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Stochastic and partial differential equations on non-smooth time-dependent domains

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Abstract

In this article, we consider non-smooth time-dependent domains whose boundary is $\mathcal{W}^{1,p}$ in time and single-valued, smoothly varying directions of reflection at the boundary. In this setting, we first prove existence and uniqueness of strong solutions to stochastic differential equations with oblique reflection. Secondly, we prove, using the theory of viscosity solutions, a comparison principle for fully nonlinear second-order parabolic partial differential equations with oblique derivative boundary conditions. As a consequence, we obtain uniqueness, and, by barrier construction and Perron's method, we also conclude existence of viscosity solutions. Our results generalize two articles by Dupuis and Ishii to time-dependent domains.

Keywords: Reflected diffusion; Skorohod problem; Oblique reflection; Time-dependent domain; Stochastic differential equations; Non-smooth domain; Viscosity solution; Parabolic partial differential equation; Comparison principle; Existence; Uniqueness

1. Introduction

In this article we establish existence and uniqueness of strong solutions to stochastic differential equations (SDE) with single-valued, smoothly varying oblique reflection at the boundary of a bounded, non-smooth time-dependent domain whose boundary is $\mathcal{W}^{1,p}$ in time.

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In the same geometric setting, we also prove a comparison principle, uniqueness and existence of viscosity solutions to partial differential equations (PDE) with oblique derivative boundary conditions.

In the SDE case, our approach is based on the Skorohod problem, which, in the form studied in this article, was first described by Tanaka [25]. Tanaka established existence and uniqueness of solutions to the Skorohod problem in convex domains with normal reflection. These results were subsequently substantially generalized by, in particular, Lions and Sznitman [20] and Saisho [24]. To the authors' knowledge, the most general results on strong solutions to reflected SDEs in time-independent domains based on the Skorohod problem are those established by Dupuis and Ishii [14]. The aim here is to generalize the SDE results mentioned above, in particular those of Case 1 in [14], to the setting of time-dependent domains.

There is, by now, a number of articles on reflected SDEs in time-dependent domains. Early results on this topic include the exhaustive study of the heat equation and reflected Brownian motion in smooth time-dependent domains by Burdzy, Chen, and Sylvester [5] and the study of reflected SDEs in smooth time-dependent domains with reflection in the normal direction by Costantini, Gobet, and El Karoui [8]. We also mention that Burdzy, Kang, and Ramanan [6] investigated the Skorohod problem in a one-dimensional, time-dependent domain and, in particular, found conditions for when there exists a solution to the Skorohod problem in the event that the two boundaries meet. Existence of weak solutions to SDEs with oblique reflection in non-smooth time-dependent domains was established by Nyström and Önskog [23] under fairly general conditions using the approach of [7]. In the article at hand, we use the approach of [14] and derive regularity conditions, under which we can obtain existence and also uniqueness of strong solutions to SDEs with oblique reflection in time-dependent domains.

Turning to the PDE case, we recall that the approach of [14] relies on the construction of test functions used earlier in Dupuis and Ishii [11] to prove the comparison principle, existence and uniqueness for fully nonlinear second-order elliptic PDEs in non-smooth time-independent domains. Here we generalize these test functions to our time-dependent setting, and obtain the corresponding results for both SDEs and PDEs in time-dependent domains. In particular, our PDE results generalize the main part of [11] to hold in the setting of fully nonlinear second-order parabolic PDEs in non-smooth time-dependent domains. Our proofs are based on the theory of viscosity solutions. The first step is to observe that the maximum principle for semicontinuous functions by Crandall and Ishii [9] holds in time-dependent domains. Using the maximum principle and the above-mentioned test functions, we prove the comparison principle by following the nowadays standard method, see Crandall, Ishii, and Lions [10] and [11]. Next, we prove existence of a unique solution to the PDE problem by means of Perron's method, the comparison principle and by constructing several explicit sub- and supersolutions (barriers) to the PDE.

To the authors' knowledge, there are no previous results on the oblique derivative problem for parabolic PDEs in non-smooth time-dependent domains. For time-independent domains, however, there are several articles in the literature. Besides [11], Dupuis and Ishii studied oblique derivative problems for fully nonlinear elliptic PDEs on domains with corners in [13]. Moreover, Barles [2] proved a comparison principle and existence of unique solutions to degenerate elliptic and parabolic boundary value problems with nonlinear Neumann type boundary conditions in bounded domains with $\mathcal{W}^{3,\infty}$ -boundary. Ishii and Sato [18] proved similar theorems for boundary value problems for some singular degenerate parabolic partial differential equations with nonlinear oblique derivative boundary conditions in bounded $\mathcal{C}^1$-domains. Further, in bounded domains with $\mathcal{W}^{3,\infty}$-boundary, Bourgoing [4] considered singular degenerate parabolic equations and equations having L^1 dependence in time.

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Concerning PDEs in the setting of time-dependent domains, we mention that Björn et al. [3] proved, among other results, a comparison principle for solutions of degenerate and singular parabolic equations with Dirichlet boundary conditions using a different technique and that Avelin [1] proved boundary estimates of solutions to the degenerate p-parabolic equation.

As a motivation for considering SDEs and PDEs in time-dependent domains, we mention that such geometries arise naturally in a wide range of applications in which the governing equation of interest is a differential equation, for example in modelling of crack propagation [22], modelling of fluids [15,16] and modelling of chemical, petrochemical and pharmaceutical processes [19].

The rest of the paper is organized as follows. In Section 2 we give preliminary definitions, notations, assumptions and also state our main results. In Section 3 we construct the test functions crucial for the proofs of both the SDE and the PDE results. Using these test functions, we prove existence of solutions to the Skorohod problem in Section 4. The results on the Skorohod problem are subsequently used, in Section 5, to prove the main results for SDEs. Finally, in Section 6, we use the theory of viscosity solutions together with the test functions derived in Section 3 to establish the PDE results.

2. Preliminaries and statement of main results

Throughout this article we will use the following definitions and assumptions. Given $n \geq 1$, $T > 0$ and a bounded, open, connected set $\Omega' \subset \mathbb{R}^{n+1}$ we will refer to

$$\Omega = \Omega' \cap ([0, T] \times \mathbb{R}^n), \tag{2.1}$$

as a time-dependent domain. Given Ω and $t \in [0, T]$, we define the time sections of Ω as $\Omega_t = \{x : (t, x) \in \Omega\}$, and we assume that

$$\Omega_t \neq \emptyset \text{ and that } \Omega_t \text{ is bounded and connected for every } t \in [0, T]. \tag{2.2}$$

Let $\partial\Omega_t$, for $t \in [0, T]$, denote the boundary of Ω_t. Let $\langle \cdot, \cdot \rangle$ and $|\cdot| = \langle \cdot, \cdot \rangle^{1/2}$ define the Euclidean inner product and norm, respectively, on $\mathbb{R}^n$ and define, whenever $a \in \mathbb{R}^n$ and $b > 0$, the sets $B(a, b) = \{x \in \mathbb{R}^n : |x - a| \leq b\}$ and $S(a, b) = \{x \in \mathbb{R}^n : |x - a| = b\}$. For any Euclidean spaces E and F, we define the following spaces of functions mapping E into F. $\mathcal{C}(E, F)$ denotes the set of continuous functions, $\mathcal{C}^k(E, F)$ denotes the set of k times continuously differentiable functions and $\mathcal{W}^{1,p}(E, F)$ denotes the Sobolev space of functions whose first order weak derivatives belong to $L^p(E)$. If we can distinguish the time variable from the spatial variables, we let $\mathcal{C}^{1,2}(E, F)$ denote the set of functions, whose elements are continuously differentiable once with respect to the time variable and twice with respect to any space variable, and by $\mathcal{C}_b^{1,2}(E, F)$ we denote the space of bounded functions in $\mathcal{C}^{1,2}(E, F)$ having bounded derivatives. Moreover, $\mathcal{BV}(E, F)$ denotes the set of functions with bounded variation. In particular, for $\eta \in \mathcal{BV}([0, T], \mathbb{R}^n)$, we let $|\eta|(t)$ denote the total variation of η over the interval $[0, t]$.

2.1. Assumptions on the domain and directions of reflection

Throughout this article we consider non-smooth time-dependent domains of the following type. Let $\Omega \subset \mathbb{R}^{n+1}$ be a time-dependent domain satisfying (2.2). The direction of reflection at $x \in \partial\Omega_t$, $t \in [0, T]$, is given by $\gamma(t, x)$ satisfying

$$\gamma \in \mathcal{C}_b^{1,2}\left(\mathbb{R}^{n+1}, B(0, 1)\right), \tag{2.3}$$

such that $\gamma(t,x) \in S(0,1)$ for all $(t,x) \in V$, where V is an open set satisfying $\Omega_t^c \subset V$ for all $t \in [0,T]$. Moreover, there is a constant $\rho \in (0,1)$ such that the exterior cone condition

$$\bigcup_{0 \le \zeta \le \rho} B(x - \zeta\gamma(t,x), \zeta\rho) \subset \Omega_t^c, \tag{2.4}$$

holds, for all $x \in \partial\Omega_t$, $t \in [0,T]$. Note that it follows from (2.4) that γ points into the domain and this is indeed the standard convention for SDEs. For PDEs, however, the standard convention is to let γ point out of the domain. To facilitate for readers accustomed with either of these conventions we, in the following, let γ point inward whenever SDEs are treated, whereas when we treat PDEs we assume the existence of a function

$$\widetilde{\gamma} \in \mathcal{C}_b^{1,2}\left(\mathbb{R}^{n+1}, B(0,1)\right), \tag{2.5}$$

defined as $\widetilde{\gamma}(t,x) = -\gamma(t,x)$, with γ as in (2.3). In particular, we have

$$\bigcup_{0 \le \zeta \le \rho} B(x + \zeta\widetilde{\gamma}(t,x), \zeta\rho) \subset \Omega_t^c, \tag{2.6}$$

for all $x \in \partial\Omega_t$, $t \in [0,T]$. Finally, regarding the temporal variation of the domain, we define $d(t,x) = \inf_{y \in \Omega_t} |x-y|$, for all $t \in [0,T]$, $x \in \mathbb{R}^n$, and assume that for some fixed $p \in (1,\infty)$ and all $x \in \mathbb{R}^n$,

$$d(\cdot, x) \in \mathcal{W}^{1,p}([0,T], [0,\infty)), \tag{2.7}$$

with Sobolev norm uniformly bounded in space. We also assume that $D_t d(t,x)$ is jointly measurable in (t,x).

Remark 2.1. A simple contradiction argument based on the exterior cone condition (2.4) for the time sections and the regularity of γ and Ω_t, shows that the time sections satisfy the interior cone condition

$$\bigcup_{0 \le \zeta \le \rho} B(x + \zeta\gamma(t,x), \zeta\rho) \subset \overline{\Omega}_t,$$

for all $x \in \partial\Omega_t$, $t \in [0,T]$. The exterior and interior cone conditions together imply that the boundary of Ω_t is Lipschitz continuous (in space) with a Lipschitz constant K_t satisfying $\sup_{t\in[0,T]} K_t < \infty$. Moreover, these conditions imply that for a suitable constant $\theta \in (0,1)$, $\theta^2 > 1 - \rho^2$, there exists $\delta > 0$ such that

$$\langle y - x, \gamma(t,x)\rangle \ge -\theta |y - x|,$$

for all $x \in \partial\Omega_t$, $y \in \overline{\Omega}_t$, $t \in [0,T]$ satisfying $|x-y| \le \delta$.

Remark 2.2. By Morrey's inequality, condition (2.7) implies the existence of a Hölder exponent $\widehat{\alpha} = 1 - 1/p \in (0,1)$ and a Hölder constant $K \in (0,\infty)$ such that, for all $s,t \in [0,T]$, $x \in \mathbb{R}^n$,

$$|d(s,x) - d(t,x)| \le K|s-t|^{\widehat{\alpha}}. \tag{2.8}$$

Remark 2.3. The assumptions imposed on the time sections of the time-dependent domain in (2.3), (2.4) coincide with those imposed on the time-independent domains in [11] and in Case 1 of [14]. For time-independent domains, existence and uniqueness results for SDE and PDE have also been obtained under the conditions given in [12] and in Case 2 of [14]. It is likely that also these results can be extended to time-dependent domains using a procedure similar to that of the article at hand, but we leave this as a topic of future research.

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Remark 2.4. Consider the function

$$l(r)=\sup_{s,t\in[0,T],\ |s-t|\le r}\ \sup_{x\in\overline{\Omega_s}}\ \inf_{y\in\overline{\Omega_t}}|x-y|,$$

introduced in [8] and frequently used in [23]. Condition (2.8) is equivalent to,

$$l(r)\le Kr^{\widehat{\alpha}},$$

which is considerably stronger than the condition $\lim_{r\to0^+}l(r)=0$ assumed in [23]. On the other hand, it was assumed in [23] that Ω_t satisfies a uniform exterior sphere condition, and this does not hold in general for domains satisfying (2.4).

2.2. *Statement of main result for SDEs*

We consider the Skorohod problem in the following form.

Definition 2.5. Given $\psi\in\mathcal{C}([0,T],\mathbb{R}^n)$, with $\psi(0)\in\overline{\Omega_0}$, we say that the pair $(\phi,\lambda)\in\mathcal{C}([0,T],\mathbb{R}^n)\times\mathcal{C}([0,T],\mathbb{R}^n)$ is a solution to the Skorohod problem for (Ω,γ,ψ) if (ψ,ϕ,λ) satisfies, for all $t\in[0,T]$,

$$\phi(t)=\psi(t)+\lambda(t),\quad \phi(0)=\psi(0),\tag{2.9}$$

$$\phi(t)\in\overline{\Omega_t},\tag{2.10}$$

$$|\lambda|(T)<\infty,\tag{2.11}$$

$$|\lambda|(t)=\int_{(0,t]}I_{\{\phi(s)\in\partial\Omega_s\}}d|\lambda|(s),\tag{2.12}$$

$$\lambda(t)=\int_{(0,t]}\widehat{\gamma}(s)\,d|\lambda|(s),\tag{2.13}$$

for some measurable function $\widehat{\gamma}:[0,T]\to\mathbb{R}^n$ satisfying $\widehat{\gamma}(s)=\gamma(s,\phi(s))$ $d|\lambda|$-a.s.

We use the Skorohod problem to construct solutions to SDEs confined to the given time-dependent domain $\overline{\Omega}$ and with direction of reflection given by γ. We shall consider the following notion of SDEs. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and let $\{\mathcal{F}_t\}_{t\ge0}$ be a filtration satisfying the usual conditions. Let m be a positive integer, let $W=(W_i)$ be an m-dimensional Wiener process and let $b:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ and $\sigma:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$ be continuous functions.

Definition 2.6. A strong solution to the SDE in $\overline{\Omega}$ driven by the Wiener process W and with coefficients b and σ, direction of reflection along γ and initial condition $x\in\overline{\Omega_0}$ is an $\{\mathcal{F}_t\}$-adapted continuous stochastic process $X(t)$ which satisfies, $\mathbb{P}$-almost surely, whenever $t\in[0,T]$,

$$X(t)=x+\int_0^t b(s,X(s))\,ds+\int_0^t\langle\sigma(s,X(s)),dW(s)\rangle+\Lambda(t),\tag{2.14}$$

where

$$X(t)\in\overline{\Omega_t},\quad |\Lambda|(t)=\int_{(0,t]}I_{\{X(s)\in\partial\Omega_s\}}d|\Lambda|(s)<\infty,\tag{2.15}$$

and where

$$\Lambda(t)=\int_{(0,t]}\widehat{\gamma}(s)\,d|\Lambda|(s),\tag{2.16}$$

for some measurable stochastic process $\widehat{\gamma} : [0, T] \to \mathbb{R}^n$ satisfying $\widehat{\gamma}(s) = \gamma(s, X(s))\ d|\Lambda|$-a.s.

Comparing Definition 2.5 with Definition 2.6, it is clear that $(X(\cdot), \Lambda(\cdot))$ should solve the Skorohod problem for $\psi(\cdot) = x + \int_0^{\cdot} b(s, X(s))\, ds + \int_0^{\cdot} \langle \sigma(s, X(s)), dW(s)\rangle$ on an a.s. pathwise basis. We assume that the coefficient functions $b(t, x)$ and $\sigma(t, x)$ satisfy the Lipschitz continuity condition

$$|b_i(t, x) - b_i(t, y)| \le K|x - y| \quad \text{and} \quad \left|\sigma_{i,j}(t, x) - \sigma_{i,j}(t, y)\right| \le K|x - y|, \tag{2.17}$$

for all $(i, j) \in \{1, \dots n\} \times \{1, \dots, m\}$, $x, y \in \mathbb{R}^n$ and for some positive constant $K \in (0, \infty)$. Our main result for SDEs is the following theorem.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a time-dependent domain satisfying (2.2) and assume that (2.3), (2.4), (2.7) and (2.17) hold. Then there exists a unique strong solution to the SDE in $\overline{\Omega}$ driven by the Wiener process W and with coefficients b and σ, direction of reflection along γ and initial condition $x \in \overline{\Omega}_0$.*

We prove Theorem 2.7 by completing the following steps. First, in Lemma 4.3, we use a penalty method to prove existence of solutions to the Skorohod problem for smooth functions. In Lemma 4.4, we then derive a compactness estimate for solutions to the Skorohod problem. Based on the compactness estimate, we are, in Lemma 4.5, able to generalize the existence result for the Skorohod problem to all continuous functions. Finally, in Section 5, we use two classes of test functions and the existence result for the Skorohod problem to obtain existence and uniqueness of strong solutions to SDEs with oblique reflection at the boundary of a bounded, time-dependent domain. Note that we are able to obtain uniqueness of the reflected SDE although the solution to the corresponding Skorohod problem need not be unique.

2.3. *Statement of main results for PDEs*

To state and prove our results for PDEs we introduce some more notation. Let Ω' be as in (2.1) and put

$$\Omega^\circ = \Omega' \cap \big((0, T) \times \mathbb{R}^n\big), \quad \widetilde{\Omega} = \overline{\Omega}' \cap \big([0, T) \times \mathbb{R}^n\big),$$
$$\partial\Omega = \left(\overline{\Omega}' \setminus \Omega'\right) \cap \big((0, T) \times \mathbb{R}^n\big).$$

We consider fully nonlinear parabolic PDEs of the form

$$u_t + F\left(t, x, u, Du, D^2u\right) = 0 \quad \text{in } \Omega^\circ. \tag{2.18}$$

Here F is a given real function on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, where $\mathbb{S}^n$ denotes the space of $n \times n$ real symmetric matrices equipped with the positive semi-definite ordering; that is, for $X, Y \in \mathbb{S}^n$, we write $X \le Y$ if $\langle (X - Y)\xi, \xi\rangle \le 0$ for all $\xi \in \mathbb{R}^n$. We also adopt the matrix norm notation

$$\|A\| = \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} = \sup\{|\langle A\xi, \xi\rangle| : |\xi| \le 1\}.$$

Moreover, u represents a real function in Ω° and Du and D^2u denote the gradient and Hessian matrix, respectively, of u with respect to the spatial variables. On the boundary we impose the oblique derivative condition to the unknown u

$$\frac{\partial u}{\partial \widetilde{\gamma}} + f(t, x, u(t, x)) = 0 \quad \text{on } \partial\Omega, \tag{2.19}$$

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where f is a real valued function on $\overline{\partial\Omega} \times \mathbb{R}$ and $\widetilde{\gamma}(t, \cdot)$ is the vector field on $\mathbb{R}^n$, oblique to $\partial\Omega_t$, introduced in (2.5) and (2.6).

Regarding the function F, we make the following assumptions.

$$F \in C\left(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n\right). \tag{2.20}$$

For some $\lambda \in \mathbb{R}$ and each $(t, x, p, A) \in \overline{\Omega} \times \mathbb{R}^n \times \mathbb{S}^n$ the function

$$r \to F(t, x, r, p, A) - \lambda r \text{ is nondecreasing on } \mathbb{R}. \tag{2.21}$$

There is a function $m_1 \in C([0, \infty))$ satisfying $m_1(0) = 0$ for which

$$\begin{aligned} &F(t, y, r, p, -Y) - F(t, x, r, p, X) \le m_1\left(|x - y|(|p| + 1) + \alpha|x - y|^2\right) \\ &\text{if} \qquad -\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \end{aligned} \tag{2.22}$$

for all $\alpha \ge 1$, $(t, x), (t, y) \in \overline{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $X, Y \in \mathbb{S}^n$, where I denotes the unit matrix of size $n \times n$. There is a neighbourhood U of $\partial\Omega$ in $\overline{\Omega}$ and a function $m_2 \in C([0, \infty))$ satisfying $m_2(0) = 0$ for which

$$|F(t, x, r, p, X) - F(t, x, r, q, Y)| \le m_2(|p - q| + \|X - Y\|), \tag{2.23}$$

for $(t, x) \in U$, $r \in \mathbb{R}$, $p, q \in \mathbb{R}^n$ and $X, Y \in \mathbb{S}^n$. Regarding the function f we assume that

$$f(t, x, r) \in C\left(\overline{\partial\Omega} \times \mathbb{R}\right), \tag{2.24}$$

and that for each $(t, x) \in \overline{\partial\Omega}$ the function

$$r \to f(t, x, r) \text{ is nondecreasing on } \mathbb{R}. \tag{2.25}$$

We remark that assumptions (2.20) and (2.22) imply the degenerate ellipticity

$$F(t, x, r, p, A + B) \le F(t, x, r, p, A) \quad \text{if } B \ge 0, \tag{2.26}$$

for $(t, x) \in \overline{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $A, B \in \mathbb{S}^n$, see Remark 3.4 in [10] for a proof. To handle the strong degeneracy allowed, we will adapt the notion of viscosity solutions [10], which we recall for problem (2.18)–(2.19) in Section 6. Let $USC(E)$ ($LSC(E)$) denote the set of upper (lower) semi-continuous functions on $E \subset \mathbb{R}^{n+1}$. Our main results for PDEs are given in the following theorems.

Theorem 2.8. *Let Ω° be a time-dependent domain satisfying* (2.2) *and assume that* (2.5)–(2.7) *and* (2.20)–(2.25) *hold. Let $u \in USC(\widetilde{\Omega})$ be a viscosity subsolution, and $v \in LSC(\widetilde{\Omega})$ be a viscosity supersolution of problem* (2.18)–(2.19) *in Ω°. If $u(0, x) \le v(0, x)$ for all $x \in \overline{\Omega}_0$, then $u \le v$ in $\widetilde{\Omega}$.*

Theorem 2.9. *Let Ω° be a time-dependent domain satisfying* (2.2) *and assume that* (2.5)–(2.7) *and* (2.20)–(2.25) *hold. Then there exists a unique viscosity solution, continuous on $\widetilde{\Omega}$, to the initial value problem*

$$\begin{aligned} u_t + F\left(t, x, u, Du, D^2u\right) &= 0 && \text{in} \quad \Omega^\circ, \\ \frac{\partial u}{\partial \widetilde{\gamma}} + f(t, x, u(t, x)) &= 0 && \text{on} \quad \partial\Omega, \\ u(0, x) &= g(x) && \text{for} \quad x \in \overline{\Omega}_0, \end{aligned} \tag{2.27}$$

where $g \in C\left(\overline{\Omega}_0\right)$.

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Theorems 2.8 and 2.9 are proved in Section 6 . The comparison principle in Theorem 2.8 is obtained using two of the test functions constructed in Section 3 together with nowadays standard techniques from the theory of viscosity solutions for fully nonlinear PDEs as described in [10]. Our proof uses ideas from the corresponding elliptic result given in [11]. The uniqueness part of Theorem 2.9 is immediate from the formulation of Theorem 2.8, which also, together with the maximum principle in Lemma 6.2, allows comparison in the setting of mixed boundary conditions, as follows.

Corollary 2.10. *Let Ω° be a time-dependent domain satisfying (2.2) and assume that (2.5)–(2.7) and (2.20)–(2.25) hold. Let $u \in USC(\widetilde{\Omega})$ be a viscosity subsolution, and $v \in LSC(\widetilde{\Omega})$ be a viscosity supersolution of (2.18) in Ω°. Suppose also that u and v satisfy the oblique derivative boundary condition (2.19) on a subset $G \subset \partial\Omega$. Then $\sup_{\widetilde{\Omega}} u - v \leq \sup_{(\partial\Omega\setminus G)\cup\overline{\Omega}_0}(u-v)^+$.*

The existence part of Theorem 2.9 is proved using Perron's method and Corollary 2.10, together with constructions of several explicit viscosity sub- and supersolutions to the problem (2.18)–(2.19).

3. Construction of test functions

In this section we show how the classes of test functions constructed in [11] for time-independent domains can be generalized to similar classes of test functions valid for time-dependent domains. Lemmas 3.1 and 3.2 provide test functions that are modifications of the square function, but which interact with the direction of γ in a suitable way. The derivations of these functions follow the lines of the derivations of the corresponding test functions in [11] with the addition that it has to be verified that the time derivative of the test functions has a certain order. Lemma 3.3 provides a non-negative test function in $\mathcal{C}^{1,2}\left(\overline{\Omega}, \mathbb{R}\right)$, whose gradient is aligned with γ at the boundary. To verify the existence of this function, the proof for the corresponding function in [11] has to be extended considerably due to the time-dependence of the domain. In particular, new methods have to be used to obtain differentiability with respect to the time variable.

The constructions of the test functions below are given with sufficient detail and for those parts of the constructions that are identical in time-dependent and time-independent domains, we refer the reader to [11]. We start by stating a straightforward extension of Lemma 4.4 in [11] from $\xi \in S(0,1)$ to $\xi \in B(0,1)$. The proof follows directly from the construction in Lemma 4.4 in [11] and is omitted. For any $\theta \in (0,1)$, there exist a function $g \in \mathcal{C}\left(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}\right)$ and positive constants χ, C such that

$$g \in \mathcal{C}^1\left(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}\right) \cap \mathcal{C}^2\left(\mathbb{R}^n \times \left(\mathbb{R}^n \setminus \{0\}\right), \mathbb{R}\right), \tag{3.1}$$

$$g(\xi, p) \geq \chi |p|^2, \quad \text{for } \xi \in B(0,1),\ p \in \mathbb{R}^n, \tag{3.2}$$

$$g(\xi, 0) = 0, \quad \text{for } \xi \in \mathbb{R}^n, \tag{3.3}$$

$$\left\langle D_p g(\xi, p), \xi \right\rangle \geq 0, \quad \text{for } \xi \in S(0,1),\ p \in \mathbb{R}^n \text{ and } \langle p, \xi \rangle \geq -\theta |p|, \tag{3.4}$$

$$\left\langle D_p g(\xi, p), \xi \right\rangle \leq 0, \quad \text{for } \xi \in S(0,1),\ p \in \mathbb{R}^n \text{ and } \langle p, \xi \rangle \leq \theta |p|, \tag{3.5}$$

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$$\left|D_\xi g\left(\xi,p\right)\right| \le C|p|^2, \quad \left|D_p g\left(\xi,p\right)\right| \le C\left|p\right|, \quad \text{for } \xi \in B\left(0,1\right), p \in \mathbb{R}^n, \tag{3.6}$$

and

$$\left\|D_\xi^2 g\left(\xi,p\right)\right\| \le C|p|^2, \quad \left\|D_\xi D_p g\left(\xi,p\right)\right\| \le C\left|p\right|, \quad \left\|D_p^2 g\left(\xi,p\right)\right\| \le C, \tag{3.7}$$

for $\xi \in B\left(0,1\right)$, $p \in \mathbb{R}^n \setminus \{0\}$. The test function provided by the following lemma will be used to assert relative compactness of solutions to the Skorohod problem in Lemma 4.4.

Lemma 3.1. *For any $\theta \in (0,1)$, there exist a function $h \in \mathcal{C}^{1,2}\left([0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}\right)$ and positive constants χ, C such that, for all $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$,*

$$h\left(t,x,p\right) \ge \chi|p|^2, \tag{3.8}$$

$$h\left(t,x,0\right) = 1, \tag{3.9}$$

$$\left\langle D_p h\left(t,x,p\right), \gamma\left(t,x\right)\right\rangle \ge 0, \quad \text{for } x \in \partial\Omega_t \text{ and } \left\langle p, \gamma\left(t,x\right)\right\rangle \ge -\theta\left|p\right|, \tag{3.10}$$

$$\left\langle D_p h\left(t,x,p\right), \gamma\left(t,x\right)\right\rangle \le 0, \quad \text{for } x \in \partial\Omega_t \text{ and } \left\langle p, \gamma\left(t,x\right)\right\rangle \le \theta\left|p\right|, \tag{3.11}$$

$$\left|D_t h\left(t,x,p\right)\right| \le C|p|^2, \quad \left|D_x h\left(t,x,p\right)\right| \le C|p|^2, \left|D_p h\left(t,x,p\right)\right| \le C\left|p\right|, \tag{3.12}$$

and

$$\left\|D_x^2 h\left(t,x,p\right)\right\| \le C|p|^2, \quad \left\|D_x D_p h\left(t,x,p\right)\right\| \le C\left|p\right|, \quad \left\|D_p^2 h\left(t,x,p\right)\right\| \le C. \tag{3.13}$$

Proof. Let $\nu \in \mathcal{C}^2\left(\mathbb{R},\mathbb{R}\right)$ be such that $\nu\left(t\right) = t$ for $t \ge 2$, $\nu\left(t\right) = 1$ for $t \le 1/2$, $\nu'\left(t\right) \ge 0$ and $\nu\left(t\right) \ge t$ for all $t \in \mathbb{R}$. Let $\theta \in (0,1)$ be given, choose $g \in \mathcal{C}\left(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}\right)$ satisfying (3.1)–(3.7) and define

$$h\left(t,x,p\right) = \nu\left(g\left(\gamma\left(t,x\right),p\right)\right).$$

The regularity of h follows easily from the regularity of g and ν and (3.3). It is straightforward to deduce properties (3.8)–(3.13) from (3.1)–(3.7) and we limit the proof to two examples, which are not fully covered in [11]. We have

$$\left|D_t h\left(t,x,p\right)\right| = \left|\nu'\left(g\left(\gamma\left(t,x\right),p\right)\right)\right| \left|D_\xi g\left(\gamma\left(t,x\right),p\right)\right| \left|\frac{\partial \gamma}{\partial t}\right| \le C|p|^2,$$

by (3.6) and the regularity of ν and γ. Moreover,

$$\begin{aligned}\left\|D_x^2 h\left(t,x,p\right)\right\| \le C(n)\Bigg(&\left|\nu''\left(g\left(\gamma\left(t,x\right),p\right)\right)\right| \left|D_\xi g\left(\gamma\left(t,x\right),p\right)\right|^2 \left\|\frac{\partial\gamma}{\partial x}\right\|^2 \\ &+ \left|\nu'\left(g\left(\gamma\left(t,x\right),p\right)\right)\right| \left\|D_\xi^2 g\left(\gamma\left(t,x\right),p\right)\right\| \left\|\frac{\partial\gamma}{\partial x}\right\|^2 \\ &+ \left|\nu'\left(g\left(\gamma\left(t,x\right),p\right)\right)\right| \left|D_\xi g\left(\gamma\left(t,x\right),p\right)\right| \max_{1\le k\le n}\left\|\frac{\partial^2\gamma_k}{\partial x^2}\right\|\Bigg).\end{aligned}$$

Since ν'' is zero unless $2 \ge g\left(\gamma\left(t,x\right),p\right) \ge \chi|p|^2$, the first term, which is of order $C|p|^4$, only contributes for small $|p|^2$ and can thus be bounded from above by $C|p|^2$. By (3.6)–(3.7), the two latter terms are also bounded from above by $C|p|^2$. □

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The test function in Lemma 3.1 is also used to verify the existence of the following test function, which will be useful in the proofs of Theorem 2.8 and Lemma 5.1.

Lemma 3.2. *For any $\theta \in (0,1)$, there exist a family $\{w_\varepsilon\}_{\varepsilon>0}$ of functions $w_\varepsilon \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ and positive constants χ, C (independent of ε) such that, for all $(t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$,*

$$w_\varepsilon(t,x,y) \geq \chi \frac{|x-y|^2}{\varepsilon}, \tag{3.14}$$

$$w_\varepsilon(t,x,y) \leq C\left(\varepsilon + \frac{|x-y|^2}{\varepsilon}\right), \tag{3.15}$$

$$\begin{aligned} &\langle D_x w_\varepsilon(t,x,y), \gamma(t,x)\rangle \leq C\frac{|x-y|^2}{\varepsilon}, \\ &\quad \text{for } x \in \partial\Omega_t,\ \langle y-x, \gamma(t,x)\rangle \geq -\theta|x-y|, \end{aligned} \tag{3.16}$$

$$\left\langle D_y w_\varepsilon(t,x,y), \gamma(t,x)\right\rangle \leq 0, \quad \text{for } x \in \partial\Omega_t,\ \langle x-y, \gamma(t,x)\rangle \geq -\theta|x-y|, \tag{3.17}$$

$$\begin{aligned} &\left\langle D_y w_\varepsilon(t,x,y), \gamma(t,y)\right\rangle \leq C\frac{|x-y|^2}{\varepsilon}, \\ &\quad \text{for } y \in \partial\Omega_t,\ \langle x-y, \gamma(t,y)\rangle \geq -\theta|x-y|, \end{aligned} \tag{3.18}$$

$$|D_t w_\varepsilon(t,x,y)| \leq C\frac{|x-y|^2}{\varepsilon}, \tag{3.19}$$

$$\left|D_y w_\varepsilon(t,x,y)\right| \leq C\frac{|x-y|}{\varepsilon}, \quad \left|D_x w_\varepsilon(t,x,y) + D_y w_\varepsilon(t,x,y)\right| \leq C\frac{|x-y|^2}{\varepsilon}, \tag{3.20}$$

and

$$D^2 w_\varepsilon(t,x,y) \leq \frac{C}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \frac{C|x-y|^2}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{3.21}$$

Proof. Let $\theta \in (0,1)$ be given and choose $h \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ as in Lemma 3.1. For all $\varepsilon > 0$, we define the function w_ε as

$$w_\varepsilon(t,x,y) = \varepsilon h\left(t, x, \frac{x-y}{\varepsilon}\right).$$

Property (3.14) follows easily from (3.8) and property (3.15) was verified in Remark 3.3 in [14]. Moreover, properties (3.16), (3.17), (3.20) and (3.21) were verified in the proof of Theorem 4.1 in [11] and (3.19) is a simple consequence of (3.12). To prove (3.18), we note that

$$\begin{aligned} \left\langle D_y w_\varepsilon(t,x,y), \gamma(t,y)\right\rangle &= -\left\langle D_p h\left(t, x, \frac{x-y}{\varepsilon}\right), \gamma(t,y)\right\rangle \\ &= -\left\langle D_p h\left(t, y, \frac{x-y}{\varepsilon}\right), \gamma(t,y)\right\rangle \\ &\quad + \left\langle D_p h\left(t, y, \frac{x-y}{\varepsilon}\right) - D_p h\left(t, x, \frac{x-y}{\varepsilon}\right), \gamma(t,y)\right\rangle. \end{aligned}$$

Moreover, if $\langle x-y,\gamma(t,y)\rangle \geq -\theta|x-y|$, then by (3.10), $\left\langle D_p h(t,y,p),\gamma(t,y)\right\rangle \geq 0$ with $p=(x-y)/\varepsilon$. Hence, for some ξ in the segment joining x and y, we obtain, with the aid of the mean value theorem and (3.13),

$$\begin{aligned}\left\langle D_y w_\varepsilon(t,x,y),\gamma(t,y)\right\rangle &\leq \left\| D_x D_p h\left(t,\xi,\frac{x-y}{\varepsilon}\right)\right\| |x-y| \\ &\leq C\left|\frac{x-y}{\varepsilon}\right| |x-y| = C\frac{|x-y|^2}{\varepsilon}. \quad \square\end{aligned}$$

We conclude this section by proving Lemma 3.3 using an appropriate Cauchy problem. The test function α in Lemma 3.3 will be crucial for the proofs of Theorems 2.8, 2.9 and Lemma 5.1.

Lemma 3.3. *There exists a nonnegative function $\alpha \in \mathcal{C}^{1,2}\left(\overline{\Omega},\mathbb{R}\right)$, which satisfies*

$$\langle D_x\alpha(t,x),\gamma(t,x)\rangle \geq 1, \tag{3.22}$$

for $x\in\partial\Omega_t$, $t\in[0,T]$. Moreover, the support of α can be assumed to lie in the neighbourhood U defined in (2.23).

Proof. Fix $s\in[0,T]$ and $z\in\partial\Omega_s$ and define $H_{s,z}$ as the hyperplane

$$H_{s,z}=\left\{x\in\mathbb{R}^n : \langle x-z,\gamma(s,z)\rangle = 0\right\}.$$

Given a function $u_0\in\mathcal{C}^2\left(H_{s,z},\mathbb{R}\right)$, such that $u_0(z)=1$, $u_0\geq 0$ and $\operatorname{supp} u_0 \subset B\left(z,\delta^2/4\right)\cap H_{s,z}$, we can use the method of characteristics to solve the Cauchy problem

$$\begin{aligned}\left\langle D_x u_{(t)}(x),\gamma(t,x)\right\rangle &= 0,\\ u_{(t)}|_{H_{s,z}} &= u_0.\end{aligned}$$

Choosing the positive constants δ and η sufficiently small, the Cauchy problem above has, for all $t\in[s-\eta,s+\eta]$, a solution $u_{(t)}\in\mathcal{C}^2(B(z,\delta),\mathbb{R})$ satisfying $u_{(t)}\geq 0$. Based on the continuity of γ and the restriction on the support of u_0, we may also assume that

$$\operatorname{supp} u_{(t)} \subset \bigcup_{\zeta\in\mathbb{R}} B(z-\zeta\gamma(s,z),\delta^2/3)\cap B(z,\delta).$$

Next, we define the combined function

$$u(t,x)=u_{(t)}(x),$$

and we claim for now that $u\in\mathcal{C}^{1,2}([s-\eta,s+\eta]\times B(z,\delta),\mathbb{R})$ and postpone the proof of this claim to the end of the proof of the lemma. By the exterior and interior cone conditions, we can, for sufficiently small δ, find $\varepsilon>0$ such that

$$\begin{aligned}&\bigcup_{\zeta>0} B(z-\zeta\gamma(s,z),\delta^2/3)\cap(B(z,\delta)\setminus B(z,\delta-2\varepsilon))\\ &\subset \bigcup_{\zeta>0} B(z-\zeta\gamma(s,z),\zeta\delta)\cap B(z,\delta)\subset\Omega_s^c,\end{aligned}$$

and such that the similar union over $\zeta<0$ belongs to Ω_s. Hence

$$\partial\Omega_s\cap\left(\operatorname{supp} u_{(t)}\setminus B(z,\delta-2\varepsilon)\right)=\emptyset,$$

and, by (2.8), it follows that if η also satisfies the constraint $\eta < (\varepsilon/K)^{1/\widehat{\alpha}}$, then

$$\partial\Omega_t \cap \left(\operatorname{supp} u_{(t)} \setminus B(z, \delta - \varepsilon)\right) = \emptyset, \tag{3.23}$$

for all $t \in [s-\eta, s+\eta]$.

Now, choose a function $\xi \in \mathcal{C}_0^{1,2}([s-\eta, s+\eta] \times B(z,\delta), \mathbb{R})$ so that $\xi(t,x) = 1$ for $t \in [s-\eta+\varepsilon, s+\eta-\varepsilon]$, $x \in B(z, \delta-\varepsilon)$ and $\xi \geq 0$, and set

$$v_{s,z}(t,x) = u(t,x)\,\xi(t,x).$$

Then $v_{s,z} \in \mathcal{C}_0^{1,2}([s-\eta, s+\eta] \times B(z,\delta), \mathbb{R})$ satisfies $v_{s,z} \geq 0$. By (3.23) and the construction of u and ξ, we obtain

$$\left\langle D_x v_{s,z}(t,x), \gamma(t,x)\right\rangle = 0 \text{ for } x \in B(z,\delta) \cap \partial\Omega_t,\, t \in [s-\eta, s+\eta].$$

Define $w_{s,z} \in \mathcal{C}^2(B(z,\delta), \mathbb{R})$ by

$$w_{s,z}(x) = \langle x - z, \gamma(s,z)\rangle + M,$$

where M is large enough so that $w_{s,z} \geq 0$. Using the continuity of γ, we can find δ and η such that $\langle \gamma(s,z), \gamma(t,x)\rangle \geq 0$ for all $(t,x) \in [s-\eta, s+\eta] \times B(z,\delta)$. Setting

$$g_{s,z}(t,x) = v_{s,z}(t,x)\, w_{s,z}(x),$$

we find that $g_{s,z} \in \mathcal{C}_0^{1,2}([s-\eta, s+\eta] \times B(z,\delta), \mathbb{R})$ satisfies $g_{s,z} \geq 0$. Moreover, using $|\gamma(t,x)| = 1$, we have

$$\begin{aligned}\left\langle D_x g_{s,z}(s,z), \gamma(s,z)\right\rangle &= v_{s,z}(s,z)\left\langle D_x w_{s,z}(z), \gamma(s,z)\right\rangle \\ &\quad + w_{s,z}(z)\left\langle D_x v_{s,z}(s,z), \gamma(s,z)\right\rangle \\ &= u(s,z)\,\xi(s,z)\,|\gamma(s,z)|^2 = 1,\end{aligned}$$

and a similar calculation shows that

$$\begin{aligned}\left\langle D_x g_{s,z}(t,x), \gamma(t,x)\right\rangle &= v_{s,z}(t,x)\left\langle D_x w_{s,z}(x), \gamma(t,x)\right\rangle \\ &\quad + w_{s,z}(x)\left\langle D_x v_{s,z}(t,x), \gamma(t,x)\right\rangle \\ &= v_{s,z}(t,x)\,\langle \gamma(s,z), \gamma(t,x)\rangle \geq 0,\end{aligned}$$

for $x \in B(z,\delta) \cap \partial\Omega_t$, $t \in [s-\eta, s+\eta]$. Now, using a standard compactness argument we conclude the existence of a nonnegative function $\alpha \in \mathcal{C}^{1,2}(\overline{\Omega}, \mathbb{R})$, which satisfies $\langle D_x\alpha(t,x), \gamma(t,x)\rangle \geq 1$ for $x \in \partial\Omega_t$, $t \in [0,T]$. Moreover, by the above construction, we can assume that the support of α lies within the neighbourhood U defined in (2.23).

It remains to prove the proposed regularity $u \in \mathcal{C}^{1,2}([s-\eta, s+\eta] \times B(z,\delta), \mathbb{R})$. The regularity in the spatial variables follows directly by construction, so it remains to show that u is continuously differentiable in the time variable. Let $x \in B(z,\delta)$ and let t and $t+h$ belong to $[s-\eta, s+\eta]$. Denote by $y(t,\cdot)$ and $y(t+h,\cdot)$ the characteristic curves through x for the vector fields $\gamma(t,\cdot)$ and $\gamma(t+h,\cdot)$, respectively, so that

$$\begin{aligned}\frac{\partial y}{\partial r}(t,r) &= \pm\gamma(t, y(t,r)), \\ y(t,0) &= x,\end{aligned}$$

and analogously for $y(t+h,\cdot)$. Choose the sign in the parametrization of $y(t,\cdot)$ so that there exists some $r(t) > 0$ such that $y(t, r(t)) = z(t) \in H_{s,z}$. Choosing the same sign in the parametrization of $y(t+h,\cdot)$ asserts the existence of some $r(t+h) > 0$ such that

$y(t+h, r(t+h)) = z(t+h) \in H_{s,z}$. Without lack of generality, we assume the sign above to be positive. Since $u(t,x) = u_0(z(t))$, where u_0 is continuously differentiable, it remains to show that the function z is continuously differentiable.

We will first show that $y(\cdot, r)$ is continuously differentiable by following an argument that can be found in e.g. [21]. Differentiating the Cauchy problem formally with respect to the time variable and introducing the function $\psi(t,r) = \frac{\partial y}{\partial t}(t,r)$, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial r}\psi(t,r) = \frac{\partial \gamma}{\partial t}(t, y(t,r)) + \frac{\partial \gamma}{\partial y}(t, y(t,r))\,\psi(t,r),\\
&\quad \psi(t,0) = 0.
\end{aligned}$$

This Cauchy problem has a unique solution, which we will next show satisfies

$$\psi(t,r) = \lim_{h\to 0}\frac{y(t+h,r) - y(t,r)}{h}, \tag{3.24}$$

so that ψ is in fact the time derivative of y (not just formally). Define

$$R(t,r,h) = \frac{y(t+h,r) - y(t,r)}{h} - \psi(t,r). \tag{3.25}$$

Now

$$\begin{aligned}
R(t,r,h) = &\int_0^r \left(\frac{\gamma(t+h, y(t+h,u)) - \gamma(t, y(t,u))}{h}\right) du\\
&- \int_0^r \left(\frac{\partial \gamma}{\partial t}(t, y(t,u)) + \frac{\partial \gamma}{\partial y}(t, y(t,u))\,\psi(t,u)\right) du.
\end{aligned}$$

By the mean value theorem

$$\begin{aligned}
&\gamma_i(t+h, y(t+h,u)) - \gamma_i(t, y(t,u))\\
&= \frac{\partial \gamma_i}{\partial t}\left(\bar{t}_i, y(t+h,u)\right) h + \frac{\partial \gamma_i}{\partial y}\left(t, \bar{y}_i\right)\left(y(t+h,u) - y(t,u)\right),
\end{aligned}$$

for some $\bar{t}_i$ between t and $t+h$, some $\bar{y}_i$ between $y(t,u)$ and $y(t+h,u)$ and all $i \in \{1,\dots,n\}$. Hence, the ith component of $R(t,r,h)$ is

$$\begin{aligned}
R_i(t,r,h) = &\int_0^r \left(\frac{\partial \gamma_i}{\partial t}\left(\bar{t}_i, y(t+h,u)\right) - \frac{\partial \gamma_i}{\partial t}(t, y(t,u))\right) du\\
&+ \int_0^r \left(\frac{\partial \gamma_i}{\partial y}\left(t, \bar{y}_i\right)\frac{y(t+h,u) - y(t,u)}{h} - \frac{\partial \gamma_i}{\partial y}(t, y(t,u))\,\psi(t,u)\right) du,
\end{aligned}$$

where the second term on the right hand side can be rewritten as

$$\int_0^r \left(\frac{\partial \gamma_i}{\partial y}\left(t, \bar{y}_i\right) R(t,u,h) + \left(\frac{\partial \gamma_i}{\partial y}\left(t, \bar{y}_i\right) - \frac{\partial \gamma_i}{\partial y}(t, y(t,u))\right)\psi(t,u)\right) du.$$

Therefore we have

$$\begin{aligned}
|R(t,r,h)| \le &\int_0^r \sum_{i=1}^n \left|\frac{\partial \gamma_i}{\partial t}\left(\bar{t}_i, y(t+h,u)\right) - \frac{\partial \gamma_i}{\partial t}(t, y(t,u))\right| du\\
&+ \int_0^r |R(t,u,h)| \sum_{i=1}^n \left|\frac{\partial \gamma_i}{\partial y}\left(t, \bar{y}_i\right)\right| du\\
&+ \int_0^r |\psi(t,u)| \sum_{i=1}^n \left|\frac{\partial \gamma_i}{\partial y}\left(t, \bar{y}_i\right) - \frac{\partial \gamma_i}{\partial y}(t, y(t,u))\right| du,
\end{aligned}$$

and by Gronwall's inequality we obtain

$$|R(t,r,h)| \le C \int_0^r \sum_{i=1}^n \left| \frac{\partial \gamma_i}{\partial t}\left(\overline{t}_i, y(t+h,u)\right) - \frac{\partial \gamma_i}{\partial t}(t, y(t,u)) \right| du$$
$$+C \int_0^r |\psi(t,u)| \sum_{i=1}^n \left| \frac{\partial \gamma_i}{\partial y}\left(t, \overline{y}_i\right) - \frac{\partial \gamma_i}{\partial y}(t, y(t,u)) \right| du, \tag{3.26}$$

for some positive constant C. Since $|\psi(t,u)|$ exists and is bounded, and since the time and space derivatives of γ are continuous, (3.26) implies boundedness of $|R(t,r,h)|$. Therefore, by (3.25) we have $|y(t+h,r) - y(t,r)| \le Ch$, for some constant C, and we can conclude that $\overline{y}_i \to y(t,u)$ and $\overline{t}_i \to t$ for all $i \in \{1,\dots,n\}$ as $h \to 0$. It follows that the differences in the integrands in (3.26) vanish as $h \to 0$ and hence $\lim_{h\to 0} R(t,r,h) = 0$. This proves (3.24) and therefore that $y(\cdot, r)$ is continuously differentiable.

Now, by the mean value theorem,

$$\begin{aligned} z_i(t+h) - z_i(t) &= y_i(t+h, r(t+h)) - y_i(t, r(t)) \\ &= y_i(t+h, r(t+h)) - y_i(t+h, r(t)) \\ &\quad + y_i(t+h, r(t)) - y_i(t, r(t)) \\ &= \frac{\partial y_i}{\partial r}(t+h, \overline{r}_i)(r(t+h) - r(t)) + \frac{\partial y_i}{\partial t}\left(\overline{t}_i, r(t)\right) h, \end{aligned}$$

for some $\overline{r}_i$ between $r(t)$ and $r(t+h)$, some $\overline{t}_i$ between t and $t+h$ and all $i \in \{1,\dots,n\}$. Since the function $r(t)$ is defined so that

$$\langle y(t, r(t)) - z, \gamma(s,z) \rangle = 0, \quad t \in (s-\eta, s+\eta),$$

it follows by the implicit function theorem and by the regularity of $y(t,r)$ that $r(t)$ is a continuously differentiable function. Hence, we conclude that $\overline{r}_i \to r(t)$ and $\overline{t}_i \to t$, all $i \in \{1,\dots,n\}$, as $h \to 0$ and therefore,

$$\lim_{h\to 0} \frac{z(t+h) - z(t)}{h} = \frac{\partial y}{\partial r}(t, r(t))\, r'(t) + \frac{\partial y}{\partial t}(t, r(t)),$$

where the right hand side is a continuous function. This proves that z is continuously differentiable and, hence, that $u \in \mathcal{C}^{1,2}([s-\eta, s+\eta] \times B(z,\delta), \mathbb{R})$. □

4. The Skorohod problem

In this section we prove existence of solutions to the Skorohod problem under the assumptions in Section 2.1. This result could be achieved using the methods in [23], but as we here assume more regularity on the direction of reflection and the temporal variation of the domain compared to the setting in [23] (and this is essential for the other sections of this article), we follow a more direct approach using a penalty method. We first note that, mimicking the proof of Lemma 4.1 in [14], we can prove the following result.

Lemma 4.1. *There is a constant $\mu > 0$ such that, for every $t \in [0,T]$, there exists a neighbourhood U_t of $\partial\Omega_t$ such that*

$$\langle D_x d(t,x), \gamma(t,x) \rangle \le -\mu, \quad \text{for a.e. } x \in U_t \setminus \overline{\Omega}_t. \tag{4.1}$$

As (4.1) holds only for almost every point in a neighbourhood of a non-smooth domain, we cannot apply (4.1) directly and will use the following mollifier approach instead. Based on the construction of the neighbourhoods $\{U_t\}_{t\in[0,T]}$ in Lemma 4.1 (see the proof of the corresponding lemma in [14] for details), there exists a constant $\overline{\beta} > 0$ such that $B\left(x, 3\overline{\beta}\right) \subset U_t$ for all $x \in \partial\Omega_t$, $t \in [0, T]$. For the value of p given in (2.7), let

$$v(t, x) = (d(t, x))^p \quad \text{and } \widetilde{v}(t, x) = (d(t, x))^{p-1}.$$

Moreover, let $\varphi_\beta \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ be a positive mollifier with support in $B(0, \beta)$, for some $\beta > 0$, and define the spatial convolutions

$$v_\beta = v * \varphi_\beta \quad \text{and } \widetilde{v}_\beta = \widetilde{v} * \varphi_\beta.$$

Lemma 4.2. *There is a constant $\kappa > 0$ such that, for sufficiently small $\beta > 0$ and every $t \in [0, T]$, there exists a neighbourhood $\widetilde{U}_t$ of $\partial\Omega_t$, $\widetilde{U}_t \supset \left\{x : d(x, \partial\Omega_t) < 2\overline{\beta}\right\}$, such that*

$$\left\langle D_x v_\beta(t, x), \gamma(t, x)\right\rangle \leq -\kappa \widetilde{v}_\beta(t, x), \quad \textit{for } x \in \widetilde{U}_t \setminus \overline{\Omega}_t. \tag{4.2}$$

Proof. For all $x \in U_t \setminus \overline{\Omega}_t$ such that $B\left(x, \overline{\beta}\right) \subset U_t$ and for all $\beta \leq \overline{\beta}$, we have

$$\left\langle D_x v_\beta(t, x), \gamma(t, x)\right\rangle = \int_{\mathbb{R}^n} \left\langle \varphi_\beta(x-y) D_y v(t, y), \gamma(t, x)\right\rangle dy$$
$$= \int_{\mathbb{R}^n} \left(\left\langle D_y v(t, y), \gamma(t, y)\right\rangle + \left\langle D_y v(t, y), \gamma(t, x) - \gamma(t, y)\right\rangle\right) \varphi_\beta(x-y)\, dy.$$

The inner product in the second term is bounded from above by

$$p(d(t, y))^{p-1} \left|D_y d(t, y)\right| L\beta,$$

where L is the Lipschitz coefficient of γ in spatial dimensions over the compact set $[0, T] \times \bigcup_{t\in[0,T]} \overline{U}_t$. By Lemma 4.1, we have, for almost every $y \in U_t \setminus \overline{\Omega}_t$, $t \in [0, T]$,

$$\left\langle D_y v(t, y), \gamma(t, y)\right\rangle = p(d(t, y))^{p-1} \left\langle D_y d(t, y), \gamma(t, y)\right\rangle \leq -p\mu(d(t, y))^{p-1},$$

and, for sufficiently small $\beta > 0$,

$$p(d(t, y))^{p-1} L\beta - p\mu(d(t, y))^{p-1} \leq -\kappa(d(t, y))^{p-1},$$

for some constant $\kappa > 0$. This proves (4.2). □

We next use a penalty method to verify the existence of a solution to the Skorohod problem for continuously differentiable functions. The following lemma generalizes Theorem 2.1 in [20] and Lemma 4.5 in [14].

Lemma 4.3. *Let $\psi \in \mathcal{C}^1([0, T], \mathbb{R}^n)$ with $\psi(0) \in \overline{\Omega}_0$. Then there exists a solution $(\phi, \lambda) \in \mathcal{W}^{1,p}([0, T], \mathbb{R}^n) \times \mathcal{W}^{1,p}([0, T], \mathbb{R}^n)$ to the Skorohod problem for (Ω, γ, ψ).*

Proof. Choose $\varepsilon > 0$ and consider the ordinary differential equation

$$\phi_\varepsilon'(t) = \frac{1}{\varepsilon} d(t, \phi_\varepsilon(t))\, \gamma(t, \phi_\varepsilon(t)) + \psi'(t), \quad \phi_\varepsilon(0) = \psi(0), \tag{4.3}$$

for $\phi_\varepsilon(t)$, which has a unique solution on $[0, T]$. Let $\kappa > 0$ and the family of neighbourhoods $\{\widetilde{U}_t\}_{t\in[0,T]}$ be as in Lemma 4.2. Choose a function $\zeta \in \mathcal{C}^\infty([0, \infty), [0, \infty))$ such that

$$\zeta(r)=\begin{cases} r, & \text{for } r\le \overline{\beta}^p/2,\\ 3\overline{\beta}^p/4, & \text{for } r\ge \overline{\beta}^p,\end{cases}$$

and $0\le \zeta'(r)\le 1$ for all $r\in[0,\infty)$. Note that if $\phi_\varepsilon(t)\notin \widetilde{U}_t\cup\overline{\Omega}_t$, then $d(t,\phi_\varepsilon(t))\ge 2\overline{\beta}$ and, as a consequence, for all $\beta\le\overline{\beta}$ it holds that $v_\beta(t,\phi_\varepsilon(t))\ge\overline{\beta}^p$ and $\zeta'\left(v_\beta(t,\phi_\varepsilon(t))\right)=0$. We next define the function $F(t)=\zeta\left(v_\beta(t,\phi_\varepsilon(t))\right)$, for $t\in[0,T]$, and investigate its time derivative. Let $D_t d$ denote the weak derivative guaranteed by (2.7) and note that

$$\begin{aligned} F'(t) &= \zeta'\left(v_\beta(t,\phi_\varepsilon(t))\right)\left(D_t v_\beta(t,\phi_\varepsilon(t))+\left\langle D_x v_\beta(t,\phi_\varepsilon(t)),\phi_\varepsilon'(t)\right\rangle\right)\\ &= \zeta'\left(v_\beta(t,\phi_\varepsilon(t))\right)\Bigg(D_t v_\beta(t,\phi_\varepsilon(t))\\ &\qquad+\left\langle D_x v_\beta(t,\phi_\varepsilon(t)),\frac{1}{\varepsilon}d(t,\phi_\varepsilon(t))\gamma(t,\phi_\varepsilon(t))+\psi'(t)\right\rangle\Bigg), \end{aligned}\tag{4.4}$$

as $\phi_\varepsilon(t)$ solves (4.3). From Lemma 4.2, we have

$$\begin{aligned} &\zeta'\left(v_\beta(t,\phi_\varepsilon(t))\right)\left\langle D_x v_\beta(t,\phi_\varepsilon(t)),\frac{1}{\varepsilon}d(t,\phi_\varepsilon(t))\gamma(t,\phi_\varepsilon(t))\right\rangle\\ &\le -\zeta'\left(v_\beta(t,\phi_\varepsilon(t))\right)\frac{\kappa}{\varepsilon}d(t,\phi_\varepsilon(t))\widetilde{v}_\beta(t,\phi_\varepsilon(t)), \end{aligned}$$

for $\phi_\varepsilon(t)\in\widetilde{U}_t\setminus\overline{\Omega}_t$ and for all other $\phi_\varepsilon(t)$ both sides vanish when $\beta\le\overline{\beta}$. Integrating the estimate for F', suppressing the s-dependence in ϕ_ε and ψ and denoting $\zeta'\left(v_\beta(s,\phi_\varepsilon)\right)$ by $\zeta'\left(v_\beta\right)$ for simplicity, we obtain, for all $t\in[0,T]$,

$$\begin{aligned} &\zeta\left(v_\beta(t,\phi_\varepsilon(t))\right)-\zeta\left(v_\beta(0,\phi_\varepsilon(0))\right)+\frac{\kappa}{\varepsilon}\int_0^t\zeta'\left(v_\beta\right)d(s,\phi_\varepsilon)\widetilde{v}_\beta(s,\phi_\varepsilon)\,ds \\ &\le\int_0^t\zeta'\left(v_\beta\right)\left|D_s v_\beta(s,\phi_\varepsilon)\right|ds+\int_0^t\zeta'\left(v_\beta\right)\left|D_x v_\beta(s,\phi_\varepsilon)\right|\left|\psi'\right|ds=I_1+I_2. \end{aligned}\tag{4.5}$$

Note that since $|D_x d|\le 1$ a.e. we have $\left|D_x v_\beta(s,\phi_\varepsilon)\right|\le p\widetilde{v}_\beta(s,\phi_\varepsilon)$, and hence, Hölder's inequality implies

$$\begin{aligned} I_2 &= \int_0^t\zeta'\left(v_\beta\right)\left|D_x v_\beta(s,\phi_\varepsilon)\right|\left|\psi'\right|ds\le p\int_0^t\zeta'\left(v_\beta\right)\widetilde{v}_\beta(s,\phi_\varepsilon)\left|\psi'\right|ds\\ &\le p\left(\int_0^t\zeta'\left(v_\beta\right)\left|\psi'\right|^p ds\right)^{1/p}\left(\int_0^t\zeta'\left(v_\beta\right)\left(\widetilde{v}_\beta(s,\phi_\varepsilon)\right)^{p/(p-1)}ds\right)^{(p-1)/p}. \end{aligned}$$

Moreover, since $|D_s v_\beta|\le p\left(v_\beta(s,\phi_\varepsilon)\right)^{(p-1)/p}\left(|D_s d|^p*\varphi_\beta\right)^{1/p}$, we also have

$$\begin{aligned} I_1 &= \int_0^t\zeta'\left(v_\beta\right)\left|D_s v_\beta(s,\phi_\varepsilon)\right|ds\\ &\le p\int_0^t\zeta'\left(v_\beta\right)\left(v_\beta(s,\phi_\varepsilon)\right)^{(p-1)/p}\left(|D_s d|^p*\varphi_\beta\right)^{1/p}ds\\ &\le p\left(\int_0^t\zeta'\left(v_\beta\right)v_\beta(s,\phi_\varepsilon)\,ds\right)^{(p-1)/p}\left(\int_0^t\zeta'\left(v_\beta\right)\left(|D_s d|^p*\varphi_\beta\right)ds\right)^{1/p}. \end{aligned}$$

Inserting the bounds for I_1 and I_2 into (4.5) yields

$$\frac{1}{p}\zeta\left(v_\beta\left(t,\phi_\varepsilon(t)\right)\right)+\frac{\kappa}{\varepsilon p}\int_0^t\zeta'\left(v_\beta\right)d\left(s,\phi_\varepsilon\right)\widetilde{v}_\beta\left(s,\phi_\varepsilon\right)ds \tag{4.6}$$
$$\leq\left(\int_0^t\zeta'\left(v_\beta\right)v_\beta\left(s,\phi_\varepsilon\right)ds\right)^{(p-1)/p}\left(\int_0^t\zeta'\left(v_\beta\right)\left(|D_sd|^p*\varphi_\beta\right)ds\right)^{1/p}$$
$$+\left(\int_0^t\zeta'\left(v_\beta\right)\left(\widetilde{v}_\beta\left(s,\phi_\varepsilon\right)\right)^{p/(p-1)}ds\right)^{(p-1)/p}\left(\int_0^t\zeta'\left(v_\beta\right)\left|\psi'\right|^pds\right)^{1/p}+\rho\left(\beta\right),$$

where $\rho(\beta)=p^{-1}\zeta\left(v_\beta\left(0,\phi_\varepsilon(0)\right)\right)\to 0$ as $\beta\to 0$. By spatial Lipschitz continuity of $d(t,x)$ we have $v_\beta\left(s,\phi_\varepsilon\right)\to v\left(s,\phi_\varepsilon\right)$ and $\widetilde{v}_\beta\left(s,\phi_\varepsilon\right)\to\widetilde{v}\left(s,\phi_\varepsilon\right)$ as $\beta\to 0$. Moreover, since d satisfies (2.7), uniformly in space, we also have

$$\int_0^t|D_sd|^pds\leq C(T)^p,$$

for some constant $C(T)$ independent of x. Therefore, by the Fubini–Tonelli theorem we can conclude, since $D_sd(t,x)$ is jointly measurable in (t,x), that

$$\int_0^t\left(|D_sd|^p*\varphi_\beta\right)ds=\int_{\mathbb{R}^n}\left(\int_0^t|D_sd|^pds\right)\varphi_\beta(x-y)dy\leq C(T)^p,$$

and so

$$\left(\int_0^t\zeta'\left(v_\beta\right)\left(|D_sd|^p*\varphi_\beta\right)ds\right)^{1/p}+\left(\int_0^t\zeta'\left(v_\beta\right)\left|\psi'\right|^pds\right)^{1/p}\leq C\left(T\right)<\infty,$$

since by construction $\left|\zeta'\left(v_\beta\right)\right|\leq 1$, and $\psi\in\mathcal{C}^1\left([0,T],\mathbb{R}^n\right)$. Thus, letting β tend to zero in (4.6), we obtain

$$\frac{1}{p}\zeta\left(v\left(t,\phi_\varepsilon(t)\right)\right)+\frac{\kappa}{\varepsilon p}\int_0^t\zeta'\left(v\left(s,\phi_\varepsilon\right)\right)v\left(s,\phi_\varepsilon\right)ds$$
$$\leq C\left(T\right)\left(\int_0^t\zeta'\left(v\left(s,\phi_\varepsilon\right)\right)v\left(s,\phi_\varepsilon\right)ds\right)^{(p-1)/p}.$$

Both terms on the left hand side are positive and each of the terms are therefore bounded from above by the right hand side. Hence

$$\frac{\kappa}{\varepsilon p}\left(\int_0^t\zeta'\left(v\left(s,\phi_\varepsilon\right)\right)v\left(s,\phi_\varepsilon\right)ds\right)^{1/p}\leq C\left(T\right),$$

and, as a consequence,

$$\zeta\left(v\left(t,\phi_\varepsilon(t)\right)\right)+\frac{\kappa}{\varepsilon}\int_0^t\zeta'\left(v\left(s,\phi_\varepsilon\right)\right)v\left(s,\phi_\varepsilon\right)ds\leq K\left(T\right)\varepsilon^{p-1}.$$

We may assume that $\varepsilon>0$ has been chosen small enough such that $v\left(t,\phi_\varepsilon(t)\right)\leq\overline{\beta}^p/2$, for all $t\in[0,T]$. Then, by the definition of ζ,

$$\frac{1}{\varepsilon^{p-1}}\left(d\left(t,\phi_\varepsilon(t)\right)\right)^p+\frac{\kappa}{\varepsilon^p}\int_0^t\left(d\left(s,\phi_\varepsilon(s)\right)\right)^pds\leq K\left(T\right), \tag{4.7}$$

for $t\in[0,T]$.

The remainder of the proof follows along the lines of the proof of Lemma 4.5 in [14], but we give the details for completeness. Relation (4.7) asserts that the sequences $\{l_\varepsilon\}_{\varepsilon>0}$ and $\{\lambda_\varepsilon\}_{\varepsilon\geq 0}$,

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where

$$l_\varepsilon(t) = \frac{1}{\varepsilon} d(t, \phi_\varepsilon(t)), \quad \lambda_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t d(s, \phi_\varepsilon(s))\,\gamma(s, \phi_\varepsilon(s))\,ds,$$

are bounded in $L^p([0,T], \mathbb{R})$ and $\mathcal{W}^{1,p}([0,T], \mathbb{R}^n)$ respectively. Thus, we may assume that l_ε and λ_ε converge weakly to $l \in L^p([0,T], \mathbb{R})$ and $\lambda \in \mathcal{W}^{1,p}([0,T], \mathbb{R}^n) \subset \mathcal{C}([0,T], \mathbb{R}^n)$, respectively, as $\varepsilon \to 0$. Moreover, from (4.3) we conclude that ϕ_ε converges weakly to $\phi \in \mathcal{W}^{1,p}([0,T], \mathbb{R}^n)$ and that $\phi(t) = \psi(t) + \lambda(t)$, $\phi(0) = \psi(0)$. This proves (2.9) and, moreover, (2.10) holds due to (4.7). By construction, $\lambda'_\varepsilon(t) = l_\varepsilon(t)\,\gamma(t, \phi_\varepsilon(t))$ and this implies that $\lambda'(t) = l(t)\,\gamma(t, \phi(t))$. Moreover, if we let $\tau = \{t \in [0,T] : \phi(t) \in \Omega_t\}$ and note that for each fixed $t \in \tau$, we have $l_\varepsilon(t) = 0$ for all sufficiently small ε and hence $l(t) = 0$ on τ. Therefore

$$|\lambda|(t) = \int_0^t |\lambda'(s)|\,ds = \int_0^t l(s)\,|\gamma(s, \phi(s))|\,ds = \int_0^t l(s)\,ds, \quad \text{for all } t \in [0,T],$$

as $|\gamma(s, \phi(s))| = 1$ for all $s \in [0,T] \setminus \tau$. This proves (2.11). In addition,

$$\lambda(t) = \int_0^t l(s)\,\gamma(s, \phi(s))\,ds = \int_0^t \gamma(s, \phi(s))\,d|\lambda|(s), \quad \text{for all } t \in [0,T],$$

which proves (2.13). It remains to verify (2.12), but this follows readily from

$$|\lambda|(t) = \int_0^t l(s)\,|\gamma(s, \phi(s))|\,ds = \int_0^t I_{\{\phi(s) \in \partial\Omega_s\}} l(s)\,ds = \int_0^t I_{\{\phi(s) \in \partial\Omega_s\}} d|\lambda|(s).$$

We have completed the proof that $(\phi, \lambda) \in \mathcal{W}^{1,p}([0,T], \mathbb{R}^n) \times \mathcal{W}^{1,p}([0,T], \mathbb{R}^n)$ solves the Skorohod problem for (Ω, γ, ψ). $\square$

The next step is to prove relative compactness of solutions to the Skorohod problem. The proof follows the proof of Lemma 4.7 in [14], but a number of changes must be made carefully to handle the time dependency of the domain.

Lemma 4.4. *Let A be a compact subset of $\mathcal{C}([0,T], \mathbb{R}^n)$. Then*

(i) *There exists a constant $L < \infty$ such that*

$$|\lambda|(T) < L,$$

for all solutions $(\psi + \lambda, \lambda)$ to the Skorohod problem for (Ω, γ, ψ) with $\psi \in A$.

(ii) *The set of ϕ, such that (ϕ, λ) solves the Skorohod problem for (Ω, γ, ψ) with $\psi \in A$, is relatively compact.*

Proof. By the compactness of $\overline{\Omega}$ and the continuity of γ, there exists a constant $c > 0$ such that for every $t \in [0,T]$ and $x \in \overline{\Omega_t} \cap V$, where V is the set defined in connection with (2.3), there exists a vector $v(t,x)$ and a set $[t, t+c] \times B(x,c)$ such that $\langle \gamma(s,y), v(t,x) \rangle > c$ for all $(s,y) \in [t, t+c] \times B(x,c)$. Without lack of generality, we may assume that $c < \delta$, for the δ introduced in Remark 2.1. Let $\psi \in A$ be given and let (ϕ, λ) be any solution to the Skorohod problem for (Ω, γ, ψ). Define T_1 to be smallest of T, c and $\inf\{t \in [0,T] : \phi(t) \notin B(\phi(0), c)\}$. Next define T_2 to be the smallest of T , $T_1 + c$ and $\inf\{t \in [T_1, T] : \phi(t) \notin B(\phi(T_1), c)\}$. Continuing in this fashion, we obtain a sequence $\{T_m\}_{m=1,2,\ldots}$ of time instants. By construction, for all $s \in [T_{m-1}, T_m)$ we have $s \in [T_{m-1}, T_{m-1} + c]$ and $\phi(s) \in B(\phi(T_{m-1}), c)$. For all m

such that $\phi(T_{m-1}) \in \overline{\Omega}_{T_{m-1}} \cap V$, we have $\langle \gamma(s, \phi(s)), v(T_{m-1}, \phi(T_{m-1})) \rangle > c$ and hence

$$\begin{aligned}
&\langle \phi(T_m) - \phi(T_{m-1}), v(T_{m-1}, \phi(T_{m-1})) \rangle \\
&- \langle \psi(T_m) - \psi(T_{m-1}), v(T_{m-1}, \phi(T_{m-1})) \rangle \\
= &\int_{T_{m-1}}^{T_m} \langle \gamma(s, \phi(s)), v(T_{m-1}, \phi(T_{m-1})) \rangle \, d|\lambda|(s) \geq c \left(|\lambda|(T_m) - |\lambda|(T_{m-1}) \right).
\end{aligned}$$

Since A is compact, the set $\{\psi(t) : t \in [0, T], \psi \in A\}$ is bounded. Moreover, since $\overline{\Omega}$ is compact and $\phi(t) \in \overline{\Omega}_t$ for all $t \in [0, T]$, there exists a constant $M < \infty$ such that

$$|\lambda|(T_m) - |\lambda|(T_{m-1}) < M.$$

Note also that, for all m such that $\phi(T_{m-1}) \notin \overline{\Omega}_{T_{m-1}} \cap V$, we have, for c sufficiently small, that $|\lambda|(T_m) - |\lambda|(T_{m-1}) = 0$.

Define the modulus of continuity of a function $f \in \mathcal{C}([0, T], \mathbb{R}^n)$ as $\|f\|_{s,t} = \sup_{s \leq t_1 \leq t_2 \leq t} |f(t_2) - f(t_1)|$ for $0 \leq s \leq t \leq T$. We next prove that there exists a positive constant R such that, for any $\psi \in A$ and $T_{m-1} \leq \tau \leq T_m$, it holds that

$$\|\lambda\|_{T_{m-1},\tau} \leq R \left(\|\psi\|_{T_{m-1},\tau}^{1/2} + \|\psi\|_{T_{m-1},\tau}^{3/2} + (\tau - T_{m-1})^{\widehat{\alpha}/2} \right), \tag{4.8}$$

where $\widehat{\alpha}$ is the Hölder exponent in Remark 2.2. As we are only interested in the behaviour during the time interval $\left[T_{m-1}, T_m\right]$, we simplify the notation by setting, without loss of generality, $T_{m-1} = 0$, $\phi(T_{m-1}) = x$, $\psi(T_{m-1}) = x$, $\lambda(T_{m-1}) = 0$ and $|\lambda|(T_{m-1}) = 0$. Let h be the function in Lemma 3.1 and let χ, C be the corresponding positive constants. Define $B_\varepsilon(t) = \varepsilon h(t, x, -\lambda(t)/\varepsilon)$ and $E(t) = e^{-(2|\lambda|(t)+t)C/\chi}$. Since $h(t, x, 0) = 1$, we get

$$\begin{aligned}
B_\varepsilon(\tau) E(\tau) &= B_\varepsilon(0) E(0) + \int_0^\tau \left(E(u) \, dB_\varepsilon(u) + B_\varepsilon(u) \, dE(u) \right) \\
&= \varepsilon + \int_0^\tau E(u) \, dB_\varepsilon(u) - \frac{2C}{\chi} \int_0^\tau B_\varepsilon(u) E(u) \, d|\lambda|(u) \\
&\quad - \frac{C}{\chi} \int_0^\tau B_\varepsilon(u) E(u) \, du,
\end{aligned}$$

where the first integral can be rewritten as

$$\begin{aligned}
\int_0^\tau E(u) \, dB_\varepsilon(u) = &\int_0^\tau E(u) \, \varepsilon D_t h(u, x, -\lambda(u)/\varepsilon) \, du \\
&- \int_0^\tau E(u) \left\langle D_p h(u, x, -\lambda(u)/\varepsilon), d\lambda(u) \right\rangle.
\end{aligned}$$

By (3.8) and (3.12), the integral involving $D_t h$ has the upper bound

$$\begin{aligned}
&\int_0^\tau E(u) \, \varepsilon D_t h(u, x, -\lambda(u)/\varepsilon) \, du \\
&\leq C\varepsilon \int_0^\tau E(u) \, |\lambda(u)/\varepsilon|^2 du \leq \frac{C}{\chi} \int_0^\tau E(u) B_\varepsilon(u) \, du.
\end{aligned}$$

Next, we would like to find an upper bound for the integral involving $D_p h$ using (3.10) in some appropriate way, but we have to be somewhat careful due to the temporal variation of the domain. Assume that $\phi(u) \in \partial \Omega_u$. If $x \notin \overline{\Omega}_u$, there exists at least one point $y_u \in \overline{\Omega}_u \cap B(x, c)$ such that $|x - y_u| = d(u, x)$. We have chosen $c < \delta$, so $\langle y_u - \phi(u), \gamma(u, \phi(u)) \rangle \geq -\theta |y_u - \phi(u)|$

holds by Remark 2.1 and, due to (3.10), we can conclude

$$I_1 := -\int_0^\tau E(u)\left\langle D_p h\left(u, \phi(u), (y_u - \phi(u))/\varepsilon\right), \gamma(u, \phi(u))\right\rangle d|\lambda|(u) \le 0,$$

since $d|\lambda|(u) = 0$ if $\phi(u) \notin \partial\Omega_u$. If $x \in \overline{\Omega}_u$, the above estimate holds with y_u replaced by x. The integral involving $D_p h$ can be decomposed into

$$-\int_0^\tau E(u)\left\langle D_p h(u, x, -\lambda(u)/\varepsilon), d\lambda(u)\right\rangle = I_1 + I_2 + I_3,$$

for I_1 as above and

$$I_2 = \int_0^\tau E(u)\left\langle D_p h(u, \phi(u), -\lambda(u)/\varepsilon) - D_p h(u, x, -\lambda(u)/\varepsilon), d\lambda(u)\right\rangle,$$

$$I_3 = \int_0^\tau E(u)\left\langle D_p h(u, \phi(u), (y_u - \phi(u))/\varepsilon) - D_p h(u, \phi(u), -\lambda(u)/\varepsilon), d\lambda(u)\right\rangle.$$

By (3.8) and (3.13), these integrals can be bounded from above by

$$\begin{aligned}
I_2 &\le \frac{C}{\varepsilon}\int_0^\tau E(u)|\lambda(u)||x - \phi(u)|\, d|\lambda|(u) \\
&\le \frac{C}{\varepsilon}\int_0^\tau E(u)\left(|\lambda(u)|^2 + |x - \psi(u)||\lambda(u)|\right) d|\lambda|(u) \\
&\le \frac{2C}{\varepsilon}\int_0^\tau E(u)\left(|\lambda(u)|^2 + |x - \psi(u)|^2\right) d|\lambda|(u) \\
&\le \frac{2C}{\chi}\int_0^\tau E(u) B_\varepsilon(u)\, d|\lambda|(u) + \frac{2C}{\varepsilon}\int_0^\tau E(u)|x - \psi(u)|^2 d|\lambda|(u),
\end{aligned}$$

and

$$\begin{aligned}
I_3 &\le \frac{C}{\varepsilon}\int_0^\tau E(u)|y_u - \phi(u) - (-\lambda(u))|\, d|\lambda|(u) \\
&= \frac{C}{\varepsilon}\int_0^\tau E(u)|y_u - \psi(u)|\, d|\lambda|(u) \\
&\le \frac{C}{\varepsilon}\int_0^\tau E(u)\left(|x - \psi(u)| + |y_u - x|\right) d|\lambda|(u) \\
&\le \frac{C}{\varepsilon}\int_0^\tau E(u)\left(|x - \psi(u)| + d(u, x)\right) d|\lambda|(u).
\end{aligned}$$

Collecting all the terms, we obtain

$$B_\varepsilon(\tau) E(\tau) \le \varepsilon + \frac{C}{\varepsilon}\int_0^\tau E(u)\left(|x - \psi(u)| + 2|x - \psi(u)|^2 + d(u, x)\right) d|\lambda|(u),$$

which implies

$$B_\varepsilon(\tau) \le \left(\frac{2C}{\varepsilon}\int_0^\tau E(u)\left(\|\psi\|_{0,\tau} + \|\psi\|_{0,\tau}^2 + K\tau^{\widehat{\alpha}}\right) d|\lambda|(u) + \varepsilon\right) e^{(2|\lambda|(\tau)+\tau)C/\chi},$$

where K and $\widehat{\alpha}$ are the constants from Remark 2.2. Now

$$\int_0^\tau E(u)\, d|\lambda|(u) \le \int_0^\tau e^{-2C|\lambda|(u)/\chi} d|\lambda|(u) \le \frac{\chi}{2C},$$

so

$$B_\varepsilon(\tau) \le \left(\frac{\chi}{\varepsilon}\left(\|\psi\|_{0,\tau} + \|\psi\|_{0,\tau}^2 + K\tau^{\widehat{\alpha}}\right) + \varepsilon\right) e^{(2|\lambda|(\tau)+\tau)C/\chi}.$$

Another application of (3.8) gives

$$|\lambda(\tau)| \le \frac{1}{2}\left(\varepsilon + \frac{1}{\varepsilon}|\lambda(\tau)|^2\right) \le \frac{\varepsilon}{2} + \frac{B_\varepsilon(\tau)}{2\chi}$$
$$\le \frac{\varepsilon}{2} + \left(\frac{1}{2\varepsilon}\left(\|\psi\|_{0,\tau} + \|\psi\|_{0,\tau}^2 + K\tau^{\widehat{\alpha}}\right) + \frac{\varepsilon}{2\chi}\right) e^{(2M+T)C/\chi}.$$

Set $\varepsilon = \max\left\{\|\psi\|_{0,\tau}^{1/2}, \tau^{\widehat{\alpha}/2}\right\}$ so that $\varepsilon \le \|\psi\|_{0,\tau}^{1/2} + \tau^{\widehat{\alpha}/2}$, $1/\varepsilon \le \|\psi\|_{0,\tau}^{-1/2}$ and $1/\varepsilon \le \tau^{-\widehat{\alpha}/2}$. Then (4.8) follows immediately from the above inequality. By (4.8) and the compactness of A, there exists a $\hat{\tau} > 0$ such that

$$\max\left\{\|\psi\|_{T_{m-1},T_{m-1}+\hat{\tau}}, \|\lambda\|_{T_{m-1},T_{m-1}+\hat{\tau}}\right\} \le c/3,$$

which implies $\|\phi\|_{T_{m-1},T_{m-1}+\hat{\tau}} \le 2c/3$. The definition of $\{T_m\}$ then implies that $T_m - T_{m-1} \ge \min\left\{\hat{\tau}, c\right\}$. This proves (i) with $L = M\left(T/\min\left\{\hat{\tau}, c\right\} + 1\right)$. Part (ii) follows from (4.8) and the bound $T_m - T_{m-1} \ge \min\left\{\hat{\tau}, c\right\}$. □

Equipped with the results above, we are now ready to state and prove the existence of solutions to the Skorohod problem. The proof is very similar to the proof of Theorem 4.8 in [14], so we only sketch the first half of the proof.

Lemma 4.5. *Let $\psi \in \mathcal{C}([0,T], \mathbb{R}^n)$ with $\psi(0) \in \overline{\Omega}_0$. Then there exists a solution (ϕ, λ) to the Skorohod problem for (Ω, γ, ψ).*

Proof. Let $\psi_n \in \mathcal{C}^1([0,T], \mathbb{R}^n)$ form a sequence of functions converging uniformly to ψ. According to Lemma 4.3, there exists a solution (ϕ_n, λ_n) to the Skorohod problem for (Ω, γ, ψ_n). By Lemma 4.4, we may assume that the sequence $\{\lambda_n\}_{n=1}^\infty$ is equibounded and equicontinuous, that is

$$\sup_n |\lambda_n|(T) \le L < \infty,$$
$$\lim_{|s-t|\to 0} \sup_n |\lambda_n(s) - \lambda_n(t)| = 0.$$

The Arzela–Ascoli theorem asserts the existence of a function $\lambda \in \mathcal{C}([0,T], \mathbb{R}^n)$ such that $\{\lambda_n\}$ converges uniformly to λ. Clearly $|\lambda|(T) \le L$. Defining the function ϕ by $\phi = \psi + \lambda$, we conclude that (2.9)–(2.11) of Definition 2.5 hold. To show properties (2.12) and (2.13) in the same definition, we define the measure μ_n on $\overline{\Omega} \times S(0,1)$ as

$$\mu_n(A) = \int_{[0,T]} I_{\{(s,\phi_n(s),\gamma(s,\phi_n(s)))\in A\}} d|\lambda_n|(s),$$

for every Borel set $A \subset \overline{\Omega} \times S(0,1)$. Introducing the notation $\overline{\Omega}_{[0,t]} := \overline{\Omega} \cap ([0,t] \times \mathbb{R}^n)$, we have, by definition and (2.13),

$$|\lambda_n|(t) = \mu_n\left(\overline{\Omega}_{[0,t]} \times S(0,1)\right),$$

and

$$\lambda_n(t) = \int_{\overline{\Omega}_{[0,t]} \times S(0,1)} \gamma \, d\mu_n(s,x,\gamma),$$

for all $t \in [0, T]$. Since $|\lambda_n|(T) \leq L < \infty$ for all n, the Banach–Alaoglu theorem asserts that a subsequence of μ_n converges to some measure μ satisfying $\mu\left(\overline{\Omega} \times S(0,1)\right) < \infty$. By weak convergence and the continuity of λ,

$$\lambda(t) = \int_{\overline{\Omega}_{[0,t]} \times S(0,1)} \gamma \, d\mu(s, x, \gamma).$$

Using the fact that (ϕ_n, λ_n) solves the Skorohod problem for (Ω, γ, ψ_n), we can draw several conclusions regarding the properties of the measure μ_n and then use weak convergence of μ_n to μ to deduce that λ satisfies (2.12) and (2.13). This procedure is carried out in the proofs of Theorem 2.8 in [7], Theorem 4.8 in [14] and Theorem 5.1 in [23], so we omit further details. $\square$

5. SDEs with oblique reflection

Using the existence of solutions (ϕ, λ) to the Skorohod problem for (Ω, γ, ψ), with $\psi \in \mathcal{C}([0,T], \mathbb{R}^n)$ and $\psi(0) \in \overline{\Omega}_0$, we can now prove existence and uniqueness of solutions to SDEs with oblique reflection at the boundary of a bounded, time-dependent domain. To this end, assume that the triple (X, Y, k) satisfies

$$Y(t) = x + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dM(s) + k(t),$$

$$X(t) \in \overline{\Omega}_t, \quad Y(t) \in \overline{\Omega}_t,$$

$$|k|(t) = \int_{(0,t]} I_{\{Y(s) \in \partial\Omega_s\}} d|k|(s) < \infty, \quad k(t) = \int_{(0,t]} \gamma(s)\, d|k|(s),$$

where $x \in \overline{\Omega}_0$ is fixed, $\gamma(s) = \gamma(s, Y(s))$ $d|k|$ -a.s. and M is a continuous $\mathcal{F}_t$-martingale satisfying

$$d\left\langle M_i, M_j \right\rangle(t) \leq C dt, \tag{5.1}$$

for some $C \in (0, \infty)$. Let $\left(X', Y', k'\right)$ be a similar triple, but with x replaced by $x' \in \overline{\Omega}_0$, and $\gamma'(s) = \gamma\left(s, Y'(s)\right) d\left|k'\right|$-a.s.

We shall prove uniqueness of solutions by a Picard iteration scheme and a crucial ingredient is then the estimate provided by the following theorem. Note that Lemma 5.1 holds for a general continuous $\mathcal{F}_t$-martingale satisfying (5.1), whereas in Theorem 2.7 we restrict our interest to M being a standard Wiener process.

Lemma 5.1. *There exists a positive constant $C < \infty$ such that*

$$E\left[\sup_{0\leq s\leq t} \left|Y(s) - Y'(s)\right|^4\right] \leq C\left(\left|x - x'\right|^4 + \int_0^t E\left[\sup_{0 \leq u \leq s} \left|X(u) - X'(u)\right|^4\right] ds\right).$$

Proof. Fix $\varepsilon > 0$, let $\lambda > 0$ be a constant to be specified later, and let $w_\varepsilon \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ and $\alpha \in \mathcal{C}^{1,2}\left(\overline{\Omega}, \mathbb{R}\right)$ be the functions defined in Lemmas 3.2–3.3. Define the stopping time

$$\tau = \inf\left\{s \in [0,T] : \left|Y(s) - Y'(s)\right| \geq \delta\right\},$$

where $\delta > 0$ is the constant from Remark 2.1. Let B denote the diameter of the smallest ball containing $\bigcup_{t\in[0,T]}\overline{\Omega}_t$. Then, assuming without loss of generality that $B/\delta \geq 1$, we have

$$E\left[\sup_{0\leq s\leq t}\left|Y(s)-Y'(s)\right|^4\right]\leq\left(\frac{B}{\delta}\right)^4 E\left[\sup_{0\leq s\leq t\wedge\tau}\left|Y(s)-Y'(s)\right|^4\right],$$

so it is sufficient to prove the theorem for $t\wedge\tau$. To simplify the notation, however, we write t in place of $t\wedge\tau$ and assume that $\left|Y(s)-Y'(s)\right|<\delta$ in the proof below.

Define, for all (t,x,y) such that $(t,x),(t,y)\in\overline{\Omega}$, the function v as

$$v(t,x,y)=e^{-\lambda(\alpha(t,x)+\alpha(t,y))}w_\varepsilon(t,x,y):=u(t,x,y)\,w_\varepsilon(t,x,y).$$

The regularity of v is inherited from that of w_ε and α. By Itō's formula we have, suppressing the s-dependence for X, X', Y and Y',

$$\begin{aligned}
&v\left(t,Y(t),Y'(t)\right)\\
&=v\left(0,x,x'\right)+\int_0^t D_s v\left(s,Y,Y'\right)ds\\
&\quad+\int_0^t\left\langle D_x v\left(s,Y,Y'\right),b(s,X)\right\rangle ds+\int_0^t\left\langle D_y v\left(s,Y,Y'\right),b\left(s,X'\right)\right\rangle ds\\
&\quad+\int_0^t\left\langle D_x v\left(s,Y,Y'\right),\sigma(s,X)\,dM(s)\right\rangle+\int_0^t\left\langle D_y v\left(s,Y,Y'\right),\sigma\left(s,X'\right)dM(s)\right\rangle\\
&\quad+\int_0^t\left\langle D_x v\left(s,Y,Y'\right),\gamma(s)\right\rangle d|k|(s)+\int_0^t\left\langle D_y v\left(s,Y,Y'\right),\gamma'(s)\right\rangle d\left|k'\right|(s)\\
&\quad+\int_0^t\operatorname{tr}\left(\begin{pmatrix}\sigma(s,X)\\ \sigma\left(s,X'\right)\end{pmatrix}^T D^2 v\left(s,Y,Y'\right)\begin{pmatrix}\sigma(s,X)\\ \sigma\left(s,X'\right)\end{pmatrix}d\langle M\rangle(s)\right).
\end{aligned}\tag{5.2}$$

We define the martingale N as

$$N(t)=\int_0^t\left\langle D_x v\left(s,Y,Y'\right),\sigma(s,X)\,dM(s)\right\rangle+\int_0^t\left\langle D_y v\left(s,Y,Y'\right),\sigma\left(s,X'\right)dM(s)\right\rangle,$$

and simplify the remaining terms in (5.2). From (3.15), (3.19) and the regularity of u, we have

$$\int_0^t D_s v\left(s,Y,Y'\right)ds\leq C(\lambda)\int_0^t\left(\varepsilon+\frac{\left|Y-Y'\right|^2}{\varepsilon}\right)ds.$$

Similarly, following the proof of Theorem 5.1 in [14], we have

$$\begin{aligned}
&\int_0^t\left\langle D_x v\left(s,Y,Y'\right),b(s,X)\right\rangle ds+\int_0^t\left\langle D_y v\left(s,Y,Y'\right),b\left(s,X'\right)\right\rangle ds\\
&\leq C(\lambda)\left(\varepsilon+\int_0^t\frac{\left|Y-Y'\right|^2}{\varepsilon}ds+\int_0^t\frac{\left|X-X'\right|^2}{\varepsilon}ds\right).
\end{aligned}\tag{5.3}$$

A simple extension of Lemma 5.7 in [14] to the time-dependent case shows that there exists a constant $K_1(\lambda)<\infty$ such that for all $t\in[0,T]$, $x,y\in\overline{\Omega}_t$, the second order derivatives of v with respect to the spatial variables satisfy

$$D^2 v(t,x,y)\leq K_1(\lambda)\left(\frac{1}{\varepsilon}\begin{pmatrix}I&-I\\-I&I\end{pmatrix}+\left(\varepsilon+\frac{|x-y|^2}{\varepsilon}\right)\begin{pmatrix}I&0\\0&I\end{pmatrix}\right).$$

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Moreover, it is an easy consequence of the Lipschitz continuity of σ that there exists a constant $K_2(\lambda) < \infty$ such that for all $t \in [0, T]$, $x, y, \xi, \omega \in \overline{\Omega}_t$,

$$\begin{pmatrix} \sigma(t,\xi) \\ \sigma(t,\omega) \end{pmatrix}^T D^2 v(t,x,y) \begin{pmatrix} \sigma(t,\xi) \\ \sigma(t,\omega) \end{pmatrix} \le K_2(\lambda)\left(\varepsilon + \frac{1}{\varepsilon}\left(|\xi-\omega|^2 + |x-y|^2\right)\right) I.$$

Consequently, the last term in (5.2) may be simplified to

$$\int_0^t \operatorname{tr}\left(\begin{pmatrix} \sigma(s,X) \\ \sigma(s,X') \end{pmatrix}^T D^2 v(s,Y,Y') \begin{pmatrix} \sigma(s,X) \\ \sigma(s,X') \end{pmatrix} d\langle M\rangle(s)\right)$$
$$\le C(\lambda)\left(\varepsilon + \int_0^t \frac{|X-X'|^2}{\varepsilon} ds + \int_0^t \frac{|Y-Y'|^2}{\varepsilon} ds\right).$$

Considering now the terms containing $|k|$ and $|k'|$, we see, following the proof of Theorem 5.1 in [14], that

$$\int_0^t \langle D_x v(s,Y,Y'), \gamma(s)\rangle d|k|(s) + \int_0^t \langle D_y v(s,Y,Y'), \gamma'(s)\rangle d|k'|(s)$$
$$\le C\int_0^t u(s,Y,Y')\frac{|Y-Y'|^2}{\varepsilon} d|k|(s) + C\int_0^t u(s,Y,Y')\frac{|Y-Y'|^2}{\varepsilon} d|k'|(s)$$
$$-\lambda\int_0^t v(s,Y,Y')\langle D_x\alpha(s,Y), \gamma(s)\rangle d|k|(s)$$
$$-\lambda\int_0^t v(s,Y,Y')\langle D_x\alpha(s,Y'), \gamma'(s)\rangle d|k'|(s).$$

Moreover, (3.14) and (3.22) give, since $d|k|(s)$ is zero unless $Y(s) \in \partial\Omega_s$,

$$-\lambda v(s,Y,Y')\langle D_x\alpha(s,Y), \gamma(s)\rangle \le -\lambda\chi u(s,Y,Y')\frac{|Y-Y'|^2}{\varepsilon},$$

so, by putting $\lambda = C/\chi$ all integrals with respect to $|k|$ and $|k'|$ vanish. Dropping the λ-dependence from the constants, (3.14) and (5.2) give

$$\frac{1}{C}\frac{|Y(t)-Y'(t)|^2}{\varepsilon} \le v(t,Y(t),Y'(t)) \le v(0,x,x') + \varepsilon + N(t)$$
$$+ \int_0^t \frac{|Y-Y'|^2}{\varepsilon} ds + \int_0^t \frac{|X-X'|^2}{\varepsilon} ds.$$

Now applying (3.15) to $v(0,x,x')$, multiplying by ε, squaring, taking supremum and expectations on both sides, we obtain

$$E\left[\sup_{0\le s\le t}|Y(s)-Y'(s)|^4\right] \le C\left(|x-x'|^4 + \varepsilon^4 + \varepsilon^2 E\left[\sup_{0\le s\le t}(N(s))^2\right]\right.$$
$$\left.+\int_0^t E\left[|X-X'|^4 + |Y-Y'|^4\right] ds\right).$$

Then proceeding as in (5.3), the Doob–Kolmogorov inequality gives

$$E\left[\sup_{0\leq s\leq t}(N(s))^2\right]\leq 4E\left[(N(t))^2\right]$$
$$\leq C\int_0^t\left(\varepsilon^2+E\left[\frac{|Y-Y'|^4}{\varepsilon^2}+\frac{|X-X'|^4}{\varepsilon^2}\right]\right)ds,$$

Letting ε tend to zero, we obtain

$$E\left[\sup_{0\leq s\leq t}|Y(s)-Y'(s)|^4\right]\leq C\left(|x-x'|^4+\int_0^t E\left[\left(|X-X'|^4+|Y-Y'|^4\right)\right]ds\right),$$

from which the requested inequality follows by a simple application of Gronwall's inequality. □

Proof of Theorem 2.7. Given Lemmas 4.5 and 5.1, the proof of Theorem 2.7 follows exactly along the lines of the proof of Corollary 5.2 in [14], which in turn follows the same outline as in [20], Theorem 4.3. Note that the main problem is verifying the adaptedness property of the solutions to the reflected SDE. This property follows from an approximation of continuous $\mathcal{F}_t$ -adapted semimartingales by bounded variation processes, for which one can show existence of unique bounded variation solutions to the Skorohod problem, and these bounded variation solutions will be $\mathcal{F}_t$ -adapted. We omit further details. □

6. Fully nonlinear second-order parabolic PDEs

In this section, we prove the results on partial differential equations. First, we recall the definition of viscosity solutions. Let $E\subset\mathbb{R}^{n+1}$ be arbitrary. If $u:E\to\mathbb{R}$, then the parabolic superjet $\mathcal{P}_E^{2,+}u(s,z)$ contains all triplets $(a,p,X)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ such that if $(s,z)\in E$ then

$$u(t,x)\leq u(s,z)+a(t-s)+\langle p,x-z\rangle+\frac{1}{2}\langle X(x-z),x-z\rangle$$
$$+o\left(|t-s|+|x-z|^2\right)\quad\text{as } E\ni(t,x)\to(s,z).$$

The parabolic subjet is defined as $\mathcal{P}_E^{2,-}u(s,z)=-\mathcal{P}_E^{2,+}(-u(s,z))$. The closures $\overline{\mathcal{P}}_E^{2,+}u(s,z)$ and $\overline{\mathcal{P}}_E^{2,-}u(s,z)$ are defined in analogue with (2.6) and (2.7) in [10]. A function $u\in USC(\widetilde{\Omega})$ is a *viscosity subsolution* of (2.18) in Ω° if, for all $(a,p,A)\in\mathcal{P}_{\widetilde{\Omega}}^{2,+}u(t,x)$, it holds that

$$a+F(t,x,u(t,x),p,A)\leq 0,\quad\text{for } (t,x)\in\Omega^\circ.$$

If, in addition, for $(t,x)\in\partial\Omega$ it holds that

$$\min\{a+F(t,x,u(t,x),p,A),\ \langle p,\widetilde{\gamma}(t,x)\rangle+f(t,x,u(t,x))\}\leq 0,\tag{6.1}$$

then u is a viscosity subsolution of (2.18)–(2.19) in $\widetilde{\Omega}$. Similarly, a function $v\in LSC(\widetilde{\Omega})$ is a *viscosity supersolution* of (2.18) in Ω° if, for all $(a,p,A)\in\mathcal{P}_{\widetilde{\Omega}}^{2,-}v(t,x)$, it holds that

$$a+F(t,x,v(t,x),p,A)\geq 0,\quad\text{for } (t,x)\in\Omega^\circ.$$

If, in addition, for $(t,x)\in\partial\Omega$ it holds that

$$\max\{a+F(t,x,v(t,x),p,A),\ \langle p,\widetilde{\gamma}(t,x)\rangle+f(t,x,v(t,x))\}\geq 0,\tag{6.2}$$

then v is a viscosity supersolution of (2.18)–(2.19) in $\widetilde{\Omega}$. A function is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution. We remark that in the definition

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of viscosity solutions above, we may replace $\mathcal{P}^{2,+}_{\widetilde{\Omega}} u(t,x)$ and $\mathcal{P}^{2,-}_{\widetilde{\Omega}} v(t,x)$ by $\overline{\mathcal{P}}^{2,+}_{\widetilde{\Omega}} u(t,x)$ and $\overline{\mathcal{P}}^{2,-}_{\widetilde{\Omega}} v(t,x)$, respectively. In the following, we often skip writing "viscosity" before subsolutions, supersolutions and solutions. Note also that, given any set $E \subset \mathbb{R}^{n+1}$ and $t \in [0,T]$, we denote, in the following, the time sections of E as $E_t = \{x : (t,x) \in E\}$.

Next we give two lemmas. The first clarifies that the maximum principle for semicontinuous functions [9,10], holds true in time-dependent domains.

Lemma 6.1. *Suppose that $\mathcal{O}^i = \widehat{\mathcal{O}}^i \cap ((0,T)\times\mathbb{R}^n)$ for $i = 1,\dots,k$ where $\widehat{\mathcal{O}}^i$ are locally compact subsets of $\mathbb{R}^{n+1}$. Assume that $u_i \in USC(\mathcal{O}^i)$ and let $\varphi : (t,x_1,\dots,x_k) \to \varphi(t,x_1,\dots,x_k)$ be defined on an open neighbourhood of $\{(t,x) : t \in (0,T)$ and $x_i \in \mathcal{O}^i_t$ for $i = 1,\dots,k\}$ and such that φ is once continuously differentiable in t and twice continuously differentiable in $(x_1,\dots,x_k)$. Suppose that $s \in (0,T)$ and $z_i \in \mathcal{O}^i_s$ and*

$$w(t,x_1,\dots,x_k) \equiv u_1(t,x_1) + \cdots + u_k(t,x_k) - \varphi(t,x_1,\dots,x_k) \le w(s,z_1,\dots,z_k),$$

for $0 < t < T$ and $x_i \in \mathcal{O}^i_t$. Assume, moreover, that there is an $r > 0$ such that for every $M > 0$ there is a C such that, for $i = 1,\dots,k$,

$$\begin{aligned} &b_i \le C, \text{ whenever } (b_i,q_i,X_i) \in \mathcal{P}^{2,+}_{\mathcal{O}^i} u_i(t,x) \text{ with } \|X_i\| \le M \text{ and} \\ &|x_i - z_i| + |t-s| + |u_i(t,x_i) - u_i(s,z_i)| + |q_i - D_{x_i}\varphi(s,z_1,\dots,z_k)| \le r. \end{aligned} \tag{6.3}$$

Then, for each $\varepsilon > 0$ there exist (b_i,X_i) such that

$$\left(b_i, D_{x_i}\varphi(s,z_1,\dots,z_k), X_i\right) \in \overline{\mathcal{P}}^{2,+}_{\mathcal{O}^i} u_i(s,z), \quad \text{for } i = 1,\dots,k,$$

$$-\left(\frac{1}{\varepsilon} + \|A\|\right) I \le \begin{pmatrix} X_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & X_k \end{pmatrix} \le A + \varepsilon A^2,$$

and

$$b_1 + \cdots + b_k = D_t\varphi(s,z_1,\dots,z_k),$$

where $A = \left(D_x^2\varphi\right)(s,z_1,\dots,z_k)$.

Proof. Following ideas from page 1008 in [9] we let K_i be compact neighbourhoods of (s,z) in $\mathcal{O}^i$ and define the extended functions $\widetilde{u}_1,\dots,\widetilde{u}_k$, $\widetilde{u}_i \in USC(\mathbb{R}^n)$ for $i = 1,\dots,k$, by

$$\widetilde{u}_i(t,x) = \begin{cases} u(t,x), & \text{if } \quad (t,x) \in K_i, \\ -\infty, & \text{otherwise.} \end{cases}$$

From the definitions of sub and superjets it follows, for $i = 1,\dots,k$, that

$$\mathcal{P}^{2,+}_{\mathbb{R}^{n+1}} \widetilde{u}_i(t,x) = \mathcal{P}^{2,+}_{\mathcal{O}^i} u_i(t,x), \tag{6.4}$$

for (t,x) in the interior of K_i relative to $\mathcal{O}^i$. Excluding the trivial case $u_i(t,x) = -\infty$, then the function $\widetilde{u}_i(t,x)$ cannot approach $u_i(s,z)$ unless $(t,x) \in K_i$ and it follows that

$$\overline{\mathcal{P}}^{2,+}_{\mathbb{R}^{n+1}} \widetilde{u}_i(t,x) = \overline{\mathcal{P}}^{2,+}_{\mathcal{O}^i} u_i(t,x). \tag{6.5}$$

Setting $\widetilde{w}(t,x_1,\dots,x_k) = \widetilde{u}_1(t,x_1) + \cdots + \widetilde{u}_k(t,x_k)$ we see that $(s,z_1,\dots,z_k)$ is also a maximum of the function $(\widetilde{w} - \varphi)(t,x_1,\dots,x_k)$. Moreover, we note that the proof of Lemma 8 in [9] still works if (27) in [9] is replaced by assumption (6.3). These facts, together with (6.4) and (6.5), allow us to complete the proof of Lemma 6.1 by using Theorem 7 in [9]. □

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Before proving the next lemma, let us note that standard arguments imply that we can assume $\lambda > 0$ in (2.21). Indeed, if $\lambda \leq 0$ then for $\bar{\lambda} < \lambda$ the functions $e^{\bar{\lambda}t}u(t,x)$ and $e^{\bar{\lambda}t}v(t,x)$ are, respectively, sub- and supersolutions of (2.18)–(2.19) with $F(t,x,r,p,X)$ and $f(t,x,r)$ replaced by

$$-\bar{\lambda}r + e^{\bar{\lambda}t}F\left(t,x,e^{-\bar{\lambda}t}r,e^{-\bar{\lambda}t}p,e^{-\bar{\lambda}t}X\right) \quad \text{and} \quad e^{\bar{\lambda}t}f\left(t,x,e^{-\bar{\lambda}t}r\right). \tag{6.6}$$

Hence, in the following proof we assume $\lambda > 0$ in (2.21). Next we prove the following version of the comparison principle.

Lemma 6.2. *Let Ω° be a time-dependent domain satisfying (2.2). Assume (2.20)–(2.22). Let $u \in USC(\widetilde{\Omega})$ be a viscosity subsolution and $v \in LSC(\widetilde{\Omega})$ a viscosity supersolution of (2.18) in Ω°. Then $\sup_{\widetilde{\Omega}} u - v \leq \sup_{\partial\Omega \cup \overline{\Omega}_0}(u-v)^+$.*

Proof. We may assume, by replacing $T > 0$ by a smaller number if necessary, that u and $-v$ are bounded from above on $\widetilde{\Omega}$. We can also assume that $\sup_{\widetilde{\Omega}} u - v$ is attained by using the well known fact that if u is a subsolution of (2.18), then so is

$$u_\beta(t,x) = u(t,x) - \frac{\beta}{T-t},$$

for all $\beta > 0$. Assume that $\sup_{\widetilde{\Omega}} u - v = u(s,z) - v(s,z) > u(t,x) - v(t,x)$ for some $(s,z) \in \Omega^\circ$ and for all $(t,x) \in \partial\Omega \cup \overline{\Omega}_0$. As in Section 5.B in [10], we use the fact that if u is a viscosity subsolution, then so is $\bar{u} = u - K$ for every constant $K > 0$. Choose $K > 0$ such that $\bar{u}(t,x) - v(t,x) \leq 0$ for all $(t,x) \in \partial\Omega \cup \overline{\Omega}_0$ and such that $\bar{u}(s,z) - v(s,z) := \delta > 0$. Using Lemma 6.1 in place of Theorem 8.3 in [10] and by observing that assumptions (2.20)–(2.22) imply (assuming $\lambda > 0$ as is possible by (6.6)) the corresponding assumptions in [10], we see that we can proceed as in the proof of Theorem 8.2 in [10] to complete the proof by deriving a contradiction. □

Proof of Theorem 2.8. In the following we may assume, by replacing $T > 0$ by a smaller number if necessary, that u and $-v$ in Theorem 2.8 are bounded from above on $\widetilde{\Omega}$. We will now produce approximations of u and v which allow us to deal only with the inequalities involving F and not the boundary conditions. To construct these approximating functions, we note that Lemma 3.3 applies with γ replaced by $\widetilde{\gamma}$ as well. Thus, there exists a $\mathcal{C}^{1,2}$ function α defined on an open neighbourhood of $\widetilde{\Omega}$ with the property that $\alpha \geq 0$ on $\widetilde{\Omega}$ and $\langle D_x\alpha(t,x), \widetilde{\gamma}(t,x)\rangle \geq 1$ for $x \in \partial\Omega_t$, $t \in [0,T]$. For $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 > 0$ we define, for $(t,x) \in \widetilde{\Omega}$,

$$\begin{aligned} u_{\beta_1,\beta_2,\beta_3}(t,x) &= u(t,x) - \beta_1\alpha(t,x) - \beta_2 - \frac{\beta_3}{T-t}, \\ v_{\beta_1,\beta_2}(t,x) &= v(t,x) + \beta_1\alpha(t,x) + \beta_2. \end{aligned} \tag{6.7}$$

Given $\beta_3, \beta_2 > 0$ there is $\beta_1 = \beta_1(\beta_2) \in (0,\beta_2)$ for which $u_{\beta_1,\beta_2,\beta_3}$ and v_{β_1,β_2} are sub- and supersolutions of (2.18)–(2.19), with $f(t,x,r)$ replaced by $f(t,x,r) + \beta_1$ and $f(t,x,r) - \beta_1$, respectively. Indeed, if $(a,p,X) \in \mathcal{P}^{2,+}_{\widetilde{\Omega}} u_{\beta_1,\beta_2,\beta_3}(t,x)$, then

$$\left(a + \beta_1\alpha_t(t,x) + \frac{\beta_3}{(T-t)^2}, p + \beta_1 D\alpha(t,x), X + \beta_1 D^2\alpha(t,x)\right) \in \mathcal{P}^{2,+}_{\widetilde{\Omega}} u(t,x). \tag{6.8}$$

Hence, if u satisfies (2.19), then $\langle p + \beta_1 D\alpha(t,x), \widetilde{\gamma}(t,x)\rangle + f(t,x,u(t,x)) \leq 0$ and since $\langle D\alpha(t,x), \widetilde{\gamma}(t,x)\rangle \geq 1$, $u_{\beta_1,\beta_2,\beta_3} \leq u$ and by (2.25) we obtain

$$\langle p, \widetilde{\gamma}(t,x)\rangle + f\left(t,x,u_{\beta_1,\beta_2,\beta_3}\right) + \beta_1 \leq 0. \tag{6.9}$$

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Using (6.8) we also see that if u satisfies (2.18) then

$$a+\beta_1\alpha_t(t,x)+\frac{\beta_3}{(T-t)^2}+F\left(t,x,u,p+\beta_1 D\alpha(t,x),X+\beta_1 D^2\alpha(t,x)\right)\le 0.$$

Using (2.21) and (2.23), assuming also that the support of α lies within U, we have

$$\begin{aligned} a+\beta_1\alpha_t(t,x)+F\left(t,x,u_{\beta_1,\beta_2,\beta_3},p,X\right)+\lambda\beta_2 \\ -m_2\left(|\beta_1 D\alpha(t,x)|+\|\beta_1 D^2\alpha(t,x)\|\right)\le 0. \end{aligned} \tag{6.10}$$

From (6.9) and (6.10) it follows that, given $\beta_2,\beta_3>0$, there exist $\beta_1\in(0,\beta_2)$ such that $u_{\beta_1,\beta_2,\beta_3}$ is a subsolution of (2.18)–(2.19) with $f(t,x,u)$ replaced by $f(t,x,u)+\beta_1$. The fact that v_{β_1,β_2} is a supersolution follows by a similar calculation.

To complete the proof of the comparison principle, it is sufficient to prove that

$$\max_{\widetilde{\Omega}}\left(u_{\beta_1,\beta_2,\beta_3}-v_{\beta_1,\beta_2}\right)\le 0,$$

holds for all $\beta_2>0$ and $\beta_3>0$. Assume that

$$\sigma=\max_{\widetilde{\Omega}}\left(u_{\beta_1,\beta_2,\beta_3}-v_{\beta_1,\beta_2}\right)>0.$$

We will derive a contradiction for any β_3 if β_2 (and hence β_1) is small enough. To simplify notation, we write, in the following, u,v in place of $u_{\beta_1,\beta_2,\beta_3}$, v_{β_1,β_2}. By Lemma 6.2, $u(0,\cdot)\le v(0,\cdot)$, upper semicontinuity of $u-v$ and boundedness from above of $u-v$, we conclude that for any $\beta_3>0$

$$\sigma=(u-v)(s,z),\quad \text{for some } z\in\partial\Omega_s \text{ and } s\in(0,T). \tag{6.11}$$

Let $\widetilde{B}((s,z),\delta)=\{(t,x):|(t,x)-(s,z)|\le\delta\}$ and define

$$E:=\widetilde{B}((s,z),\delta)\cap\widetilde{\Omega}.$$

By Remark 2.1, there exists $\theta\in(0,1)$ such that

$$\langle x-y,\widetilde{\gamma}(t,x)\rangle\ge-\theta|x-y|,\quad \text{for all } (t,x)\in E\setminus\Omega^\circ \text{ and } (t,y)\in E. \tag{6.12}$$

By decreasing δ if necessary, we may assume that (2.23) holds in E. From now on, we restrict our attention to events in the set E. By Lemma 3.2 we obtain, for any $\theta\in(0,1)$, a family $\{w_\varepsilon\}_{\varepsilon>0}$ of functions $w_\varepsilon\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R})$ and positive constants χ, C (independent of ε) such that (3.14), (3.15), (3.19)–(3.21) as well as

$$\langle D_x w_\varepsilon(t,x,y),\widetilde{\gamma}(t,x)\rangle\ge -C\frac{|x-y|^2}{\varepsilon},\quad \text{if } \langle x-y,\widetilde{\gamma}(t,x)\rangle\ge-\theta|x-y|, \tag{6.13}$$

$$\left\langle D_y w_\varepsilon(t,x,y),\widetilde{\gamma}(t,y)\right\rangle\ge -C\frac{|x-y|^2}{\varepsilon},\quad \text{if } \langle y-x,\widetilde{\gamma}(t,y)\rangle\ge-\theta|x-y|, \tag{6.14}$$

hold. Note that (6.13) and (6.14) are direct analogues to (3.16) and (3.18) but with γ replaced by $\widetilde{\gamma}$.

Let $\varepsilon>0$ be given and define

$$\Phi(t,x,y)=u(t,x)-v(t,y)-\varphi(t,x,y),$$

where

$$\varphi(t,x,y)=w_\varepsilon(t,x,y)+f(s,z,u(s,z))\langle y-x,\widetilde{\gamma}(s,z)\rangle+\beta_1|x-z|^2+(t-s)^2.$$

Let $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$ be a maximum point of Φ. From (3.14) and (3.15) we have

$$\sigma - C\varepsilon \leq \Phi(s,z,z) \leq \Phi(t_\varepsilon, x_\varepsilon, y_\varepsilon) \leq u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon) - \chi \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \\ - f(s,z,u(s,z)) \langle y_\varepsilon - x_\varepsilon, \widetilde{\gamma}(s,z)\rangle - \beta_1 |x_\varepsilon - z|^2 - (t_\varepsilon - s)^2. \tag{6.15}$$

From this we first see that

$$|x_\varepsilon - y_\varepsilon| \to 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

Therefore, using the upper semi-continuity of $u - v$ and (6.15) we also obtain

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \to 0, \qquad x_\varepsilon, y_\varepsilon \to z, \qquad t_\varepsilon \to s, \\ u(t_\varepsilon, x_\varepsilon) \to u(s,z), \qquad v(t_\varepsilon, y_\varepsilon) \to v(s,z), \tag{6.16}$$

as $\varepsilon \to 0$. In the following we assume ε to be so small that $(t_\varepsilon, x_\varepsilon) \in E$

We introduce the notation

$$\bar{p} = D_x \varphi(t_\varepsilon, x_\varepsilon, y_\varepsilon) = D_x w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) - f(s,z,u(s,z))\widetilde{\gamma}(s,z) + 2\beta_1(x_\varepsilon - z), \\ \bar{q} = D_y \varphi(t_\varepsilon, x_\varepsilon, y_\varepsilon) = D_y w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) + f(s,z,u(s,z))\widetilde{\gamma}(s,z),$$

and observe that

$$\langle \bar{p}, \widetilde{\gamma}(t_\varepsilon, x_\varepsilon)\rangle + f(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon)) \\ = \langle D_x w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon), \widetilde{\gamma}(t_\varepsilon, x_\varepsilon), \rangle + f(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon)) \\ - f(s,z,u(s,z)) \langle \widetilde{\gamma}(s,z), \widetilde{\gamma}(t_\varepsilon, x_\varepsilon)\rangle + 2\beta_1 \langle x_\varepsilon - z, \widetilde{\gamma}(t_\varepsilon, x_\varepsilon)\rangle, \tag{6.17}$$

and

$$-\langle \bar{q}, \widetilde{\gamma}(t_\varepsilon, y_\varepsilon)\rangle + f(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, x_\varepsilon)) \\ = -\langle D_y w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon), \widetilde{\gamma}(t_\varepsilon, y_\varepsilon)\rangle + f(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon)) \\ - f(s,z,u(s,z)) \langle \widetilde{\gamma}(s,z), \widetilde{\gamma}(t_\varepsilon, y_\varepsilon)\rangle. \tag{6.18}$$

Using (2.3), (2.24), (2.25) and (6.16)–(6.18) we see that if ε is small enough, then

$$\langle D_x w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon), \widetilde{\gamma}(t_\varepsilon, x_\varepsilon)\rangle \geq -\frac{\beta_1}{2} \\ \Longrightarrow \langle \bar{p}, \widetilde{\gamma}(t_\varepsilon, x_\varepsilon)\rangle + f(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon)) + \beta_1 > 0, \\ \langle D_y w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon), \widetilde{\gamma}(t_\varepsilon, y_\varepsilon)\rangle \geq -\frac{\beta_1}{2} \\ \Longrightarrow -\langle \bar{q}, \widetilde{\gamma}(t_\varepsilon, y_\varepsilon)\rangle + f(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, x_\varepsilon)) - \beta_1 < 0. \tag{6.19}$$

Moreover, from (6.12)–(6.14), we also have

$$\langle D_x w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon), \widetilde{\gamma}(t_\varepsilon, x_\varepsilon)\rangle \geq -C\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}, \quad \text{if } x_\varepsilon \in \partial\Omega_{t_\varepsilon}, \\ \langle D_y w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon), \widetilde{\gamma}(t_\varepsilon, y_\varepsilon)\rangle \geq -C\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}, \quad \text{if } y_\varepsilon \in \partial\Omega_{t_\varepsilon}. \tag{6.20}$$

Using (6.19) and (6.20), it follows by the definition of viscosity solutions that if ε is small enough, say $0 < \varepsilon < \varepsilon_{\beta_1}$, then

$$a + F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), \bar{p}, X) \leq 0 \leq -b + F(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon), -\bar{q}, -Y), \tag{6.21}$$

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whenever

$$(a, \bar{p}, X) \in \overline{\mathcal{P}}^{2,+}_{\tilde{\Omega}} u(t_\varepsilon, x_\varepsilon) \quad \text{and} \quad (-b, -\bar{q}, -Y) \in \overline{\mathcal{P}}^{2,-}_{\tilde{\Omega}} v(t_\varepsilon, y_\varepsilon).$$

We next intend to use Lemma 6.1 to show the existence of such matrices X, Y and numbers a, b. Hence, we have to verify condition (6.3). To do so, we observe that (6.19) holds true with $\bar{p}$ and $\bar{q}$ replaced by any p and q satisfying $|\bar{p} - p| \le r$ and $|\bar{q} - q| \le r$ if we choose $r = r(\varepsilon)$ small enough. It follows that also (6.21) holds with these p and q and we can conclude

$$a \le -F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), p, X) \le C \quad \text{and} \quad b \le F(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon), -q, -Y) \le C,$$

for some $C = C(\varepsilon)$ whenever (a, p, X) and (b, q, Y) are as in (6.3). Hence, condition (6.3) holds and Lemma 6.1 gives the existence of $X, Y \in \mathbb{S}^n$ and $a, b \in \mathbb{R}$ such that

$$\begin{aligned}
&-\left(\frac{1}{\varepsilon} + \|A\|\right) I \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le A + \varepsilon A^2, \\
&(a, \bar{p}, X) \in \overline{\mathcal{P}}^{2,+}_{\tilde{\Omega}} u(t_\varepsilon, x_\varepsilon), \quad (-b, -\bar{q}, -Y) \in \overline{\mathcal{P}}^{2,-}_{\tilde{\Omega}} v(t_\varepsilon, y_\varepsilon), \\
&a + b = D_t \varphi(t_\varepsilon, x_\varepsilon, y_\varepsilon) = D_t w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) + 2(t_\varepsilon - s),
\end{aligned} \tag{6.22}$$

where $A = D^2_{x,y}\left(w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) + \beta_1 |x_\varepsilon - z|^2\right)$. Using (2.21), (3.19) and (6.21) we obtain, by recalling that we can assume $\lambda > 0$ in (2.21), that

$$\begin{aligned}
0 &\ge D_t w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) + 2(t_\varepsilon - s) \\
&\quad + F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), \bar{p}, X) - F(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon), -\bar{q}, -Y) \\
&\ge -C\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} + 2(t_\varepsilon - s) + \lambda(u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon)) \\
&\quad + F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), \bar{p}, X) - F(t_\varepsilon, y_\varepsilon, u(t_\varepsilon, x_\varepsilon), -\bar{q}, -Y).
\end{aligned}$$

Next, assumption (2.23) gives

$$\begin{aligned}
0 &\ge -C\bar{s} + 2(t_\varepsilon - s) + \lambda(u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon)) \\
&\quad + F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), -\bar{q}, X - C\bar{s}I) - F(t_\varepsilon, y_\varepsilon, u(t_\varepsilon, x_\varepsilon), -\bar{q}, -Y + C\bar{s}I) \\
&\quad - m_2(|\bar{p} + \bar{q}| + C\bar{s}) - m_2(C\bar{s}),
\end{aligned} \tag{6.23}$$

where we use the notation $\bar{s} = |x_\varepsilon - y_\varepsilon|^2/\varepsilon$. Note that since the eigenvalues of εA^2 are given by $\varepsilon\lambda^2$, where λ is an eigenvalue to A, and since λ is bounded, $A + \varepsilon A^2 \le CA$. Hence, by (3.21) we obtain

$$A + \varepsilon A^2 \le \frac{C}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C\bar{s} I_{2n},$$

and since $\|A\| \le C/\varepsilon$ for some large C, we also conclude that (6.22) implies

$$-\frac{C}{\varepsilon} I_{2n} \le \begin{pmatrix} X - C\bar{s}I & 0 \\ 0 & Y - C\bar{s}I \end{pmatrix} \le \frac{C}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Using the above inequality, assumption (2.22), (6.23), the definition of $\bar{q}$ and (3.20) we have

$$\begin{aligned}
0 &\ge -C\bar{s} + 2(t_\varepsilon - s) + \lambda(u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon)) \\
&\quad - m_1(C|x_\varepsilon - y_\varepsilon| + 2C\bar{s}) - m_2(|\bar{p} + \bar{q}| + C\bar{s}) - m_2(C\bar{s}),
\end{aligned}$$

when $0 < \varepsilon < \varepsilon_{\beta_1}$ and $u(t_\varepsilon, x_\varepsilon) \ge v(t_\varepsilon, y_\varepsilon)$. Sending first ε and then β_2 to zero (the latter implies $\beta_1 \to 0$) and using (3.20) we obtain a contradiction. This completes the proof of the comparison principle in Theorem 2.8. □

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Using the same methodology as in the proof of Theorem 2.8, we are now able to prove the comparison principle for mixed boundary conditions stated in Corollary 2.10. This result will be an important ingredient in the proof of Theorem 2.9.

Proof of Corollary 2.10. If u is a viscosity subsolution, then so is $u - K$ for all $K > 0$. It thus suffices to prove that if $u \leq v$ on $(\partial\Omega \setminus G) \cup \overline{\Omega}_0$, then $u \leq v$ in $\widetilde{\Omega}$. If $G = \partial\Omega$, then this implication and its proof is identical to Theorem 2.8. If $G \subset \partial\Omega$ is arbitrary, then we know by assumption that $u \leq v$ on $\partial\Omega \setminus G$ and so the point (s, z) defined in (6.11) must belong to the set G where the boundary condition is satisfied. Hence, we can follow the proof of Theorem 2.8 and conclude that $u \leq v$ in $\widetilde{\Omega}$. □

Proof of Theorem 2.9. We will prove existence using Perron's method. In particular, we show that the supremum of all subsolutions to the initial value problem given by (2.27) is indeed a solution to the same problem. To ensure that the supremum is taken over a nonempty set, we need to find at least one subsolution to the problem. We also need to know that the supremum is finite. This is obtained by producing a supersolution, which, due to the comparison principle, provides an upper bound for the supremum.

To find the supersolution, let, for some constants A and B to be chosen later,

$$\widehat{v} = A\alpha(t,x) + B, \quad \text{for } (t,x) \in \widetilde{\Omega},$$

where $\alpha(t,x)$ is the function guaranteed by Lemma 3.3. By (2.24), (2.25) and the boundedness of Ω°, we can find $A > 0$ such that

$$\langle D\widehat{v}(t,x), \widetilde{\gamma}(t,x)\rangle + f(t,x,\widehat{v}(t,x)) \geq A + f(t,x,0) \geq 0,$$

for $(t,x) \in \partial\Omega$. Moreover, since the support of α lies in U, we have, with λ and m_2 defined in (2.21) and (2.23),

$$\begin{aligned}
&D_t\widehat{v}(t,x) + F\left(t,x,\widehat{v}(t,x), D\widehat{v}(t,x), D^2\widehat{v}(t,x)\right)\\
&\geq -A\sup_U\{|D_t\alpha(t,x)|\} + B\lambda + F(t,x,0,0,0)\\
&\quad - \sup_U m_2\left(A\left(|D\alpha(t,x)| + \|D^2\alpha(t,x)\|\right)\right).
\end{aligned}$$

By (2.20), the boundedness of Ω° and by recalling that we can assume $\lambda > 0$, we see that taking B large enough, $\widehat{v}$ is a classical supersolution of (2.27). Hence, using (2.26) and Proposition 7.2 in [10], $\widehat{v}$ is also a viscosity supersolution. Next, we observe that $\check{u} = -\widehat{v}$ is a viscosity subsolution $\check{u}$ to the problem given by (2.27).

We now apply Perron's method by defining our solution candidate as

$$\widetilde{w} := \sup\{w(x) : w \in USC(\widetilde{\Omega}) \text{ is a viscosity subsolution of (2.27)}\}.$$

In the following we let u^* and u_* denote the upper and lower semicontinuous envelopes of a function u, respectively. By the comparison principle and by construction we obtain

$$\check{u}_* \leq \widetilde{w}_* \leq \widetilde{w}^* \leq \widehat{v}^* \quad \text{on } \widetilde{\Omega}. \tag{6.24}$$

Let us assume for the moment that $\widetilde{w}^*$ satisfies the initial condition of being a subsolution and that $\widetilde{w}_*$ satisfies the initial condition of being a supersolution, that is

$$\widetilde{w}^*(0,x) \leq g(x) \leq \widetilde{w}_*(0,x), \quad \text{for all } x \in \overline{\Omega}_0. \tag{6.25}$$

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We can then proceed as in [10] (see also [2] and [17]) to show that $\widetilde{w}^*$ is a viscosity subsolution and $\widetilde{w}_*$ is a viscosity supersolution of the initial value problem in (2.27). Using the comparison principle again, we then have $\widetilde{w}_* \geq \widetilde{w}^*$ and so by (6.24) $\widetilde{w}_* = \widetilde{w}^*$ is the requested viscosity solution. To complete the proof of Theorem 2.9, it hence suffices to prove (6.25). This will be achieved by constructing families of explicit viscosity sub- and supersolutions.

We first show that the subsolution candidate $\widetilde{w}^*$ satisfies the initial conditions for all $x \in \Omega_0$. To this end, we define, for arbitrary $z \in \Omega_0$ and $\varepsilon > 0$, the barrier function

$$V_{z,\varepsilon}(t,x) = g(z) + \varepsilon + B|x-z|^2 + Ct, \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^n,$$

where B and C are constants, which may depend on z and ε, to be chosen later. We first observe that, by continuity of g and boundedness of Ω_0, we can, for any $\varepsilon > 0$, choose B so large that $V_{z,\varepsilon}(0,x) \geq g(x)$, for all $x \in \overline{\Omega}_0$. Moreover, since $\widetilde{w}$ is bounded on $\overline{\Omega}$, we conclude, by increasing B and C if necessary, that we also have

$$V_{z,\varepsilon}(t,x) \geq \widetilde{w}(t,x), \quad \text{for } (t,x) \in \partial\Omega \cup \overline{\Omega}_0.$$

A computation shows that, for z, ε, B given, we can choose the constant C so large that $V_{z,\epsilon}$ is a classical supersolution of (2.18) in $[0,\infty) \times \mathbb{R}^n$. Hence, by (2.26), $V_{z,\epsilon}$ is also a continuous viscosity supersolution of (2.18) in Ω°. By the maximum principle in Lemma 6.2 applied to $V_{z,\varepsilon}$ and each component in the definition of $\widetilde{w}$, we obtain

$$V_{z,\varepsilon}(t,x) \geq \widetilde{w}(t,x), \quad \text{for } (t,x) \in \widetilde{\Omega}. \tag{6.26}$$

It follows that $\widetilde{w}^* \leq V^*_{z,\varepsilon} = V_{z,\varepsilon}$ in this set and hence the initial condition in Ω_0 follows since for any $x \in \Omega_0$

$$\widetilde{w}^*(0,x) \leq \inf_{\varepsilon,z} V_{z,\varepsilon}(0,x) = g(x). \tag{6.27}$$

To prove that the supersolution candidate $\widetilde{w}_*$ satisfies the initial condition in Ω_0, we proceed similarly by studying a family of subsolutions of the form

$$U_{z,\epsilon}(t,x) = g(z) - B|x-z|^2 - \varepsilon - Ct.$$

We next prove that $\widetilde{w}^*$ satisfies the boundary conditions for each $x \in \partial\Omega_0$. In this case the barriers above will not work as we cannot ensure that they exceed $\widetilde{w}^*$ on $\partial\Omega$. Instead, we will construct barriers that are sub- and supersolutions only locally, near the boundary, during a short time interval. These local barriers are useful due to the maximum principle for mixed boundary conditions proved in Corollary 2.10. To construct the local barriers, fix $\widehat{z} \in \partial\Omega_0$ and let $z(t)$ be the Hölder continuous function

$$z(t) = \widehat{z} - K\widetilde{\gamma}(0,\widehat{z})\, t^{\widehat{\alpha}},$$

where $\widehat{\alpha}$ is the Hölder exponent from (2.8) and K is a constant depending on the Hölder constant and the shape of the exterior cones in (2.4). It follows that $z(t)$ stays inside of Ω for a short time and that $z(0) = \widehat{z}$. Consider, for $\varepsilon > 0$, the barrier function

$$\widetilde{V}_{\varepsilon,\widehat{z}}(t,x) = g(\widehat{z}) + A(\alpha(t,x) - \alpha(0,\widehat{z})) + e^{(\widehat{C}/\chi)\alpha(t,x)} w_\varepsilon(t,x,z(t)) + B + Ct^{\widehat{\alpha}},$$

whenever $(t,x) \in [0,T] \times \mathbb{R}^n$, where $\widehat{C}$ and χ are the constants from Lemma 3.2 and A, B and C are constants to be chosen later, possibly depending on $\widehat{z}$ and ε. We first show that for any choice of A, we can find B such that

$$g(x) \leq \widetilde{V}_{\varepsilon,\widehat{z}}(0,x), \quad \text{for all } x \in \overline{\Omega}_0 \quad \text{and} \inf_{\varepsilon} \widetilde{V}_{\varepsilon,\widehat{z}}(0,\widehat{z}) = g(\widehat{z}). \tag{6.28}$$

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Indeed, to prove the left inequality in (6.28), observe that by (3.14) we have $\chi|x-\widehat{z}|^2/\varepsilon \le w_\varepsilon(0, x, \widehat{z})$. Moreover, by the continuity of $g(\cdot) - A\alpha(0, \cdot)$ in $\overline{\Omega}_0$, we can find B, depending on ε and A, so that

$$g(x) - g(\widehat{z}) - A(\alpha(0, x) - \alpha(0, \widehat{z})) \le B + \chi\frac{|x-\widehat{z}|^2}{\varepsilon}.$$

This proves the left inequality in (6.28). Finally, it is no restriction to assume that $B \to 0$ as $\varepsilon \to 0$, and this implies the right inequality in (6.28).

We next show that $\widetilde{V}_{\varepsilon,\widehat{z}}$ satisfies the boundary condition in a small neighbourhood of $\widehat{z}$ in $\partial\Omega$. To do so, let $E_{\widehat{z}} = (0, \kappa) \times B(\widehat{z}, \rho)$ for some $\kappa, \rho > 0$ to be chosen. We intend to find κ , ρ, A and C such that

$$\langle D_x\widetilde{V}_{\varepsilon,\widehat{z}}(t, x), \widetilde{\gamma}(t, x)\rangle + f\left(t, x, \widetilde{V}_{\varepsilon,\widehat{z}}(t, x)\right) \ge 0, \quad \text{for } (t, x) \in E_{\widehat{z}} \cap \partial\Omega. \tag{6.29}$$

First, observe that α is differentiable in time on $\overline{\Omega}$. Therefore, by taking C large enough and by using (6.28) we ensure that

$$\widetilde{V}_{\varepsilon,\widehat{z}}(t, x) \ge g(\widehat{z}), \quad \text{for } (t, x) \in \overline{\Omega}.$$

In general, the choice of C will depend on A, but it is evident from the next inequality that this will not give rise to circular reasoning. By (2.25) and the boundedness of $\overline{\Omega}$, we can choose A so that

$$f\left(t, x, \widetilde{V}_{\varepsilon,\widehat{z}}(t, x)\right) \ge f(t, x, g(\widehat{z})) \ge -A, \quad \text{for } (t, x) \in \overline{\Omega}.$$

Thus, the boundary condition in (6.29) will follow if we can prove

$$\langle D_x\widetilde{V}_{\varepsilon,\widehat{z}}(t, x), \widetilde{\gamma}(t, x)\rangle \ge A, \quad \text{for } (t, x) \in E_{\widehat{z}} \cap \partial\Omega. \tag{6.30}$$

To this end, choose ρ and κ so small that

$$\langle x - z(t), \widetilde{\gamma}(t, x)\rangle \ge -\theta|x - z(t)| \quad \text{whenever } x \in B(\widehat{z}, \rho) \cap \partial\Omega_t,\ t \in [0, \kappa]. \tag{6.31}$$

Inequality (6.13) then holds with $y = z(t)$ for all $(t, x) \in E_{\widehat{z}} \cap \partial\Omega$. Together with the properties of α, this gives

$$\begin{aligned}
&\langle D_x\widetilde{V}_{\varepsilon,\widehat{z}}(t, x), \widetilde{\gamma}(t, x)\rangle \\
&= A\langle D_x\alpha(t, x), \widetilde{\gamma}(t, x)\rangle + e^{(\widehat{C}/\chi)\alpha(t,x)} \\
&\cdot\left\langle D_x w_\varepsilon(t, x, z(t)) + w_\varepsilon(t, x, z(t))\frac{\widehat{C}}{\chi}D_x\alpha(t, x), \widetilde{\gamma}(t, x)\right\rangle \\
&\ge A - \widehat{C}\frac{|x-z(t)|^2}{\varepsilon} + \chi\frac{|x-z(t)|^2}{\varepsilon}\frac{\widehat{C}}{\chi} = A, \quad \text{for } (t, x) \in E_{\widehat{z}} \cap \partial\Omega.
\end{aligned}$$

This proves (6.30) and hence the boundary condition (6.29) follows.

We now show that for C large enough, $\widetilde{V}_{\varepsilon,\widehat{z}}$ is a supersolution to (2.18), that is

$$\begin{aligned}
&D_t\widetilde{V}_{\varepsilon,\widehat{z}}(t, x) + F\left(t, x, \widetilde{V}_{\varepsilon,\widehat{z}}(t, x), D_x\widetilde{V}_{\varepsilon,\widehat{z}}(t, x), D_x^2\widetilde{V}_{\varepsilon,\widehat{z}}(t, x)\right) \ge 0, \\
&\quad \text{for } (t, x) \in \Omega^0.
\end{aligned} \tag{6.32}$$

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With D_s and D_η denoting differentiation with respect to the first and third arguments of w_ε, respectively, we have

$$\begin{aligned} D_t \widetilde{V}_{\varepsilon,\widehat{z}}(t,x) = {} & A D_t \alpha(t,x) + e^{(\widehat{C}/\chi)\alpha(t,x)} \frac{\widehat{C}}{\chi} D_t \alpha(t,x)\, w_\varepsilon(t,x,z(t)) + e^{(\widehat{C}/\chi)\alpha(t,x)} \\ & \cdot \left(D_s w_\varepsilon(t,x,z(t)) - 2K\widehat{\alpha} \left\langle D_\eta w_\varepsilon(t,x,z(t)), \widetilde{\gamma}(0,\widehat{z}) \right\rangle t^{\widehat{\alpha}-1} \right) \\ & + C\widehat{\alpha} t^{\widehat{\alpha}-1}. \end{aligned} \tag{6.33}$$

Moreover, by (2.21) with $\lambda = 0$ and by (2.23) we have

$$\begin{aligned} & F\left(t, x, \widetilde{V}_{\varepsilon,\widehat{z}}(t,x), D_x \widetilde{V}_{\varepsilon,\widehat{z}}(t,x), D_x^2 \widetilde{V}_{\varepsilon,\widehat{z}}(t,x)\right) \\ & \geq F(t,x,g(\widehat{z}),0,0) - \sup_{\Omega} m_2 \left(|D_x \widetilde{V}_{\varepsilon,\widehat{z}}(t,x)| + \| D_x^2 \widetilde{V}_{\varepsilon,\widehat{z}}(t,x) \| \right). \end{aligned} \tag{6.34}$$

By (3.19)–(3.21), (6.33) and (6.34), we can find C so that (6.32) is satisfied. Hence, using (2.26) and Proposition 7.2. in [10], $\widetilde{V}_{\varepsilon,z}$ is a viscosity supersolution in Ω which satisfies the boundary condition (2.19) on $E_{\widehat{z}} \cap \partial\Omega$ in the viscosity sense.

We now perform the localized comparison. From the construction of $\widetilde{w}$, it is clear that $\widetilde{w}(0,x) \leq g(x)$, for all $x \in \overline{\Omega}_0$. Combined with the left inequality in (6.28), this yields

$$\widetilde{V}_{\varepsilon,\widehat{z}}(0,x) \geq \widetilde{w}(0,x), \quad \text{for } x \in \overline{\Omega}_0. \tag{6.35}$$

Moreover, for some constant K depending on g, α, $\widehat{z}$, A, κ and ρ, we have

$$\widetilde{V}_{\varepsilon,\widehat{z}}(t,x) \geq -K + \chi \frac{|x - z(t)|^2}{\varepsilon} + B, \quad \text{for } (t,x) \in (\partial E_{\widehat{z}} \setminus \partial\Omega) \cap \left([0,\kappa) \times \mathbb{R}^n\right).$$

Since $\widetilde{w}$ is bounded, we can conclude, by increasing B if necessary, that

$$\widetilde{V}_{\varepsilon,\widehat{z}}(t,x) \geq \widetilde{w}(t,x), \quad \text{for } (t,x) \in (\partial E_{\widehat{z}} \setminus \partial\Omega) \cap \left([0,\kappa) \times \mathbb{R}^n\right). \tag{6.36}$$

Now, let κ be so small that for some $\widetilde{\varepsilon} > 0$, it holds that

$$|x - z(t)| > \widetilde{\varepsilon} > 0 \quad \text{whenever } (t,x) \in (\partial E_{\widehat{z}} \setminus \partial\Omega) \cap \left([0,\kappa) \times \mathbb{R}^n\right). \tag{6.37}$$

This choice is possible by the definition of $z(t)$ and by the properties of the domain. Inequality (6.37) implies that it is no restriction to assume that $B \to 0$ as $\varepsilon \to 0$, which is necessary. By means of (6.29), (6.35) and (6.36), we can use Corollary 2.10 to make comparison in $E_{\widehat{z}} \cap \overline{\Omega}$ of the supersolution $\widetilde{V}_{\varepsilon,\widehat{z}}$ with each subsolution in the definition of $\widetilde{w}$. Hence

$$\widetilde{V}_{\varepsilon,\widehat{z}}(t,x) \geq \widetilde{w}(t,x), \quad \text{for } (t,x) \in \overline{E}_{\widehat{z}} \cap \overline{\Omega},$$

and, as a consequence, $\widetilde{V}_{\varepsilon,\widehat{z}} = \widetilde{V}^*_{\varepsilon,\widehat{z}} \geq \widetilde{w}^*$ in $\overline{E}_{\widehat{z}} \cap \overline{\Omega}$. Thus, for any $x \in \partial\Omega_0$,

$$\widetilde{w}^*(0,x) \leq \inf_{\varepsilon,\widehat{z}} \widetilde{V}_{\varepsilon,\widehat{z}}(0,x) = g(x).$$

To prove that $\widetilde{w}_*$ satisfies the initial condition on $\partial\Omega_0$, we proceed similarly by constructing a family of subsolutions of the form

$$\tilde{U}_{\varepsilon,\widehat{z}}(t,x) = g(\widehat{z}) - A(\alpha(t,x) - \alpha(0,\widehat{z})) - e^{(\widehat{C}/\chi)\alpha(t,x)} w_\varepsilon(t,x,z(t)) - B - Ct^{\widehat{\alpha}}.$$

This completes the proof of Theorem 2.9. □

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