$

Finally, a Borel–Cantelli's lemma-type argument completes the proof. □

Next, we shall state a theorem which is a generalization of Theorem 2.2. Due to the fact that Theorem 2.2 appears as a particular case of this general result, we could have established only this last one. However, for the sake of clarity, we have preferred to describe first the simpler one, and then show the general one.

We shall assume the following generalized coercivity condition (CC)$'$:

There exist constants $\alpha > 0$, $\lambda \in \mathbb{R}$, $\mu > 0$, $0 \leqslant \sigma < 1$ and non-negative continuous functions $\gamma(t)$, $\tau(t)$, $t \in \mathbb{R}_+$, such that

$$2\langle A(t,v),v\rangle + ||B(t,v)||_2^2 \leqslant -\alpha||v||^p + \lambda|v|^2 + \tau(t)\mathrm{e}^{-\mu t}|v|^{2\sigma} + \gamma(t)\mathrm{e}^{-\mu t}, \quad v \in V, \tag{2.20}$$

where $p > 1$, and for arbitrary $\delta > 0$, $\gamma(t)$ and $\tau(t)$ satisfy $\tau(t)=\mathrm{o}(\mathrm{e}^{\delta t})$ and $\gamma(t)=\mathrm{o}(\mathrm{e}^{\delta t})$, as $t \to \infty$.

Remark 2.2. The same comments concerning the compatibility of (2.20) with the existence of the strong solutions of (1.1) as in Remark 2.1 once more remains true. This follows immediately from the fact that $h^{2\sigma} \leqslant 1 + h^2$ for all $h \in \mathbb{R}$ and $0 \leqslant \sigma < 1$.

Theorem 2.4. *Assume assumptions* (CC)$'$, (b.2) *and* (b.3) *hold. Let* X_t *be a global strong solution to Eq.* (1.1). *Then there exist constants* $\tau > 0$, $C > 0$ *such that*

$$E|X_t|^2 \leqslant C \cdot \mathrm{e}^{-\tau t}, \quad \forall t \geqslant 0, \tag{2.21}$$

if either one of the hypotheses (i) *or* (ii) *in Theorem* 2.2 *holds.*

Proof. By a similar argument to that one in the proof of Theorem 2.2, we can get

$$\begin{aligned}
\mathrm{e}^{(\mu-\delta)t}E|X_t|^2 &\leqslant E|X_0|^2 + (\mu-\delta-\nu)\int_0^t \mathrm{e}^{(\mu-\delta)s}E|X_s|^2\,\mathrm{d}s \\
&\quad + \int_0^t \tau(s)\mathrm{e}^{-\delta s}E|X_s|^{2\sigma}\,\mathrm{d}s + \int_0^t \gamma(s)\mathrm{e}^{-\delta s}\,\mathrm{d}s \\
&\leqslant E|X_0|^2 + (\mu-\delta-\nu)\int_0^t \mathrm{e}^{(\mu-\delta)s}E|X_s|^2\,\mathrm{d}s \\
&\quad + \int_0^t \tau(s)\mathrm{e}^{-(\delta+\sigma(\mu-\delta))s}(\mathrm{e}^{(\mu-\delta)s}E|X_s|^2)^\sigma\,\mathrm{d}s \\
&\quad + \int_0^t \gamma(s)\mathrm{e}^{-\delta s}\,\mathrm{d}s,
\end{aligned} \tag{2.22}$$

where $\nu = (\alpha - \lambda\beta^2)/\beta^2$.

If $\mu - v \leqslant 0$, it follows

$$e^{(\mu-\delta)t}E|X_t|^2 \leqslant E|X_0|^2 + \int_0^t \tau(s)e^{-(\delta+\sigma(\mu-\delta))s}(e^{(\mu-\delta)s}E|X_s|^2)^\sigma \, ds + \int_0^t \gamma(s)e^{-\delta s} \, ds.$$

Now, an extended Gronwall-type lemma from Mao (1994) (in fact, Corollary 7.5 in Chapter 1, p. 27), immediately yields

$$\begin{aligned} e^{(\mu-\delta)t}E|X_t|^2 \leqslant \Bigg[& \left(E|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s} \, ds\right)^{1-\sigma} \\ & + (1-\sigma)\int_0^t \tau(s)e^{-(\delta+\sigma(\mu-\delta))s} \, ds\Bigg]^{1/(1-\sigma)} \end{aligned}$$

which implies that there exists a positive constant $K(\delta) > 0$ such that

$$E|X_t|^2 \leqslant K(\delta) \cdot e^{-(\mu-\delta)t}. \tag{2.23}$$

On the other hand, if $\mu - v > 0$, it is always possible to choose a suitable $\delta > 0$ such that $\mu - v - \delta > 0$. Then, by virtue of Gronwall's lemma we easily derive from (2.22) that

$$E|X_t|^2 \leqslant \left[E|X_0|^2 + \int_0^t \tau(s)e^{-\delta s}(E|X_s|^2)^\sigma \, ds + \int_0^t \gamma(s)e^{-\delta s} \, ds\right] e^{-vt}.$$

Once again, the extended Gronwall-type lemma from Mao (1994) immediately implies

$$\begin{aligned} E|X_t|^2 & \leqslant e^{-vt}\left\{\left(E|X_0|^2 + \int_0^t \gamma(s)e^{-\delta s} \, ds\right)^{1-\sigma} + (1-\sigma)\int_0^t \tau(s)e^{-\delta s} \, ds\right\}^{1/(1-\sigma)} \\ & \equiv C(\delta) \cdot e^{-vt} \end{aligned}$$

and the proof is complete. □

In a similar manner as in the proof of Theorem 2.3, we could also prove the following result.

Theorem 2.5. *Assume that the hypotheses in Theorem* 2.4 *hold. Then there exist positive constants* M, ε *and a subset* $N_0 \subset \Omega$ *with* $P(N_0) = 0$ *such that, for each* $\omega \notin N_0$, *there exists a positive random number* $T(\omega)$ *such that*

$$|X_t|^2 \leqslant M \cdot e^{-\varepsilon t}, \quad \forall t \geqslant T(\omega). \tag{2.24}$$

3. Examples

In this section, we consider some stochastic partial differential equations, in order to illustrate our theory.

Example 3.1. Firstly, we consider the following semilinear stochastic partial differential equation, which models the heat production by an exothermic reaction taking place inside a rod of length π whose ends are maintained at 0° and whose sides are insulated (see Haussmann (1978) for a similar situation in the linear case):

$$\begin{aligned} \mathrm{d}Y_t(x) &= \left[\frac{\partial^2 Y_t(x)}{\partial x^2} + r_0 Y_t(x)\right] \mathrm{d}t + \alpha(Y_t(x))\,\mathrm{d}W_t, \quad t > 0,\ x \in (0,\pi), \\ Y_0(x) &= y_0(x), \quad Y_t(0) = Y_t(\pi) = 0,\ t \geqslant 0. \end{aligned} \tag{3.1}$$

Here W_t is a real standard Wiener process (so, $K=\mathbb{R}$ and $Q=1$), $r_0 \in \mathbb{R}$, and $\alpha(\cdot): \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $\alpha(0)=0$. We can set this problem in our formulation by taking $H = L^2[0,\pi]$, $V = W_0^{1,2}([0,\pi])$ (a Sobolev space with elements satisfying the boundary conditions above), $K = \mathbb{R}$, $A(t,u) = (\mathrm{d}^2/\mathrm{d}x^2)u(x) + r_0 u(x)$, and $B(t,u) = \alpha(u)$.

Clearly, the operator B satisfies (b.2) and (b.3). On the other hand, it is easy to deduce for arbitrary $u \in V$ that

$$2\langle A(t,u), u\rangle + ||B(t,u)||_2^2 \leqslant -2||u||^2 + 2r_0|u|^2 + k^2|u|^2,$$

where k is the Lipschitz constant for the function α, and the norm in V is given by $||u||^2 = \int_0^\pi (u'(x))^2\,\mathrm{d}x$.

Therefore, it follows that hypothesis (b) in Theorems 2.2 and 2.3 is fulfilled provided $(k^2 + 2r_0)\beta^2 < 2$ (observe that we can set $\beta = \pi/\sqrt{2}$ in this case).

Consequently, we easily deduce that the strong solution of the equation is the mean square and almost surely exponentially stable.

Remark 3.1. Observe that Theorem 1.2 can also be applied to this situation since our operators satisfy $A(t,0) = 0$ and $B(t,0) = 0$.

Nevertheless, it happens that under some circumstances, additional heat is applied to the system in order to drive it to a desired state, if possible. This can be modeled by introducing some time-dependent terms in the equation. In our case, if we suppose that the additional heat applied in each point is the same (so it is given by a function $h(t)$, independent of x), we can consider several possibilities according to the term in which this function can appear (the diffusion, the drift or both of them).

Thus, we can study the following problems:

$$\begin{aligned} \mathrm{d}Y_t(x) &= \left[\frac{\partial^2 Y_t(x)}{\partial x^2} + r_0 Y_t(x)\right] \mathrm{d}t + (\alpha(Y_t(x)) + h(t))\,\mathrm{d}W_t, \quad t > 0,\ x \in (0,\pi), \\ Y_0(x) &= y_0(x), \quad Y_t(0) = Y_t(\pi) = 0,\ t \geqslant 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \mathrm{d}Y_t(x) &= \left[\frac{\partial^2 Y_t(x)}{\partial x^2} + r_0 Y_t(x) + h(t)\right] \mathrm{d}t + \alpha(Y_t(x))\,\mathrm{d}W_t, \quad t > 0,\ x \in (0,\pi), \\ Y_0(x) &= y_0(x), \quad Y_t(0) = Y_t(\pi) = 0,\ t \geqslant 0 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \mathrm{d}Y_t(x) &= \left[\frac{\partial^2 Y_t(x)}{\partial x^2} + r_0 Y_t(x) + h(t)\right] \mathrm{d}t + (\alpha(Y_t(x)) + h(t))\,\mathrm{d}W_t, \\ &\qquad t > 0,\ x \in (0,\pi), \\ Y_0(x) &= y_0(x), \quad Y_t(0) = Y_t(\pi) = 0,\ t \geqslant 0. \end{aligned} \tag{3.4}$$

For instance, in the case of (3.2), taking into account that the inequality $2ab \leqslant \varepsilon a^2 + \varepsilon^{-1} b^2$ holds for $a, b \in \mathbb{R}$ and $\varepsilon > 0$, it can be easily deduced that

$$2\langle A(t,u),u\rangle + ||B(t,u)||_2^2 \leqslant -2||u||^2 + (2r_0 + (1+\varepsilon)k^2)|u|^2 + \pi(1+\varepsilon^{-1})h(t)^2. \tag{3.5}$$

Thus, if $(k^2 + 2r_0)\beta^2 < 2$, we can choose a positive constant $\varepsilon > 0$ small enough such that $(k^2(1+\varepsilon) + 2r_0)\beta^2 < 2$. If, in addition, $h(t)$ is of subexponential type, i.e. $h(t) = \gamma(t)e^{-\mu t}$ with $\mu > 0$ and γ satisfying the conditions in (CC), the hypotheses in Theorems 2.2 and 2.3 are satisfied again.

Remark 3.2. Observe that Theorem 1.2 cannot be applied to this occasion since the coercivity condition there does not hold.

Now, problems (3.3) and (3.4) can be analyzed by applying Theorems 2.4 and 2.5. For instance, in the case of problem (3.3) we can obtain

$$2\langle A(t,u),u\rangle + ||B(t,u)||_2^2 \leqslant -2||u||^2 + (2r_0 + k^2)|u|^2 + \sqrt{\pi}\,|h(t)|\,|u|,$$

where $|h(t)|$ denotes the absolute value of $h(t)$. Thus, if $h(t)$ is of subexponential type as above, the hypotheses in Theorems 2.4 and 2.5 are fulfilled by taking $\sigma = \frac{1}{2}$, provided that $(2r_0 + k^2)\beta^2 < 2$.

Lastly, let us simply come back to an example investigated in Caraballo and Real (1994).

Example 3.2. Let $D = [0,1]$ and $2 < p < +\infty$, $r > 0$, and consider the following:

$$\begin{aligned} dX_t(x) &= \left[\frac{\partial}{\partial x}\left(\left|\frac{\partial X_t(x)}{\partial x}\right|^{p-2}\frac{\partial X_t(x)}{\partial x}\right) - a(x)X_t(x)\right] dt + g(X_t(x))\,dW_t, \\ & \qquad t > 0,\ x \in D \\ X_0(x) &= x_0(x), \quad x \in D, \qquad X_t(0) = X_t(1) = 0, \quad \text{a.s.}, \end{aligned} \tag{3.6}$$

where $a \in L^\infty(D)$ satisfies $a(x) \geqslant \tilde{a} > 0$ a.s., $x \in D$ and $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with constant $k > 0$ such that $k^2 < 2\tilde{a}$ and $g(0) = 0$. W_t is a standard real Wiener process.

Let $H = L^2(D)$, $V = W_0^{1,p}(D)$ be the Sobolev space with elements satisfying the above boundary conditions. At the moment, $A(t,u)$ is nonlinear, $B(t,u) = g(u)$, for all $u \in V$.

It is easy to check that in this case (2.2) holds with $\gamma(s)=0$, $\lambda=-\varepsilon < 0$, $p > 2$, $\alpha=2$, where $\varepsilon > 0$ is such that $k^2 < 2\tilde{a} - \varepsilon$. Using Theorem 2.2, we easily obtain the required exponential stability.

4. Remarks and conclusions

We have proved some results which, in particular, extend the theory developed by Caraballo and Real (1994). In fact, our results can be applied to a number of examples where the criteria in that paper do not hold, since the coercivity condition assumed there requires a uniform bound on the operators. We no longer require the condition $A(t,0)=B(t,0)=0$ from Caraballo and Real (1994); nevertheless, even in this case (when $X_t \equiv 0$ is solution to (1.1)) our theory improves that which obtains exponential stability in the mean square and almost surely of the trivial solution to (1.1). However, the results proved in Section 2 are stronger still. Indeed, what we have shown is that, under the assumptions in Theorem 2.2 (or Theorem 2.4) the strong solution to (1.1) exponentially converges in the mean square (and almost surely) to zero even if $X_t \equiv 0$ is *not* a solution of Eq. (1.1).

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