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Asymptotic properties of estimators in a stable Cox–Ingersoll–Ross model

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Abstract

We study the estimation of a stable Cox–Ingersoll–Ross model, which is a special subcritical continuous-state branching process with immigration. The exponential ergodicity and strong mixing property of the process are proved by a coupling method. The regular variation properties of distributions of the model are studied. The key is to establish the convergence of some point processes and partial sums associated with the model. From those results, we derive the consistency and central limit theorems of the conditional least squares estimators (CLSEs) and the weighted conditional least squares estimators (WCLSEs) of the drift parameters based on low frequency observations. The theorems show that the WCLSEs are more efficient than the CLSEs and their errors have distinct decay rates $n^{-(\alpha-1)/\alpha}$ and $n^{-(\alpha-1)/\alpha^2}$, respectively, as the sample sizes n goes to infinity. The arguments depend heavily on the recent results on the construction and characterization of the model in terms of stochastic equations.

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1. Introduction

The *Cox–Ingersoll–Ross model* (CIR-model) introduced by Cox et al. [10] has been used widely in the financial world. This model has many appealing advantages. In particular, it is mean-reverting and remains non-negative. Let $a > 0$, $b > 0$ and $\sigma > 0$ be given constants. The classical CIR-model is a non-negative diffusion process $\{X(t) : t \geq 0\}$ defined by

$$dX(t) = (a - bX(t))dt + \sigma\sqrt{X(t)}dB(t), \tag{1.1}$$

where $\{B(t) : t \geq 0\}$ is a standard Brownian motion. The process defined by (1.1) has continuous sample paths and light tailed marginal distributions. The restrictions $a > 0$ and $b > 0$ of the parameters come from the mean-reverting assumption in mathematical finance.

It is well-known that many financial processes exhibit discontinuous sample paths and heavy tailed distributions. This phenomenon has been pointed out by many authors such as Mandelbrot [36] and Fama [20]. Thus α-stable processes as generalizations of the Brownian motion have often been used in mathematical finance. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ be a filtered probability space satisfying the usual hypotheses. A natural generalization of (1.1) is the stochastic differential equation

$$dX_t = (a - bX_t)dt + \sigma\sqrt[\alpha]{X_{t-}}dZ_t, \tag{1.2}$$

where $\{Z_t : t \geq 0\}$ is a spectrally positive stable $(\mathscr{F}_t)$-Lévy process with index $1 < \alpha \leq 2$. For $\alpha = 2$, we understand the noise as a standard Brownian motion, so (1.2) reduces to (1.1). When $1 < \alpha < 2$, we assume it is a stable process with Lévy measure

$$\nu_\alpha(dz) := \frac{1_{\{z>0\}}dz}{\alpha\Gamma(-\alpha)z^{\alpha+1}}. \tag{1.3}$$

By a result of Fu and Li [22], there is a pathwise unique non-negative strong solution $\{X_t : t \geq 0\}$ to (1.2). We refer to this process as a *stable Cox–Ingersoll–Ross model* (SCIR-model). We shall see that the discontinuous SCIR-model indeed captures the important heavy tail property in the sense of infinite variance. This model was considered in [7, p. 134] as a time-changed α-stable process. The reader may refer to Borkovec and Klüppelberg [5], Embrechts et al. [19, Section 7.6] and Fasen et al. [21] for other similar modifications of the CIR-model. In the recent work of Jing et al. [29], some statistical evidence from high-frequency data has been provided to support the application of pure jump models alone in financial modeling. The SCIR-model is a particular form of the so-called *continuous-state branching processes with immigration* (CBI-processes), which arise as scaling limits of *Galton–Watson branching processes with immigration* (GWI-processes); see, e.g., [30]. The general CBI-processes were also constructed and studied in terms of stochastic integral equations in [14,15,22,34]. From the results in [22,30], it is clear that (1.2) essentially gives the most general form of a discontinuous CBI-process driven by a single Lévy process.

The estimation for stochastic processes based on the minimization of a sum of squared deviations about conditional expectations was developed in [31]. They applied their results to the *conditional least squares estimators* (CLSEs) of the offspring and immigration means of subcritical GWI-processes. Their estimators are essentially the same as those studied by Quine [42,43]. By the results of Klimko and Nelson [31] and Quine [42,43], under a finite third moment condition, as the sample size n goes to infinity, the errors of the CLSEs decay at rate $n^{-1/2}$ and they are asymptotically Gaussian; see also the earlier work of Heyde and Seneta [23,24]. The asymptotic properties of CLSEs of GWI-processes with general offspring laws were studied in [50,51]. The estimation problems of the CIR-model defined by (1.1) were studied by Overbeck and Rydén [40], who proposed some CLSEs and proved Gaussian central limit theorems for them; see also [39].

Based on the idea of Nelson [38], the *weighted conditional least squares estimators* (WCLSEs) of the offspring and immigration means of GWI-processes were proposed by Wei and Winnicki [52], who proved some self-normalized central limit theorems for the estimators in the subcritical, critical and supercritical cases. The limiting distributions of Wei and Winnicki [52] are Gaussian in the subcritical and supercritical cases. Wei and Winnicki [52] observed that the WCLSEs are more efficient than the CLSEs in the sense that they have smaller asymptotic variances. The reader may refer to de la Peña et al. [16] for recent developments in self-normalized limit theorems and their statistical applications. The asymptotics of the WCLSEs of the drift parameters of general CBI-processes was studied in [25] under the finite variance condition, which are not satisfied by the SCIR-model.

In this paper, we consider the estimation problem for the drift coefficients (b, a) of the SCIR-model using low frequency observations at equidistant time points $\{k\Delta : k = 0, 1, \dots, n\}$ from a single realization $\{X_t : t \geq 0\}$. For simplicity, we take $\Delta = 1$, but all the results presented below can be modified to the general case. We also introduce the parameters

$$\gamma = e^{-b}, \qquad \rho = ab^{-1}(1-\gamma). \tag{1.4}$$

By applying Itô's formula to (1.2), for any $t \geq r \geq 0$ we have

$$X_t = e^{-b(t-r)}X_r + a\int_r^t e^{-b(t-s)}ds + \sigma\int_r^t e^{-b(t-s)}X_{s-}^{1/\alpha}dZ_s. \tag{1.5}$$

From (1.5) we obtain the first order autoregressive equation

$$X_k = \rho + \gamma X_{k-1} + \varepsilon_k, \tag{1.6}$$

where

$$\varepsilon_k = \sigma\int_{k-1}^k e^{-b(k-s)}X_{s-}^{1/\alpha}dZ_s. \tag{1.7}$$

It is easy to see that

$$\varepsilon_k = X_k - \mathbf{E}(X_k|\mathscr{F}_{k-1}), \quad k \geq 1$$

is a sequence of martingale differences. See also [23,24,31,40] for similar considerations. The CLSEs of (γ, ρ) and (b, a) can be given by minimizing the sum of squares

$$\sum_{k=1}^n \varepsilon_k^2 = \sum_{k=1}^n [X_k - \mathbf{E}(X_k|\mathscr{F}_{k-1})]^2 = \sum_{k=1}^n (X_k - \gamma X_{k-1} - \rho)^2. \tag{1.8}$$

In particular, the estimators of (b, a) are given by

$$\hat{b}_n = -\log\frac{\sum_{k=1}^n X_{k-1}\sum_{k=1}^n X_k - n\sum_{k=1}^n X_{k-1}X_k}{\left(\sum_{k=1}^n X_{k-1}\right)^2 - n\sum_{k=1}^n X_{k-1}^2} \tag{1.9}$$

and

$$\hat{a}_n = \frac{\hat{b}_n\big(\sum_{k=1}^{n} X_k - e^{-\hat{b}_n}\sum_{k=1}^{n} X_{k-1}\big)}{n(1-e^{-\hat{b}_n})}. \tag{1.10}$$

We also consider the WCLSEs of (γ, ρ) and (b, a) following Wei and Winnicki [52] and Huang et al. [25]. Those are obtained by minimizing the weighted sum

$$\sum_{k=1}^{n} \frac{\varepsilon_k^2}{X_{k-1}+1} = \sum_{k=1}^{n} \frac{[X_k - \gamma(X_{k-1}+1) - (\rho-\gamma)]^2}{X_{k-1}+1}. \tag{1.11}$$

The advantage of considering the above quantity is it does not fluctuate too much even when the values of the samples $X_k, k = 0, 1, \ldots, n$, are large. The resulting WCLSEs of (b, a) are given by

$$\tilde{b}_n = -\log \frac{\sum_{k=1}^{n} X_k \sum_{k=1}^{n} \frac{1}{X_{k-1}+1} - n\sum_{k=1}^{n} \frac{X_k}{X_{k-1}+1}}{\sum_{k=1}^{n}(X_{k-1}+1)\sum_{k=1}^{n}\frac{1}{X_{k-1}+1} - n^2} \tag{1.12}$$

and

$$\tilde{a}_n = \frac{\tilde{b}_n\big(\sum_{k=1}^{n} X_k - e^{-\tilde{b}_n}\sum_{k=1}^{n} X_{k-1}\big)}{n(1-e^{-\tilde{b}_n})}. \tag{1.13}$$

The main purpose of this paper is to study the asymptotic properties of the WCLSEs and the CLSEs of the SCIR-model defined above. We show that the estimators are consistent and obey some central limit theorems. It turns out that the WCLSEs are more efficient than the CLSEs with different convergence rates. More precisely, for $1 < \alpha \le 2$ we prove that the sequence $n^{(\alpha-1)/\alpha}(\tilde{b}_n - b, \tilde{a}_n - a)$ converges to an α-stable random vector as $n \to \infty$. This extends the results on general CBI-processes established in [25] under the finite second moment condition. For $1 < \alpha < (1+\sqrt{5})/2$, we show that $n^{(\alpha-1)/\alpha^2}(\hat{b}_n - b, \hat{a}_n - a)$ converges to a nontrivial limit as $n \to \infty$. Then the errors of the WCLSEs and the CLSEs have distinct decay rates $n^{-(\alpha-1)/\alpha}$ and $n^{-(\alpha-1)/\alpha^2}$, respectively. This is interesting and different from the situation of CBI-processes or GWI-processes with finite variance observed in [25,40,52], where the errors of the WCLSEs and the CLSEs have the same decay rate $n^{-1/2}$ as $n \to \infty$. It is somewhat unfortunate that our approach to the central limit theorem of the CLSEs only works for $1 < \alpha < (1+\sqrt{5})/2$. Since the relevant distributions are actually not well-defined otherwise, we do not think one can remove the restriction by a simple modification of the approach. As a consequence, the complete characterization of the asymptotics of the CLSEs is left as an open problem. From the viewpoint of theoretical completeness, we certainly expect a complete solution of the problem. However, since one can always choose the more efficient WCLSEs instead of the CLSEs, the incompleteness of our results does not really cause much inconvenience from the viewpoint of applications.

The proofs of our limit theorems are rather different from and more difficult than those in the Gaussian case and depend heavily on the construction and characterization of CBI-processes in terms of the stochastic equations established in [14,15,22,34]. These also stimulate the study of a number of interesting properties of CBI-processes. We first prove the exponential ergodicity and strong mixing property of the subcritical CBI-process by using the coupling method developed in [9,8]. We also study the regular variation properties of the distributions of the SCIR-model. Then we prove the convergence of some point processes defined from the discrete observations. Based on those results, the most important step of our approach is to establish the convergence of some normalized partial sums. More precisely, we shall prove limit theorems of the sequences

$$\frac{1}{n^{1/\alpha}}\Big(\sum_{k=1}^n \varepsilon_k, \sum_{k=1}^n \frac{\varepsilon_k}{X_{k-1}+1}\Big), \qquad \Big(\frac{1}{n^{2/\alpha}}\sum_{k=1}^n X_{k-1}^2, \frac{1}{n^{(\alpha+1)/\alpha^2}}\sum_{k=1}^n X_{k-1}\varepsilon_k\Big).$$

The techniques of point processes in the study of limit theorems have been developed extensively by Basrak and Segers [3], Davis and Hsing [12], Davis and Mikosch [13] among others. We also make use of the results of Hult and Lindskog [27] on the extremal behavior of Lévy stochastic integrals.

The paper is organized as follows. In Section 2, we prove the exponential ergodicity of some subcritical CBI-processes, which implies the strong mixing property of the SCIR-model. Section 3 is devoted to the regular variation properties of some random sequences defined from the model. The limit theorems of random point processes and partial sums are established in Section 4. Based on those theorems, the asymptotic properties of the estimators are proved in Section 5. Some basic concepts and technical results on regular variations are reviewed in the Appendix.

Notation. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{R} = (-\infty, \infty)$, $\bar{\mathbb{R}} = [-\infty, \infty]$ and $\bar{\mathbb{R}}_0^d = \bar{\mathbb{R}}^d \setminus \{\mathbf{0}\}$, where $\mathbf{0} = (0, 0, \ldots, 0)$. Let $C_0^+(\bar{\mathbb{R}}_0^d)$ be the collection of non-negative continuous functions on $\bar{\mathbb{R}}_0^2$ with compact support. Let $M(\bar{\mathbb{R}}_0^d)$ be the class of Radon point measures on $\bar{\mathbb{R}}_0^d$ furnished with the topology of vague convergence. We use C with or without subscripts to denote non-negative constants whose values are not important.

2. CBI-processes and ergodicity

In this section, we prove some simple properties of CBI-processes. In particular, we prove a subcritical CBI-process is exponentially ergodic and strongly mixing. The results are essentially useful in the study of the stable limit theorems of the partial sums related to the CLSEs and the WCLSEs. They should also be interesting on their own right.

We start with an important special case of the CBI-process. Let $\sigma \geq 0$ and b be constants and $(u \wedge u^2)m(du)$ a finite measure on $(0, \infty)$. For $z \geq 0$ set

$$\phi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (e^{-zu} - 1 + zu)m(du).$$

A Markov process with state space $\mathbb{R}_+ := [0, \infty)$ is called a *continuous-state branching process* (CB-process) with *branching mechanism* ϕ if it has transition semigroup $(Q_t)_{t\geq 0}$ given by

$$\int_0^\infty e^{-\lambda y} Q_t(x, dy) = e^{-x v_t(\lambda)}, \tag{2.1}$$

where $t \mapsto v_t(\lambda)$ is the unique non-negative solution of

$$\frac{\partial}{\partial t} v_t(\lambda) = -\phi(v_t(\lambda)), \qquad v_0(\lambda) = \lambda. \tag{2.2}$$

The CB-process is called *critical*, *subcritical* or *supercritical* as $b = 0, b > 0$ or $b < 0$, respectively. From (2.2) we obtain the following semigroup property:

$$v_{r+t}(\lambda) = v_r(v_t(\lambda)), \quad r, t, \lambda \geq 0.$$

Taking the derivatives of both sides of (2.2) one can see $u_t := (d/d\lambda)v_t(0)$ solves the equation $(d/dt)u_t = -bu_t$, and so $u_t = e^{-bt}$ for $t \geq 0$. Then differentiating both sides of (2.1) gives

$$\int_0^\infty y Q_t(x, dy) = xe^{-bt}, \quad t, x \geq 0.$$

By Jensen's inequality, we have $v_t(\lambda) \leq \lambda e^{-bt}$ for $t, \lambda \geq 0$.

It is easy to see that $(Q_t)_{t\geq 0}$ is a Feller transition semigroup, so it has a Hunt realization. Let $X = (\Omega, \mathscr{G}, \mathscr{G}_t, X_t, \mathbf{Q}_x)$ be a Hunt realization of the CB-process. The hitting time $\tau_0 = \inf\{t \geq 0 : X_t = 0\}$ is called the *extinction time* of X. It follows from Theorem 3.5 of Li [33] that for $t \geq 0$ the limit $\bar{v}_t =\uparrow \lim_{\lambda\to\infty} v_t(\lambda)$ exists in $(0, \infty]$, and

$$\mathbf{Q}_x(\tau_0 \leq t) = \mathbf{Q}_x(X_t = 0) = \exp\{-x\bar{v}_t\}. \tag{2.3}$$

By Theorem 3.8 of Li [33], we have $\bar{v}_t < \infty$ for all $t > 0$ if and only if the following condition holds:

Condition 2.1. *There is some constant $\theta > 0$ such that $\phi(z) > 0$ for $z > \theta$ and*

$$\int_\theta^\infty \phi(z)^{-1} dz < \infty.$$

Let $t \mapsto v_t(\lambda)$ be defined by (2.2). A Markov process with state space $\mathbb{R}_+$ is called a *CBI-process* with *branching mechanism* ϕ and *immigration rate* $a \geq 0$ if it has transition semigroup $(P_t)_{t\geq 0}$ given by

$$\int_0^\infty e^{-\lambda y} P_t(x, dy) = \exp\Big\{-xv_t(\lambda) - a\int_0^t v_s(\lambda)ds\Big\}. \tag{2.4}$$

By differentiating both sides of (2.4) we obtain

$$\int_0^\infty y P_t(x, dy) = xe^{-bt} + a\int_0^t e^{-bs} ds = xe^{-bt} + ab^{-1}(1 - e^{-bt}), \tag{2.5}$$

where $b^{-1}(1 - e^{-bt}) = t$ when $b = 0$ by convention.

Proposition 2.2. *Suppose that $b > 0$. Then the transition semigroup $(P_t)_{t\geq 0}$ has a unique stationary distribution μ, which is given by*

$$L_\mu(\lambda) = \int_0^\infty e^{-\lambda x} \mu(dx) = \exp\Big\{-a\int_0^\lambda z\phi(z)^{-1} dz\Big\}, \quad \lambda \geq 0. \tag{2.6}$$

Moreover, we have

$$\int_0^\infty x\mu(dx) = \frac{d}{d\lambda} L_\mu(\lambda)\Big|_{\lambda=0+} = \frac{a}{b}. \tag{2.7}$$

Proof. By Theorem 3.20 of Li [33], there is a probability measure μ on $\mathbb{R}_+$ so that $\lim_{t\to\infty} P_t(x,\cdot) = \mu$ by weak convergence for any $x \geq 0$. Then $(P_t)_{t\geq 0}$ has the unique stationary distribution μ. The expression (2.6) of the Laplace transform follows from a formula in [33, p. 67]. By differentiating (2.6) we obtain (2.7). □

A realization of the CBI-process can be constructed as the strong solution to a stochastic integral equation. Let $W(ds,du)$ be a time–space Gaussian white noise on $(0,\infty)^2$ with intensity $dsdu$ and $N_1(ds,dz,du)$ a Poisson random measure on $(0,\infty)^3$ with intensity $dsm(dz)du$. Let $\tilde{N}_1(ds,dz,du) = N_1(ds,dz,du) - dsm(dz)du$ denote the compensated measure. Then for each $x \geq 0$ there is a pathwise unique non-negative strong solution to the following stochastic equation:

$$Y_t(x) = x + \int_0^t (a - bY_s(x))ds + \sigma \int_0^t \int_0^{Y_{s-}(x)} W(ds,du) + \int_0^t \int_0^\infty \int_0^{Y_{s-}(x)} z\tilde{N}_1(ds,dz,du). \tag{2.8}$$

The solution $\{Y_t(x), t \geq 0\}$ is a CBI-process with branching mechanism ϕ and immigration rate a. See Theorem 3.1 of Dawson and Li [15] or Theorem 2.1 of Li and Ma [34]. A slightly different formulation of the process was given in [14].

Proposition 2.3. *Suppose that Condition 2.1 holds. For $x, y \geq 0$, let $T_{x,y} := \inf\{t \geq 0 : Y_t(x) - Y_t(y) = 0\}$. Then we have $\mathbf{P}\{T_{x,y} < \infty\} = 1$ and*

$$\mathbf{P}\{T_{x,y} \leq t\} = \exp\{-|x-y|\bar{v}_t\}, \quad t \geq 0. \tag{2.9}$$

Moreover, we have $Y_t(x) = Y_t(y)$ for all $t \geq T_{x,y}$.

Proof. It suffices to consider the case of $y \geq x \geq 0$. By Theorem 3.2 of Dawson and Li [15], we have $\mathbf{P}\{Y_t(x) \geq Y_t(y) \geq 0 \text{ for all } t \geq 0\} = 1$ and $\{Y_t(x) - Y_t(y) : t \geq 0\}$ is a CB-process with branching mechanism ϕ; see also Remark 2.1(iv) of Li and Ma [34]. Then (2.9) follows from (2.3). The pathwise uniqueness of (2.8) implies that $Y_t(x) = Y_t(y)$ for all $t \geq T_{x,y}$. By Corollary 3.9 of Li [33] we have $\mathbf{P}\{T_{x,y} < \infty\} = 1$. □

The above proposition provides a successful coupling of the CBI-processes. This has many important implications. The coupling method goes back to Doeblin [17]. We refer the reader to Chen [9,8] for systematical study of the method and its applications in the theory of Markov processes. In particular, we shall use the above coupling to prove the strong Feller property and exponential ergodicity of the CBI-process following the arguments in Section 5.3 of Chen [9]. The later implies a strong mixing property, which is necessary in the study of central limit theorems of the estimators defined in the introduction. Write $f \in b\mathscr{B}(\mathbb{R}_+)$ if f is a bounded measurable function on $\mathbb{R}_+$.

Theorem 2.4. *Under Condition 2.1, the transition semigroup $(P_t)_{t\geq 0}$ given by (2.4) has the strong Feller property. Moreover, for any $t > 0$ and $x, y \geq 0$, we have*

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{var}} \leq 2(1 - e^{-\bar{v}_t|x-y|}) \leq 2\bar{v}_t|x-y|, \tag{2.10}$$

where $\|\cdot\|_{\mathrm{var}}$ denotes the total variation norm.

Proof. Clearly, the strong Feller property follows from (2.10). Then it suffices to prove this inequality. Let $f \in b\mathscr{B}(\mathbb{R}_+)$ satisfy $\|f\|_\infty \le 1$. Then

$$|P_t f(x) - P_t f(y)| \le \mathbf{E}[|f(Y_t(x)) - f(Y_t(y))| 1_{\{T_{x,y}>t\}}] \le 2\mathbf{P}(T_{x,y} > t).$$

By Proposition 2.3 we have $\mathbf{P}(T_{x,y} > t) = 1 - e^{-\bar{v}_t|x-y|} \le \bar{v}_t|x-y|$. That gives (2.10). □

Theorem 2.5. *Suppose that $b > 0$ and Condition 2.1 is satisfied. Then the transition semigroup $(P_t)_{t\ge0}$ is exponentially ergodic. More precisely, for any $x, t \ge 0$, we have*

$$\|P_t(x,\cdot) - \mu(\cdot)\|_{\mathrm{var}} \le 2[1 \wedge (x + ab^{-1})\bar{v}_1 e^{b(1-t)}], \tag{2.11}$$

where μ is given by (2.6).

Proof. Since the total variation norm is a special case of the so-called Wasserstein distance, the exponential ergodicity is essentially a consequence of the arguments in Section 5.3 of Chen [9]. In fact, by Theorem 2.4 one can see

$$\begin{aligned}
\|P_t(x,\cdot) - \mu(\cdot)\|_{\mathrm{var}} &= \left\| \int_0^\infty [P_t(x,\cdot) - P_t(y,\cdot)]\mu(dy) \right\|_{\mathrm{var}} \\
&\le \int_0^\infty \|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{var}} \mu(dy) \\
&\le 2\int_0^\infty (1 \wedge |x-y|\bar{v}_t)\mu(dy) \\
&\le 2\int_0^\infty [1 \wedge (x+y)\bar{v}_t]\mu(dy) \\
&= 2[1 \wedge (x + ab^{-1})\bar{v}_t],
\end{aligned}$$

where the last equality follows by (2.7). By Corollary 3.11 of Li [33, p. 61], for $t \ge 1$, we have $\bar{v}_t = v_{t-1}(\bar{v}_1)$ and so $\bar{v}_t \le e^{b(1-t)}\bar{v}_1$ for $t \ge 1$. Then we obtain desired result. □

Under the conditions of Theorem 2.5, for any finite set $\{t_1 < t_2 < \cdots < t_n\} \subset \mathbb{R}$ we can define the probability measure $\mu_{t_1,t_2,\ldots,t_n}$ on $\mathbb{R}_+^n$ by

$$\mu_{t_1,t_2,\ldots,t_n}(dx_1, dx_2, \ldots, dx_n) = \mu(dx_1)P_{t_2-t_1}(x_1, dx_2) \cdots P_{t_n-t_{n-1}}(x_{n-1}, dx_n). \tag{2.12}$$

It is easy to see that $\{\mu_{t_1,t_2,\ldots,t_n} : t_1 < t_2 < \cdots < t_n \in \mathbb{R}\}$ is a consistent family. By Kolmogorov's theorem, there is a stochastic process $\{Y_t : t \in \mathbb{R}\}$ with finite-dimensional distributions given by (2.12). This process is a (strictly) stationary Markov process with one-dimensional marginal distribution μ and transition semigroup $(P_t)_{t\ge0}$. Since $(P_t)_{t\ge0}$ is a Feller semigroup, the process $\{Y_t : t \in \mathbb{R}\}$ has a càdlàg modification.

Remark 2.6. Let $\{Y_t : t \in \mathbb{R}\}$ be a Markov process with finite-dimensional distributions given by (2.6). Then it follows from Theorem 2.5 that it is also strongly mixing with geometric rate, that is, as $t \to \infty$,

$$\pi_t := \sup_{A\in\sigma\{Y_s, s\le0\}} \sup_{B\in\sigma\{Y_s, s>t\}} \left|\mathbf{P}(A\cap B) - \mathbf{P}(A)\mathbf{P}(B)\right|$$

decays to zero exponentially; see, e.g., [37, p. 516] or [6, p. 112]. □

Now let us consider a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ satisfying the usual hypotheses. Let $\{Z_t : t \geq 0\}$ be a spectrally positive α-stable Lévy process. For $\alpha = 2$, we understand the process as a standard Brownian motion; and for $1 < \alpha < 2$, we assume it is a stable process with Lévy measure $\nu_\alpha(dz)$ given by (1.3). By Theorem 6.2 of Fu and Li [22], for any initial value X_0, which is a non-negative $\mathscr{F}_0$-measurable random variable, there is a unique non-negative strong solution $\{X_t : t \geq 0\}$ to (1.2). The existence and uniqueness of this solution also follows from Corollary 6.3 of Fu and Li [22] by a time change. Let f be a bounded continuous function on $\mathbb{R}$ with bounded continuous derivatives up to the second order. For $\alpha = 2$, we can use Itô's formula to see that

$$f(X_t) = f(X_r) + \int_r^t Lf(X_s)ds + M_t(f), \quad t \geq r, \tag{2.13}$$

where $\{M_t(f) : t \geq r\}$ is a martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq r}$ and

$$Lf(x) = (a - bx)f'(x) + \frac{\sigma^2}{2}xf''(x), \quad x \geq 0.$$

When $1 < \alpha < 2$, by the Lévy–Itô representation of $\{Z_t\}$, we can rewrite (1.2) into the integral form:

$$X_t = X_r + \int_r^t (a - bX_s)ds + \sigma \int_r^t \int_0^\infty X_{s-}^{1/\alpha} z\tilde{N}(ds, dz), \quad t \geq r, \tag{2.14}$$

where $\tilde{N}(ds, dz)$ is a compensated Poisson random measure on $(0, \infty)^2$ with intensity $ds\nu_\alpha(dz)$. By Itô's formula one can see that (2.13) still holds for $1 < \alpha < 2$ with the operator L defined by

$$Lf(x) = (a - bx)f'(x) + \frac{\sigma^\alpha x}{\alpha\Gamma(-\alpha)} \int_0^\infty [f(x + y) - f(x) - yf'(x)]\frac{dy}{y^{\alpha+1}}.$$

By Theorem 9.30 of Li [33], for any $1 < \alpha \leq 2$, we can identify the SCIR-model as a subcritical CBI-process with immigration rate a and branching mechanism

$$\phi(\lambda) = b\lambda + \frac{\sigma^\alpha}{\alpha}\lambda^\alpha, \quad \lambda \geq 0. \tag{2.15}$$

It follows from Proposition 2.2 that the SCIR-model has the unique stationary distribution μ with Laplace transform given by

$$L_\mu(\lambda) = \int_0^\infty e^{-\lambda x}\mu(dx) = \exp\left\{-\int_0^\lambda \frac{\alpha a dz}{\alpha b + \sigma^\alpha z^{\alpha-1}}\right\}, \quad \lambda \geq 0. \tag{2.16}$$

Let $\mathbf{P}_x$ denote the law of the SCIR-model $\{X_t : t \geq 0\}$ defined by (1.2) with $X_0 = x \geq 0$ and let $\mathbf{E}_x$ denote the corresponding expectation.

Proposition 2.7. *Let $1 < \alpha < 2$ and let $\{X_t\}$ be the SCIR-model defined by (2.14). Then for any $0 < \beta < \alpha$, there is a constant $C \geq 0$ so that, for $t, T \geq 0$,*

$$\mathbf{E}_x\left(\left|\int_0^t e^{-b(t-s)}X_{s-}^{1/\alpha}dZ_s\right|^\beta\right) \leq C(1 + x^{\beta/\alpha}e^{-\beta bt/\alpha})$$

and

$$\mathbf{E}_x\left(\sup_{0\leq t\leq T}\left|\int_0^t e^{-b(t-s)}X_{s-}^{1/\alpha}dZ_s\right|^\beta\right) \leq C(x^{\beta/\alpha}e^{\beta b(1-1/\alpha)T} + e^{\beta bT}).$$

Proof. By Remark A.8, we have

$$\mathbf{E}_x\Big(\Big|\int_0^t e^{-b(T-s)} X_{s-}^{1/\alpha} dZ_s\Big|^\beta\Big) \le C_1 e^{-\beta bt}\mathbf{E}_x\Big[\Big(\int_0^t e^{\alpha bs} X_s ds\Big)^{\beta/\alpha}\Big]$$

and

$$\mathbf{E}_x\Big(\sup_{0\le t\le T}\Big|\int_0^t e^{-b(t-s)} X_{s-}^{1/\alpha} dZ_s\Big|^\beta\Big) \le C_1 \mathbf{E}_x\Big[\Big(\int_0^T e^{\alpha bs} X_s ds\Big)^{\beta/\alpha}\Big].$$

By Hölder's inequality and (2.5) it is easy to see

$$\begin{aligned}\mathbf{E}_x\Big[\Big(\int_0^t e^{\alpha bs} X_s ds\Big)^{\beta/\alpha}\Big] &\le \Big[\int_0^t \mathbf{E}_x(e^{\alpha bs} X_s) ds\Big]^{\beta/\alpha}\\ &\le \Big[\int_0^t e^{\alpha bs}(xe^{-bs} + ab^{-1}) ds\Big]^{\beta/\alpha}\\ &\le C_2(x^{\beta/\alpha} e^{\beta b(1-1/\alpha)t} + e^{\beta bt}).\end{aligned}$$

Then we have the desired inequalities. □

Proposition 2.8. *Let $1 < \alpha < 2$ and let $\{X_t\}$ be the SCIR-model defined by (2.14). Then for any $0 < \beta < \alpha$, there is a constant $C \ge 0$ and a locally bounded function $T \mapsto C(T) \ge 0$ so that, for $t, T \ge 0$,*

$$\mathbf{E}_x(X_t^\beta) \le C(1 + x^\beta e^{-\beta bt/\alpha})$$

and

$$\mathbf{E}_x\Big(\sup_{0\le t\le T} X_t^\beta\Big) \le C(T)(1 + x^\beta).$$

Proof. Using (1.5) with $r = 0$ and an elementary inequality, we have

$$\mathbf{E}_x(X_t^\beta) \le C_1\mathbf{E}_x\Big[x^\beta e^{-\beta bt} + a^\beta b^{-\beta} + \sigma^\beta \mathbf{E}_x\Big(\Big|\int_0^t e^{-b(t-s)} X_{s-}^{1/\alpha} dZ_s\Big|^\beta\Big)\Big]$$

and

$$\mathbf{E}_x\Big(\sup_{0\le t\le T} X_t^\beta\Big) \le C_1\Big[x^\beta + a^\beta b^{-\beta} + \sigma^\beta \mathbf{E}_x\Big(\sup_{0\le t\le T}\Big|\int_0^t e^{-b(t-s)} X_{s-}^{1/\alpha} dZ_s\Big|^\beta\Big)\Big].$$

Then the results follow by Proposition 2.7. □

3. Regular variation

In this section, we assume $1 < \alpha < 2$. We shall study the regular variation properties of some random sequences associated with the SCIR-model. The approach to be given uses heavily the stochastic equations (1.2) and (2.8). Some necessary concepts and technical results are reviewed in the last section of the paper. Recall that $\mathbf{P}_x$ denotes the law of the SCIR-model $\{X_t : t \ge 0\}$ defined by (1.2) with $X_0 = x \ge 0$ and $\mathbf{E}_x$ denotes the corresponding expectation.

Proposition 3.1. *Fix $t > 0$. For any $x \ge 0$, we have, as $u \to \infty$,*

$$\mathbf{P}_x(X_t > u) \sim \frac{\sigma^\alpha t}{\alpha\Gamma(-\alpha)}[q_\alpha(t) + p_\alpha(t)x]u^{-\alpha},$$

where

$$p_\alpha(t) = \frac{1}{b(\alpha-1)}[e^{-bt} - e^{-\alpha bt}], \qquad q_\alpha(t) = \frac{a}{b}\Big[\frac{1}{\alpha b}(1-e^{-\alpha bt}) - p_\alpha(t)\Big]. \tag{3.1}$$

Proof. In view of (1.5), the extremal behavior of X_t is determined by a stochastic integral. Then, using Remark A.7, we have, as $u \to \infty$,

$$\begin{aligned}\mathbf{P}_x(X_t > u) &\sim \mathbf{P}_x\Big(\sigma\int_0^t e^{-b(t-s)}\sqrt[\alpha]{X_{s-}}dZ_s > u\Big)\\ &\sim \sigma^\alpha \mathbf{P}_x(Z_t > u)\int_0^t e^{-\alpha b(t-s)}\mathbf{E}_x(X_s)ds.\end{aligned}$$

Based on (2.5), it is easy to compute

$$\mathbf{E}_x\Big(\int_0^t e^{-\alpha b(t-s)}X_s ds\Big) = q_\alpha(t) + p_\alpha(t)x. \tag{3.2}$$

By Remark A.6 we have $\mathbf{P}_x(Z_t > u) \sim t/(\alpha\Gamma(-\alpha)u^\alpha)$. Then the desired result follows. □

Proposition 3.2. *For any $K > 0$, we have*

$$\lim_{u\to\infty}\sup_{x\in[0,K]}\Big|u^\alpha \mathbf{P}_x(X_1 > u) - \frac{\sigma^\alpha}{\alpha\Gamma(-\alpha)}(q_\alpha + p_\alpha x)\Big| = 0$$

and

$$\lim_{u\to\infty}\sup_{x\in[0,K]}\Big|u^{\alpha-1}\mathbf{E}_x(X_1 1_{\{X_1>u\}}) - \frac{\sigma^\alpha}{(\alpha-1)\Gamma(-\alpha)}(q_\alpha + p_\alpha x)\Big| = 0$$

where $p_\alpha = p_\alpha(1)$ and $q_\alpha = q_\alpha(1)$ are defined by (3.1).

Proof. Let $\{X_t(x)\}$ be the SCIR-model defined by (1.2) with initial value $X_0 = x$. By Theorem 5.5 of Fu and Li [22], the random function $x \mapsto X_t(x)$ is increasing, so $x \mapsto \mathbf{P}_x(X_t > u)$ is increasing for any $t, u \geq 0$; see also (2.8). Then the first convergence holds by Proposition 3.1 and Dini's theorem. The second convergence follows similarly by Proposition A.2. □

In the sequel of this section, let us consider a stationary càdlàg realization $\{X_t : t \in \mathbb{R}\}$ of the SCIR-model with one-dimensional marginal distribution μ given by (2.16). By a modification of the arguments in the proofs of Theorems 9.31 and 9.32 Li [33] one can see that, on an extension of the probability space, there is a compensated Poisson random measure $\tilde{N}(ds, dz)$ on $\mathbb{R}\times(0,\infty)$ with intensity $ds\nu_\alpha(dz)$ so that (2.14) is satisfied for all $t \geq r \in \mathbb{R}$. For any integer $k \in \mathbb{Z}$ let

$$\mathbf{I}_k = \varepsilon_k\big(1, (1+X_{k-1})^{-1}\big), \qquad \mathbf{H}_k = X_{k-1}\big(X_{k-1}^{1/\alpha}, \varepsilon_k\big), \tag{3.3}$$

where

$$\varepsilon_k = \sigma\int_{k-1}^k\int_0^\infty e^{-b(k-s)}X_{s-}^{1/\alpha}z\tilde{N}(ds, dz). \tag{3.4}$$

It is easy to see that the above sequences are stationary. We are going to prove that they are jointly regularly varying.

Proposition 3.3. *Let μ be the stationary distribution of the SCIR-model given by (2.16). For any $t \geq 0$, we have, as $x \to \infty$,*

$$\mu(x,\infty) \sim -\frac{a\sigma^\alpha}{\alpha^2 b^2}\Gamma(1-\alpha)^{-1}x^{-\alpha} = \frac{a\sigma^\alpha}{\alpha^3 b^2}\Gamma(-\alpha)^{-1}x^{-\alpha}.$$

Consequently, for any $0 < r < \alpha$, we have

$$\int_0^\infty x^r\mu(dx) = \int_0^\infty \mu(y^{1/r},\infty)dy < \infty.$$

Proof. The tail behavior of X_t is closely related with the asymptotics of its Laplace transform. By (2.16), as $\lambda \to 0$,

$$\begin{aligned}
L_\mu(\lambda) &= 1 - \frac{a}{b}\int_0^\lambda \frac{\alpha b dz}{\alpha b + \sigma^\alpha z^{\alpha-1}} + O(\lambda^2) \\
&= 1 - \frac{a}{b}\lambda + \frac{a}{b}\int_0^\lambda \frac{\sigma^\alpha z^{\alpha-1}dz}{\alpha b + \sigma^\alpha z^{\alpha-1}} + O(\lambda^2) \\
&= 1 - \frac{a}{b}\lambda + \frac{a}{b^2}\int_0^\lambda \frac{b\sigma^\alpha z^{\alpha-1}dz}{\alpha b + \sigma^\alpha z^{\alpha-1}} + O(\lambda^2) \\
&= 1 - \frac{a}{b}\lambda + \frac{a}{\alpha b^2}\int_0^\lambda \sigma^\alpha z^{\alpha-1}dz - \frac{a}{\alpha b^2}\int_0^\lambda \frac{\sigma^{2\alpha} z^{2(\alpha-1)}dz}{\alpha b + \sigma^\alpha z^{\alpha-1}} + O(\lambda^2) \\
&= 1 - \lambda\int_0^\infty x\mu(dx) + \frac{a\sigma^\alpha}{\alpha^2 b^2}\lambda^\alpha - O(\lambda^{2\alpha-1}) + O(\lambda^2),
\end{aligned}$$

where we have used (2.7) for the last equality. Then the result follows by Theorem 8.1.6 of Bingham et al. [4]. □

Lemma 3.4. *Let $\tilde{N}(ds,dz)$ be the compensated Poisson random measure in (2.14) and let*

$$z(t) = \sigma\int_0^t\int_0^1 e^{-b(t-s)}X_{s-}^{1/\alpha}z\tilde{N}(ds,dz). \tag{3.5}$$

Then, for any $1 \leq r < \alpha^2$ and any $T \geq 0$, we have

$$\mathbf{E}\Big[\sup_{0\leq t\leq T}|z(t)|^r\Big] < \infty.$$

Proof. We follow an idea in the proof of Lemma 5.5 of Hult and Lindskog [27]. Let $q = 1/(1-p^{-1})$ for any $p \in (1,\alpha^2/r)$. By the Burkholder–Davis–Gundy inequality and Hölder's inequality, we have

$$\begin{aligned}
\mathbf{E}\Big[\sup_{0\leq t\leq T}|z(t)|^r\Big] &\leq \sigma^r\mathbf{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t\int_0^1 e^{bs}X_{s-}^{1/\alpha}z\tilde{N}(ds,dz)\Big|^r\Big] \\
&\leq C_1\mathbf{E}\Big[\Big(\int_0^T\int_0^1 e^{2bs}X_{s-}^{2/\alpha}z^2N(ds,dz)\Big)^{r/2}\Big] \\
&\leq C(T)\mathbf{E}\Big[\sup_{0\leq s\leq T}X_s^{r/\alpha}\Big(\int_0^T\int_0^1 z^2N(ds,dz)\Big)^{r/2}\Big] \\
&\leq C(T)\Big(\mathbf{E}\Big[\sup_{0\leq s\leq T}X_s^{rp/\alpha}\Big]\Big)^{1/p}\Big(\mathbf{E}\Big[\Big(\int_0^T\int_0^1 z^2N(ds,dz)\Big)^{rq/2}\Big]\Big)^{1/q}.
\end{aligned}$$

The first expectation on the right-hand side is finite by Proposition 2.8. By Theorem 34 of Protter [41, p. 25], the second expectation is also finite. □

Lemma 3.5. *Suppose that $\{A_n\} \subset \mathscr{F}_0$ is a sequence of events so that $\mathbf{P}(A_n) \to 0$ as $n \to \infty$. Then, for any $x > 0$ and $T \geq 0$, we have*

$$\lim_{n\to\infty} n\mathbf{P}\Big(A_n, \sup_{0\le t\le T}\Big|\int_0^t e^{-b(t-s)} X_{s-}^{1/\alpha} dZ_s\Big| > a_n x\Big) = 0. \tag{3.6}$$

Proof. The right-hand side of (3.6) is bound above by $J_1 + J_2 + J_3$, where

$$J_1 = n\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \int_0^1 e^{-b(t-s)} X_{s-}^{1/\alpha} z\tilde{N}(ds, dz)\Big| > a_n x/3\Big),$$

$$J_2 = n\mathbf{P}\left(\sup_{0\le t\le T} \int_0^t e^{-b(t-s)} X_s^{1/\alpha} ds \int_1^\infty z\nu_\alpha(dz)\right| > a_n x/3\Bigg),$$

and

$$J_3 := n\mathbf{P}\left(A_n, \sup_{0\le t\le T} \int_0^t \int_1^\infty e^{-b(t-s)} X_{s-}^{1/\alpha} zN(ds, dz)\right| > a_n x/3\Bigg).$$

Let $z(t)$ be defined by (3.5). Then, for any $1 \le r < \alpha^2$, we have

$$J_1 \le \frac{C_1 n}{n^{r/\alpha} x^r} \mathbf{E}\Big[\sup_{0\le t\le T} |z(t)|^r\Big], \qquad J_2 \le \frac{C_2 n}{n^{r/\alpha} x^r} \mathbf{E}\Big[\Big(\int_0^T X_{s-}^{1/\alpha} ds\Big)^r\Big],$$

where the two expectations are finite by Proposition 2.8 and Lemma 3.4. Then $J_1 \to 0$ and $J_2 \to 0$ as $n \to \infty$. By introducing the Lévy process

$$\xi(t) := \int_0^t \int_1^\infty zN(ds, dz), \qquad t \ge 0,$$

for any $K \ge 1$, we have

$$J_3 \le n\mathbf{P}(A_n, K^{1/\alpha}\xi(T) > a_n x/6) + n\mathbf{P}\Big(\int_0^T X_{s-}^{1/\alpha} 1_{\{X_{s-}>K\}} d\xi(s) > a_n x/6\Big).$$

By Remark A.6 and the property of independent increments of $\{\xi(t)\}$, it follows that

$$\lim_{n\to\infty} n\mathbf{P}(A_n, K^{1/\alpha}\xi(T) > a_n x/6) = \lim_{n\to\infty} n\mathbf{P}(A_n)\mathbf{P}(K^{1/\alpha}\xi(T) > a_n x/6) = 0.$$

By Remarks A.6 and A.7,

$$\lim_{n\to\infty} n\mathbf{P}\Big(\int_0^T X_{s-}^{1/\alpha} 1_{\{X_{s-}>K\}} d\xi(s) > a_n x/6\Big) = C_3 x^{-\alpha} \int_0^T \mathbf{E}[X_s 1_{\{X_s>K\}}] ds,$$

which tends to zero as $K \to \infty$. Then we have the desired result. □

Theorem 3.6. *The sequence $\{X_k\}$ is jointly regularly varying with index α. More precisely, as $x \to \infty$ we have*

$$\mathbf{P}(X_0 > x) \sim \frac{a\sigma^\alpha}{\alpha^3 b^2} \Gamma(-\alpha)^{-1} x^{-\alpha} \tag{3.7}$$

and, for any integer $k \geq 1$,

$$\mathbf{P}\big(X_0^{-1}(X_0, \dots, X_k) \in \cdot | X_0 > x\big) \xrightarrow{\text{w}} \delta_{(1,e^{-b},\dots,e^{-bk})}(\cdot). \tag{3.8}$$

Proof. By Proposition 3.3 we have the asymptotics (3.7). It suffices to show (3.8) holds when $x \to \infty$ along the sequence $a_n := n^{1/\alpha}$. Let $\mathbf{X} = (X_0, X_1, \dots, X_k)$ and $\bar{\mathbf{X}} = (X_0, X_0e^{-b}, \dots, X_0e^{-bk})$. Let $z(t)$ be defined by (3.5). For any $\delta > 0$, we can use (1.5) to see

$$\begin{aligned}
\mathbf{P}\big(\|\mathbf{X} - \bar{\mathbf{X}}\| > a_n\delta | X_0 > a_n\big) &\leq \mathbf{P}\left(ak + \max_{1\leq j\leq k}\left|\int_0^j e^{-b(j-s)} X_{s-}^{1/\alpha} dZ_s\right| > a_n\delta \,\middle|\, X_0 > a_n\right) \\
&\leq Cn\mathbf{P}\Big(X_0 > a_n, \max_{1\leq j\leq k}\left|\int_0^j e^{-b(j-s)} X_{s-}^{1/\alpha} dZ_s\right| > a_n\delta - ak\Big) \\
&\leq Cn\sum_{j=1}^k \mathbf{P}\Big(X_0 > a_n, \left|\int_0^j e^{-b(j-s)} X_{s-}^{1/\alpha} dZ_s\right| > (a_n\delta - ak)/k\Big).
\end{aligned}$$

By Lemma 3.5 it is easy to see the right-hand tends to zero as $n \to \infty$. By an equivalent form of regular variation given in [44, p. 69], we conclude that, as $n \to \infty$,

$$\begin{aligned}
\mathbf{P}(\mathbf{X}/a_n \in \cdot | X_0 > a_n) &\sim \mathbf{P}(\bar{\mathbf{X}}/a_n \in \cdot | X_0 > a_n) \\
&\xrightarrow{\text{v}} \alpha \int_1^\infty 1_{\{z(1,e^{-b},\dots,e^{-bk})\in\cdot\}} \frac{dz}{z^{\alpha+1}}.
\end{aligned}$$

Then (3.8) follows by the continuous mapping theorem. By Corollary 3.2 in [3], the sequence $\{X_k\}$ is jointly regularly varying with index α. □

Proposition 3.7. *The sequence $\{\mathbf{I}_k\}$ defined by (3.3) is jointly regularly varying with index α. More precisely, as $x \to \infty$ we have*

$$\mathbf{P}(\|\mathbf{I}_1\| > x) \sim \frac{\mathbf{E}(G)}{\alpha^2\Gamma(-\alpha)} x^{-\alpha} = \frac{a\sigma^\alpha(1 - e^{-\alpha b})}{\alpha^3 b^2\Gamma(-\alpha)} x^{-\alpha} \tag{3.9}$$

and, for any integer $k \geq 1$,

$$\begin{aligned}
&\mathbf{P}\big(\|\mathbf{I}_1\|^{-1}(\mathbf{I}_1, \dots, \mathbf{I}_k) \in \cdot | \|\mathbf{I}_1\| > x\big) \\
&\quad\xrightarrow{\text{w}} \mathbf{E}(G)^{-1}\mathbf{E}\big[G; ((1, (1 + X_0)^{-1}), \mathbf{0}, \dots, \mathbf{0}) \in \cdot\big],
\end{aligned} \tag{3.10}$$

where

$$G = \sigma^\alpha \int_0^1 e^{-\alpha b(1-t)} X_t dt. \tag{3.11}$$

Proof. Clearly, it suffices to show that (3.9) and (3.10) hold when $x \to \infty$ along the sequence $a_n := n^{1/\alpha}$. For $0 \leq s \leq k$, let $\mathbf{Z}_s = (Z_s, Z_s)$ and $\mathbf{\Phi}_s = (\phi_1(s), \phi_2(s))$, where

$$\phi_1(s) = \sum_{j=1}^k 1_{(j-1,j]}(s) e^{-b(j-s)} \sigma X_{s-}^{1/\alpha}, \qquad \phi_2(s) = \sum_{j=1}^k 1_{(j-1,j]}(s) \frac{e^{-b(j-s)}\sigma X_{s-}^{1/\alpha}}{1 + X_{j-1}}.$$

We consider the process

$$(\mathbf{\Phi} \cdot \mathbf{Z})_t := \Big(\int_0^t \phi_1(s) dZ_s, \int_0^t \phi_2(s) dZ_s\Big).$$

Choose some $\delta \in (\alpha, \alpha^2)$. By Proposition 3.3, we have $\mathbf{E}[X_0^{\delta/\alpha}] < \infty$. Then Proposition 2.8 implies that

$$\mathbf{E}\Big[\sup_{0\le s\le k} \|\mathbf{\Phi}_s\|^\delta\Big] \le \mathbf{E}\Big[\sup_{0\le s\le k} \sigma^\delta X_s^{\delta/\alpha}\Big] \le \sigma^\delta C(k)[1+\mathbf{E}(X_0^{\delta/\alpha})] < \infty.$$

By Theorem 3.4 in [27] and Remark A.6, as $n \to \infty$,

$$n\mathbf{P}(a_n^{-1}(\mathbf{\Phi}\cdot\mathbf{Z}) \in \cdot) \xrightarrow{\hat{\mathrm{w}}} Q(\cdot) := k\mathbf{E}[\nu_\alpha\{x \in \mathbb{R}_+ : x\mathbf{\Phi}_\tau 1_{[\tau,k]} \in \cdot\}], \tag{3.12}$$

where ν_α is defined by (1.3) and τ is uniformly distributed on $[0, k]$ and independent of $\mathbf{\Phi}$. In view of (3.12), we have

$$n\mathbf{P}(a_n^{-1}(\mathbf{\Phi}\cdot\mathbf{Z})_1 \in \cdot) \xrightarrow{\mathrm{v}} Q(\{\mathbf{y} \in \mathbb{D}^2[0,k] : \mathbf{y}_1 \in \cdot\}).$$

By Definition A.1 it follows that

$$Q(\{\mathbf{y} \in \mathbb{D}^2[0,k] : \|\mathbf{y}_1\| > r\}) = r^{-\alpha} Q(\{\mathbf{y} \in \mathbb{D}^2[0,k] : \|\mathbf{y}_1\| > 1\}).$$

Now define the functions $h_0, h_1, h_2 : \mathbb{D}^2[0,k] \to \mathbb{R}^{2k}$ by

$$h_0(\mathbf{y}) = (\mathbf{y}_1, \mathbf{y}_2 - \mathbf{y}_1, \ldots, \mathbf{y}_k - \mathbf{y}_{k-1}),$$
$$h_1(\mathbf{y}) = 1_{\{\|\mathbf{y}_1\|>1\}}, \qquad h_2(\mathbf{y}) = h_0(\mathbf{y})h_1(\mathbf{y}).$$

Let $\mathrm{Disc}(h_i)$ be the set of discontinuities of h_i $(i = 0, 1, 2)$. By (3.12) it is easy to see that $Q(\mathrm{Disc}(h_0)) = Q(\mathrm{Disc}(h_1)) = 0$, so $Q(\mathrm{Disc}(h_2)) = 0$. Moreover, for any $B \in \mathscr{B}(\mathbb{R}^{2k})$ bounded away from $\mathbf{0}$ the set $h_2^{-1}(B) \in \mathscr{B}(\mathbb{D}^2[0,k])$ is bounded away from $\mathbf{0}$. Applying the continuous mapping theorem, we obtain as $n \to \infty$,

$$n\mathbf{P}\big(\|\mathbf{I}_1\| > a_n,\ a_n^{-1}(\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_k) \in \cdot\big) \xrightarrow{\mathrm{v}} Q \circ h_2^{-1}(\cdot)$$

on $\mathbb{R}^{2k} \setminus \{\mathbf{0}\}$, where

$$\begin{aligned} Q\circ h_2^{-1}(\cdot) &= k\mathbf{E}\Big[\nu_\alpha\Big\{x \in \mathbb{R}_+ : \|x\mathbf{\Phi}_\tau 1_{[\tau,k]}(1)\| > 1,\ x\big(\mathbf{\Phi}_\tau 1_{[\tau,k]}(1), \mathbf{0}, \ldots, \mathbf{0}\big) \in \cdot\Big\}\Big] \\ &= k\mathbf{E}\Big[\nu_\alpha\Big\{x \in \mathbb{R}_+ : \|x\mathbf{\Phi}_\tau\| > 1, \tau \le 1,\ x\big(\mathbf{\Phi}_\tau 1_{[\tau,k]}(1), \mathbf{0}, \ldots, \mathbf{0}\big) \in \cdot\Big\}\Big] \\ &= k\mathbf{E}\Big[\nu_\alpha\Big\{x \in \mathbb{R}_+ : x\|\mathbf{\Phi}_\tau\| > 1, x\big(\mathbf{\Phi}_\tau, \mathbf{0}, \ldots, \mathbf{0}\big) \in \cdot\Big\}1_{[0,1]}(\tau)\Big]. \end{aligned}$$

Let $E = \{(\mathbf{y}_1, \ldots, \mathbf{y}_k) \in \mathbb{R}^{2k} : \|\mathbf{y}_1\| > 0\}$. Define the injection $f : E \to (0,\infty) \times E$ by

$$f(\mathbf{y}_1, \ldots, \mathbf{y}_k) = \big(\|\mathbf{y}_1\|, \mathbf{y}_1/\|\mathbf{y}_1\|, \ldots, \mathbf{y}_k/\|\mathbf{y}_1\|\big).$$

Then we have as $n \to \infty$,

$$n\mathbf{P}\big(\|\mathbf{I}_1\| > a_n,\ \|\mathbf{I}_1\|^{-1}(\mathbf{I}_1, \ldots, \mathbf{I}_k) \in \cdot\big) \xrightarrow{\mathrm{w}} Q \circ h_2^{-1} \circ f^{-1}(\cdot)$$

on E, where

$$\begin{aligned} Q\circ h_2^{-1}\circ f^{-1}(\cdot) &= k\mathbf{E}\Big[\nu_\alpha\Big\{x \in \mathbb{R}_+ : x\|\mathbf{\Phi}_\tau\| > 1, \big(\mathbf{\Phi}_\tau/\|\mathbf{\Phi}_\tau\|, \mathbf{0}, \ldots, \mathbf{0}\big) \in \cdot\Big\}1_{[0,1]}(\tau)\Big] \\ &= k\mathbf{E}\Big[\nu_\alpha\Big\{x \in \mathbb{R}_+ : x\|\mathbf{\Phi}_\tau\| > 1\Big\}1_{[0,1]}(\tau), \big(\mathbf{\Phi}_\tau/\|\mathbf{\Phi}_\tau\|, \mathbf{0}, \ldots, \mathbf{0}\big) \in \cdot\Big] \\ &= \frac{k}{\alpha\Gamma(-\alpha)}\mathbf{E}\Big[\int_{\|\mathbf{\Phi}_\tau\|^{-1}}^\infty \frac{dz}{z^{\alpha+1}} 1_{[0,1]}(\tau), \big((1, (1+X_0)^{-1}), \mathbf{0}, \ldots, \mathbf{0}\big) \in \cdot\Big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{k}{\alpha^2 \Gamma(-\alpha)} \mathbf{E}\big[\|\mathbf{\Phi}_\tau\|^\alpha 1_{[0,1]}(\tau), \big((1, (1+X_0)^{-1}), \mathbf{0}, \ldots, \mathbf{0}\big) \in \cdot\big] \\
&= \frac{1}{\alpha^2 \Gamma(-\alpha)} \int_0^1 \mathbf{E}\big[\|\mathbf{\Phi}_s\|^\alpha, \big((1, (1+X_0)^{-1}), \mathbf{0}, \ldots, \mathbf{0}\big) \in \cdot\big] ds \\
&= \frac{1}{\alpha^2 \Gamma(-\alpha)} \mathbf{E}\big[G, \big((1, (1+X_0)^{-1}), \mathbf{0}, \ldots, \mathbf{0}\big) \in \cdot\big].
\end{aligned}
$$

In particular, as $n \to \infty$ we have

$$
n\mathbf{P}(\|\mathbf{I}_1\| > a_n) \longrightarrow \frac{\mathbf{E}(G)}{\alpha^2 \Gamma(-\alpha)}.
$$

By (2.7), we have $\mathbf{E}(X_t) = a/b$. It follows that

$$
\mathbf{E}(G) = \frac{a\sigma^\alpha}{b} \int_0^1 e^{-\alpha b(1-t)} dt = \frac{a\sigma^\alpha}{\alpha b^2}(1 - e^{-\alpha b}).
$$

Then we have (3.9) and (3.10). By Corollary 3.2 in [3], the sequence $\{\mathbf{I}_k\}$ is jointly regularly varying with index α. □

Remark 3.8. For $k = 1, 2, \ldots$ define

$$
V_k = \sigma \int_{k-1}^k e^{-b(k-s)} e^{-b(s-k+1)/\alpha} dZ_s. \tag{3.13}
$$

Then the sequence $\{V_k\}$ is i.i.d. with the same distribution as

$$
\sigma \Big(\frac{e^{-b} - e^{-\alpha b}}{(\alpha - 1)b} \Big)^{1/\alpha} Z_1,
$$

which is regularly varying with index α.

Lemma 3.9. *Let $\{V_k\}$ be defined by (3.13) and let $\bar{\mathbf{H}}_k = X_{k-1}^{(\alpha+1)/\alpha}(1, V_k)$. Then, for any $0 < r < \alpha^3/(\alpha^2 + 1)$, we have*

$$
\mathbf{E}\big[\|\mathbf{H}_k - \bar{\mathbf{H}}_k\|^r\big] < \infty.
$$

Proof. Since $0 < r < \alpha^3(\alpha^2 + 1)^{-1} < \alpha$, by Remark A.8 and Hölder's inequality,

$$
\begin{aligned}
\mathbf{E}\big[\|\mathbf{H}_k - \bar{\mathbf{H}}_k\|^r\big] &= \mathbf{E}\Big[\Big| \int_{k-1}^k \sigma e^{-b(k-s)} X_{k-1} \Big(X_{s-}^{1/\alpha} - X_{k-1}^{1/\alpha} e^{-b(s-k+1)/\alpha} \Big) dZ_s \Big|^r\Big] \\
&\le C\mathbf{E}\Big[\Big(\int_{k-1}^k e^{-\alpha b(k-s)} X_{k-1}^\alpha \big| X_s - X_{k-1} e^{-b(s-k+1)} \big| ds \Big)^{r/\alpha}\Big] \\
&\le C\mathbf{E}\Big\{ X_{k-1}^r \Big[\mathbf{E}_{X_{k-1}} \Big(\int_0^1 |X_s - X_0 e^{-bs}| ds \Big) \Big]^{r/\alpha} \Big\} \\
&\le C\mathbf{E}\Big\{ X_{k-1}^r \Big[\int_0^1 \mathbf{E}_{X_{k-1}} \big(|X_s - X_0 e^{-bs}| \big) ds \Big]^{r/\alpha} \Big\} \\
&\le C\mathbf{E}\Big\{ X_{k-1}^r \Big[\int_0^1 \mathbf{E}_{X_{k-1}} \Big(\Big| \int_0^s e^{-b(s-u)} X_{u-}^{1/\alpha} dZ_u \Big| \Big) ds \Big]^{r/\alpha} \Big\}
\end{aligned}
$$

$$\leq C\mathbf{E}\Big\{X_{k-1}^r\Big[\int_0^1 \mathbf{E}_{X_{k-1}}\Big(\Big|\int_0^s e^{-b(s-u)}X_{u-}^{1/\alpha}dZ_u\Big|^r\Big)ds\Big]^{1/\alpha}\Big\}$$
$$\leq C\mathbf{E}\big[X_{k-1}^r(1+X_{k-1}^{r/\alpha^2})\big],$$

which is finite by Proposition 2.8. □

Proposition 3.10. *The sequence $\{\mathbf{H}_k\}$ defined by (3.3) is jointly regularly varying with index $\alpha^2/(\alpha+1)$. Let $\{V_k\}$ be defined by (3.13). Then we have, as $x \to \infty$,*

$$\mathbf{P}(\|\mathbf{H}_1\| > x) \sim \mathbf{E}\big[1 \vee |V_1|^{\alpha^2/(\alpha+1)}\big]\frac{a\sigma^\alpha}{\alpha^3 b^2 \Gamma(-\alpha)}x^{-\alpha^2/(\alpha+1)} \tag{3.14}$$

and, for any integer $k \geq 1$,

$$\mathbf{P}\big(\|\mathbf{H}_1\|^{-1}(\mathbf{H}_1,\ldots,\mathbf{H}_k) \in \cdot\,|\,\|\mathbf{H}_1\| > x\big) \xrightarrow{\mathrm{w}} \frac{\mathbf{E}\big[1 \vee |V_1|^{\alpha^2/(\alpha+1)}; (\mathbf{\Theta}_1,\ldots,\mathbf{\Theta}_k) \in \cdot\big]}{\mathbf{E}\big[1 \vee |V_1|^{\alpha^2/(\alpha+1)}\big]}, \tag{3.15}$$

where

$$\mathbf{\Theta}_j = e^{-b(j-1)(\alpha+1)/\alpha}(1 \vee |V_1|)^{-1}(1, V_j).$$

Proof. We only need to show (3.14) and (3.15) hold when $x \to \infty$ along the sequence $c_n := n^{(\alpha+1)/\alpha^2}$. By Proposition 3.3 it follows that $X_0^{(\alpha+1)/\alpha}$ is regularly varying with the index $\alpha^2/(\alpha+1)$. More precisely, as $n \to \infty$,

$$\mathbf{P}\big(X_0^{(\alpha+1)/\alpha} > x\big) \sim \frac{a\sigma^\alpha}{\alpha^3 b^2 \Gamma(-\alpha)}x^{-\alpha^2/(\alpha+1)}. \tag{3.16}$$

By Remark 3.8, we have $\mathbf{E}[|V_k|^r] < \infty$ for any $0 < r < \alpha$. Note that $\|\bar{\mathbf{H}}_k\| = X_{k-1}^{(\alpha+1)/\alpha}(1 \vee V_k)$. By (3.16) and Breiman's Lemma, as $n \to \infty$,

$$\mathbf{P}(\|\bar{\mathbf{H}}_1\| > x) \sim \mathbf{E}\big[1 \vee |V_1|^{\alpha^2/(\alpha+1)}\big]\frac{a\sigma^\alpha}{\alpha^3 b^2 \Gamma(-\alpha)}x^{-\alpha^2/(\alpha+1)}; \tag{3.17}$$

see, e.g., [45, p. 231]. Then $\|\bar{\mathbf{H}}_1\|$ is regularly varying with the index $\alpha^2/(\alpha+1)$. By Lemma 3.9, for any $0 < r < \alpha^3/(\alpha^2+1)$, we have $\mathbf{E}[\|\mathbf{H}_1 - \bar{\mathbf{H}}_1\|^r] < \infty$. Now let $\mathbf{H} = (\mathbf{H}_1,\ldots,\mathbf{H}_k)$ and $\bar{\mathbf{H}} = (\bar{\mathbf{H}}_1,\ldots,\bar{\mathbf{H}}_k)$. By Markov's inequality and (3.17) we have, as $x \to \infty$,

$$\frac{\mathbf{P}(\|\mathbf{H} - \bar{\mathbf{H}}\| > x)}{\mathbf{P}(\|\bar{\mathbf{H}}_1\| > x)} \leq \frac{x^{-r}}{\mathbf{P}(\|\bar{\mathbf{H}}_1\| > x)}\sum_{j=1}^k \mathbf{E}\big[\|\mathbf{H}_j - \bar{\mathbf{H}}_j\|^r\big] \to 0.$$

As in the proof of Lemma 3.12 in [28], we obtain (3.14) from (3.17). From the above relation we also have, as $n \to \infty$,

$$\mathbf{P}(c_n^{-1}\mathbf{H} \in \cdot\,|\,\|\mathbf{H}_1\| > c_n) \sim \mathbf{P}(c_n^{-1}\bar{\mathbf{H}} \in \cdot\,|\,\|\bar{\mathbf{H}}_1\| > c_n). \tag{3.18}$$

Let $\tilde{\mathbf{H}}_k = (X_0 e^{-b(k-1)})^{(\alpha+1)/\alpha}(1, V_k)$. For $\delta > 0$ and $K > 1$ we have

$$\mathbf{P}\big(\|\tilde{\mathbf{H}}_k - \bar{\mathbf{H}}_k\| > c_n\delta | X_0 > a_n\big) \leq \mathbf{P}\big(\|\tilde{\mathbf{H}}_k - \bar{\mathbf{H}}_k\| > c_n\delta, |V_k| \leq K | X_0 > a_n\big)$$
$$+\mathbf{P}\big(\|\tilde{\mathbf{H}}_k - \bar{\mathbf{H}}_k\| > c_n\delta, |V_k| > K | X_0 > a_n\big).$$

Let J_1 and J_2 denote the two terms on the right-hand side. Then

$$J_1 \le \mathbf{P}\big(K|(X_0 e^{-b(k-1)})^{(\alpha+1)/\alpha} - X_{k-1}^{(\alpha+1)/\alpha}| > c_n\delta | X_0 > a_n\big).$$

By Theorem 3.6 and the continuous mapping theorem, we have $J_1 \to 0$ as $n \to \infty$. Since X_0 is independent of V_k, we have

$$J_2 \le \mathbf{P}(|V_k| > K|X_0 > a_n) = \mathbf{P}(|V_k| > K),$$

which tends to zero as $K \to \infty$. Then the regular variation property of X_0 implies, for any $\zeta > 0$,

$$\lim_{x\to\infty} \mathbf{P}(\|\tilde{\mathbf{H}}_k - \bar{\mathbf{H}}_k\| > x^{(\alpha+1)/\alpha}\zeta | X_0 > x) = 0. \tag{3.19}$$

See, e.g., [45, p. 14] for a similar method. By (3.7) we have $\mathbf{P}(K^{\alpha/(\alpha+1)}X_0 > a_n) \sim h_1 n^{-1}$ as $n \to \infty$ for some constant $h_1 > 0$. By (3.19) and the multiplicative formula it is easy to see

$$\lim_{n\to\infty} n\mathbf{P}(\|\tilde{\mathbf{H}}_k - \bar{\mathbf{H}}_k\| > c_n\delta, K^{\alpha/(\alpha+1)}X_0 > a_n) = 0.$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty} n\mathbf{P}(\|\tilde{\mathbf{H}}_k - \bar{\mathbf{H}}_k\| > c_n\delta, \|\bar{\mathbf{H}}_1\| > c_n) \\
&\quad = \limsup_{n\to\infty} n\mathbf{P}(\|\tilde{\mathbf{H}}_k - \bar{\mathbf{H}}_k\| > c_n\delta, X_0^{(\alpha+1)/\alpha}(1 \vee V_1) > c_n) \\
&\quad \le \lim_{n\to\infty} n\mathbf{P}(X_0^{(\alpha+1)/\alpha}|V_1| 1_{\{|V_1|>K\}} > c_n) \\
&\quad = \lim_{n\to\infty} n\mathbf{E}[|V_1|^{\alpha^2/(\alpha+1)} 1_{\{|V_1|>K\}}]\mathbf{P}(X_0^{(\alpha+1)/\alpha} > c_n) \\
&\quad = C_1\mathbf{E}[|V_1|^{\alpha^2/(\alpha+1)} 1_{\{|V_1|>K\}}],
\end{aligned}$$

where we have used Breiman's Lemma again for the second equality. The right hand side goes to zero as $K \to \infty$. But, by (3.17) there is a constant $h_2 > 0$ so that $\mathbf{P}(\|\bar{\mathbf{H}}_1\| > c_n) \sim h_2 n^{-1}$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \mathbf{P}(\|\tilde{\mathbf{H}}_k - \bar{\mathbf{H}}_k\| > c_n\delta | \|\bar{\mathbf{H}}_1\| > c_n) = 0.$$

Let $\tilde{\mathbf{H}} = (\tilde{\mathbf{H}}_1, \ldots, \tilde{\mathbf{H}}_k)$. We have

$$\mathbf{P}(\|\tilde{\mathbf{H}} - \bar{\mathbf{H}}\| > c_n\delta | \|\tilde{\mathbf{H}}_1\| > c_n) \le \sum_{j=1}^{k} \mathbf{P}(\|\tilde{\mathbf{H}}_j - \bar{\mathbf{H}}_j\| > c_n\delta | \|\tilde{\mathbf{H}}_1\| > c_n) \to 0.$$

Since $\tilde{\mathbf{H}}_1 = \bar{\mathbf{H}}_1$, by the above relation, we have as $n \to \infty$,

$$\begin{aligned}
\mathbf{P}(c_n^{-1}\bar{\mathbf{H}} \in \cdot | \|\bar{\mathbf{H}}_1\| > c_n) &\sim \mathbf{P}(c_n^{-1}\tilde{\mathbf{H}} \in \cdot | \|\tilde{\mathbf{H}}_1\| > c_n) \\
&\sim h_2^{-1} n\mathbf{P}(\|\tilde{\mathbf{H}}_1\| > c_n, c_n^{-1}\tilde{\mathbf{H}} \in \cdot).
\end{aligned} \tag{3.20}$$

By Proposition 3.3 one can see, as $n \to \infty$,

$$n\mathbf{P}(c_n^{-1}X_0^{(\alpha+1)/\alpha} \in \cdot) \xrightarrow{\mathrm{v}} \nu(\cdot) := C_2 \int_0^\infty \frac{1_{\{u\in\cdot\}}du}{u^{1+\alpha^2/(\alpha+1)}}.$$

Note that X_0 is independent of V_k for $k \geq 1$. By the extended Breiman's Lemma, as $n \to \infty$,

$$n\mathbf{P}(c_n^{-1}\tilde{\mathbf{H}} \in \cdot) \xrightarrow{\text{v}} \mathbf{E}\big[\nu\{u : u(\Xi_1, \ldots, \Xi_k) \in \cdot\}\big]$$

where $\Xi_j = e^{-b(j-1)(\alpha+1)/\alpha}(1, V_j)$; see Theorem 3.1 of Hult and Lindskog [27]. By (3.18) and (3.20), we have

$$\mathbf{P}(c_n^{-1}\mathbf{H} \in \cdot \,|\, \|\mathbf{H}_1\| > c_n)$$
$$\xrightarrow{\text{v}} C_3 \int_0^\infty \mathbf{E}\big[\nu\{u > (1 \vee V_1)^{-1} : u(\Xi_1, \ldots, \Xi_k) \in \cdot\}\big] \frac{du}{u^{1+\alpha^2/(\alpha+1)}}.$$

Then (3.15) follows by an application of the continuous mapping theorem. By Corollary 3.2 in [3], the sequence $\{\mathbf{H}_k\}$ is jointly regularly varying with index α. $\square$

Remark 3.11. It follows from Theorem 2.5 and Remark 2.6 that the process $\{X_t\}$ is strongly mixing with geometric rate. From (1.6) and (3.3) we see $\mathbf{I}_k$ and $\mathbf{H}_k$ are measurable with respect to $\sigma(X_{k-1}, X_k)$. Then $\{\mathbf{I}_k\}$ and $\{\mathbf{H}_k\}$ are also strongly mixing with geometric rate, and thus satisfies the mixing condition $\mathscr{A}(a_n)$ defined in Remark A.3.

4. Convergence of partial sums

In this section, we will first prove some convergence results on the point processes associated with the stationary sequences $\{\mathbf{I}_k\}$ and $\{\mathbf{H}_k\}$ defined by (3.3). Then we derive the limit theorems of suitably normalized partial sums for those sequences. The results will play essential roles in the proofs of the asymptotics of the estimators of the SCIR-model. The techniques used in this section have been developed extensively by Basrak and Segers [3], Davis and Hsing [12], Davis and Mikosch [13] among others. For i.i.d. random variables, the idea goes back to Davis [11] and LePage et al. [32]. Let $a_n = n^{1/\alpha}$ and $c_n = n^{(\alpha+1)/\alpha^2} = a_n^{(\alpha+1)/\alpha}$ for $n \geq 1$. We shall assume $1 < \alpha < 2$ except for Theorem 4.5.

Lemma 4.1. *Let $r_n = [n^\delta]$ for any $0 < \delta < 1$. Then for any $x > 0$ we have*

$$\lim_{m\to\infty} \limsup_{n\to\infty} n \sum_{k=m}^{r_n} \mathbf{P}(|\varepsilon_k| > a_n x, |\varepsilon_1| > a_n x) = 0. \tag{4.1}$$

Proof. Let $\mathbf{E}_x$ denote the expectation of $\{X_t : t \geq 0\}$ given $X_0 = x$. Take a constant $r \in (\delta, 1)$. For $k \geq 2$, we can use Markov's inequality and Proposition 2.7 to see

$$\begin{aligned}
\mathbf{P}(|\varepsilon_k| > a_n x, |\varepsilon_1| > a_n x) &= \frac{\sigma^{r\alpha}}{(a_n x)^{r\alpha}} \mathbf{E}\Big[1_{\{|\varepsilon_1| > a_n x\}} \Big|\int_{k-1}^k e^{-b(k-s)} X_{s-}^{1/\alpha} dZ_s\Big|^{r\alpha}\Big] \\
&= \frac{\sigma^{r\alpha}}{(a_n x)^{r\alpha}} \mathbf{E}\Big[1_{\{|\varepsilon_1| > a_n x\}} \mathbf{E}_{X_{k-1}}\Big(\Big|\int_0^1 e^{-b(1-s)} X_{s-}^{1/\alpha} dZ_s\Big|^{r\alpha}\Big)\Big] \\
&\leq \frac{C_1\sigma^{r\alpha}}{(a_n x)^{r\alpha}} \mathbf{E}\big[1_{\{|\varepsilon_1| > a_n x\}}(1 + X_{k-1}^r)\big] \\
&\leq \frac{C_2\sigma^{r\alpha}}{(a_n x)^{r\alpha}} \mathbf{E}\big[1_{\{|\varepsilon_1| > a_n x\}}(1 + X_1^r e^{-rb(k-2)/\alpha})\big],
\end{aligned}$$

where the last inequality follows from Proposition 2.8. In view of (1.6), we have

$$n\sum_{k=m}^{r_n}\mathbf{P}(|\varepsilon_k|>a_nx,|\varepsilon_1|>a_nx)\leq J_1+J_2+J_3,$$

where

$$J_1=\frac{C_3\sigma^{r\alpha}n}{(a_nx)^{r\alpha}}(r_n-m+1)\mathbf{P}\big(|\varepsilon_1|>a_nx\big),$$
$$J_2=\frac{C_4\sigma^{r\alpha}n}{(a_nx)^{r\alpha}}\sum_{k=m}^{\infty}e^{-rb(k-2)/\alpha}\mathbf{E}\big(|\varepsilon_1|^r1_{\{|\varepsilon_1|>a_nx\}}\big),$$
$$J_3=\frac{C_5\sigma^{r\alpha}n}{(a_nx)^{r\alpha}}\sum_{k=m}^{\infty}e^{-rb(k-2)/\alpha}\mathbf{E}\big(X_0^r1_{\{|\varepsilon_1|>a_nx\}}\big).$$

By Proposition 3.7, we have $\mathbf{P}(|\varepsilon_1|>a_nx)=O(n^{-1})$. It follows that $J_1=O(n^{\delta-r})$ as $n\to\infty$. By Proposition A.2 one can see, as $n\to\infty$,

$$\mathbf{E}\big(|\varepsilon_1|^r1_{\{|\varepsilon_1|>a_nx\}}\big)\sim\frac{\alpha}{\alpha-r}(a_nx)^r\mathbf{P}\big(|\varepsilon_1|>a_nx\big)$$
$$\sim\frac{\alpha x^r}{\alpha-r}n^{r/\alpha}\mathbf{P}\big(|\varepsilon_1|>a_nx\big).$$

Thus we have $J_2=O(n^{r/\alpha-r})$ as $n\to\infty$. By Markov's inequality and Proposition 2.7,

$$\begin{aligned}\mathbf{E}\big(X_0^r1_{\{|\varepsilon_1|>a_nx\}}\big)&\leq\frac{1}{(a_nx)^{\alpha(1-r)}}\mathbf{E}\big(X_0^r|\varepsilon_1|^{\alpha(1-r)}\big)\\&=\frac{1}{(a_nx)^{\alpha(1-r)}}\mathbf{E}\big\{X_0^r[\mathbf{E}_{X_0}(|\varepsilon_1|^{\alpha(1-r)})]\big\}\\&\leq\frac{C_6}{(a_nx)^{\alpha(1-r)}}\mathbf{E}(X_0^r+X_0).\end{aligned}$$

It follows that

$$\lim_{m\to\infty}\limsup_{n\to\infty}J_3\leq\lim_{m\to\infty}C_7\sum_{k=m}^{\infty}e^{-r\alpha b(k-1)}=0.$$

Then we have (4.1). □

Proposition 4.2. *Let G be defined by (3.11). Then we have, as $n\to\infty$,*

$$\eta_n:=\sum_{k=1}^{n}\delta_{a_n^{-1}\mathbf{I}_k}\xrightarrow{\mathrm{d}}\eta\quad\text{on }M(\bar{\mathbb{R}}_0^2),\tag{4.2}$$

where η is a point process on $\bar{\mathbb{R}}_0^2$ with the Laplace functional $\mathbf{E}[e^{-\eta(f)}]$, $f\in C_0^+(\bar{\mathbb{R}}_0^2)$ given by

$$\exp\Big\{-\frac{1}{\alpha\Gamma(-\alpha)}\int_0^\infty\mathbf{E}\Big[\Big(1-\exp\Big\{-f\Big(y,\frac{y}{1+X_0}\Big)\Big\}\Big)G\Big]\frac{dy}{y^{\alpha+1}}\Big\}.\tag{4.3}$$

Proof. By Remarks 3.11 and A.3, the sequence $\{\mathbf{I}_k\}$ satisfies the mixing condition $\mathscr{A}(a_n)$ with $r_n=[n^\delta]$ for any $0<\delta<1$. Since $\{\varepsilon_k\}$ is a stationary sequence, we have

$$\begin{aligned} &n\mathbf{P}\Big(\max_{m\le|k|\le r_n}|\varepsilon_k|>a_nx,|\varepsilon_1|>a_nx\Big)\\ &\quad\le n\sum_{k=m}^{r_n}\Big[\mathbf{P}(|\varepsilon_k|>a_nx,|\varepsilon_1|>a_nx)+\mathbf{P}(|\varepsilon_{-k}|>a_nx,|\varepsilon_1|>a_nx)\Big]\\ &\quad= n\sum_{k=m}^{r_n}\Big[\mathbf{P}(|\varepsilon_k|>a_nx,|\varepsilon_1|>a_nx)+\mathbf{P}(|\varepsilon_1|>a_nx,|\varepsilon_{k+2}|>a_nx)\Big]\\ &\quad\le 2n\sum_{k=m}^{r_n+2}\mathbf{P}(|\varepsilon_k|>a_nx,|\varepsilon_1|>a_nx). \end{aligned}$$

The right hand side tends to zero as $n\to\infty$ by Lemma 4.1. By Proposition 3.7 we have, as $n\to\infty$,

$$\mathbf{P}\Big(\max_{m\le|k|\le r_n}|\varepsilon_k|>a_nx\,|\,|\varepsilon_1|>a_nx\Big)\to 0.$$

By Proposition 3.7 we have $n\mathbf{P}(\|\mathbf{I}_1\|>(cn)^{1/\alpha})\to 1$ as $n\to\infty$, where

$$c=\frac{a\sigma^\alpha(1-e^{-\alpha b})}{\alpha^3b^2\Gamma(-\alpha)}.$$

By Theorem 4.5 in [3] we have (4.2) with the Laplace functional $\mathbf{E}[e^{-\eta(f)}]$ given by

$$\exp\bigg\{-\frac{1}{\mathbf{E}(G)}\int_0^\infty\mathbf{E}\Big[\Big(1-\exp\Big\{-f\Big(c^{1/\alpha}v,\frac{c^{1/\alpha}v}{1+X_0}\Big)\Big\}\Big)G\Big]d(-v^{-\alpha})\bigg\}.$$

This clearly coincides with (4.3). $\square$

Based on the above theorem, we now study the convergence of some partial sums associated with the sequence $\{\mathbf{I}_k\}$ defined by (3.3). To do so, let us introduce some notation. For any $B\in\mathscr{B}(\mathbb{R}_+)$ define

$$U_{1,n}(B)=\sum_{k=1}^n\varepsilon_k1_B(|\varepsilon_k|),\qquad U_{2,n}(B)=\sum_{k=1}^n\frac{\varepsilon_k}{1+X_{k-1}}1_B\Big(\Big|\frac{\varepsilon_k}{1+X_{k-1}}\Big|\Big).\tag{4.4}$$

Then we define $\tilde U_{j,n}(B)=U_{j,n}(B)-\mathbf{E}[U_{j,n}(B)]$ for $j=1,2$.

Lemma 4.3. *For any $\delta>0$ we have*

$$\lim_{z\to0}\limsup_{n\to\infty}\mathbf{P}\big(a_n^{-1}|\tilde U_{1,n}(0,a_nz]|>\delta\big)=0.$$

Proof. Since $\mathbf{E}(\varepsilon_k)=\mathbf{E}(\varepsilon_k|\mathscr{F}_{k-1})=0$, we have

$$\begin{aligned} a_n^{-1}\tilde U_{1,n}(0,a_nz]&=a_n^{-1}\sum_{k=1}^n[\varepsilon_k1_{\{|\varepsilon_k|\le a_nz\}}-\mathbf{E}(\varepsilon_k1_{\{|\varepsilon_k|\le a_nz\}})]\\ &=a_n^{-1}\sum_{k=1}^n[\varepsilon_k1_{\{|\varepsilon_k|\le a_nz\}}-\mathbf{E}(\varepsilon_k1_{\{|\varepsilon_k|\le a_nz\}}|\mathscr{F}_{k-1})]\\ &\quad-a_n^{-1}\sum_{k=1}^n\Big[\mathbf{E}(\varepsilon_k1_{\{|\varepsilon_k|>a_nz\}}|\mathscr{F}_{k-1})-\mathbf{E}(\varepsilon_k1_{\{|\varepsilon_k|>a_nz\}})\Big]. \end{aligned}$$

Let J_1 and J_2 denote the two terms on the right-hand side. By Proposition 3.7 one can see that ε_1^2 is regularly varying with index $\alpha/2$. Then by Proposition A.2 it follows that, as $n \to \infty$,

$$\begin{aligned}\mathbf{Var}(J_1) &= a_n^{-2} \sum_{k=1}^{n} \mathbf{E}\big\{\big[\varepsilon_k 1_{\{|\varepsilon_k| \le a_n z\}} - \mathbf{E}(\varepsilon_k 1_{\{|\varepsilon_k| \le a_n z\}} | \mathscr{F}_{k-1})\big]^2\big\} \\ &\le n a_n^{-2} \mathbf{E}(\varepsilon_1^2 1_{\{|\varepsilon_1| \le a_n z\}}) = n a_n^{-2} \mathbf{E}(\varepsilon_1^2 1_{\{\varepsilon_1^2 \le a_n^2 z^2\}}) \sim C z^{2-\alpha},\end{aligned}$$

which goes to zero as $z \to 0$. Now we discuss the asymptotics of J_2. Observe that, for $u > \gamma x + \rho$, we have $|X_1 - \gamma x - \rho| > u$ if and only if $X_1 > u + \gamma x + \rho$. It follows that

$$\begin{aligned}u^{\alpha-1} \mathbf{E}_x\big(|X_1 - \varepsilon_1| 1_{\{|\varepsilon_1| > u\}}\big) &= u^{\alpha-1} \mathbf{E}_x\big(|\gamma x + \rho| 1_{\{|X_1 - \gamma x - \rho| > u\}}\big) \\ &= (\gamma x + \rho) u^{\alpha-1} \mathbf{P}_x(X_1 > u + \gamma x + \rho).\end{aligned}$$

Using Proposition 3.1 we see the right-hand side tends to zero uniformly in $x \in [0, K]$ as $u \to \infty$. By Proposition 3.2, we have

$$\begin{aligned}\lim_{u\to\infty} u^{\alpha-1} \mathbf{E}_x\big[|\varepsilon_1| 1_{\{|\varepsilon_1| > u\}}\big] &= \lim_{u\to\infty} u^{\alpha-1} \mathbf{E}_x\big[\varepsilon_1 1_{\{|X_1 - \gamma x - \rho| > u\}}\big] \\ &= \lim_{u\to\infty} u^{\alpha-1} \mathbf{E}_x[X_1 1_{\{X_1 > u + \gamma x + \rho\}}] \\ &= \frac{\sigma^\alpha}{(\alpha-1)\Gamma(-\alpha)}(q_\alpha + p_\alpha x),\end{aligned}$$

and the convergence is uniform in $x \in [0, K]$. It follows that, as $n \to \infty$, we have almost surely

$$\begin{aligned}&a_n^{-1} \sum_{k=1}^{n} 1_{\{X_{k-1} \le K\}} \mathbf{E}\big[\varepsilon_k 1_{\{|\varepsilon_k| > a_n z\}} | \mathscr{F}_{k-1}\big] = a_n^{-1} \sum_{k=1}^{n} 1_{\{X_{k-1} \le K\}} \mathbf{E}_{X_{k-1}}\big[\varepsilon_1 1_{\{|\varepsilon_1| > a_n z\}}\big] \\ &= \frac{\sigma^\alpha z^{1-\alpha}}{(\alpha-1)\Gamma(-\alpha) n} \sum_{k=1}^{n} 1_{\{X_{k-1} \le K\}} (q_\alpha + p_\alpha X_{k-1}) + o(1) \\ &= \frac{\sigma^\alpha z^{1-\alpha}}{(\alpha-1)\Gamma(-\alpha) n} \sum_{k=1}^{n} 1_{\{X_{k-1} \le K\}} (q_\alpha + p_\alpha X_{k-1}) + o(1) \\ &= \frac{\sigma^\alpha z^{1-\alpha}}{(\alpha-1)\Gamma(-\alpha)} \mathbf{E}\big[1_{\{X_0 \le K\}} (q_\alpha + p_\alpha X_0)\big] + o(1),\end{aligned} \tag{4.5}$$

where the last equality holds from the ergodic theorem. Similarly, we have

$$n a_n^{-1} \mathbf{E}\big[1_{\{X_0 \le K\}} \varepsilon_1 1_{\{|\varepsilon_1| > a_n z\}}\big] = \frac{\sigma^\alpha z^{1-\alpha}}{(\alpha-1)\Gamma(-\alpha)} \mathbf{E}\big[1_{\{X_0 \le K\}} (q_\alpha + p_\alpha X_0)\big] + o(1). \tag{4.6}$$

Then (4.5) and (4.6) cancel asymptotically as $n \to \infty$. Observe that

$$\begin{aligned}\mathbf{P}(1_{\{X_0 > K\}} |\varepsilon_1| > u) &= \mathbf{P}\Big(\sigma \int_0^1 1_{\{X_0 > K\}} e^{-b(1-s)} X_{s-}^{1/\alpha} dZ_s > u\Big) \\ &\quad + \mathbf{P}\Big(\sigma \int_0^1 1_{\{X_0 > K\}} e^{-b(1-s)} X_{s-}^{1/\alpha} d(-Z_s) > u\Big).\end{aligned}$$

By Remark A.7, as $u \to \infty$,

$$\mathbf{P}(1_{\{X_0 > K\}} |\varepsilon_1| > u) \sim C(K)\big[\mathbf{P}(Z_1 > u) + \mathbf{P}(-Z_1 > u)\big] = C(K) u^{-\alpha},$$

where

$$C(K) = \sigma^\alpha \mathbf{E}\Big[\int_0^1 1_{\{X_0>K\}} e^{-\alpha b(1-s)} X_s ds\Big] \le C_1 \mathbf{E}[1_{\{X_0>K\}}(1+X_0)].$$

Then, by Proposition A.2, as $n \to \infty$,

$$\begin{aligned}
&a_n^{-1}\mathbf{E}\Big\{\sum_{k=1}^n \big[1_{\{X_{k-1}>K\}}\mathbf{E}(|\varepsilon_k|1_{\{|\varepsilon_k|>a_n z\}}|\mathscr{F}_{k-1}) + \mathbf{E}(1_{\{X_{k-1}>K\}}|\varepsilon_k|1_{\{|\varepsilon_k|>a_n z\}})\big]\Big\}\\
&\quad = 2a_n^{-1}\sum_{k=1}^n \mathbf{E}\big(1_{\{X_{k-1}>K\}}|\varepsilon_k|1_{\{1_{\{X_{k-1}>K\}}|\varepsilon_k|>a_n z\}}\big)\\
&\quad = a_n^{-1} n\mathbf{E}\big[1_{\{X_0>K\}}|\varepsilon_1|1_{\{1_{\{X_0>K\}}|\varepsilon_1|>a_n z\}}\big]\\
&\quad = C_2 n\mathbf{P}\big(1_{\{X_0>K\}}|\varepsilon_1| > a_n z\big) = C_2 C(K) z^{-\alpha}.
\end{aligned}$$

The right hand side goes to zero as $K \to \infty$. That gives the desired result. □

Lemma 4.4. *For any $\delta > 0$ we have*

$$\lim_{z\to 0}\limsup_{n\to\infty} \mathbf{P}\big(a_n^{-1}|\tilde{U}_{2,n}(0, a_n z]| > \delta\big) = 0.$$

Proof. It is simple to see that

$$u^{\alpha-1}\mathbf{E}_x\Big[\frac{\varepsilon_1}{x+1}1_{\{|\varepsilon_1|>(x+1)u\}}\Big] = \frac{[u(x+1)]^{\alpha-1}}{(x+1)^\alpha}\mathbf{E}_x[\varepsilon_1 1_{\{|\varepsilon_1|>(x+1)u\}}],$$

where $u(x+1) > u$ and $(x+1)^{-\alpha} \le 1$. Thus as $u \to \infty$, uniformly for $x \in [0, K]$,

$$u^{\alpha-1}\mathbf{E}_x\Big[\frac{\varepsilon_1}{x+1}1_{\{|\varepsilon_1|>(x+1)u\}}\Big] \to \frac{\sigma^\alpha(q_\alpha + p_\alpha x)}{(\alpha-1)\Gamma(-\alpha)(x+1)^\alpha}.$$

The remaining argument is similar to the proof of Lemma 4.3. □

Theorem 4.5. *Let $U_{1,n} = U_{1,n}(0,\infty)$ and $U_{2,n} = U_{2,n}(0,\infty)$. Then for $1 < \alpha \le 2$ we have, as $n \to \infty$,*

$$a_n^{-1}(U_{1,n}, U_{2,n}) \xrightarrow{\mathrm{d}} (U_1, U_2) \quad \text{on } \mathbb{R}^2,$$

where (U_1, U_2) is the α-stable random vector with characteristic function given by

$$\mathbf{E}\big[\exp\{i(\lambda_1 U_1 + \lambda_2 U_2)\}\big] = \exp\Big\{\frac{\sigma^\alpha}{\alpha}\mathbf{E}\Big[\Big(\lambda_1 + \frac{\lambda_2}{1+X_0}\Big)^\alpha\big(q_\alpha + p_\alpha X_0\big)\Big]e^{-i\pi\alpha/2}\Big\}, \tag{4.7}$$

and $p_\alpha = p_\alpha(1)$ and $q_\alpha = q_\alpha(1)$ are defined by (3.1).

Proof. A proof of the result in the Gaussian case ($\alpha = 2$) was given in [25, p. 1106] for a more general model, so we only consider the case $1 < \alpha < 2$. Fix $z > 0$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and define the function on $\mathbb{R}^2$ by $f_{\lambda,z}(x_1, x_2) = \lambda_1 x_1 1_{\{|x_1|>z\}} + \lambda_2 x_2 1_{\{|x_2|>z\}}$. Then we have

$$\eta_n(f_{\lambda,z}) = a_n^{-1}\sum_{j=1}^2 \lambda_j U_{j,n}(a_n z, \infty).$$

It is easy to see that the mapping from $M(\mathbb{R}^2)$ into $\mathbb{R}$ defined by

$$N := \sum_{k=1}^{\infty} \delta_{(x_{1,k},x_{2,k})} \mapsto N(f_{\lambda,z})$$

is a.s. continuous with respect to the distribution of the limit point process η in Proposition 4.2. By the continuous mapping theorem, as $n \to \infty$, we have $\eta_n(f_{\lambda,z}) \xrightarrow{\mathrm{d}} \eta(f_{\lambda,z})$, and hence

$$\mathbf{E}\Big[\exp\Big\{ia_n^{-1}\sum_{j=1}^{2}\lambda_j U_{j,n}(a_n z,\infty)\Big\}\Big] = \mathbf{E}\big[\exp\{i\eta_n(f_{\lambda,z})\}\big] \to \mathbf{E}\big[\exp(i\eta(f_{\lambda,z}))\big],$$

where the right-hand side is given by

$$\exp\Big\{\frac{1}{\alpha\Gamma(-\alpha)}\int_0^{\infty}\mathbf{E}\Big[\Big(\exp\Big\{i\lambda_1 y 1_{\{y>z\}} + \frac{i\lambda_2 y 1_{\{y>z(1+X_0)\}}}{1+X_0}\Big\} - 1\Big)G\Big]\frac{dy}{y^{\alpha+1}}\Big\}.$$

By Propositions A.2 and 3.7, as $n \to \infty$,

$$\begin{aligned} a_n^{-1}\mathbf{E}[U_{1,n}(a_n z,\infty)] &= n a_n^{-1}\mathbf{E}(\varepsilon_1 1_{\{|\varepsilon_1|>a_n z\}}) \sim \frac{\alpha n z}{\alpha-1}\mathbf{P}(|\varepsilon_1|>a_n z) \\ &\sim \frac{1}{\alpha(\alpha-1)\Gamma(-\alpha)}\mathbf{E}(G)z^{1-\alpha} \\ &= \frac{1}{\alpha\Gamma(-\alpha)}\int_0^{\infty}\mathbf{E}(G)y1_{\{y>z\}}\frac{dy}{y^{\alpha+1}}. \end{aligned}$$

By Proposition A.2 and Remarks A.6 and A.7, as $n \to \infty$,

$$\begin{aligned} a_n^{-1}\mathbf{E}[U_{2,n}(a_n z,\infty)] &= n a_n^{-1}\mathbf{E}\Big[\frac{\varepsilon_1}{1+X_0}1_{\{|\frac{\varepsilon_1}{1+X_0}|>a_n z\}}\Big] \\ &\sim \frac{\alpha n z}{\alpha-1}\mathbf{P}\Big(\Big|\frac{\varepsilon_1}{1+X_0}\Big|>a_n z\Big) \\ &\sim \frac{1}{\alpha(\alpha-1)\Gamma(-\alpha)}\mathbf{E}\Big[\frac{G}{(1+X_0)^{\alpha}}\Big]z^{1-\alpha} \\ &= \frac{1}{\alpha\Gamma(-\alpha)}\int_0^{\infty}\mathbf{E}\Big[\frac{G}{(1+X_0)^{\alpha}}\Big]y1_{\{y>z\}}\frac{dy}{y^{\alpha+1}}. \end{aligned}$$

Consequently, for fixed z, as $n \to \infty$,

$$\mathbf{E}\Big[\exp\Big\{ia_n^{-1}\sum_{j=1}^{2}\lambda_j \tilde{U}_{j,n}(a_n z,\infty)\Big\}\Big] = \mathbf{E}\big[\exp\{i\tilde{\eta}_n(f_{\lambda,z})\}\big]$$

converges to

$$\begin{aligned} \exp\Big\{\frac{1}{\alpha\Gamma(-\alpha)}\int_0^{\infty}\mathbf{E}\Big[\Big(&\exp\Big\{i\lambda_1 y1_{\{y>z\}} + \frac{i\lambda_2 y}{1+X_0}1_{\{y>z(1+X_0)\}}\Big\} - 1 \\ &- i\lambda_1 y1_{\{y>z\}} - \frac{i\lambda_2 y}{1+X_0}1_{\{y>z(1+X_0)\}}\Big)G\Big]\frac{dy}{y^{\alpha+1}}\Big\}. \end{aligned}$$

As $z \to 0$, the above quantity tends to

$$\exp\Big\{\frac{1}{\alpha\Gamma(-\alpha)}\int_0^\infty \mathbf{E}\Big[\Big(e^{i\lambda_1 y+\frac{i\lambda_2 y}{1+X_0}} - 1 - i\lambda_1 y - \frac{i\lambda_2 y}{1+X_0}\Big)G\Big]\frac{dy}{y^{\alpha+1}}\Big\}$$
$$= \exp\Big\{\frac{1}{\alpha\Gamma(-\alpha)}\int_0^\infty (e^{iz}-1-iz)\mathbf{E}\Big[\Big(\lambda_1+\frac{\lambda_2}{1+X_0}\Big)^\alpha G\Big]\frac{dz}{z^{\alpha+1}}\Big\}.$$

By Corollary 14.11 of Sato [49] and (3.2) one can see this coincides with (4.7). Since $\mathbf{E}(U_{j,n}) = \mathbf{E}[U_{j,n}(0,\infty)] = 0$, by the above calculations and Lemmas 4.3 and 4.4, first as $n \to \infty$, then as $z \to 0$,

$$\mathbf{E}\Big[\exp\Big\{ia_n^{-1}\sum_{j=1}^2 \lambda_j U_{j,n}\Big\}\Big] = \mathbf{E}\Big[\exp\Big\{ia_n^{-1}\sum_{j=1}^2 \lambda_j \tilde{U}_{j,n}(0,\infty)\Big\}\Big]$$

converges to (4.7). That gives the desired result. □

Remark 4.6. The proof of the convergence of $a_n^{-1}(U_{1,n}, U_{2,n})$ in the Gaussian case ($\alpha = 2$) given in [25] is based on the standard ergodic theory and the martingale convergence theorem. The approach to the non-Gaussian case ($1 < \alpha < 2$) given above is much more involved and uses heavily the theory of regular variations and the convergence of point processes developed in this and the last sections. The above theorem plays the key role in the proof of the limit theorem of the WCLSEs.

Lemma 4.7. *Let $r_n = [n^\delta]$ with $0 < \delta < 1$. Then we have*

$$\lim_{m\to\infty}\limsup_{n\to\infty} n\mathbf{P}\Big(\max_{-r_n\le k\le -m}\|\mathbf{H}_k\| > c_n x, X_0 > a_n x\Big) = 0.$$

Proof. Since $\{(\mathbf{H}_k, X_k) : k \in \mathbb{Z}\}$ is a stationary sequence, by (1.5), it is easy to see

$$n\mathbf{P}\Big(\max_{-r_n\le k\le -m}\|\mathbf{H}_k\| > c_n x, X_0 > a_n x\Big) = n\mathbf{P}\Big(\max_{m-r_n\le k\le 0}\|\mathbf{H}_k\| > c_n x, X_m > a_n x\Big)$$
$$\le n\mathbf{P}\Big(e^{-bm}X_0 + a\int_0^m e^{-b(m-s)}ds > a_n x/2\Big)$$
$$+ n\mathbf{P}\Big(A_n, \sigma\Big|\int_0^m e^{-b(m-s)}X_{s-}^{1/\alpha}dZ_s\Big| > a_n x/2\Big),$$

where

$$A_n = \Big(\max_{-r_n\le k\le 0}\|\mathbf{H}_k\| > c_n x\Big).$$

By Proposition 3.10 it is easy to see that

$$\lim_{n\to\infty}\mathbf{P}(A_n) = \lim_{n\to\infty}\sum_{k=0}^{r_n}\mathbf{P}(\|\mathbf{H}_{-k}\| > c_n x)$$
$$= \lim_{n\to\infty}(r_n+1)\mathbf{P}(\|\mathbf{H}_0\| > c_n x) = 0.$$

Then Lemma 3.5 implies that

$$\lim_{n\to\infty} n\mathbf{P}\Big(A_n, \sigma\Big|\int_0^m e^{-b(m-s)}X_{s-}^{1/\alpha}dZ_s\Big| > a_n x/2\Big) = 0.$$

From Proposition 3.3 it follows that

$$\lim_{n\to\infty} n\mathbf{P}\Big(e^{-bm}X_0 + a\int_0^m e^{-b(m-s)}ds > a_n x/2\Big) = Ce^{-\alpha bm}x^{-\alpha},$$

which goes to zero as $m \to \infty$. Then we have the desired result. □

Lemma 4.8. *There exists $\delta \in (0, 1)$ so that for $r_n = [n^\delta]$ we have*

$$\lim_{m\to\infty} \limsup_{n\to\infty} n\mathbf{P}\Big(\max_{m\le k\le r_n} \|\mathbf{H}_k\| > c_n x, X_0 > a_n x\Big) = 0.$$

Proof. Recall that X_0 is regularly varying with index α. It is easy to see that

$$\begin{aligned} n\mathbf{P}\Big(\max_{m\le k\le r_n} \|\mathbf{H}_k\| > c_n x, X_0 > a_n x\Big) &\le n\sum_{k=m}^{r_n} \mathbf{P}(\|\mathbf{H}_k\| > c_n x, X_0 > a_n x) \\ &\le n\sum_{k=m}^{r_n} \mathbf{P}(\|\bar{\mathbf{H}}_k - \mathbf{H}_k\| > c_n x/2) \\ &\quad + n\sum_{k=m}^{r_n} \mathbf{P}(\|\bar{\mathbf{H}}_k\| > c_n x/2, X_0 > a_n x). \end{aligned}$$

Let J_1 and J_2 denote the two terms on the right-hand side. By Lemma 3.9, we can choose $r_n = [n^\delta]$ for sufficiently small $\delta \in (0, 1)$, and thus

$$\begin{aligned} \limsup_{n\to\infty} J_1 &\le \limsup_{n\to\infty} \frac{Cnr_n}{x^r c_n^r}\mathbf{E}(\|\bar{\mathbf{H}}_1 - \mathbf{H}_1\|^r) \\ &\le \limsup_{n\to\infty} \frac{Cn^{1+\delta}}{x^r c_n^r}\mathbf{E}(\|\bar{\mathbf{H}}_1 - \mathbf{H}_1\|^r) = 0. \end{aligned}$$

By Proposition A.2, we have

$$\mathbf{E}[X_0 1_{\{X_0 > a_n x\}}] \sim \frac{\alpha a_n x}{\alpha - 1}\mathbf{P}(X_0 > a_n x) \sim C(a_n x)^{1-\alpha}.$$

By Remark 3.8, we have $E[|1 \vee V_k|^{\alpha/(\alpha+1)}] < \infty$. Note that (X_0, X_{k-1}) is independent of V_k for $k \ge 2$. Then for some constant $\delta \in (0, 1/\alpha)$,

$$\begin{aligned} J_2 &\le \frac{2^{\alpha/(\alpha+1)}n}{a_n x^{\alpha/(\alpha+1)}} \sum_{k=m}^{r_n} \mathbf{E}\big[X_{k-1}|1 \vee V_k|^{\alpha/(\alpha+1)}; X_0 > a_n x\big] \\ &\le \frac{Cn}{a_n x^{\alpha/(\alpha+1)}} \sum_{k=m}^{r_n} \mathbf{E}\big[|1 \vee V_k|^{\alpha/(\alpha+1)}\big]\mathbf{E}\big[1_{\{X_0 > a_n x\}}\mathbf{E}_{X_0}(X_{k-1})\big] \\ &\le \frac{Cn}{a_n x^{\alpha/(\alpha+1)}} \sum_{k=m}^{r_n} \mathbf{E}\Big\{1_{\{X_0 > a_n x\}}\big[X_0 e^{-b(k-1)} + ab^{-1}(1 - e^{-b(k-1)})\big]\Big\} \\ &\le \frac{Cn}{a_n x^{\alpha/(\alpha+1)}}\mathbf{E}[X_0 1_{\{X_0 > a_n x\}}] \sum_{k=m}^{r_n} e^{-b(k-1)} + \frac{Cnr_n}{a_n x^{\alpha/(\alpha+1)}}\mathbf{P}(X_0 > a_n x). \end{aligned}$$

It follows that

$$\limsup_{n\to\infty} J_2 \le \frac{Cx^{1-\alpha}}{x^{\alpha/(\alpha+1)}} \sum_{k=m}^{\infty} e^{-b(k-1)},$$

which goes to zero as $m \to \infty$. $\square$

Lemma 4.9. *Let $\bar{\mathbf{H}}_k = X_{k-1}^{(\alpha+1)/\alpha}(1, V_k)$. Then there exists $\delta \in (0,1)$ so that for $r_n = [n^\delta]$ we have*

$$\lim_{m\to\infty} \limsup_{n\to\infty} n\mathbf{P}\Big(\max_{m\le|k|\le r_n} \|\mathbf{H}_k\| > c_n x, \|\bar{\mathbf{H}}_1\| > c_n x\Big) = 0.$$

Proof. For any $K > 1$, we have

$$\begin{aligned} n\mathbf{P}\Big(\max_{m\le|k|\le r_n} \|\mathbf{H}_k\| > c_n x, \|\bar{\mathbf{H}}_1\| > c_n x\Big) &\le n\mathbf{P}\big(X_0^{(\alpha+1)/\alpha}|V_1|1_{\{|V_1|>K\}} > c_n x/2\big) \\ &\quad + n\mathbf{P}\Big(\max_{m\le k\le r_n} \|\mathbf{H}_k\| > c_n x, KX_0^{(\alpha+1)/\alpha} > c_n x/2\Big) \\ &\quad + n\mathbf{P}\Big(\max_{-r_n\le k\le -m} \|\mathbf{H}_k\| > c_n x, KX_0^{(\alpha+1)/\alpha} > c_n x/2\Big). \end{aligned}$$

Observe that $X_0^{(\alpha+1)/\alpha}$ is regularly varying with index $\alpha^2/(\alpha+1)$. By Remark 3.8, we have $\mathbf{E}[|V_1|^b] < \infty$ for some $b > \alpha^2/(\alpha+1)$. It follows from Breiman's Lemma that

$$\begin{aligned} &\lim_{n\to\infty} n\mathbf{P}\big(X_0^{(\alpha+1)/\alpha}|V_1|1_{\{|V_1|>K\}} > c_n x/2\big) \\ &\quad = \lim_{n\to\infty} n\mathbf{E}(|V_1|^{\alpha^2/(\alpha+1)}1_{\{|V_1|>K\}})\mathbf{P}(X_0^{(\alpha+1)/\alpha} > c_n x/2) \\ &\quad = Cx^{-\alpha^2/(\alpha+1)}\mathbf{E}(|V_1|^{\alpha^2/(\alpha+1)}1_{\{|V_1|>K\}}). \end{aligned}$$

The right-hand side goes to zero as $K \to \infty$. Then the result follows by Lemmas 4.7 and 4.8. $\square$

Proposition 4.10. *Let $\{V_j\}$ be defined by (3.13). Then we have, as $n \to \infty$,*

$$\xi_n := \sum_{k=1}^{n} \delta_{c_n^{-1}\mathbf{H}_k} \xrightarrow{\mathrm{d}} \xi \quad \text{on } M(\bar{\mathbb{R}}_0^2), \tag{4.8}$$

where ξ is a point process on $\bar{\mathbb{R}}_0^2$ with Laplace functional $\mathbf{E}[e^{-\xi(f)}]$, $f \in C_0^+(\bar{\mathbb{R}}_0^2)$ given by

$$\begin{aligned} \exp\Big\{&-\frac{a\sigma^\alpha}{\alpha^2 b^2 \Gamma(-\alpha)} \int_0^\infty \mathbf{E}\Big(1 - \exp\big\{-f\big(y^{(\alpha+1)/\alpha}(1, V_1)\big)\big\}\Big) \\ &\times \mathbf{E}\Big[\exp\Big\{-\sum_{j=2}^{\infty} f\big(y^{(\alpha+1)/\alpha}e^{-b(j-1)(\alpha+1)/\alpha}(1, V_j)\big)\Big\}\Big] \frac{dy}{y^{\alpha+1}}\Big\}. \end{aligned} \tag{4.9}$$

Proof. By Remarks 3.11 and A.3, the sequence $\{\mathbf{H}_k\}$ satisfies the mixing condition $\mathscr{A}(a_n)$ with $r_n = [n^\delta]$ for any $0 < \delta < 1$. It is easy to see that

$$\begin{aligned} n\mathbf{P}\Big(\max_{m\le|k|\le r_n} \|\mathbf{H}_k\| > c_n x, \|\mathbf{H}_1\| > c_n x\Big) &\le n\mathbf{P}\big(\|\bar{\mathbf{H}}_1 - \mathbf{H}_1\| > c_n x/2\big) \\ &\quad + n\mathbf{P}\Big(\max_{m\le|k|\le r_n} \|\mathbf{H}_k\| > c_n x, \|\bar{\mathbf{H}}_1\| > c_n x/2\Big). \end{aligned}$$

By Lemma 3.9, we have $\mathbf{E}[\|\mathbf{H}_k - \bar{\mathbf{H}}_k\|^r] < \infty$ for some $r > \alpha^2/(\alpha+1)$. Then Markov's inequality implies that

$$\limsup_{n\to\infty} n\mathbf{P}\big(\|\bar{\mathbf{H}}_1 - \mathbf{H}_1\| > c_n x/2\big) \le \limsup_{n\to\infty} 2^r (c_n x)^{-r}\mathbf{E}\big[\|\mathbf{H}_1 - \bar{\mathbf{H}}_1\|^r\big] = 0.$$

By Lemma 4.9 we have

$$\lim_{m\to\infty}\limsup_{n\to\infty} n\mathbf{P}\Big(\max_{m\le|k|\le r_n}\|\mathbf{H}_k\| > c_n x, \|\mathbf{H}_1\| > c_n x\Big) = 0,$$

where $r_n = [n^\delta]$ for some $\delta \in (0,1)$. Let

$$h = \frac{a\sigma^\alpha}{\alpha^3 b^2 \Gamma(-\alpha)}\mathbf{E}[1 \vee |V_1|^{\alpha^2/(\alpha+1)}].$$

By Proposition 3.10 we have, as $n\to\infty$,

$$n\mathbf{P}\{\|\mathbf{H}_1\| > (hn)^{(\alpha+1)/\alpha^2}\} = n\mathbf{P}\{\|\mathbf{H}_1\| > h^{(\alpha+1)/\alpha^2} c_n\} \to 1.$$

Observe also that

$$\xi_n(f) = \sum_{k=1}^n f(c_n^{-1}\mathbf{H}_k) = \sum_{k=1}^n f(h^{(\alpha+1)/\alpha^2}(h^{(\alpha+1)/\alpha^2}c_n)^{-1}\mathbf{H}_k).$$

Let $\mathbf{\Theta}_i$ be defined as in Proposition 3.10. Then we can use Theorem 4.5 in [3] to obtain (4.8) with $\mathbf{E}[e^{-\xi(f)}]$ given by

$$\begin{aligned}
&\exp\bigg\{-\frac{1}{\mathbf{E}(1\vee|V_1|^{\alpha^2/(\alpha+1)})}\int_0^\infty \mathbf{E}\Big[\exp\Big\{-\sum_{j=2}^\infty f(h^{(\alpha+1)/\alpha^2}v\mathbf{\Theta}_i)\Big\}\\
&\quad\times\Big(1-\exp\big\{-f(h^{(\alpha+1)/\alpha^2}v\mathbf{\Theta}_1)\big\}\Big)(1\vee|V_1|^{\alpha^2/(\alpha+1)})\Big]d(-v^{-\alpha^2/(\alpha+1)})\bigg\}\\
&= \exp\bigg\{-\frac{a\sigma^\alpha}{\alpha^3 b^2\Gamma(-\alpha)}\int_0^\infty \mathbf{E}\Big[\exp\Big\{-\sum_{j=2}^\infty f(y^{(\alpha+1)/\alpha}(1\vee|V_1|)\mathbf{\Theta}_i)\Big\}\\
&\quad\times\Big(1-\exp\big\{-f(y^{(\alpha+1)/\alpha}(1\vee|V_1|)\mathbf{\Theta}_1)\big\}\Big)\Big]d(-y^{-\alpha})\bigg\},
\end{aligned}$$

which can be rewritten as (4.9). □

From the above theorem, we can derive some limit theorem of partial sums associated with the sequence $\{\mathbf{H}_k\}$ defined by (3.3). For $B \in \mathscr{B}(\mathbb{R}_+)$ define

$$S_{1,n}(B) = \sum_{k=1}^n X_{k-1}^2 1_B(X_{k-1}), \qquad S_{2,n}(B) = \sum_{k=1}^n X_{k-1}\varepsilon_k 1_B(|X_{k-1}\varepsilon_k|). \tag{4.10}$$

Lemma 4.11. *For any $\delta > 0$ we have*

$$\lim_{z\to0}\limsup_{n\to\infty}\mathbf{P}\big(c_n^{-2}|S_{1,n}(0,c_n z)| > \delta\big) = 0.$$

Proof. By Theorem 3.6, it is easy to see that X_0^2 is regularly varying with index $\alpha/2 < 1$. Using Proposition A.2 and Theorem 3.6, we have, as $n \to \infty$,

$$\mathbf{E}\big[c_n^{-2} S_{1,n}(0, c_n z)\big] = \frac{1}{c_n^2}\sum_{k=1}^{n}\mathbf{E}\big[X_{k-1}^2 1_{\{X_{k-1}<c_n z\}}\big] \sim \frac{n\alpha z^2}{2-\alpha}\mathbf{P}(X_0 > c_n z) \sim C z^{2-\alpha}.$$

The right-hand side tends to zero as $z \to 0$. Then we have the desired result. □

Lemma 4.12. *Suppose that $1 < \alpha < (1+\sqrt{5})/2$. Then for any $\delta > 0$ we have*

$$\lim_{z\to 0}\limsup_{n\to\infty}\mathbf{P}\big(c_n^{-1}|S_{2,n}(0, c_n z)| > \delta\big) = 0.$$

Proof. By Proposition 3.10, we see $X_0\varepsilon_1$ is regularly varying with index $\alpha^2/(\alpha+1)$. Under the condition $1 < \alpha < (1+\sqrt{5})/2$, we have $\alpha^2/(\alpha+1) < 1$. By Propositions A.2 and 3.10, as $n \to \infty$,

$$\begin{aligned}\mathbf{E}\big[c_n^{-1}|S_{2,n}(0, c_n z)|\big] &\le \frac{1}{c_n}\sum_{k=1}^{n}\mathbf{E}\big[|X_{k-1}\varepsilon_k| 1_{\{|X_{k-1}\varepsilon_k|<c_n z\}}\big] = \frac{n}{c_n}\mathbf{E}\big[|X_0\varepsilon_1| 1_{\{|X_0\varepsilon_1|<c_n z\}}\big] \\ &\sim \frac{\alpha^2 n z}{\alpha+1-\alpha^2}\mathbf{P}(|X_0\varepsilon_1| > c_n z) \sim C z^{1-\alpha^2/(\alpha+1)}.\end{aligned}$$

The right-hand side tends to zero as $z \to 0$. That gives the result; see also [12, p. 896]. □

Theorem 4.13. *Let V_1 be defined by (3.13). Let $S_{1,n} = S_{1,n}(0,\infty)$ and $S_{2,n} = S_{2,n}(0,\infty)$. If $1 < \alpha < (1+\sqrt{5})/2$, then we have, as $n \to \infty$,*

$$(a_n^{-2} S_{1,n}, c_n^{-1} S_{2,n}) \xrightarrow{\mathrm{d}} (S_1, S_2) \quad on\ \mathbb{R}^2,$$

where (S_1, S_2) has characteristic function $\mathbf{E}[\exp\{i\lambda_1 S_1 + i\lambda_2 S_2\}]$ given by

$$\begin{aligned}\exp\bigg\{&-\frac{a\sigma^\alpha}{\alpha^2 b^2 \Gamma(-\alpha)}\int_0^\infty \mathbf{E}\Big(1-\exp\big\{i\lambda_1 y^2 + i\lambda_2 y^{(\alpha+1)/\alpha} V_1\big\}\Big) \\ &\times \mathbf{E}\Big[\exp\Big\{\frac{ie^{-2b}\lambda_1 y^2}{1-e^{-2b}} + \frac{ie^{-b(\alpha+1)/\alpha}\lambda_2 y^{(\alpha+1)/\alpha} V_2}{(1-e^{-b(\alpha+1)})^{1/\alpha}}\Big\}\Big]\frac{dy}{y^{\alpha+1}}\bigg\}. \end{aligned} \tag{4.11}$$

Proof. We first remark that the integral on the right-hand side of (4.11) is well-defined. In fact, by Remarks A.6 and 3.8, we have, as $x \to \infty$,

$$\mathbf{P}(V_1 \ge x) \sim C_1 x^{-\alpha} + o(x^{-\alpha}), \qquad \mathbf{P}(V_1 \le -x) = o(x^{-\alpha}).$$

By Theorems 8.1.10 and 8.1.11 in [4], we have, as $\lambda \to 0$,

$$\mathbf{E}[1-\cos(\lambda V_1)] \sim C_2\lambda^\alpha + o(\lambda^\alpha), \qquad \mathbf{E}[\sin(\lambda V_1)] \sim C_3\lambda^\alpha + o(\lambda^\alpha).$$

It follows that $\mathbf{E}(1-e^{i\lambda V_1}) = \mathbf{E}(1-e^{i\lambda V_2}) \sim c\lambda^\alpha$ as $\lambda \to 0$. Then the integral in (4.11) converges. Fix any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $z > 0$, define the function on $\mathbb{R}_+ \times \mathbb{R}$ by

$$g_{\lambda,z}(x_1, x_2) = \lambda_1 x_1^{2\alpha/(\alpha+1)} 1_{\{x_1 > z^{(\alpha+1)/\alpha}\}} + \lambda_2 x_2 1_{\{|x_2|>z\}}.$$

It is easy to check that

$$\xi_n(g_{\lambda,z}) := \int_{\mathbb{R}_+\times\mathbb{R}} g_{\lambda,z}\, d\xi_n = \lambda_1 a_n^{-2} S_{1,n}(a_n z, \infty) + \lambda_2 c_n^{-1} S_{2,n}(c_n z, \infty).$$

On the other hand, one can see the mapping from $M(\bar{\mathbb{R}}_0^2)$ into $\mathbb{R}$ defined by

$$N := \sum_{k=1}^{\infty} \delta_{(x_{1,k},x_{2,k})} \mapsto N(g_{\lambda,z}) := \int_{\mathbb{R}_+\times\mathbb{R}} g_{\lambda,z} dN$$

is a.s. continuous with respect to distribution of the limit process ξ in Proposition 4.10. By the continuous mapping theorem, as $n \to \infty$, we have $\xi_n(g_{\lambda,z}) \xrightarrow{\mathrm{d}} \xi(g_{\lambda,z})$, and hence

$$\mathbf{E}[\exp(i\xi_n(g_{\lambda,z}))] \to \mathbf{E}[\exp(i\xi(g_{\lambda,z}))],$$

where the right hand side is equal to

$$\begin{aligned}
&\exp\bigg\{-\frac{a\sigma^\alpha}{\alpha^2 b^2 \Gamma(-\alpha)} \int_0^\infty \mathbf{E}\Big[\exp\Big\{i\lambda_1 y^2 \sum_{j=2}^{\infty} e^{-2b(j-1)} 1_{\{ye^{-b(j-1)}>z\}}\Big\} \\
&\quad\times \exp\Big\{i\lambda_2 y^{(\alpha+1)/\alpha} \sum_{j=2}^{\infty} e^{-b(j-1)(\alpha+1)/\alpha} V_j 1_{\{|e^{-b(j-1)(\alpha+1)/\alpha} V_j|>z\}}\Big\}\Big] \\
&\quad\times \mathbf{E}\Big(1-\exp\{i\lambda_1 y^2 + i\lambda_2 y^{(\alpha+1)/\alpha} V_1\}\Big)\frac{dy}{y^{\alpha+1}}\bigg\}.
\end{aligned}$$

Then we can use dominated convergence theorem to see that, as $z \to 0$, the above quantity goes to

$$\begin{aligned}
&\exp\bigg\{-\frac{a\sigma^\alpha}{\alpha^2 b^2 \Gamma(-\alpha)} \int_0^\infty \mathbf{E}\Big(1-\exp\{i\lambda_1 y^2 + i\lambda_2 y^{(\alpha+1)/\alpha} V_1\}\Big) \\
&\quad\times \mathbf{E}\Big[\exp\Big\{i\sum_{j=2}^{\infty}\big(\lambda_1 y^2 e^{-2b(j-1)} + \lambda_2 y^{(\alpha+1)/\alpha} e^{-b(j-1)(\alpha+1)/\alpha} V_j\big)\Big\}\Big]\frac{dy}{y^{\alpha+1}}\bigg\}.
\end{aligned}$$

Since the sequence $\{V_1, V_2, \ldots\}$ is i.i.d., the above quantity is equal to (4.11). Note that $\mathbf{E}(V_1) = 0$. Then the theorem follows by Lemmas 4.11 and 4.12. □

5. Asymptotics of the estimators

In this section, we investigate the asymptotics of the estimators for the SCIR-model. The results are presented in a number of theorems. Let us consider a stationary càdlàg realization $\{X_t : t \in \mathbb{R}\}$ of the SCIR-model with one-dimensional marginal distribution μ given by (2.16). In fact, we shall first study the asymptotics of the estimators of the parameters (γ, ρ) defined in (1.4). Their CLSEs can be obtained by minimizing the sum of squares in (1.8). They are given by

$$\hat{\gamma}_n = \frac{\sum_{k=1}^{n} X_{k-1} \sum_{k=1}^{n} X_k - n\sum_{k=1}^{n} X_{k-1}X_k}{\big(\sum_{k=1}^{n} X_{k-1}\big)^2 - n\sum_{k=1}^{n} X_{k-1}^2} \tag{5.1}$$

and

$$\hat{\rho}_n = \frac{1}{n}\Big[\sum_{k=1}^{n} X_k - \hat{\gamma}_n \sum_{k=1}^{n} X_{k-1}\Big]. \tag{5.2}$$

By minimizing the weighted sum in (1.11), we obtain the WCLSEs of the parameters:

$$\tilde{\gamma}_n = \frac{\sum_{k=1}^{n} X_k \sum_{k=1}^{n} \frac{1}{X_{k-1}+1} - n \sum_{k=1}^{n} \frac{X_k}{X_{k-1}+1}}{\sum_{k=1}^{n} (X_{k-1}+1) \sum_{k=1}^{n} \frac{1}{X_{k-1}+1} - n^2} \tag{5.3}$$

and

$$\tilde{\rho}_n = \frac{1}{n}\Big[\sum_{k=1}^{n} X_k - \tilde{\gamma}_n \sum_{k=1}^{n} X_{k-1}\Big]. \tag{5.4}$$

In view of Proposition 2.3 and the above expressions, in the discussions of the above estimators it suffices to consider a stationary realization $\{X_t : t \geq 0\}$ of the SCIR-model.

Lemma 5.1. *We have, as $n \to \infty$,*

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{\text{a.s.}} \frac{a}{b}, \qquad \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+X_{k-1}} \xrightarrow{\text{a.s.}} \lambda, \tag{5.5}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} \frac{X_k}{X_{k-1}+1} \xrightarrow{\text{a.s.}} \rho\lambda + \gamma(1-\lambda), \tag{5.6}$$

where

$$\lambda = \mathbf{E}\Big(\frac{1}{1+X_0}\Big).$$

Proof. By Theorem 2.5, the process $\{X_t\}$ is exponentially ergodic and thus strongly mixing, so the tail σ-algebra of the process is trivial; see, e.g., [18, p. 351]. Recall that $\mathbf{E}(X_0) = a/b$. In view of (1.6), we have

$$\mathbf{E}\Big(\frac{X_1}{1+X_0}\Big) = \rho\mathbf{E}\Big(\frac{1}{1+X_0}\Big) + \gamma\mathbf{E}\Big(\frac{X_0}{1+X_0}\Big) = \rho\lambda + \gamma(1-\lambda).$$

Then the result follows by Birkhoff's ergodic theorem; see, e.g., [18, p. 341]. □

Proposition 5.2. *The estimators $(\tilde{\rho}_n, \tilde{\gamma}_n)$ are strongly consistent and, as $n \to \infty$, $n^{(\alpha-1)/\alpha}\big(\tilde{\gamma}_n - \gamma, \tilde{\rho}_n - \rho\big)$ converges in distribution to*

$$\kappa^{-1}(U_1, U_2)\begin{pmatrix} \lambda & \lambda - 1 \\ -1 & ab^{-1} \end{pmatrix} = \kappa^{-1}\big(\lambda U_1 - U_2, (\lambda - 1)U_1 + ab^{-1}U_2\big),$$

where $\kappa = (1 + ab^{-1})\lambda - 1$ and (U_1, U_2) is an α-stable random vector with characteristic function given by (4.7).

Proof. In view of (5.3) and (5.4), the results of Lemma 5.1 imply that

$$\tilde{\gamma}_n \xrightarrow{\text{a.s.}} \frac{ab^{-1}\lambda - \rho\lambda - \gamma(1-\lambda)}{(1+ab^{-1})\lambda - 1} = \frac{ab^{-1}\gamma\lambda - \gamma(1-\lambda)}{(1+ab^{-1})\lambda - 1} = \gamma$$

and

$$\tilde{\rho}_n \xrightarrow{\text{a.s.}} \frac{a}{b}(1-\gamma) = \rho.$$

Those give the strong consistency of $\tilde{\rho}_n$ and $\tilde{\gamma}_n$. A simple calculation based on (1.6), (5.3) and (5.4) shows that

$$n^{(\alpha-1)/\alpha}\big(\tilde{\gamma}_n - \gamma, \tilde{\rho}_n - \rho\big) = \kappa_n^{-1}\mathbf{U}_n\mathbf{B}_n,$$

where

$$\kappa_n = \frac{1}{n^2}\sum_{k=1}^{n}(X_{k-1}+1)\sum_{k=1}^{n}\frac{1}{X_{k-1}+1} - 1,$$

$$\mathbf{B}_n = \frac{1}{n}\begin{pmatrix} \displaystyle\sum_{k=1}^{n}\frac{1}{X_{k-1}+1} & \displaystyle -\sum_{k=1}^{n}\frac{X_{k-1}}{X_{k-1}+1} \\ -n & \displaystyle\sum_{k=1}^{n}X_{k-1} \end{pmatrix},$$

and

$$\mathbf{U}_n = \frac{1}{n^{1/\alpha}}\left(\sum_{k=1}^{n}\varepsilon_k, \sum_{k=1}^{n}\frac{\varepsilon_k}{X_{k-1}+1}\right).$$

From (5.5) it follows that $\kappa_n \xrightarrow{\text{a.s.}} \kappa$ and

$$\mathbf{B}_n \xrightarrow{\text{a.s.}} \mathbf{B} := \begin{pmatrix} \lambda & \lambda - 1 \\ -1 & ab^{-1} \end{pmatrix}.$$

By Theorem 4.5, we have $\mathbf{U}_n \xrightarrow{\text{d}} (U_1, U_2)$. Then we have the desired convergence. $\square$

Theorem 5.3. *The estimators $(\tilde{b}_n, \tilde{a}_n)$ are strongly consistent and as $n \to \infty$, $n^{(\alpha-1)/\alpha}(\tilde{b}_n - b, \tilde{a}_n - a)$ converges in distribution to*

$$\kappa^{-1}\big(e^b(U_2 - \lambda U_1), (1-e^{-b})^{-1}[a\lambda + b(\lambda-1)]U_1 + ab^{-1}e^b(U_2 - \lambda U_1)\big),$$

where $\kappa = (1+ab^{-1})\lambda - 1$ and (U_1, U_2) is an α-stable random vector with characteristic function given by (4.7).

Proof. The strong consistency of $\tilde{b}_n$ and $\tilde{a}_n$ follows from that of $\tilde{\rho}_n$ and $\tilde{\gamma}_n$. By the relations in (1.4), we have, as $n \to \infty$,

$$(\tilde{\gamma}_n - \gamma) = e^{-\tilde{b}_n} - e^{-b} = -(\tilde{b}_n - b)e^{-b} + o(\tilde{b}_n - b) \tag{5.7}$$

and

$$\begin{aligned} \tilde{a}_n - a &= \frac{\tilde{\rho}_n\tilde{b}_n}{1-e^{-\tilde{b}_n}} - \frac{\rho b}{1-e^{-b}} = \frac{\tilde{\rho}_n\tilde{b}_n(1-e^{-b}) - \rho b(1-e^{-\tilde{b}_n})}{(1-e^{-\tilde{b}_n})(1-e^{-b})} \\ &= \frac{b(\tilde{\rho}_n - \rho)}{1-e^{-b}} - \frac{ae^b(\tilde{\gamma}_n - \gamma)}{b} + \frac{a(\tilde{\gamma}_n - \gamma)}{1-e^{-b}} + o(\tilde{b}_n - b). \end{aligned} \tag{5.8}$$

Then the desired convergence follows from Proposition 5.2. $\square$

Proposition 5.4. *The estimators $(\hat{\rho}_n, \hat{\gamma}_n)$ are weakly consistent. Moreover, if $1 < \alpha < (1 + \sqrt{5})/2$, then, as $n \to \infty$,*

$$n^{(\alpha-1)/\alpha^2}(\hat{\gamma}_n - \gamma, \hat{\rho}_n - \rho) \xrightarrow{\mathrm{d}} S_1^{-1} S_2(1, -ab^{-1}), \tag{5.9}$$

where (S_1, S_2) has characteristic function given by (4.11).

Proof. By (1.6) and (5.1) we have

$$\hat{\gamma}_n - \gamma = \frac{\sum_{k=1}^n X_{k-1} \sum_{k=1}^n \varepsilon_k - n \sum_{k=1}^n X_{k-1}\varepsilon_k}{\left(\sum_{k=1}^n X_{k-1}\right)^2 - n \sum_{k=1}^n X_{k-1}^2}.$$

Then using (5.2) we get

$$\hat{\rho}_n - \rho = \frac{\sum_{k=1}^n X_{k-1} \sum_{k=1}^n X_{k-1}\varepsilon_k - \sum_{k=1}^n X_{k-1}^2 \sum_{k=1}^n \varepsilon_k}{\left(\sum_{k=1}^n X_{k-1}\right)^2 - n \sum_{k=1}^n X_{k-1}^2}.$$

By Theorem 4.13 it is easy to see that

$$\frac{\sum_{k=1}^n X_{k-1}\varepsilon_k}{\sum_{k=1}^n X_{k-1}^2} \xrightarrow{\mathrm{p}} 0.$$

Then we have $\hat{\rho}_n - \rho \xrightarrow{\mathrm{p}} 0$ and $\hat{\gamma}_n - \gamma \xrightarrow{\mathrm{p}} 0$, giving the weak consistency of $(\hat{\rho}_n, \hat{\gamma}_n)$. Take any constant $0 < \delta < [1 \wedge (\alpha - 1)^2]/\alpha^2$. From the above relations it follows that

$$n^{(\alpha-1)/\alpha^2}\big(\hat{\gamma}_n - \gamma, \hat{\rho}_n - \rho\big) = T_n^{-1}\mathbf{S}_n\mathbf{A}_n,$$

where

$$T_n = \frac{1}{n^{1+2/\alpha}}\left(\sum_{k=1}^n X_{k-1}\right)^2 - \frac{1}{n^{2/\alpha}}\sum_{k=1}^n X_{k-1}^2,$$

$$\mathbf{A}_n = \begin{pmatrix} \frac{1}{n^{1+1/\alpha^2-\delta}}\sum_{k=1}^n X_{k-1} & -\frac{1}{n^{1+1/\alpha^2-\delta}}\sum_{k=1}^n X_{k-1}^2 \\ -1 & \frac{1}{n}\sum_{k=1}^n X_{k-1} \end{pmatrix},$$

and

$$\mathbf{S}_n = \left(\frac{1}{n^{1/\alpha+\delta}}\sum_{k=1}^n \varepsilon_k, \frac{1}{n^{(\alpha+1)/\alpha^2}}\sum_{k=1}^n X_{k-1}\varepsilon_k\right).$$

If $\alpha^2 < \alpha + 1$, by (5.5) and Theorem 4.13, we have $T_n \xrightarrow{\text{d}} -S_1$ and

$$\mathbf{A}_n \xrightarrow{\text{d}} \mathbf{A} := \begin{pmatrix} 0 & 0 \\ -1 & ab^{-1} \end{pmatrix}.$$

By Theorems 4.5 and 4.13 we have $\mathbf{S}_n \xrightarrow{\text{d}} (0, S_2)$. Then (5.9) holds. □

Theorem 5.5. *The estimators $(\hat{b}_n, \hat{a}_n)$ are weakly consistent. Moreover, if $1 < \alpha < (1+\sqrt{5})/2$, then, as $n \to \infty$,*

$$n^{(\alpha-1)/\alpha^2}(\hat{b}_n - b, \hat{a}_n - a) \xrightarrow{\text{d}} -e^b(1, ab^{-1})S_1^{-1}S_2,$$

where (S_1, S_2) has the characteristic function given by (4.11).

Proof. The weak consistency of $\hat{b}_n$ and $\hat{a}_n$ follows from that of $\hat{\rho}_n$ and $\hat{\gamma}_n$. The relations (5.7) and (5.8) still hold when the "checks" are replaced by "hats". Then the desired result follows from Proposition 5.4. □

Remark 5.6. By the results of Huang et al. [25], Overbeck and Rydén [40] and Wei and Winnicki [52], for CBI-processes or GWI-processes with finite variance the sequences $(\hat{b}_n - b, \hat{a}_n - a)$ and $(\tilde{b}_n - b, \tilde{a}_n - a)$ have the same magnitude $\sqrt{n}$ and they both have Gaussian limit distributions. In other words, for those models the WCLSEs are not much more efficient than the CLSEs. On the other hand, by Theorems 5.3 and 5.5, the sequence $(\tilde{b}_n - b, \tilde{a}_n - a)$ has convergence rate $n^{(\alpha-1)/\alpha}$ while the sequence $(\hat{b}_n - b, \hat{a}_n - a)$ has rate $n^{(\alpha-1)/\alpha^2}$ for $1 < \alpha < (1+\sqrt{5})/2$. Then the SCIR-model provides an example whose WCLSEs are more efficient than the CLSEs with different convergence rates. It is somewhat unfortunate that our approach to the central limit theorem of the CLSEs only works for $1 < \alpha < (1+\sqrt{5})/2$. Since (4.11) does not define a characteristic function otherwise, it seems the restriction cannot be removed by a simple modification of the approach.

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Appendix

In this last section, we review some concepts and technical results on regularly varying stochastic processes, which have been used in the preceding proofs. Most of the results can be found in [1,3,26,27,44,46]. The reader may also refer to Samorodnitsky and Grigoriu [48] for results on the tail behavior of solutions to certain stochastic differential equations driven by Lévy processes. Let "$|\cdot|$" be any norm on $\mathbb{R}^d$.

Definition A.1. A d-dimensional random vector $\mathbf{X}$ is said to be *regularly varying* if there exists a Radon measure η on $\mathbb{R}^d$, finite on sets of the form $\{x \in \mathbb{R}^d : |x| \geq r\}$, and a sequence $\{a_n\}$

satisfying $a_n \to \infty$ such that, as $n \to \infty$,

$$n\mathbf{P}(a_n^{-1}\mathbf{X} \in \cdot) \xrightarrow{\text{v}} \eta(\cdot). \tag{A.1}$$

The above sequential form of the condition is the same as saying there exists a (necessarily regularly varying) function $t \mapsto g(t)$ such that, as $n \to \infty$,

$$g(t)\mathbf{P}(t^{-1}\mathbf{X} \in \cdot) \xrightarrow{\text{v}} \eta(\cdot).$$

It is known that the condition implies the existence of a constant $\alpha > 0$ such that $\eta(rA) = r^{-\alpha}\eta(A)$ for all $r > 0$ and all $A \in \mathscr{B}(\mathbb{R}^d)$ bounded away from $\mathbf{0}$, where $rA = \{rx : x \in A\}$. In this case, we say $\mathbf{X}$ is regularly varying *with index* $\alpha > 0$.

The next proposition follows immediately from Karamata's theorem; see, e.g., [46, pp. 25 and 36].

Proposition A.2. *Let ξ be a positive regularly varying random variable with index $\alpha > 0$. Then we have:*

(i) *If $\alpha > 1$, then as $x \to \infty$,*

$$\mathbf{E}(\xi 1_{\{\xi > x\}}) \sim \frac{\alpha}{\alpha - 1} x\mathbf{P}(\xi > x).$$

(ii) *If $0 < \alpha < 1$, then as $x \to \infty$,*

$$\mathbf{E}(\xi 1_{\{\xi < x\}}) \sim \frac{\alpha}{1 - \alpha} x\mathbf{P}(\xi > x).$$

Remark A.3. Suppose that $\{\mathbf{Y}_k\}$ is a stationary sequence of regularly varying random vectors. Let $\{a_n\}$ be taken such that $n\mathbf{P}(|\mathbf{Y}_0| > a_n) \to 1$ as $n \to \infty$. By Lemma 2.3.9 of Basrak [1], the strong mixing condition implies the mixing condition $\mathscr{A}(a_n)$, i.e. there exists a sequence of non-negative integers r_n such that $r_n \to \infty$, $l_n = [n/r_n] \to \infty$ as $n \to \infty$ and

$$\mathbf{E}\exp\Big\{-\sum_{k=1}^{n} f(\mathbf{Y}_k/a_n)\Big\} - \Big(\mathbf{E}\exp\Big\{-\sum_{k=1}^{r_n} f(\mathbf{Y}_k/a_n)\Big\}\Big)^{l_n} \to 0$$

for every $f \in C_0^+(\bar{\mathbb{R}}_0)$. See also Section 3.4.3 of Basrak et al. [2] for similar arguments. In fact, it was pointed out in Remark 2.3.10 of Basrak [1] that we can choose $r_n = [n^\delta]$ for any $0 < \delta < 1$ if $\{\mathbf{Y}_k\}$ is strongly mixing with geometric rate.

Definition A.4. A sequence of random variables $\{\mathbf{X}_k : k \in \mathbb{Z}\}$ in $\mathbb{R}^d$ is called *jointly regularly varying* if all the vectors of the form $(\mathbf{X}_1, \dots, \mathbf{X}_l)$ are regularly varying.

The concept of regular variations can also be defined for continuous time stochastic processes. Let $T \geq 0$ and let $\mathbb{D}^d[0, T] := \mathbb{D}([0, T], \mathbb{R}^d)$ be the space of all $\mathbb{R}^d$-valued càdlàg functions on $[0, T]$ equipped with Skorokhod topology; see [18, p. 353]. In the sequel, we use the norm $\|\mathbf{x}\| := \max_i |x_i|$ for $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$. Let

$$\mathbb{S}^d[0, T] = \Big\{\mathbf{y} \in \mathbb{D}^d[0, T] : \sup_{0 \leq t \leq T} \|\mathbf{y_t}\| = 1\Big\}.$$

Definition A.5. A stochastic process $\mathbf{Y} = \{\mathbf{Y}_t : 0 \leq t \leq T\}$ with sample path in $\mathbb{D}^d[0, T]$ is said to be *regularly varying* if there exist a measure Q on $\mathbb{D}^d[0, T]$, finite on sets bounded away from $\mathbf{0}$, and a sequence $\{a_n\}$ satisfying $a_n \to \infty$ such that for any set $B \in \mathscr{B}(\mathbb{D}^d[0,T])$ bounded away from $\mathbf{0}$ with $Q(\partial B) = 0$, as $n \to \infty$,

$$n\mathbf{P}(a_n^{-1}\mathbf{Y} \in B) \to Q(B).$$

The above property implies there is a constant $\alpha > 0$ such that $Q(uB) = u^{-\alpha}Q(B)$ for all $u > 0$ and all $B \in \mathscr{B}(\mathbb{D}^d[0,T])$ bounded away from $\mathbf{0}$. In this situation, we say $\mathbf{Y}$ is regularly varying *with index* $\alpha > 0$.

The convergence in the above definition can be formulated for general boundedly finite measures on $\bar{\mathbb{D}}_0^d[0,T] = (0,\infty] \times \mathbb{S}^d[0,T]$. We shall denote the convergence by "$\xrightarrow{\hat{\mathrm{w}}}$". The reader may refer to Hult and Lindskog [26] for more details.

Remark A.6. Let $0 < \alpha < 2$ and let $\{Z_t : t \geq 0\}$ be a one-dimensional α-stable process with Lévy measure $\nu(dz)$. It follows from Lemma 2.1 of Hult and Lindskog [27] that, as $n \to \infty$,

$$n\mathbf{P}(n^{-1/\alpha}Z_t \in \cdot) \xrightarrow{\mathrm{v}} t\nu(\cdot).$$

Remark A.7. Let $0 < \alpha < 2$ and $T \geq 1$. Suppose that $\{Z_t : 0 \leq t \leq T\}$ is a one-dimensional Lévy process such that $X = Z_1$ satisfies (A.1) with $\eta(z,\infty) = cz^{-\alpha}$ for some $c > 0$. Let $\{Y_t : 0 \leq t \leq T\}$ be a non-negative predictable càglàd process satisfying $\sup_{0\leq t\leq T} Y_t > 0$ a.s. and $\mathbf{E}[\sup_{0\leq t\leq T} Y_t^{\alpha+\delta}] < \infty$ for some $\delta > 0$. By Theorem 3.4 and Example 3.1 in [27], for any $z > 0$ and $0 \leq t \leq T$, we have, as $n \to \infty$,

$$n\mathbf{P}\Big(a_n^{-1}\int_0^t Y_{s-}dZ_s > z\Big) \to \eta(z,\infty)\int_0^t \mathbf{E}(Y_s^\alpha)ds = cz^{-\alpha}\int_0^t \mathbf{E}(Y_s^\alpha)ds.$$

Remark A.8. Suppose that $\{Z_t\}$ is an α-stable Lévy process with $0 < \alpha \leq 2$ and $\{y(t)\}$ is a predictable process satisfying, a.s.,

$$\int_0^T |y(t)|^\alpha dt < \infty, \quad T \geq 0.$$

It was proved in [35, p. 649] that for any $0 < r < \alpha$ there exists a constant $C = C(r,\alpha) \geq 0$ such that

$$\mathbf{E}\Big[\sup_{t\leq T}\Big|\int_0^t y(s)dZ_s\Big|^r\Big] \leq C\mathbf{E}\Big[\Big(\int_0^T |y(t)|^\alpha dt\Big)^{r/\alpha}\Big].$$

The above result can be regarded as a generalization of Theorem 3.2 of Rosinski and Woyczynski [47], where the symmetric case was considered.

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