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Remarks on non-linear noise excitability of some stochastic heat equations[☆]

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Abstract

We consider nonlinear parabolic SPDEs of the form $\partial_t u = \Delta u + \lambda \sigma(u) \dot{w}$ on the interval $(0, L)$, where \dot{w} denotes space–time white noise, σ is Lipschitz continuous. Under Dirichlet boundary conditions and a linear growth condition on σ , we show that the expected L^2 -energy is of order $\exp[\text{const} \times \lambda^4]$ as $\lambda \rightarrow \infty$. This significantly improves a recent result of Khoshnevisan and Kim. Our method is very different from theirs and it allows us to arrive at the same conclusion for the same equation but with Neumann boundary condition. This improves over another result in Khoshnevisan and Kim.

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1. Introduction

The main objective of this paper is to study the effect of noise on the solutions to various stochastic heat equations. Fix $L > 0$ and consider the following

$$\begin{cases} \partial_t u_t(x) = \Delta u_t(x) + \lambda \sigma(u_t(x)) \dot{w}(t, x), \\ u_t(0) = 0, \quad u_t(L) = 0, \end{cases} \quad (1.1)$$

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where \dot{w} denotes the space time white noise on $(0, \infty) \times (0, L)$. λ is a positive number called the *noise intensity*. Here and throughout this paper, the initial function $u_0 : [0, L] \rightarrow \mathbf{R}_+$ is a nonrandom and non-negative function which is strictly positive on a set of positive measures in $[0, L]$. The function $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is continuous with $\sigma(0) = 0$ and

$$l_\sigma := \inf_{x \in \mathbf{R} \setminus \{0\}} \left| \frac{\sigma(x)}{x} \right| \quad \text{and} \quad L_\sigma := \sup_{x \in \mathbf{R} \setminus \{0\}} \left| \frac{\sigma(x)}{x} \right|,$$

where $0 < l_\sigma \leq L_\sigma < \infty$.

Our study is motivated by a recent paper of Khoshnevisan and Kim [7] where the authors initiated the study of the effect of λ on the energy of the solution. In the case of linear multiplicative noise, λ can also be thought of as the inverse temperature and the solution u can be regarded as the partition function of a continuous time and space random polymer; see [2] and the references therein.

A mild solution to (1.1) is any u which is adapted to the filtration generated by the white noise and satisfies the following evolution equation

$$u_t(x) = (\mathcal{G}_D u)_t(x) + \lambda \int_0^L \int_0^t p_D(t-s, x, y) \sigma(u_s(y)) w(ds dy), \quad (1.2)$$

where

$$(\mathcal{G}_D u)_t(x) := \int_0^L u_0(y) p_D(t, x, y) dy,$$

and $p_D(t, x, y)$ denotes the Dirichlet heat kernel. As usual, (1.2) will be the starting point of most of our analysis. We will shortly describe the results of [7] in a bit more detail but let us mention that existence and uniqueness is not an issue for us. It is well known that the above equation has a unique mild solution satisfying

$$\sup_{x \in [0, L]} \sup_{t \in [0, T]} E|u_t(x)|^k < \infty \quad \text{for all } T > 0 \text{ and } k \in [2, \infty]. \quad (1.3)$$

For more information about existence-uniqueness, see [3] or [11] for more information.

To describe our results in a precise manner, we adopt some notations and definitions from [7,8]. We begin by defining the *energy of the solution at time t* by

$$\mathcal{E}_t(\lambda) := \sqrt{E \left(\|u_t\|_{L^2[0, L]}^2 \right)}. \quad (1.4)$$

One of the main results in [7] states that as λ gets large, $\mathcal{E}_t(\lambda)$ grows at most like $\exp(\text{const} \times \lambda^4)$ but at least like $\exp(\text{const} \times \lambda^2)$. This current project grew out of trying to understand this discrepancy. The following indices were introduced in [8] to capture the *super exponential* growth just mentioned.

Definition 1.1. The *upper excitation index* of u at time t is given by

$$\bar{e}(t) := \limsup_{\lambda \rightarrow \infty} \frac{\log \log \mathcal{E}_t(\lambda)}{\log \lambda}.$$

Definition 1.2. The lower excitation index of u at time t is given by

$$\underline{e}(t) := \liminf_{\lambda \rightarrow \infty} \frac{\log \log \mathcal{E}_t(\lambda)}{\log \lambda}.$$

When $\bar{e}(t)$ and $\underline{e}(t)$ are equal, we simply refer to the common value as the *noise excitation index* of the solution at time t , which is required to be strictly positive. We are now ready to state the first main result of the paper.

Theorem 1.3. The noise excitation index of the solution to (1.1) is 4.

Estimating the lower excitation index is the main contribution of this paper and our approach requires two new ideas which we now describe.

1. We use two renewal inequalities which give the desired upper and lower bounds on the energy. The use of renewal theoretic ideas was introduced in [4] but here we use it in a different manner.
2. To arrive at these renewal inequalities, we make use of the idea that for small times and away from the boundary, the Dirichlet heat kernel behaves similarly to the Gaussian heat kernel. This idea has been the subject of intense investigations for decades now; see [1,9]. Since we are working in spatial dimension one, we provide complete analytic proofs of the main estimates we need.

It is also interesting to note that in [7], the bound on the upper index was the harder part of the proof. Here the complete opposite is true; the lower bound is much harder and requires the second point mentioned above which is entirely novel. As far as we know, Gaussian estimates for Dirichlet Laplacian have never been used in the study of stochastic partial differential equations. Using these two ideas, we were able to improve the bound on the lower index.

It turns out that our method can be adapted to study the same stochastic PDE but with Neumann boundary condition. We now describe our main findings in this context. Consider the following equation

$$\begin{cases} \partial_t u_t(x) = \Delta u_t(x) + \lambda \sigma(u_t(x)) \dot{w}(t, x), \\ \partial_x u_t(0) = 0, \quad \partial_x u_t(L) = 0. \end{cases} \quad (1.5)$$

Here we stress the fact that as opposed to [7], we do not require our initial function to be bounded below. Any bounded nonrandom non negative initial function which is nonzero on a set of positive measures will be enough. A mild solution to (1.5) is any u which solves the following evolution equation

$$u_t(x) = (\mathcal{G}_N u)_t(x) + \lambda \int_0^L \int_0^t p_N(t-s, x, y) \sigma(u_s(y)) w(ds dy), \quad (1.6)$$

where

$$(\mathcal{G}_N u)_t(x) := \int_0^L u_0(y) p_N(t, x, y) dy$$

and $p_N(t, x, y)$ is the Neumann heat kernel. It is well known that (1.5) has a unique mild solution satisfying (1.3). We refer to [6,3,11] for more information about various technicalities. To state

our main result for (1.5), we set the following notations,

$$\mathcal{I}_t(\lambda) := \inf_{x \in [0, L]} \mathbb{E}|u_t(x)|^2 \quad (1.7)$$

and

$$\mathcal{S}_t(\lambda) := \sup_{x \in [0, L]} \mathbb{E}|u_t(x)|^2, \quad (1.8)$$

where u_t is the solution to (1.5).

Theorem 1.4. Fix $t > 0$, then

$$\liminf_{\lambda \rightarrow \infty} \frac{\log \log \mathcal{I}_t(\lambda)}{\log \lambda} = \limsup_{\lambda \rightarrow \infty} \frac{\log \log \mathcal{S}_t(\lambda)}{\log \lambda} = 4.$$

An immediate consequence of the above is the following.

Corollary 1.5. The noise excitation index of the solution to (1.5) is 4.

Our technique seems to be suited for the study of a wider class of stochastic equations. If the Laplacian in say (1.1) were replaced by the fractional Dirichlet Laplacian of order α and the white noise were replaced by a colored noise with Riesz Kernel of order β , we conjecture that the non-linear excitation index is $2\alpha/(\alpha - \beta)$. This is currently under investigation and will be the subject of [5].

We end this introduction with the plan of the article. Section 2 contains the renewal type inequalities. Section 3 contains the relevant Dirichlet heat estimates and the proof of Theorem 1.3. Section 4 contains the corresponding estimates for the Neumann heat kernel as well as the proof of Theorem 1.4 and its corollary.

2. Some estimates

This section will be devoted to the renewal-type inequalities mentioned in the introduction. The perceptive reader will recognize that the presence of the square root inside the integrals is motivated by the Gaussian heat kernel. The ideas behind the proof of the following proposition are extensions of the proof of Gronwall's inequality, but we could not any exact reference in the literature for the second estimate in that proposition.

Proposition 2.1. Suppose that $f(t)$, is a non-negative integrable function on $0 \leq t \leq T$ satisfying

$$f(t) \leq a + bk \int_0^t \frac{f(s)}{\sqrt{t-s}} ds \quad \text{for all } k > 0 \text{ and } 0 \leq t \leq T, \quad (2.1)$$

where a and b are positive constants and $T < \infty$. Then for each $0 < t \leq T$, we have

$$\limsup_{k \rightarrow \infty} \frac{\log \log f(t)}{\log k} \leq 2.$$

On the other hand, if instead of (2.1), $f(t)$ satisfies the following

$$f(t) \geq a + bk \int_0^t \frac{f(s)}{\sqrt{t-s}} ds \quad \text{for all } k > 0 \text{ and } 0 \leq t \leq T, \quad (2.2)$$

then for each $0 < t \leq T$, we have

$$\liminf_{k \rightarrow \infty} \frac{\log \log f(t)}{\log k} \geq 2.$$

Proof. We prove the second part of the proposition only. The proof of the first part can be deduced from Gronwall's inequality. We iterate inequality (2.2) once to obtain

$$\begin{aligned} f(t) &\geq a + bk \int_0^t \frac{f(s)}{\sqrt{t-s}} ds \\ &= a + abk\sqrt{t} + b^2k^2 \int_0^t \int_0^s \frac{f(l)}{\sqrt{(t-s)(s-l)}} dl ds. \end{aligned}$$

We change the order of integration in the above double integral to find that

$$\begin{aligned} \int_0^t \int_0^s \frac{f(l)}{\sqrt{(t-s)(s-l)}} dl ds &= \int_0^t \int_l^t \frac{f(l)}{\sqrt{(t-s)(s-l)}} ds dl \\ &= c_1 \int_0^t f(l) dl, \end{aligned}$$

where c_1 is some positive constant. This together with the above inequality gives

$$\begin{aligned} f(t) &\geq c_2 + c_2k\sqrt{t} + c_2k^2 \int_0^t f(l) dl \\ &\geq c_2 + c_2k^2 \int_0^t f(l) dl, \end{aligned}$$

for some positive constant c_2 . We now note that $f(t) \geq g(t)$ where g satisfies the following ordinary differential equation $g'(t) = c_2k^2g(t)$ with initial condition $g(0) = c_2$. Since $g(t) = c_2e^{c_2k^2t}$, the result is proved. \square

3. The Dirichlet equation

We start off with a result which gives a lower bound on the Dirichlet heat kernel in terms of the Gaussian heat kernel. This is borrowed from [10]. But we give a proof here for the sake of completeness. Recall that from the method of images, we have the following representation,

$$p_D(t, x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[e^{-\frac{|x-(y+2nL)|^2}{4t}} - e^{-\frac{|x-(-y+2nL)|^2}{4t}} \right]. \quad (3.1)$$

Lemma 3.1. Suppose that $x, y \in (0, L)$ and set $\epsilon := \min\{x, y, L-x, L-y\}$, then we have

$$p_D(t, x, y) \geq (1 - 2e^{-\epsilon^2/t})p(t, x, y).$$

Proof. The proof involves rewriting (3.1) in a suitable way and making use of the following observation. For $n \geq 1$ and $x, y \in (0, L)$

$$|x + y + 2nL| \geq |x - y + 2nL| \quad (3.2)$$

and

$$|-(x + y) + 2(n + 1)L| \geq |2nL - (x - y)|. \quad (3.3)$$

1 **Q3** We can now write

$$\begin{aligned}
 & \sum_{n=-\infty}^{\infty} \left[e^{-\frac{|x-(y+2nL)|^2}{4t}} - e^{-\frac{|x-(-y+2nL)|^2}{4t}} \right] = e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x+y|^2}{4t}} - e^{-\frac{|x+y-2L|^2}{4t}} \\
 & + \sum_{n=1}^{\infty} \left[e^{-\frac{|x-y-2nL|^2}{4t}} + e^{-\frac{|x-y+2nL|^2}{4t}} - e^{-\frac{|x+y-2(n+1)L|^2}{4t}} - e^{-\frac{|x+y+2nL|^2}{4t}} \right].
 \end{aligned}$$

4 We now use (3.2) and (3.3) together with (3.1) to conclude that

$$\begin{aligned}
 p_D(t, x, y) & \geq \frac{1}{\sqrt{4\pi t}} \left[e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x+y|^2}{4t}} - e^{-\frac{|x+y-2L|^2}{4t}} \right] \\
 & \geq \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} \left[1 - e^{-\frac{xy}{t}} - e^{-\frac{(L-x)(L-y)}{t}} \right].
 \end{aligned}$$

7 This and the definition of ϵ essentially finish the proof. \square

8 A consequence of the above lemma is that away from the boundary, the Dirichlet heat kernel
 9 behaves pretty much like the Gaussian one provided that time is small enough. This is intuitively
 10 clear from the probabilistic point of view. The Dirichlet heat kernel is the transition density of
 11 a Brownian motion conditioned to be killed at the boundary. Starting from an interior point, the
 12 behavior of the killed Brownian motion in a very short period of time is not very different than
 13 that of a regular Brownian motion.

14 **Corollary 3.2.** Fix $\epsilon > 0$, then there exists $t_0 > 0$ depending on ϵ such that for all $t \leq t_0$ and all
 15 $x, y \in [\epsilon, L - \epsilon]$, we have

$$p_D(t, x, y) \geq \frac{1}{2} p(t, x, y).$$

17 **Proof.** Fix $\epsilon > 0$. For $t \leq \frac{\epsilon^2}{\ln 4}$, we have $1 - 2e^{-\epsilon^2/t} \geq \frac{1}{2}$. The result then follows from the above
 18 lemma. \square

19 Another starting point for the proof of the above result could be the following. Let τ denote
 20 the first exit time of a regular Brownian motion from $(0, L)$. Since $p(t, x, y)$ is the transition
 21 probability of this Brownian motion, we have

$$p_D(t, x, y) = P^x(\tau > t | B_t = y) p(t, x, y).$$

23 We also have

$$p_D(t, x, y) = p(t, x, y) - E^x(p(t - \tau, X_D, y); \tau < t),$$

25 where X_D is the position of the Brownian motion when it hits the boundary, making the following
 26 clear.

27 **Lemma 3.3.** For all $x, y \in (0, L)$ and $t > 0$, the following holds

$$p_D(t, x, y) \leq p(t, x, y). \quad (3.4)$$

29 The next result says that provided we stay from the boundary, we can find a suitable lower
 30 bound on the growth of the second moment of the solution of (1.1). To state our result, we

introduce the following notation. For $\epsilon > 0$,

$$\mathcal{I}_{\epsilon,t}(\lambda) := \inf_{x \in [\epsilon, L-\epsilon]} \mathbb{E}|u_t(x)|^2, \quad (3.5)$$

where u is the solution to (1.1).

Proposition 3.4. Fix $\epsilon > 0$. Then there exists $t_0 > 0$ such that for all $t \leq t_0$, we have

$$\liminf_{\lambda \rightarrow \infty} \frac{\log \log \mathcal{I}_{\epsilon,t}(\lambda)}{\log \lambda} \geq 4.$$

Proof. Using the mild formulation and Ito's isometry, we have

$$\mathbb{E}|u_t(x)|^2 = |(\mathcal{G}_D u)_t(x)|^2 + \lambda^2 \int_0^t \int_0^L p_D^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy. \quad (3.6)$$

We now fix an $\epsilon > 0$ sufficiently small and let t_0 be defined as in the proof of Corollary 3.2. We bound the first term on the right hand side of the above display first. Recall that $(\mathcal{G}_D u)_t(x)$ solves the deterministic heat equation, that is (1.1) with $\lambda = 0$. Provided we stay away from the boundary, it is bounded below by a constant for $t \leq t_0$. In other words for $x \in [\epsilon, L - \epsilon]$ and $t \leq t_0$, we have $|(\mathcal{G}_D u)_t(x)|^2 \geq c_1$ for some positive constant c_1 depending on t_0 . We now look at the second term. Using Corollary 3.2, we obtain

$$\begin{aligned} & \lambda^2 \int_0^L \int_0^t p_D^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy \\ & \geq \frac{\lambda^2 l_\sigma^2}{4} \int_\epsilon^{L-\epsilon} \int_0^t p^2(t-s, x, y) \mathbb{E}|(u_s(y))|^2 ds dy \\ & \geq \frac{\lambda^2 l_\sigma^2}{4} \int_0^t \mathcal{I}_{\epsilon,t}(\lambda) \int_\epsilon^{L-\epsilon} p^2(t-s, x, y) dy ds. \end{aligned}$$

We now estimate the innermost integral appearing in the above line. For fixed t, s and $x \in [\epsilon, L - \epsilon]$, set $D := [\epsilon, L - \epsilon] \cap \{y : |y - x| \leq \sqrt{t-s}\}$. Hence for $y \in D$, we have $p(t-s, x, y) \geq c_2/\sqrt{t-s}$, for some constant c_2 . We therefore have

$$\int_\epsilon^{L-\epsilon} p^2(t-s, x, y) dy \geq c_3 \int_D \frac{1}{t-s} dy \geq \frac{c_4}{\sqrt{t-s}},$$

for some constants c_3 and c_4 . Consequently

$$\lambda^2 \int_0^L \int_0^t p_D^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy \geq \frac{\lambda^2 l_\sigma^2 c_4}{4} \int_0^t \frac{\mathcal{I}_{\epsilon,s}(\lambda)}{\sqrt{t-s}} ds.$$

We now combine the above estimates to yield the following inequality

$$\mathcal{I}_{\epsilon,t}(\lambda) \geq c_5 + \frac{\lambda^2 l_\sigma^2 c_4}{4} \int_0^t \frac{\mathcal{I}_{\epsilon,s}(\lambda)}{\sqrt{t-s}} ds. \quad (3.7)$$

The proof now follows from an application of Proposition 2.1. \square

We are now ready to prove Theorem 1.3.

3.1. Proof of Theorem 1.3

We will first show that $\bar{e}(t) \leq 4$. This will be done in one step. We will then show that $\underline{e}(t) \geq 4$. We will do so in two steps. We first prove the bound for small times and then extend it to all times.

Proof of the upper bound.

The proof of the upper bound is already known (see [7]) but we give a much simpler proof here. We start off with the mild formulation, take the second moment and then integrate to obtain

$$\begin{aligned} \int_0^L \mathbb{E}|u_t(x)|^2 dx &= \int_0^L |(\mathcal{G}_D u)_t(x)|^2 dx \\ &\quad + \lambda^2 \int_0^L \int_0^L \int_0^t p_D^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy dx. \end{aligned} \quad (3.8)$$

For fixed t , the first term is a bounded function, so that we have $\int_0^L |(\mathcal{G}_D u)_t(x)|^2 dx \leq c_1$.

We now turn our attention to the second term. Using (3.4) and the semigroup property of the heat kernel, we end up with

$$\begin{aligned} &\lambda^2 \int_0^L \int_0^L \int_0^t p_D^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy dx \\ &\leq \lambda^2 L_\sigma^2 \int_0^L \int_0^t p(2(t-s), y, y) \mathbb{E}|u_s(y)|^2 ds dy \\ &\leq c_3 \lambda^2 L_\sigma^2 \int_0^t \frac{\mathcal{E}_s^2(\lambda)}{\sqrt{t-s}} ds. \end{aligned}$$

Combining the above estimates, we obtain

$$\mathcal{E}_t^2(\lambda) \leq c_4 + c_4 \lambda^2 L_\sigma^2 \int_0^t \frac{\mathcal{E}_s^2(\lambda)}{\sqrt{t-s}} ds.$$

The upper bound is thus proved after an application of Proposition 2.1.

Proof of the lower bound.

Step 1: We first prove the lower bound for $t \leq t_0$ where t_0 is some positive number. Being the solution to the deterministic heat equation, $(\mathcal{G}_D u)_t(x)$, $t \leq t_0$ is uniformly bounded below by a constant depending on t_0 . So the first term of (3.8) is thus uniformly bounded below for $t \leq t_0$.

To find a lower bound on (3.8), we use the lower bound on σ as well as the Markov property of killed Brownian motion to find that

$$\begin{aligned} &\lambda^2 \int_0^L \int_0^L \int_0^t p_D^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy dx \\ &\geq \lambda^2 l_\sigma^2 \int_0^L \int_0^L \int_0^t p_D^2(t-s, x, y) \mathbb{E}|u_s(y)|^2 ds dx dy \\ &= \lambda^2 l_\sigma^2 \int_0^L \int_0^t p_D(2(t-s), y, y) \mathbb{E}|u_s(y)|^2 dy ds. \end{aligned} \quad (3.9)$$

Fix $\epsilon > 0$. As in the proof of Proposition 3.4, we have that the above is

$$\begin{aligned} &\geq \lambda^2 l_\sigma^2 \int_\epsilon^{L-\epsilon} \int_0^t p_D(2(t-s), y, y) \mathbb{E}|u_s(y)|^2 ds dy \\ &\geq \lambda^2 l_\sigma^2 \int_0^t \mathcal{I}_{\epsilon,s}(\lambda) \int_\epsilon^{L-\epsilon} p_D(2(t-s), y, y) dy ds \\ &\geq \lambda^2 l_\sigma^2 \int_0^{t/2} \mathcal{I}_{\epsilon,s}(\lambda) \int_\epsilon^{L-\epsilon} p_D(2(t-s), y, y) dy ds. \end{aligned}$$

The next step is to bound the inner integral appearing in the last display. Corollary 3.2 shows that there exists a constant c_5 such that

$$\begin{aligned} \int_\epsilon^{L-\epsilon} p_D(2(t-s), y, y) dy &\geq \frac{1}{2} \int_\epsilon^{L-\epsilon} p(2(t-s), y, y) dy \\ &\geq c_5 \end{aligned}$$

if $t \leq t_0$, where t_0 depends on ϵ . We combine the above estimates to obtain

$$\int_0^L \mathbb{E}|u_t(x)|^2 dx \geq c_6 + \lambda^2 l_\sigma^2 c_7 \int_0^{t/2} \mathcal{I}_{\epsilon,s}(\lambda) ds.$$

We now note that (3.7) actually means that $\mathcal{I}_{\epsilon,s}(\lambda)$ grows at least like $\exp(c_8 \lambda^4)$. Some calculus then finishes the proof for $t \leq t_0$.

Step 2: We now show that the lower bound holds for any $t > 0$. We assume that $t > t_0$ (t_0 is chosen in Step 1), otherwise there is nothing else to prove. Let t_1 be a small constant to be chosen later. We write $t = t - t_1 + t_1$ and set $T = t - t_1$ for notational convenience. As we have seen before the mild formulation of the solution yields

$$\mathbb{E}|u_t(x)|^2 = |(\mathcal{G}_D u)_t(x)|^2 + \lambda^2 \int_0^L \int_0^t p_D^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy.$$

Using $t = T + t_1$, the above equation can be rewritten as

$$\begin{aligned} \mathbb{E}|u_{T+t_1}(x)|^2 &= |(\mathcal{G}_D u)_{T+t_1}(x)|^2 + \lambda^2 \int_0^L \int_0^T p_D^2(T+t_1-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy \\ &\quad + \lambda^2 \int_0^L \int_0^{t_1} p_D^2(t_1-s, x, y) \mathbb{E}|\sigma(u_{T+s}(y))|^2 ds dy. \end{aligned}$$

We now set

$$v_s(x) := u_{T+s}(x)$$

and integrate over $x \in [0, L]$ to get

$$\begin{aligned} \int_0^L \mathbb{E}|v_{t_1}(x)|^2 dx &\geq \int_0^L |(\mathcal{G}_D u)_{T+t_1}(x)|^2 dx \\ &\quad + \lambda^2 \int_0^L \int_0^L \int_0^{t_1} p_D^2(t_1-s, x, y) \mathbb{E}|\sigma(v_s(y))|^2 ds dy dx. \end{aligned}$$

Let us now choose $t_1 = \frac{t_0}{2}$. Since T is fixed, the first term is positive. The second term is exactly as in (3.9). Since $t_1 < t_0$, all the arguments in Step 1 continue to hold with u replaced by v . We have therefore proved the result for all $t > 0$. \square

4. The Neumann equation

We begin this section with two estimates on the Neumann heat kernel. First, recall that by the method of images, we have

$$p_N(t, x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[e^{-\frac{|x-y-2nL|^2}{4t}} + e^{-\frac{|x+y-2nL|^2}{4t}} \right]. \quad (4.1)$$

All the terms on the right hand side of the above display are positive, so by considering $n = 0$, we have

Lemma 4.1. *For all $x, y \in [0, L]$ and $t > 0$, the following holds*

$$p_N(t, x, y) \geq p(t, x, y). \quad (4.2)$$

A little more work shows the following.

Lemma 4.2. *Let $T > 0$, then for all $x, y \in [0, L]$ and $t \leq T$, the following holds*

$$p_N(t, x, y) \leq c_T p(t, x, y), \quad (4.3)$$

where c_T is some constant depending on $T > 0$.

4.1. Proof of Theorem 1.4

Proof. We begin by proving the lower bound first. We start off with the mild formulation and take second moment to end up with

$$\mathbb{E}|u_t(x)|^2 = |(\mathcal{G}_N u)_t(x)|^2 + \lambda^2 \int_0^L \int_0^t p_N^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy. \quad (4.4)$$

We bound the first term on the right hand side of the above display first. Since $(\mathcal{G}_N u)_t(x)$ solves the corresponding deterministic problem, we have that for fixed $t > 0$, $(\mathcal{G}_N u)_t(x)$ is bounded below by a positive constant depending on t . We now deal with the second term. We will again use (4.2) as well as the definition of $\mathcal{I}_t(\lambda)$ (see (1.7)).

$$\begin{aligned} & \lambda^2 \int_0^L \int_0^t p_N^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 ds dy \\ & \geq \lambda^2 l_\sigma^2 \int_0^L \int_0^t p_N^2(t-s, x, y) \mathbb{E}|u_s(y)|^2 ds dy \\ & \geq \lambda^2 l_\sigma^2 \int_0^t \mathcal{I}_s(\lambda) \int_0^L p_N^2(t-s, x, y) dy ds \\ & \geq \lambda^2 l_\sigma^2 \int_0^t \mathcal{I}_s(\lambda) p_N(2(t-s), x, x) ds \\ & \geq \frac{\lambda^2 l_\sigma^2}{\sqrt{4\pi}} \int_0^t \frac{\mathcal{I}_s(\lambda)}{\sqrt{t-s}} ds. \end{aligned}$$

Combining the above inequalities, we obtain

$$\mathcal{I}_t(\lambda) \geq c_1 + \frac{\lambda^2 l_\sigma^2}{\sqrt{4\pi}} \int_0^t \frac{\mathcal{I}_s(\lambda)}{\sqrt{t-s}} ds$$

for some constant c_1 . An application of (2.2) yields the lower bound stated in the theorem. We now prove the upper bound. Our starting point is (4.4). Finding an upper bound on the first term is straight forward since the initial condition is a bounded function. For the second term, we need a bit more work.

$$\begin{aligned}
 & \lambda^2 \int_0^L \int_0^t p_N^2(t-s, x, y) E|\sigma(u_s(y))|^2 ds dy \\
 & \leq \lambda^2 L_\sigma^2 \int_0^L \int_0^t p_N^2(t-s, x, y) E|u_s(y)|^2 ds dy \\
 & \leq \lambda^2 L_\sigma^2 \int_0^t \mathcal{S}_s(\lambda) \int_0^L p_N^2(t-s, x, y) dy ds \\
 & = \lambda^2 L_\sigma^2 \int_0^t \mathcal{S}_s(\lambda) p_N(2(t-s), x, x) ds \\
 & \leq c_T \frac{\lambda^2 L_\sigma^2}{\sqrt{4\pi}} \int_0^t \frac{\mathcal{S}_s(\lambda)}{\sqrt{t-s}} ds.
 \end{aligned}$$

With this inequality, (4.4) reduces to

$$\mathcal{S}_t(\lambda) \leq c_2 + \frac{c_T \lambda^2 L_\sigma^2}{\sqrt{4\pi}} \int_0^t \frac{\mathcal{S}_s(\lambda)}{\sqrt{t-s}} ds.$$

An application of (2.1) now yields the desired result. \square

4.2. Proof of Corollary 1.5

The proof of Corollary 1.5 is straightforward.

Proof. Note that

$$\|u_t\|_{L^2[0,L]}^2 \leq \mathcal{S}_t^2(\lambda)L,$$

from which the upper bound follows. As for the lower bound, we have

$$\mathcal{I}_t^2(\lambda)L \leq \|u_t\|_{L^2[0,L]}^2. \quad \square$$

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