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Einstein relation for reversible random walks in random environment on $\mathbb{Z}$

Hoang-Chuong Lam [a,b,*], Jerome Depauw [c]

[a] *Department of Mathematics, College of Natural Sciences, Can Tho University, Can Tho City, Viet Nam*
[b] *Department of Electrical and Computer Engineering, Ben Gurion University of the Negev, Beer Sheva, Israel*
[c] *Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Université François Rabelais, Parc de Grandmont, F-37000 Tours, France*

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Abstract

The aim of this paper is to consider reversible random walk in a random environment in one dimension and prove the Einstein relation for this model. It says that the derivative at 0 of the effective velocity under an additional local drift equals the diffusivity of the model without drift (Theorem 1.2). Our method here is very simple: we solve the Poisson equation $(P_\omega - I)g = f$ and then use the pointwise ergodic theorem in Wiener (1939) [10] to treat the limit of the solutions to obtain the desired result. There are analogous results for Markov processes with discrete space and for diffusions in random environment.

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1. Introduction

The definition of a Random walk in Random environment involves two ingredients: The environment which is randomly chosen but remains fixed throughout the time evolution; and the

* Corresponding author at: Department of Mathematics, College of Natural Sciences, Can Tho University, Can Tho City, Viet Nam.

E-mail addresses: lhchuong@ctu.edu.vn (H.-C. Lam), jerome.depauw@univ-tours.fr (J. Depauw).

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random walk whose transition probabilities are determined by the environment. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. The space Ω is interpreted as the space of environments. For each $\omega \in \Omega$, we define the random walk in the environment ω as the (time-homogeneous) Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ on $\mathbb{Z}^d$ with certain (random) transition probabilities

$$p(\omega, x, y) = \mathbb{P}_\omega\{X_1 = y | X_0 = x\},$$

where the probability measure $\mathbb{P}_\omega$ determines the distribution of the random walk in a given environment ω. In the case that the random walk has the initial condition $X_0 = x$,

$$\mathbb{P}^x_\omega\{X_0 = x\} = 1.$$

The probability measure $\mathbb{P}^x_\omega$, which denotes the distribution of the random walk in a given environment ω with the initial position of the walk at x, is referred to as *the Quenched law*.

By averaging the Quenched probability $\mathbb{P}^x_\omega$ further, with respect to the environment distribution, we obtain *the Annealed measure* $\mathbf{P}^x = \mathbb{P} \times \mathbb{P}^x_\omega$, which determines the probability law of the random walk in random environment

$$\mathbf{P}^x(A) = \int_\Omega \mathbb{P}^x_\omega(A)\mathbb{P}(\mathrm{d}\omega) = \mathbb{E}\left\{\mathbb{P}^x_\omega(A)\right\}.$$

For more information on the random walk in random environment, the reader can refer to [1–3, 6,9,11].

We now consider again the model for the random walk in random environment as in [3]. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and T be an invertible measure preserving transformation on Ω which is ergodic. More precisely, T acts on Ω by

$$\begin{aligned} T : \Omega \times \mathbb{Z} &\longrightarrow \Omega \\ (\omega, k) &\longmapsto T^k\omega, \end{aligned}$$

which is jointly measurable and satisfies

- For any $k, h \in \mathbb{Z} : T^{k+h} = T^k T^h$ and $T^0\omega = \omega$,
- T preserves the measure $\mu : \mu(T^k A) = \mu(A)$ for any $k \in \mathbb{Z}$,
- T is ergodic: If, for all $A \in \mathcal{A}$, $T^k A = A$ (up to null sets) for all $k \in \mathbb{Z}$ then $\mu(A) = 0$ or 1.

On the lattice $\mathbb{Z}$ we assume that the conductivity of the edge between $\{k, k+1\}$ is equal to $c(T^k\omega)$, where c is a positive measurable function on Ω. Fix $\omega \in \Omega$, we consider a random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ with $X_0 = 0$ and with a transition probability $p(\omega, k, h)$ which is given by

$$\begin{aligned} p(\omega; k, k+1) &= \mathbb{P}^0_\omega\{X_{n+1} = k+1 | X_n = k\} = c(T^k\omega)/\bar{c}(T^k\omega), \\ p(\omega; k, k-1) &= \mathbb{P}^0_\omega\{X_{n+1} = k-1 | X_n = k\} = c(T^{k-1}\omega)/\bar{c}(T^k\omega), \end{aligned}$$

where $\bar{c} = c + c \circ T^{-1}$. The random walk is reversible since for all adjacent vertices x, y in $\mathbb{Z}$ we have $\bar{c}(T^x\omega)p(\omega; x, y) = \bar{c}(T^y\omega)p(\omega; y, x)$. The corresponding Markov operator $f \longmapsto P_\omega f$ is defined by

$$P_\omega f(k) = \frac{1}{\bar{c}(T^k\omega)}\left[c(T^{k-1}\omega)f(k-1) + c(T^k\omega)f(k+1)\right].$$

When c is integrable, but c^{-1} is not, Y. Derriennic and M. Lin have proved, in an unpublished work, the Annealed Limit Theorem: $\lim_{n\to+\infty} n^{-1}\mathbb{E}_\omega(X_n^2) = 0$ in μ-measure, where $\mathbb{E}_\omega$ denotes the expectation relative to the randomness of the walk, the environment being fixed. For the

Quenched version, recently in [3] J. Depauw and J.-M. Derrien considered a non negative solution f, defined on $\mathbb{Z}$, of the Poisson equation $(P_\omega - I)f = 1$ with $f(0) = 0$ in order to obtain the limit of the variance of the reversible random walk $(X_n)_{n\geq 0}$. That is,

Theorem 1.1 (*Depauw and Derrien, [3]*)**.** *For almost all environments ω,*

$$\lim_{n\to+\infty} \mathbb{E}_\omega \left\{ \frac{X_n^2}{n} \right\} = \left[\int \frac{1}{c}\, d\mu \int c\, d\mu \right]^{-1}. \tag{1}$$

This limit is null if at least one of the integrals is $+\infty$ and it is denoted by σ^2.

In [7], H.-C. Lam then generalized Theorem 1.1 and established the Quenched Central Limit Theorem. Their proofs do not involve a martingale construction and c is only required to be positive. In the case when at least one of c and c^{-1} is not integrable, $\lim_{n\to+\infty} n^{-1}\mathbb{E}_\omega(X_n^2) = 0$ and $X_n/\sqrt{n}$ converges to 0 as $n \to +\infty$.

In the sequel, we will study the following model. Fixed environment $\omega \in \Omega$ and fixed number $\lambda \neq 0$, the conductances of the edges $[k, k+1]$ are equal to $e^\lambda c(T^k\omega)$. The number λ is called the "drift" of the model. We consider a random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ with $X_0 = 0$ and the corresponding Markov operator $f \longmapsto P_{\lambda,\omega} f$ which is defined by

$$P_{\lambda,\omega} f(k) = \frac{1}{\pi(T^k\omega)} \left[e^{-\lambda} c(T^{k-1}\omega) f(k-1) + e^\lambda c(T^k\omega) f(k+1) \right],$$

where $\pi = e^\lambda c + e^{-\lambda} c \circ T^{-1}$. $\mathbb{E}_{\lambda,\omega}$ will denote the expectation relative to the space (Ω, μ), the environment being fixed in the case $\lambda \neq 0$. The aim of the present paper is to prove the *Quenched Einstein relation* for the last model with the drift λ. It is adapted from J. Depauw and J.-M. Derrien [3]. For the Einstein relation, the reader can refer to [4,5,8]. The following definitions can be found in [4].

Definition 1.1. The Quenched diffusivity of a random walk X_n without drift is defined by

$$\kappa = \lim_{n\to+\infty} \mathbb{E}_\omega \left\{ \frac{X_n^2}{n} \right\}.$$

Remark 1.1. When the model is without drift $\lambda = 0$, from (1) for almost all environment ω we have $\kappa = \sigma^2$ if c and $c^{-1} \in L^1(\mu)$, and $\kappa = 0$ if not.

Definition 1.2. The Quenched effective drift of X_n is defined by

$$d_\omega(\lambda) = \lim_{n\to+\infty} \mathbb{E}_{\lambda,\omega} \left\{ \frac{X_n}{n} \right\}.$$

Remark 1.2. When the model is without drift $\lambda = 0$, then $d_\omega(0) = 0$.

We now state our main theorem which is a version of the Einstein relation.

Theorem 1.2 (*Einstein Relation*)**.** *For almost all environment ω, the function $\lambda \longmapsto d_\omega(\lambda)$ has a derivative at $\lambda = 0$ which satisfies*

$$\lim_{\lambda\to 0} \frac{d_\omega(\lambda)}{\lambda} = \kappa = \sigma^2, \tag{2}$$

if $c \in L^p(\mu)$ and $c^{-1} \in L^q(\mu)$ with $p, q \in [1, +\infty]$ such that $1/p + 1/q = 1$.

For the L^p-integrability assumptions ($p \geq 1$) our model works without using either of the classical assumptions of uniform ellipticity and independence on the conductances. They are used for proving $H_\lambda/c \in L^1(\mu)$ where H_λ is defined as in (10). We can then apply successfully the pointwise ergodic theorem in Wiener [10] for the function H_λ/c to obtain $d_\omega(\lambda)$. This is the key to the proof of Theorem 1.2. However we do not know if it works for the case where at least one of c and c^{-1} is not integrable! We will see in the proof of this theorem that $d_\omega(\lambda)$ is defined *a.s* and does not depend on ω. So, it will be denoted by $d(\lambda)$ in the sequel.

Remark 1.3. The Einstein relation for reversible diffusions in random environment is discussed in a recent paper of Gantert, Mathieu, Piatnitski [4]. This paper is in $\mathbb{R}^d$ but assumes uniform ellipticity of the diffusion coefficients, boundedness of the drift and finite range dependence.

This paper is organized as follows. We will prove Theorem 1.2 in Section 2. In Section 3 there is an analogue to a Markov process with continuous time and discrete space, and the diffusion in random environment.

2. Proof of Theorem 1.2

Our method here is adapted from [3]. Fix $\omega \in \Omega$, we first consider the Poisson equation on $\mathbb{Z}$

$$\begin{cases} (P_{\lambda,\omega} - I) f \equiv 1, \\ \quad f(0) = 0. \end{cases}$$

This equation has a particular solution f which depends on λ and it will be denoted by f_λ. By the definition of $P_{\lambda,\omega}$ one has $\mathbb{E}_{\lambda,\omega}\{f_\lambda(X_n)\} = n$ for any $n \geq 0$. Furthermore, if the limit of $f_\lambda(m)/m$ exists and finite for *a.a* ω then we can treat the limit of $\mathbb{E}_{\lambda,\omega}\{X_n\}/n$ to obtain $d(\lambda)$ as in [3]. Theorem 1.2 will be proved by Propositions 2.1 and 2.3. We begin with the following elementary lemmas.

Lemma 2.1. *Let u_n and v_n be two sequences of positive real numbers. Assume that* $\lim_{n\to+\infty} n^{-1}\sum_{\ell=1}^{n} u_\ell = u$ *and* $\lim_{n\to+\infty} v_n = v$ *then for each* $\alpha = 0, 1, \ldots$

$$\lim_{n\to+\infty} \frac{1}{n^{\alpha+1}} \sum_{\ell=1}^{n} \ell^\alpha u_\ell v_\ell = \frac{uv}{\alpha+1}. \tag{3}$$

Proof. Firstly, we prove the case $\alpha = 0$, we will show that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{\ell=1}^{n} u_\ell v_\ell = uv. \tag{4}$$

For any $\varepsilon > 0$ the inequalities

$$\begin{aligned} \left| \frac{1}{n} \sum_{\ell=1}^{n} u_\ell v_\ell - uv \right| &\leq \left| \frac{1}{n} \sum_{\ell=1}^{n} u_\ell (v_\ell - v) \right| + \left| \frac{1}{n} \sum_{\ell=1}^{n} (u_\ell - u) v \right| \\ &\leq \frac{1}{n} \sum_{\ell=1}^{n} u_\ell \left| v_\ell - v \right| + v \left| \frac{1}{n} \sum_{\ell=1}^{n} u_\ell - u \right| < \varepsilon \end{aligned}$$

hold for all large enough n which completes (4).

Now assume that (3) is true for $\alpha \geq 0$, we claim that it holds also for $\alpha + 1$ that is

$$\lim_{n\to+\infty} \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n} \ell^{\alpha+1} u_\ell v_\ell = \frac{uv}{\alpha+2}. \tag{5}$$

Put $W_n = \sum_{\ell=1}^{n} \ell^{\alpha} u_\ell v_\ell$, using summation by parts formula

$$\frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n} \ell^{\alpha+1} u_\ell v_\ell = -\frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n-1} W_\ell + \frac{1}{n^{\alpha+1}} W_n = -I_1 + I_2.$$

By the assumption $\lim_{n\to+\infty} I_2 = \lim_{n\to+\infty} W_n/n^{\alpha+1} = uv/(\alpha+1)$, and then for any $\varepsilon > 0$ the inequalities

$$\begin{aligned}\left| I_1 - \frac{uv}{(\alpha+1)(\alpha+2)} \right| &\leq \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n-1} \ell^{\alpha+1} \left| \frac{W_\ell}{\ell^{\alpha+1}} - \frac{uv}{\alpha+1} \right| \\ &\quad + \left| \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n-1} \ell^{\alpha+1} - \frac{1}{\alpha+2} \right| \frac{uv}{\alpha+1} \\ &< \varepsilon\end{aligned}$$

hold for all large enough n. So,

$$\lim_{n\to+\infty} \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n} \ell^{\alpha+1} u_\ell v_\ell = -\frac{uv}{(\alpha+1)(\alpha+2)} + \frac{uv}{\alpha+1} = \frac{uv}{\alpha+2}$$

which completes (5). ■

Lemma 2.2. *Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers and let A_n be a partial sum $A_n = \sum_{i=0}^{n} a_i$. Assume that $\lim_{n\to\infty} A_n/n = L$ then*

$$\sum_{\ell=0}^{+\infty} a_\ell \ell \rho^\ell < +\infty \tag{6}$$

where $0 < \rho < 1$. Furthermore

$$\lim_{\rho\to 1^-} (1-\rho) \sum_{\ell=0}^{+\infty} a_\ell \rho^\ell = L. \tag{7}$$

Proof. Using summation by parts formula

$$\sum_{\ell=0}^{n} a_\ell \ell \rho^\ell = (1-\rho) \sum_{\ell=0}^{n-1} A_\ell \ell \rho^\ell - \sum_{\ell=0}^{n-1} A_\ell \rho^{\ell+1} + A_n n \rho^n.$$

Since $\lim_{n\to\infty} A_n/n = L$ then $\lim_{n\to\infty} A_n n \rho^n = 0$, $\sum_{\ell=0}^{+\infty} A_\ell \ell \rho^\ell$ and $\sum_{\ell=0}^{+\infty} A_\ell \rho^{\ell+1}$ converge by the D'Alembert criterion, which completes (6).

Eq. (7) means that the existence of Cesaro means implies the existence of Abel means. We recall the proof of the classical result. We have

$$(1-\rho) \sum_{\ell=0}^{+\infty} a_\ell \rho^\ell - L = (1-\rho)^2 \sum_{\ell=0}^{\infty} \left(\frac{A_\ell}{\ell} - \frac{1}{\rho} L \right) \ell \rho^\ell. \tag{8}$$

For any $\varepsilon > 0$ there exists $N > 0$ such that for any $n \geq N$ one has $|A_n/n - L| < \varepsilon$ and $\lim_{n\to\infty} A_n\rho^n = 0$. Then the absolute value of the right hand side of (8) is bounded by

$$(1-\rho)^2 \sum_{\ell=0}^{N-1} \left| \frac{A_\ell}{\ell} - \frac{1}{\rho} L \right| \ell\rho^\ell + (1-\rho)L + \varepsilon$$

which completes (7). ■

Proposition 2.1. *For almost all environment ω and for $\lambda > 0$ we have*

$$\lim_{\lambda\to 0^+} \frac{d(\lambda)}{\lambda} = \lim_{\lambda\to 0^+} \frac{1}{\lambda} \left[\lim_{n\to+\infty} \mathbb{E}_{\lambda,\omega} \left\{ \frac{X_n}{n} \right\} \right] = \sigma^2.$$

Proof. Fix $\omega \in \Omega$ we consider a function f_λ, defined on $\mathbb{Z}$, such that $(P_{\lambda,\omega} - I) f_\lambda \equiv 1$ and $f_\lambda(0) = 0$. For example, we can take

$$f_\lambda(m) = \begin{cases} \displaystyle\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)e^{2\ell\lambda}} \sum_{s=-\infty}^{\ell} \pi(T^s\omega)e^{(2s-1)\lambda}, & \text{if } m \geq 1 \\ 0, & \text{if } m = 0 \\ \displaystyle -\sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} e^{2\ell\lambda} \sum_{s=-\infty}^{-\ell} \pi(T^s\omega)e^{(2s-1)\lambda}, & \text{if } m \leq -1. \end{cases}$$

Replacing m by X_n and taking the expectation, one has

$$\mathbb{E}_{\lambda,\omega}\{f_\lambda(X_n)\} = n \quad \forall n \geq 0. \tag{9}$$

The formula (9) can be rewritten as

$$\mathbb{E}_{\lambda,\omega} \left\{ \frac{f_\lambda(X_n)}{X_n} \times \frac{X_n}{n} \right\} = 1.$$

We will see that the limit of $f_\lambda(m)/m$ exists as $m \to \infty$ and then, as $X_n \to \infty$ as $n \to +\infty$, the limit of $\mathbb{E}_{\lambda,\omega}\{X_n/n\}$ will exist as $n \to +\infty$.

In the next step we will compute the limit of $f_\lambda(m)/m$. The pointwise ergodic theorem is a limiting statement $n^{-1} \sum_{k=0}^{n-1} \pi(T^{-k}\omega) = \int_\Omega \pi \, \mathrm{d}\mu$. For the rest of this section we assume that $\rho = e^{-2\lambda}$. If we put

$$H_\lambda(\omega) = \sqrt{\rho} \sum_{k=0}^{+\infty} \pi(T^{-k}\omega)\rho^k \tag{10}$$

then Lemma 2.2 ensures that $H_\lambda(\omega)$ is finite and one has also

$$\lim_{\lambda\to 0^+} (1 - e^{-2\lambda}) H_\lambda(\omega) = \lim_{\rho\to 1^-} (1-\rho) H_\lambda(\omega) = \int_\Omega \pi \, \mathrm{d}\mu. \tag{11}$$

Lemma 2.3. *With the function f_λ defined as above, we have*

$$\lim_{m\to\pm\infty} \frac{f_\lambda(m)}{m} = \int_\Omega \frac{H_\lambda}{c} \mathrm{d}\mu.$$

This limit is strictly positive and it is denoted by L_λ.

Proof. We will prove this for $m > 0$, the other case is left to the reader. By the definition of function f_λ one has

$$f_\lambda(m) = \sum_{\ell=0}^{m-1} \frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=-\infty}^{\ell}\pi(T^s\omega)\rho^{-s} = \sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=-\infty}^{\ell}\pi(T^s\omega)\rho^{\ell-s}.$$

Since $T^s = T^{s-\ell}\circ T^\ell$ one has

$$f_\lambda(m) = \sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)}\left[\sqrt{\rho}\sum_{s=-\infty}^{\ell}\pi(T^{s-\ell}\omega)\rho^{\ell-s}\right]\circ T^\ell.$$

Replacing $\ell - s$ by k one obtains

$$f_\lambda(m) = \sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)}\left[\sqrt{\rho}\sum_{k=0}^{+\infty}\pi(T^{-k}\omega)\rho^k\right]\circ T^\ell = \sum_{\ell=0}^{m-1}\frac{H_\lambda}{c}\circ T^\ell(\omega).$$

By hypothesis $c \in L^p(\mu)$ we have $\pi \in L^p(\mu)$. Using Holder's inequality one can show that $H_\lambda \in L^p(\mu)$. Again, using Holder's inequality and the hypothesis that $1/c \in L^q(\mu)$ we conclude that $H_\lambda/c \in L^1(\mu)$. The proof of Lemma 2.3 is thus complete by using the pointwise ergodic theorem for the function H_λ/c. ■

From Lemma 2.3 for any $\varepsilon > 0$ there exists $M > 0$ such that for any $|m| > M$ then

$$\left|\frac{1}{L_\lambda}\frac{f_\lambda(m)}{m} - 1\right| < \varepsilon. \tag{12}$$

We now combine (9) and (12) to compute the limit of $\mathbb{E}_{\lambda,\omega}\{X_n/n\}$. If we decompose $\Omega = \{|X_n| \le M\} \cup \{|X_n| > M\}$ then the inequalities

$$\begin{aligned}\left|\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\right\} - \frac{1}{L_\lambda}\right| &\le \frac{1}{n}\mathbb{E}_{\lambda,\omega}\left\{\left|X_n - \frac{f_\lambda(X_n)}{L_\lambda}\right|\mathbf{1}_{\{|X_n|\le M\}}\right\} \\ &\quad + \mathbb{E}_{\lambda,\omega}\left\{\left|1 - \frac{1}{L_\lambda}\frac{f_\lambda(X_n)}{X_n}\right|\frac{|X_n|}{n}\mathbf{1}_{\{|X_n|>M\}}\right\} \\ &< \varepsilon + \varepsilon\sqrt{\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\right\}}\end{aligned} \tag{13}$$

hold for all large enough n. We see that if $\mathbb{E}_{\lambda,\omega}\left\{X_n^2/n^2\right\}$ is bounded then the limit of $\mathbb{E}_{\lambda,\omega}\{X_n/n\}$ is equal to $1/L_\lambda$.

Proposition 2.2. *For almost all environment ω and for $\lambda > 0$ we have*

$$\lim_{n\to+\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\right\} = \frac{1}{L_\lambda^2}. \tag{14}$$

Proof. Again, fix $\omega \in \Omega$ we consider a function $g_\lambda \geq 0$, defined on $\mathbb{Z}$, such that $(P_{\lambda,\omega} - I)g_\lambda \equiv f_\lambda$ and $g_\lambda(0) = 0$. For example, we can take

$$g_\lambda(m) = \begin{cases} \displaystyle\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)e^{2\ell\lambda}} \sum_{s=-\infty}^{\ell} \pi(T^s\omega)e^{(2s-1)\lambda} f_\lambda(s), & \text{if } m \geq 1 \\ 0, & \text{if } m = 0 \\ \displaystyle -\sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} e^{2\ell\lambda} \sum_{s=-\infty}^{-\ell} \pi(T^s\omega)e^{(2s-1)\lambda} f_\lambda(s), & \text{if } m \leq -1 \end{cases}$$

then $(P_{\lambda,\omega} - I)g_\lambda(m) = f_\lambda(m)$ for any $m \in \mathbb{Z}$. Replacing m by X_n and taking the expectation, one has

$$\mathbb{E}_{\lambda,\omega}\{g_\lambda(X_n)\} = \frac{n(n-1)}{2}, \quad \forall n \geq 0. \tag{15}$$

The formula (15) can be rewritten as

$$\mathbb{E}_{\lambda,\omega}\left\{\frac{g_\lambda(X_n)}{X_n^2} \times \frac{X_n^2}{n^2}\right\} \sim \frac{1}{2},$$

where $f \sim g$ means $\lim_{n\to+\infty} f(n)/g(n) = 1$. We will see also that the limit of $g_\lambda(m)/m^2$ exists as $m \to \infty$ and then, as $X_n \to \infty$ as $n \to +\infty$, the limit of $\mathbb{E}_{\lambda,\omega}\left\{X_n^2/n^2\right\}$ will exist as $n \to +\infty$.

In the next step we will compute the limit of $g_\lambda(m)/m^2$ by using Lemmas 2.1–2.3.

Lemma 2.4. *With the function g_λ defined as above we have*

$$\lim_{m\to\pm\infty} \frac{g_\lambda(m)}{m^2} = \frac{1}{2}L_\lambda^2.$$

Proof. We will prove this for $m > 0$, the other case is left to the reader. Put

$$\xi_1 = \sum_{\ell=0}^{m-1} \frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho} \sum_{s=-\infty}^{0} \pi(T^s\omega)\rho^{-s} f_\lambda(s),$$
$$\xi_2 = \sum_{\ell=0}^{m-1} \frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho} \sum_{s=1}^{\ell} \pi(T^s\omega)\rho^{-s} f_\lambda(s),$$
$$\xi_3 = \sum_{\ell=0}^{m-1} \frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho} \sum_{s=1}^{\ell} \pi(T^s\omega)\rho^{-s} s.$$

By the definition of function g_λ, we have $g_\lambda(m) = \xi_1 + \xi_2$. We will prove that

$$\lim_{m\to+\infty} \frac{\xi_1}{m^2} = 0 \tag{16}$$

and

$$\lim_{m\to+\infty} \frac{\xi_2}{m^2} = \frac{1}{2}L_\lambda^2. \tag{17}$$

By (6) and $\lim_{s\to\infty} f_\lambda(s)/s = L_\lambda$ then $\sum_{s=-\infty}^{0} \pi(T^s\omega)\rho^{-s} f_\lambda(s)$ is finite which completes (16).

Proof of (17). Replacing $\ell - s$ by k we obtain

$$\xi_3 = \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell \omega)} \sqrt{\rho} \sum_{s=1}^{\ell} \pi(T^s \omega) \rho^{\ell - s} s = \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell \omega)} \sqrt{\rho} \sum_{k=0}^{\ell-1} \pi(T^{\ell-k}\omega)\rho^k (\ell - k).$$

Since $T^{\ell-k} = T^{-k} \circ T^\ell$, one has

$$\xi_3 = \sum_{\ell=0}^{m-1} \frac{\ell}{c(T^\ell \omega)} \left[\sqrt{\rho} \sum_{k=0}^{\ell-1} \pi(T^{-k}\omega)\rho^k \right] \circ T^\ell - \sum_{\ell=0}^{m-1} \frac{\sqrt{\rho}}{c(T^\ell \omega)} \left[\sum_{k=0}^{\ell-1} \pi(T^{-k}\omega) k \rho^k \right] \circ T^\ell.$$

If we put

$$G_\lambda(\omega) = \sqrt{\rho} \sum_{k=0}^{\infty} \pi(T^{-k}\omega) k \rho^k$$

then $G_\lambda \in L^p(\mu)$ and so $G_\lambda / c \in L^1(\mu)$ by Holder's inequality. Now, using the pointwise ergodic theorem we see that when m goes to infinity $m^{-2} \sum_{\ell=0}^{m-1} [G_\lambda(\omega)/c(\omega)] \circ T^\ell$ goes to 0. It follows that

$$\lim_{m \to +\infty} \frac{1}{m^2} \sum_{\ell=0}^{m-1} \frac{\sqrt{\rho}}{c(T^\ell \omega)} \left[\sum_{k=0}^{\ell-1} \pi(T^{-k}\omega) k \rho^k \right] \circ T^\ell = 0.$$

On the other hand, for each $\ell = 0, 1, \ldots$

$$\left| \sqrt{\rho} \sum_{k=0}^{\ell-1} \pi(T^{\ell-k}\omega)\rho^k - H_\lambda(T^\ell \omega) \right| = \rho^\ell H_\lambda(\omega).$$

Taking $a_i = c^{-1}(T^i \omega)$ in (6) one has $\lim_{m \to +\infty} m^{-2} \sum_{\ell=0}^{m-1} c^{-1}(T^\ell \omega) \ell \rho^\ell H_\lambda(\omega) = 0$. Hence we have

$$\lim_{m \to +\infty} \frac{1}{m^2} \sum_{\ell=0}^{m-1} \frac{\ell}{c(T^\ell \omega)} \left| \sqrt{\rho} \sum_{k=0}^{\ell-1} \pi(T^{\ell-k}\omega)\rho^k - H_\lambda(T^\ell \omega) \right| = 0.$$

So,

$$\lim_{m \to +\infty} \frac{\xi_3}{m^2} = \lim_{m \to +\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} \frac{H_\lambda}{c} \circ T^\ell(\omega) \left(\frac{\ell}{m} \right).$$

By Lemma 2.1, this limit is equal to $\frac{1}{2} L_\lambda$.

Moreover, since $\lim_{s \to \infty} f_\lambda(s)/s = L_\lambda$ then $\lim_{m \to +\infty} \sup_{s \le m} m^{-1} |f_\lambda(s) - s L_\lambda| = 0$. Replacing $\ell - s$ by k in both definitions of ξ_2 and ξ_3 one has

$$\left| \frac{\xi_2}{m^2} - \frac{\xi_3}{m^2} L_\lambda \right| \le \left(\frac{1}{m} \sum_{\ell=0}^{m-1} \frac{H_\lambda(T^\ell \omega)}{c(T^\ell \omega)} \right) \sup_{s \le m} \frac{1}{m} |f_\lambda(s) - s L_\lambda| .$$

It follows that this tends to 0 when m goes to infinity, and then

$$\lim_{m \to +\infty} \frac{\xi_2}{m^2} = \lim_{m \to +\infty} \frac{\xi_3}{m^2} L_\lambda = \frac{1}{2} L_\lambda^2$$

which completes (17). ■

By Lemma 2.4, for any $\varepsilon' > 0$, there exists $M' > 0$ such that for any $|m| > M'$ then

$$\left| \frac{m^2}{g_\lambda(m)} - \frac{2}{L_\lambda^2} \right| < \varepsilon'/2. \tag{18}$$

We now combine (15) and (18) to compute the limit of $\mathbb{E}_{\lambda,\omega}\{X_n/n\}$. If we decompose $\Omega = \left\{|X_n| \le M'\right\} \cup \left\{|X_n| > M'\right\}$ then the inequality

$$\left| \mathbb{E}_{\lambda,\omega}\left\{ \frac{X_n^2}{n^2} \right\} - \frac{1}{L_\lambda^2} \right| < \varepsilon'$$

holds for all large enough n. ■

We have thus proved that $\lim_{n\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{X_n^2/n^2\right\} = L_\lambda^{-2}$. From (13), we obtain

$$\lim_{n\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{ \frac{X_n}{n} \right\} = \frac{1}{L_\lambda} = \left[\int_\Omega \frac{H_\lambda}{c} \mathrm{d}\mu \right]^{-1} = d(\lambda).$$

This limit does not depend on ω. Finally, by using L^p-convergence stated in (11) one has

$$\begin{aligned} \lim_{\lambda\to 0^+} \frac{d(\lambda)}{\lambda} &= \lim_{\lambda\to 0^+} \frac{(1-e^{-2\lambda})}{\lambda} \left[\int_\Omega \frac{(1-e^{-2\lambda})H_\lambda}{c} \mathrm{d}\mu \right]^{-1} \\ &= \lim_{\lambda\to 0^+} \frac{(1-e^{-2\lambda})}{\lambda(e^{-\lambda}+e^{\lambda})} \left[\int_\Omega c\, \mathrm{d}\mu \int_\Omega \frac{1}{c}\, \mathrm{d}\mu \right]^{-1} \\ &= \sigma^2 \end{aligned}$$

which completes the proof of Proposition 2.1. ■

Proposition 2.3. *For almost all environment ω and for $\lambda < 0$ we have*

$$\lim_{\lambda\to 0^-} \frac{d(\lambda)}{\lambda} = \sigma^2.$$

Proof. The proof of this proposition is very similar to Proposition 2.1 where we modify the functions f_λ and g_λ, defined on $\mathbb{Z}$, as follows

$$f_\lambda(m) = \begin{cases} -\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)e^{2\ell\lambda}} \sum_{s=\ell}^{+\infty} \pi(T^s\omega)e^{(2s+1)\lambda}, & \text{if } m \ge 1 \\ 0, & \text{if } m = 0 \\ \sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} e^{2\ell\lambda} \sum_{s=-\ell}^{+\infty} \pi(T^s\omega)e^{(2s+1)\lambda}, & \text{if } m \le -1 \end{cases}$$

and

$$g_\lambda(m) = \begin{cases} -\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)e^{2\ell\lambda}} \sum_{s=\ell}^{+\infty} \pi(T^s\omega)e^{(2s+1)\lambda} f_\lambda(s), & \text{if } m \ge 1 \\ 0, & \text{if } m = 0 \\ \sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} e^{2\ell\lambda} \sum_{s=-\ell}^{+\infty} \pi(T^s\omega)e^{(2s+1)\lambda} f_\lambda(s), & \text{if } m \le -1 \end{cases}$$

where ω is fixed. ■

Remark 2.1. We have proved that for almost all $\omega \lim_{n\to+\infty} \mathbb{E}_{\lambda,\omega}\{X_n/n\} = d(\lambda)$ and $\lim_{n\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{X_n^2/n^2\right\} = d(\lambda)^2$ with $\lambda \neq 0$. This implies that for ω a.s X_n/n converges to $d(\lambda)$ in probability.

Remark 2.2. By the argument as in [1], we can show that X_n/n converges a.s. Indeed, we will prove this for $\lambda > 0$ and the other case is left to the reader. Following the notation of [1] we introduce

$$\alpha_i(\omega) = \mathbb{P}^0_\omega\{X_{n+1} = i+1 | X_n = i\} = e^\lambda c(T^i\omega)/\pi(T^i\omega),$$
$$\beta_i(\omega) = \mathbb{P}^0_\omega\{X_{n+1} = i-1 | X_n = i\} = e^{-\lambda} c(T^{i-1}\omega)/\pi(T^i\omega),$$
$$\rho_i = \rho_i(\omega) = \beta_i(\omega)/\alpha_i(\omega),$$

and $S = S(\omega) = 1 + \sum_{k=1}^{+\infty} \rho_1\rho_2\ldots\rho_k$ one has

$$S = c(\omega)\sum_{k=0}^{+\infty} \frac{1}{c(T^k\omega)} e^{-2\lambda k} \in L^1(\mu)$$

by Holder's inequality. Theorem 4.1 in [1] ensures that

$$\lim_{n\to+\infty} \frac{X_n}{n} = v, \quad \mu \text{ a.s } \omega,$$

where $v^{-1} = \int_\Omega (1+\rho_0(\omega))S(\omega)\mathrm{d}\mu(\omega)$. By calculating the integrand in the last expression

$$e^{-\lambda}\sum_{k=0}^{+\infty} \frac{\pi(\omega)e^{-2\lambda k}}{c(T^k\omega)} = e^{-\lambda}\sum_{k=0}^{+\infty}\left[\frac{\pi(T^{-k}\omega)e^{-2\lambda k}}{c(T^k\omega)}\right]\circ T^k.$$

Since T preserves the measure μ one obtains

$$v = \left[\int_\Omega \frac{H_\lambda}{c}\mathrm{d}\mu\right]^{-1} = d(\lambda), \quad \mu \text{ a.s } \omega,$$

where $H_\lambda = e^{-\lambda}\sum_{k=0}^{+\infty} \pi(T^{-k}\omega)e^{-2\lambda k}$. However, we do not know if Alili's method works for continuous time process or for diffusion!

3. An analogue to a continuously time process

In this section, we will discuss two theorems as the continuous versions of Theorem 1.2 in one dimension. These theorems can be proved by solving Poisson's equation as in Theorem 1.2.

3.1. Markov process with discrete space

We first consider Markov process $(X_t)_{t\geq 0}$ with continuous time on $\mathbb{Z}$ and the initial condition $X_0 = 0$, the infinitesimal generator is defined by

$$L_{\lambda,\omega} f(k) = e^{-\lambda} c(T^{k-1}\omega) f(k-1) + e^\lambda c(T^k\omega) f(k+1) - \pi(T^k\omega) f(k),$$

where $\pi = e^\lambda c + e^{-\lambda} c\circ T^{-1}$.

Theorem 3.1. *For almost all environment ω,*

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\left[\lim_{t\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\}\right] = 2\left[\int_\Omega \frac{1}{c}\mathrm{d}\mu\right]^{-1},$$

if c^{-1} is integrable.

Proof. We consider this theorem for $\lambda > 0$ and the other case is left to the reader. Fixed ω, we solve two Poisson's equations on $\mathbb{Z}$

$$L_{\lambda,\omega} f_\lambda \equiv 1, \qquad f_\lambda(0) = 0$$

and

$$L_{\lambda,\omega} g_\lambda \equiv f_\lambda, \qquad g_\lambda(0) = 0$$

in order to obtain the particular solutions as follows

$$f_\lambda(m) = \begin{cases} \displaystyle\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)e^{2\ell\lambda}} \sum_{s=-\infty}^{\ell} e^{(2s-1)\lambda}, & \text{if } m \geq 1 \\ 0, & \text{if } m = 0 \\ \displaystyle -\sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} e^{2\ell\lambda} \sum_{s=-\infty}^{-\ell} e^{(2s-1)\lambda}, & \text{if } m \leq -1 \end{cases}$$

and

$$g_\lambda(m) = \begin{cases} \displaystyle\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)e^{2\ell\lambda}} \sum_{s=-\infty}^{\ell} e^{(2s-1)\lambda} f_\lambda(s), & \text{if } m \geq 1 \\ 0, & \text{if } m = 0 \\ \displaystyle -\sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} e^{2\ell\lambda} \sum_{s=-\infty}^{-\ell} e^{(2s-1)\lambda} f_\lambda(s), & \text{if } m \leq -1. \end{cases}$$

The hypothesis $c^{-1} \in L^{-1}(\mu)$ ensures that for almost all ω

$$\lim_{m\to\pm\infty} \frac{f_\lambda(m)}{m} = \frac{e^{-\lambda}}{1-e^{-2\lambda}} \int_\Omega \frac{1}{c} \mathrm{d}\mu,$$

and

$$\lim_{m\to\pm\infty} \frac{g_\lambda(m)}{m^2} = \frac{1}{2}\left(\frac{e^{-\lambda}}{1-e^{-2\lambda}} \int_\Omega \frac{1}{c} \mathrm{d}\mu\right)^2.$$

We then obtain

$$\lim_{t\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\right\} = d(\lambda)^2 \quad \text{and} \quad \lim_{t\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\} = d(\lambda), \tag{19}$$

where $d(\lambda) = (e^\lambda - e^{-\lambda})\left[\int_\Omega \frac{1}{c}\mathrm{d}\mu\right]^{-1}$. It follows that

$$\lim_{\lambda\to0^+} \frac{1}{\lambda}\left[\lim_{t\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\}\right] = 2\left[\int_\Omega \frac{1}{c}\mathrm{d}\mu\right]^{-1}. \quad \blacksquare$$

Remark 3.1. From (19) we deduce that for ω a.s X_t/t converges to $d(\lambda)$ in probability.

3.2. Diffusion in random environment

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space equipped with an ergodic flow $(T_x)_{x\in\mathbb{R}}$. We consider two random variables $a, b > 0$ such that the functions $x \mapsto a(T_x\omega)$ and $x \mapsto b(T_x\omega)$ are continuous.

In this section we study, for fixed ω and drift $\lambda \neq 0$, the process with the infinitesimal generator defined by

$$L_{\lambda,\omega} f(x) = \frac{1}{2e^{\lambda x} a(T_x\omega)} \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\lambda x} b(T_x\omega) \frac{\mathrm{d}f}{\mathrm{d}x} \right),$$

and the initial condition $X_0 = 0$.

The associated process satisfies the stochastic differential equation

$$\mathrm{d}X_t = \sigma_\omega(X_t)\mathrm{d}B_t + \mu_\omega(X_t)\mathrm{d}t, \tag{20}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion, the coefficient of diffusion $\sigma^2_\omega(x) = b(T_x\omega)/a(T_x\omega)$ and the drift $\mu_\omega(x) = \left[2e^{\lambda x} a(T_x\omega)\right]^{-1} \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\lambda x} b(T_x\omega)\right)$.

Theorem 3.2. *Suppose that, for almost every $\omega \in \Omega$, the functions $\sigma^2_\omega(x)$ and $\mu_\omega(x)$ are local Lipschitz. Then, for almost all $\omega \in \Omega$, the solution $(X_t)_{t\geq 0}$ of (20) satisfies*

$$\lim_{\lambda\to 0} \frac{1}{\lambda} \left[\lim_{t\to+\infty} \frac{\mathbb{E}_{\lambda,\omega}\{X_t\}}{t} \right] = \left[\int_\Omega a \,\mathrm{d}\mu \int_\Omega \frac{1}{b} \,\mathrm{d}\mu \right]^{-1},$$

if $a \in L^p(\mu)$ and $b^{-1} \in L^q(\mu)$ with $p, q \in [1, +\infty]$ such that $1/p + 1/q = 1$.

Proof. We consider this theorem for $\lambda > 0$ and the other case is left to the reader. Fixed ω, we solve two Poisson's equations on $\mathbb{R}$

$$L_{\lambda,\omega} f_\lambda \equiv 1, \qquad f_\lambda(0) = 0$$

and

$$L_{\lambda,\omega} g_\lambda \equiv f_\lambda, \qquad g_\lambda(0) = 0$$

in order to obtain the particular solutions as follows

$$f_\lambda(x) = \begin{cases} \displaystyle\int_{v=0}^{x} \frac{1}{e^{\lambda v} b(T_v\omega)} \int_{u=-\infty}^{v} 2e^{\lambda u} a(T_u\omega)\,\mathrm{d}u\,\mathrm{d}v, & \text{if } x \geq 0 \\ \displaystyle -\int_{v=x}^{0} \frac{1}{e^{\lambda v} b(T_v\omega)} \int_{u=-\infty}^{v} 2e^{\lambda u} a(T_u\omega)\,\mathrm{d}u\,\mathrm{d}v, & \text{if } x < 0 \end{cases}$$

and

$$g_\lambda(x) = \begin{cases} \displaystyle\int_{v=0}^{x} \frac{1}{e^{\lambda v} b(T_v\omega)} \int_{u=-\infty}^{v} 2e^{\lambda u} a(T_u\omega) f_\lambda(u)\,\mathrm{d}u\,\mathrm{d}v, & \text{if } x \geq 0 \\ \displaystyle -\int_{v=x}^{0} \frac{1}{e^{\lambda v} b(T_v\omega)} \int_{u=-\infty}^{v} 2e^{\lambda u} a(T_u\omega) f_\lambda(u)\,\mathrm{d}u\,\mathrm{d}v, & \text{if } x < 0. \end{cases}$$

The hypotheses $a \in L^p(\mu)$ and $b^{-1} \in L^q(\mu)$ ensure that for almost all ω

$$\lim_{x\to\pm\infty} \frac{f_\lambda(x)}{x} = \int_\Omega \frac{H_\lambda}{b} \mathrm{d}\mu,$$

and

$$\lim_{x\to\pm\infty} \frac{g_\lambda(x)}{x^2} = \frac{1}{2} \left(\int_\Omega \frac{H_\lambda}{b} \mathrm{d}\mu \right)^2,$$

where $H_\lambda(\omega) = 2\int_{t=0}^{+\infty} e^{-\lambda t} a(T_{-t}\omega)\,\mathrm{d}t$. We then obtain

$$\lim_{t\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\right\} = d(\lambda)^2 \quad \text{and} \quad \lim_{t\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\} = d(\lambda), \tag{21}$$

where $d(\lambda) = \left[\int_\Omega H_\lambda/b\,\mathrm{d}\mu\right]^{-1}$. It follows that

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda}\left[\lim_{t\to+\infty} \frac{\mathbb{E}_{\lambda,\omega}\{X_t\}}{t}\right] = \left[\int_\Omega a\,\mathrm{d}\mu \int_\Omega \frac{1}{b}\,\mathrm{d}\mu\right]^{-1}. \quad \blacksquare$$

Remark 3.2. From (21) we deduce that for ω a.s X_t/t converges to $d(\lambda)$ in probability.

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