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On a covariance structure of some subset of self-similar Gaussian processes

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Abstract

We introduce a class of self-similar Gaussian processes and provide sufficient and necessary conditions for a member of the class to admit a unique small scale limit in the Skorokhod space. The class includes several well known processes. An example of application to the problem of estimation is given.

Keywords: Gaussian process; Covariance; Small scale limit; Tangent process

1. Introduction

In 1923 Norbert Wiener (see, e.g., Karatzas and Shreve [18], Ch. 2, Section 11) provided a rigorous mathematical construction of a centred Gaussian process $W = (W_t)_{t \geq 0}$ describing random movement of particles in suspension. This phenomenon was previously observed in 1828 by the Scottish botanist Robert Brown and attracted the considerable attention of the scientific community. Named in Wiener's honour and nowadays interchangeably termed as (ordinary) Brownian motion or Wiener process, W appeared to be well suited to serve as a basis for a huge amount of models used in different fields of stochastic applications, including those in physics, biology, financial markets, engineering, limit functional theory, etc. To gain insight into this phenomenon, recall that a centred Gaussian process is completely determined by its covariance function. In case of W, the latter is given by (here and in the rest part of the paper, $\wedge$ stands for a minimum)

$$R_W(s,t) = \mathrm{E}\, W_s W_t = s \wedge t.$$

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Its form implies the following properties of W: almost sure equality to 0 at the origin, continuity of the paths, independence and stationarity of increments, Markov property, self-similarity. Although all these properties are important in stochastic modelling and explain previously mentioned popularity of W, the last one is central to the present paper. Therefore, we remind a definition. A real-valued process $(X_t)_{t\geq 0}$ is self-similar with index $\gamma > 0$, provided, for any $c > 0$, it holds $(X_{ct})_{t\geq 0} \overset{d}{=} c^{\gamma}(X_t)_{t\geq 0}$, where $\overset{d}{=}$ denotes equality in distribution. Thus, the covariance of W yields that it is self-similar with $\gamma = 1/2$.

What does it mean, and how it may be described in the context of centred self-similar Gaussian processes? Though self-similarity, by definition, may be interpreted in terms of the time scale zooming, there is no one definite answer. Nonetheless, there are many facts to provide a more detailed view of the picture. In order to depict main (as it seems to us) features, we proceed further with our short historical account.

At the end of the 30s of the 20th century, Kolmogorov looked for a model of turbulence in liquids. As a consequence, in 1940 he introduced [19] the first very well-known and widespread family of self-similar Gaussian processes, encompassing W as a separate case, namely, a family of fractional Brownian motions (further on we use abbreviation fBm; the name and related terminology, appearing in the sequel without additional comments, originates from seminal paper [23] of Mandelbrot and Van Ness). A centred Gaussian process $B^H = (B_t^H)_{t\geq 0}$, depending on some fixed $H \in (0, 1)$, is called a fBm with Hurst index H if its covariance function

$$R_{B^H}(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

Note that H is the self-similarity index of B^H, and that $B^{1/2} = W$.

As with the case of ordinary Brownian motion W, family $\{B^H \mid H \in (0, 1)\}$ became a very popular one, and many models driven by W were translated to the more general setting of $\{B^H \mid H \in (0, 1)\}$.

To gain a concrete example, consider fractional version of the Ornstein–Uhlenbeck process $(X_t)_{t\geq 0}$, investigated by Cheridito et al. [10] and described by a stochastic differential equation

$$\mathrm{d}X_t = \lambda X_t + \sigma \mathrm{d}B_t^H,$$

where $\lambda, \sigma \in (0, \infty)$ are some fixed model defining parameters. One of the possible applications of this stochastic differential equation is a modelling of interest rates. Ability to choose H provides additional flexibility as compared to an initial version driven by $W = B^{1/2}$.

Applications of such kind spawned very intensive research of dependence of properties of B^H on H. Though it still takes place in various forms, much are known. Here are some basic facts featuring a role of the self-similarity index H (Mishura [25] and Nourdin [27] provide much more details):

(a) for any fixed $T, \varepsilon > 0$, there exists a non-negative r.v. $G_{T,\varepsilon}$ having finite moments $\mathrm{E}\, G_{T,\varepsilon}^p$ of any order $p > 0$ and such that $|B_t^H - B_s^H| \leq |t-s|^{H-\varepsilon} G_{T,\varepsilon}$ a.s. for all $t, s \in [0, T]$;

(b) for $\forall t > 0$, power variations of B^H satisfy asymptotic relationship

$$\sum_{j=1}^{2^n} |B^H_{\frac{jt}{2^n}} - B^H_{\frac{(j-1)t}{2^n}}|^p \xrightarrow{P} \begin{cases} 0, \ p > \dfrac{1}{H}; \\ \mathrm{E}\,|B_t^H|^{\frac{1}{H}}, \ p = \dfrac{1}{H}; \\ \infty, \ p < \dfrac{1}{H}; \end{cases}$$

(c) fBm has stationary increments, and the correlation function $\rho_n^H = \mathrm{Corr}(X_0^H, X_n^H)$, $n \geq 0$, of the Fractional noise sequence $X_j^H = B_{j+1}^H - B_j^H$, $j \geq 0$, is summable for $H < \frac{1}{2}$, and is unsummable for $H > \frac{1}{2}$;
(d) for $H \neq 1/2$, fBm is neither a semimartingale, nor a Markov process.

Properties (a)–(b) show that H controls the smoothness of the trajectories, which are Hölder continuous of order $H - \varepsilon$ for any fixed ε. Property (c) demonstrates an impact of H on the strength of dependence between outputs in time. The case $H > 1/2$ corresponds to phenomenon referred in the literature as the long range dependence or the long memory (a good review is given by Samorodnitsky [30]; Beran et al. [6] provide a comprehensive account), which is quite hot topic nowadays, since in practice one meets a lot of processes exhibiting such type of dependence. $H < 1/2$ corresponds to the opposite type, i.e., short range dependence, which also plays an important role in applications, yet it is usually easier to handle analytically. Finally, (d) demonstrates that analytical tractability of B^H having $H \neq 1/2$ is much more challenging than that of W.

Though fBms' family provides considerable flexibility, it is not the only one family of popular centred self-similar Gaussian processes met in applications. Below we provide three more families of such processes and explain one particular unifying interrelationship as well as the relationship with the fBm. Finally, we explicate the purpose of the present paper for which, as we hope, the introductory part above provides sufficient context highlighting the meaning of objects involved.

1. Riemann–Liouville process $RL^H = (RL_t^H)_{t\geq 0}$ is a centred Gaussian process with a covariance function

$$R_{RL^H}(s,t) = \frac{\int_0^{s\wedge t} ((t-v)(s-v))^{H-1/2}\,\mathrm{d}v}{\Gamma^2\left(H+\frac{1}{2}\right)}. \tag{1.1}$$

The corresponding family $\{RL^H \mid H \in (0,1)\}$ depends on a single parameter H, which is also a self-similarity index.

2. Sub-fractional Brownian motion (sfBm) $S^H = (S_t^H)_{t\geq 0}$ was introduced by Bojdecki et al. [9] in the context of occupation time fluctuations of branching particle systems. Covariance of the latter process is given by

$$R_{S^H}(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \tag{1.2}$$

and the corresponding family is again indexed by the self-similarity index $H \in (0,1)$.

3. Bi-fractional Brownian motion (bfBm) $B^{H,K} = (B_t^{H,K})_{t\geq 0}$ is indexed by two parameters: $H \in (0,1)$, $K \in (0,1]$. It was introduced by Houdré and Villa [17], and it has the following covariance function:

$$R_{B^{H,K}}(s,t) = 2^{-K}\left(\left(s^{2H}+t^{2H}\right)^K - |s-t|^{2HK}\right). \tag{1.3}$$

The self-similarity index of $B^{H,K}$ is equal to HK.

Setting $K = 1$, one obtains previously announced relationship with the fBm: $B^H = B^{H,1}$, $H \in (0,1)$. To see the above mentioned interrelationship, we proceed as follows.

Let $\gamma \in (0,1)$, $\sigma \in (0,\infty)$, and $l : [0,\infty) \to \mathbb{R}$ be fixed. Assume that l is measurable, and that $l(0) = 1$. Consider a centred real-valued self-similar Gaussian process $(X_t)_{t\geq 0}$ with $X_0 \equiv 0$,

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self-similarity index γ, and a covariance function given by

$$R(s,t) = \sigma^2 (s \wedge t)^{2\gamma} l\left(\frac{|s-t|}{s \wedge t}\right), s \wedge t > 0. \tag{1.4}$$

It is straightforward to check that the above exemplary families admit such representation. The corresponding quantities are as follows.

- For the case of RL,

$$\begin{aligned} &\gamma = H \in (0,1), \quad \sigma^2 = \frac{1}{2H\Gamma^2\left(H+\frac{1}{2}\right)}, \\ &l(u) = 2H\int_0^1 ((v+u)v)^{H-1/2}\,\mathrm{d}v. \end{aligned} \tag{1.5}$$

- For the case of sfBm,

$$\begin{aligned} &\gamma = H \in (0,1), \quad \sigma^2 = 2 - 2^{2H-1}, \\ &l(u) = (2-2^{2H-1})^{-1}\left(1 + (1+u)^{2H} - \frac{1}{2}\left((2+u)^{2H} + u^{2H}\right)\right). \end{aligned} \tag{1.6}$$

- For the case of bfBm,

$$\gamma = HK \in (0,1), \quad \sigma^2 = 1, \quad l(u) = 2^{-K}\left(\left(1+(1+u)^{2H}\right)^K - u^{2HK}\right). \tag{1.7}$$

There are many works devoted for investigation of the properties of these families. Here is an exemplary list: Alòs et al. [2], Bojdecki et al. [9], Houdré and Villa [17], Lei and Nualart [21], Lim [22], Marinucci and Robinson [24], Russo and Tudor [29], Tudor [31], Tudor and Xiao [32]. Such popularity is the first reason to study a class of centred self-similar Gaussian processes having the covariance given by (1.4).

Next, recall that $\{B^H \mid H \in (0,1)\}$ is the only family of the centred self-similar Gaussian processes having members with stationary increments. Hence, excluding it from the above one, we end up with the class of Gaussian processes with non-stationary increments; additionally, in certain cases, the covariance is suitable for modelling of long-range dependence. Therefore, it is interesting from both practical and theoretical point of view. Moreover, the structure of the covariance function R is completely determined by the self-similarity parameter γ and the function l. It is apparent that different properties of the members of the class can be expressed in terms of the analytic properties of l and the restrictions on the range of γ. Since l depends on a single variable, such characterization appeals to be well suited for applications. Hence one more reason for investigations.

The current paper aims to identify members of the class admitting small scale limit. The concept was introduced by R. L. Dobrushin [14]. It is defined as follows. One says that a process $X = (X_t)_{t\geq 0}$ admits a small scale limit (ssl) at $t_0 \in [0,\infty)$, whenever there exists a normalization $a_{t_0} : (0,\infty) \to (0,\infty)$, $a_{t_0}(u) \to 0+0$, $u \to 0+0$, and a process $Y^{t_0} = (Y^{t_0}_\tau)_{\tau\geq 0}$, such that

$$\left(\frac{X_{t_0} - X_{t_0+\tau u}}{a_{t_0}(u)}\right)_{\tau \geq 0} \xrightarrow{\text{fdd}} (Y^{t_0}_\tau)_{\tau\geq 0}, u \to 0+0, \tag{1.8}$$

where fdd stands for a convergence of finite dimensional distributions. An existence of such limit is quite important property, having both theoretical and practical applications. For the theoretical ones, we refer to Falconer [15,16]. For the practical ones, consider Bardet and Surgailis [3].

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They develop limit theorems targeting ssl setting and provide reasonable amount of statistical estimation examples.

One more thing to note is that, under certain assumptions (see Falconer [15,16]), Gaussian process admits ssls only in the class of fBms (up to the positive multiplier). In the forthcoming part of the paper, we provide sufficient and necessary conditions on l ensuring that X, having covariance given by (1.4), admits such ssl at each $t > 0$. Moreover, it turns out that self-similarity, which is present in our case, enables to replace fdd convergence above by the stronger one—namely, weak convergence in the Skorokhod space $D[0, \infty)$. This is the core result of the paper.

Since, for any $H \in (0, 1)$, the fBm B^H is also in the class considered, it has itself as such limit at each $t > 0$. Hence, the other members of the class having ssl equal to B^H (again, up to the positive constant) are alike in this limiting sense, and one can expect that they share certain properties, resembling the corresponding ones of the B^H. Our main result provides a reflection of this thought in terms of the value of γ and the structure of l.

The remaining part of the paper is split into two sections. Section 2 is devoted for the statement of the main result. It also contains relevant comments and several examples of applications, implied by an existence of ssl. The proofs are given in Section 3.

2. Results

Our main result is contained in the first two theorems given below. Before proceeding to the statement, we provide several comments regarding the notions.

- Whenever it is possible and no confusion occurs, we omit time argument for the process and denote it by a single letter, e.g., X is used instead of $(X_t)_{t\geq 0}$. The time argument always appears as a lower subscript; upper ones are left for the parameters upon which the process depends.
- In all the rest part of the paper, $\xrightarrow{d}$ denotes weak convergence in $D[0, \infty)$ when used with process type arguments. In case of random variables, it denotes a common weak convergence. $\mathcal{F}_D$ denotes the set of random elements of $D[0, \infty)$.
- Let $A \in (0, \infty)$, $f : [0, A] \to \mathbb{R}$. Then we define

$$\begin{aligned} \Delta f_{t,u} &= f(t+u) - f(t),\ \Delta^{(2)} f_{t,u} = \Delta f_{t+u,u} - \Delta f_{t,u} \\ &= f(t+2u) - 2f(t+u) + f(t), \end{aligned}$$

  provided $t, u \geq 0$ are such that $t + 2u \in [0, A]$.
- For any real valued f, g, notion $f \sim g, u \to u_0$, means that $f(u) = g(u)(1 + o(1)), u \to u_0$; the same applies to one sided limits.
- $\mathrm{D}(X)$ denotes a variance of a random variable X.
- Notion $\stackrel{\text{def}}{=}$ stands for "by definition". It is used to define quantities on the flow, when it is not explicitly stated that we define a new quantity.
- Up to now, the fBm B^H was defined for $H \in (0, 1)$ as the centred Gaussian process with the covariance function

$$R_{B^H}(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

  It is convenient to extend this notion and to allow H to attain value 1. Then B^1 is defined by

$$B_t^1 = tZ,\ Z \sim \mathcal{N}(0,1), t \geq 0.$$

It is obvious that $\lim_{H\to 1-0} R_{B^H} = R_{B^1}$. The latter relationship justifies the introduced extension.

Theorem 2.1. *Let $(X_t)_{t\geq 0}$ be Gaussian with a covariance defined by (1.4), and let*

$$l(u) = \frac{1}{2}\left(1 + (1+u)^{2\gamma} - (u^\kappa L(u))^2\right),$$

with some fixed $\kappa \in (0,1]$ and $L : (0,\infty) \to (0,\infty)$ slowly varying at zero. Fix $u, t \in (0,\infty)$, and define a random process $(Z_\tau^{t,u})_{\tau\geq 0}$ by

$$Z_\tau^{t,u} = \frac{X_t - X_{t+u\tau}}{u^\kappa L(u)}, \tau \geq 0.$$

Then $Z^{t,u} \xrightarrow{d} c_t B^\kappa$, $u \to 0+0$, where $c_t = \sigma t^{\gamma-\kappa}$.

Theorem 2.2. *Let $(X_t)_{t\geq 0}$ be Gaussian with a covariance defined by (1.4). Assume that, for all $t \in (0,\infty)$ and some fixed $A > 0$, there exist random process $(Y_\tau^t)_{\tau\geq 0}$ and $a_t : (0,A) \to (0,\infty)$ such that:*

(y1) *Y_1^t is non-degenerate;*
(y2) *$Y^t \in \mathcal{F}_D$;*
(a1) *$a_t(u) \xrightarrow[u\to 0+0]{} 0+0$;*
(a2) *$\left(\frac{X_t - X_{t+\tau u}}{a_t(u)}\right)_{\tau\geq 0} \xrightarrow{d} Y^t$, $u \to 0+0$.*

Then there exist $\kappa \in (0,1]$ and $L : (0,\infty) \to [0,\infty)$ such that:

(i) *L is slowly varying at zero;*
(ii) *$a_t(u) \sim c_t u^\kappa L(u)$, $u \to 0+0$, with $c_t = \sigma t^{\gamma-\kappa}\sqrt{\frac{\mathrm{D}(Y_1^1)}{\mathrm{D}(Y_t^1)}}$;*
(iii) *$l(u) = \frac{1}{2}\left(1 + (1+u)^{2\gamma} - (u^\kappa L(u))^2\right)$, $u > 0$;*
(iv) *for $\forall t$, Y^t is a constant multiple of B^κ, and it is a unique (up to a constant multiplier) small scale limit of X at t.*

Summarizing contents of the above theorems, we arrive to the conclusion that the centred Gaussian process having covariance (1.4) admits a ssl if and only if function l is of the form given in Theorem 2.1. Quantity κ is the self-similarity index of the corresponding ssl. Proposition 2.1 below shows that $\kappa = \gamma$ for all processes mentioned in the introductory section. That is, it coincides with their self-similarity index. Note, however, that this is not true in general. To get a plain example, fix $H \in (0, \frac{1}{2})$ and take $R(s,t) = R_W(s,t)R_{B^H}(s,t)$. Simultaneous application of the forthcoming Proposition 2.1 and Theorem 2.1 implies that a centred Gaussian process X having covariance $R(s,t)$ admits the ssl with $\kappa = H$; yet its self-similarity index γ equals to $\frac{1}{2} + H$. Keeping this in mind, we proceed to our examples mentioned in the very end of the introductory section.

Under assumptions that the model under consideration involves the process X, which belongs to the Gaussian class defined by covariance (1.4), and which admits the ssl, it may be of interest to estimate κ.

Once the structure of R is fully specified, there are many ways to do this. One can start from plain maximum-likelihood approach. In case of failure, one can switch to other alternatives, which are more suitable or convenient for the context considered. The list includes more general M-estimators, L-estimators, Z-estimators, Bayes estimators, power-variation based estimators, etc. Turning to particular examples of the introductory section, literature is abundant, and it offers many opportunities (especially for the case of the fBm). One can assume that: X is observed directly, or that it is involved into some more complex model (consider, e.g., stochastic differential equation for the Ornstein–Uhlenbeck process, mentioned in the introductory section); the data is collected at equally or unequally spaced time-points, within a fixed or infinitely growing time interval; observations belong to one particular trajectory, or there are several independent copies of these; etc. We do not aim to provide a review; therefore, we will not dive into details, and we will not provide any references treating particular models of the introductory section.

Talking about the partially specified structure of R, the things go differently. While searching the literature, we have found out that several authors considered classes of Gaussian processes spanning functional form of the covariance function considered by us or at least intersecting one of ours. Here is the chronological list: Dahlhaus and Polonik (2006) [13], Bégyn (2007) [5], Coeurjolly (2008) [12], Bardet and Surgailis (2011, 2013) [3,4], Kubilius (2015) [20], Norvaiša (2015) [26]. Below are brief comments.

Assuming appropriate discrete observational setting, a model investigated by Dahlhaus and Polonik [13] could be employed to fit the covariance structure considered by us. Nonetheless, providing our illustrative example below, we did not take this opportunity into account because of the following reasons. First of all, results of Dahlhaus and Polonik [13] are presented in inconvenient to us spectral setting, and the corresponding time domain representation is quite artificial and unnatural in our context. Secondly, results of Bardet and Surgailis [4] cover these of Dahlhaus and Polonik [13], and they also generalize these of Bardet and Surgailis [3] (see Example 5.1), taken by us as a basis for illustration of the utility of ssl property. Summing up, we did not expect any real benefit from adoption results of Dahlhaus and Polonik [13]: neither in ease of interpretation or analytical tractability, nor in technical performance.

The functional form of covariance considered by Coeurjolly [12] spans (1.4), however, the author imposes stationarity of increments.

Results of Kubilius [20], Norvaiša [26] and Bégyn [5] seem closest in the spirit of setting and validity to apply immediately. Kubilius [20] and Norvaiša [26] impose asymptotic covariance constraints satisfied by the covariance considered by us. However, the form of assumed covariance explicates only self-similarity index γ. Therefore, such quantity as κ is not present at all. Hence, by making use of their results, we can estimate κ consistently only under the assumption $\kappa = \gamma$. It is also important to note that both authors do not offer confidence limits, and their focus is mainly on point estimators. Bégyn [5] does. However, there is the same problem of explication of γ alone. Additionally, the author imposes differentiability of covariance. In general, the latter should not hold in our case.

In a view of this short survey, we provide illustrative Theorem 2.3, based on the result of Bardet and Surgailis [3], which seems to be the most general (together with their result [4]) from the above and best suited for our case. Theorem 2.3 applies to any values of γ and κ. It may be adopted in the context of stochastic differential equations driven by processes from the class considered. This makes sense since power variational methods usually work well here. Nonetheless, we point out that it is of illustrative nature, rather than being of real benefit, and that, in our opinion, one can do better when dealing with particular models. Some related discussion is given after the statement. By proving this theorem, we pursued two goals.

First of all, we aimed to demonstrate the use of ssl property. It comes into play due to the fact that, deriving their results presented in [3,4], the authors, in fact, targeted the setting under which ssl exists, i.e., the one of ours.

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Secondly, we aimed to demonstrate difficulties arising even in such case when the structure of the covariance is quite well explicated. By inspecting the proof, one can clearly see the restrictions laid on by the general "ready to apply" result of [3].

In order to state the theorem, we describe the statistical model first. Let $T > 0$ be fixed. Assume we have observations coming from one particular trajectory of a centred Gaussian X having covariance given by (1.4). These are collected at discrete time grid lying within $[0, T]$. We aim to estimate κ. To achieve this, we can make use of Theorem 2.3 and Corollary 2.1.

Theorem 2.3. *Assume that conditions of Theorem 2.1 hold. Moreover, let L satisfy the following additional constraints:*

(L1) $L(0) \overset{\text{def}}{=} \lim_{u\to 0+0} L(u)$ *exists, is positive and finite;*
(L2) $L(u) = L(0) + o(\sqrt{u}),\ u \to 0+0$*;*
(L3) *there exist fixed* $c \in (0, \infty)$, $\zeta \in (\frac{1}{2}, \infty)$ *and* $\varepsilon \in (0, \frac{1}{2})$ *such that, for* $\forall k \in \{2, \dots, n-2\}$, $n \geq 3$, *and for* $\forall u \in \left(0, \frac{\varrho}{n}\right]$,

$$\frac{\left|\Delta^{(2)} p_{ku,u} - 2(1+u)^{2\gamma} \Delta^{(2)} p_{(k-1)\frac{u}{1+u}, \frac{u}{1+u}} + (1+2u)^{2\gamma} \Delta^{(2)} p_{(k-2)\frac{u}{1+2u}, \frac{u}{1+2u}}\right|}{u^{2\kappa}} \leq ck^{-\zeta},$$

where $p(u) = (u^{\kappa} L(u))^2$ *and* $\varrho = \frac{1-\varepsilon}{\varepsilon}$.
Then, for $T_0 = T_0(\varepsilon, T) = \varepsilon T$ *and* $\delta_n = \delta_n(\varepsilon, T) = \frac{1-\varepsilon}{n} T$,

(i) $R_n^T \overset{\text{def}}{=} \frac{1}{n-2} \sum_{k=0}^{n-3} \psi\left(\Delta^{(2)} X_{T_0+k\delta_n, \delta_n}, \Delta^{(2)} X_{T_0+(k+1)\delta_n, \delta_n}\right) \xrightarrow{a.s} \Lambda(\kappa) = \lambda(r(\kappa))$, *where*

$$\psi(x, y) = \frac{|x+y|}{|x|+|y|}, \tag{2.1}$$

$$\lambda(r) = \frac{1}{\pi}\left(\arccos(-r) + \sqrt{\frac{1+r}{1-r}} \ln\left(\frac{2}{1+r}\right)\right), \tag{2.2}$$

$$r(x) = \operatorname{corr}(\Delta^{(2)} B_{0,1}^x, \Delta^{(2)} B_{1,1}^x) = \frac{-7 - 9^x + 4^{x+1}}{2(4-4^x)},\ x \in (0, 1); \tag{2.3}$$

(ii) $\sqrt{n}\left(R_n^T - \Lambda(\kappa)\right) \xrightarrow{d} \mathcal{N}(0, \Sigma(\kappa))$, *where*

$$\Sigma(x) = \sum_{k\in\mathbb{Z}} \operatorname{Cov}\left(\psi(\Delta^{(2)} B_{0,1}^x, \Delta^{(2)} B_{1,1}^x), \psi(\Delta^{(2)} B_{k,1}^x, \Delta^{(2)} B_{k+1,1}^x)\right). \tag{2.4}$$

Corollary 2.1. $\widehat{\kappa}_n \overset{\text{def}}{=} \Lambda^{-1}(R_n^T) \xrightarrow{a.s} \kappa; \sqrt{n}(\widehat{\kappa}_n - \kappa) \xrightarrow{d} \mathcal{N}(0, \Sigma(\kappa)(\Lambda'(\Lambda^{-1}(\kappa)))^2)$.

Now, after the statement, we can provide several additional remarks explicating weakness of the theorem and expressing some speculations regarding asymptotic performance. We also point out directions of possible improvements.

It is common to assume that one observes a trajectory of the process within $[0, T]$. Theorem 2.3 therefore states that one should discard the data coming from $[0, \varepsilon T]$. The requirement seems pretty strange and could be treated as an artificial condition, imposed by the method of proving of the CLT. On the other hand, note that, with the fBm being an exception, the process under consideration is the one with non-stationary increments. Consequently, its behaviour at the start of evolution is expected to be unpleasant, and the stable one appears only after some time has passed. Moreover, even discarding the portion of data from $[0, \varepsilon T]$ (if such does exist) and applying theorem only to data from $[\varepsilon T, T]$, one still retains the usual rate of convergence in CLT. Thus, it is very likely that the improvements of shrinkage of asymptotic confidence interval are possible only up to a constant multiplier, with the order of shrinkage remaining $n^{-\frac{1}{2}}$. Practical superiority of estimating statistics based on data from $[0, T]$, rather than $[\varepsilon T, T]$, is also questionable because of the reasons mentioned above. That is, convergence to asymptotic distribution may be slower and/or more unstable, giving a real gain only for very large datasets. Moreover, as it was mentioned previously, a lot depends on the initial modelling context. In order to address these questions, simulation studies similar to that of Coeurjolly [11] (conducted for the case of the fBm) are needed. For this, however, some time should pass, since, to our best knowledge, the setting considered here is a new one. Consequently, κ has first to be recognized as an important quantity on its own. Talking about an improved version of Theorem 2.3, which is based on data from $[0, T]$, our advice is to take careful inspection (undone by us) of results of Bardet and Surgailis [4]. We are inclined to think that such inspection should yield a remedy, and we see here an open room for those interested in a challenge of such kind.

We finish this section by one more illustrative statement. It shows that all families of processes listed in the introduction possess ssls having self-similarity index equal to the self-similarity index of the original process.

Proposition 2.1. *For $\forall H \in (0, 1)$ and for $\forall K \in (0, 1)$, covariances of processes (S_t^H), $(B_t^{H,K})$ and (LR_t^H) admit representation with l as in Theorem 2.1. The defining quantities are as follows:*

- $L^2_{S^H}(u) = \frac{1}{2-2^{2H-1}}\left(1+\left(\frac{2}{u}\right)^{2H}\left(\left(1+\frac{u}{2}\right)^{2H} - \frac{1+(1+u)^{2H}}{2}\right)\right)$, $\kappa = H$;
- $L^2_{B^{H,K}}(u) = 2^{1-K}\left[1+\left(\frac{1}{u}\right)^{2HK}\left(2^{K-1}(1+(1+u)^{2HK}) - (1+(1+u)^{2H})^K\right)\right]$, $\kappa = HK$;
- $L^2_{LR^H}(u) = 2H\int_0^{1/u}\left[v^{2H-1} + (v+1)^{2H-1} - 2(v(1+v))^{H-1/2}\right]dv$, $\kappa = H$.

Moreover, Corollary 2.1 applies to all classes of processes as well, provided that $\kappa < \frac{3}{4}$.

3. Proofs

Proof of Theorem 2.1. By self similarity of X,

$$\left(Z_\tau^{t,u}\right) \overset{d}{=} t^\gamma\left(\frac{X_1 - X_{1+\frac{u}{t}\tau}}{u^\kappa L(u)}\right) = t^{\gamma-\kappa}\left(\frac{X_1 - X_{1+\frac{u}{t}\tau}}{\left(\frac{u}{t}\right)^\kappa L\left(\frac{u}{t}\right)}\right)\frac{L\left(\frac{u}{t}\right)}{L(u)} = t^{\gamma-\kappa}\frac{L\left(\frac{u}{t}\right)}{L(u)}(Z_\tau^{1,\frac{u}{t}}).$$

Therefore, taking into account slow variation of L, it suffices to prove the theorem for $t = 1$. We accomplish this by checking that conditions given in Pollard [28], Ch. VI, Lemma 9 and Theorem 10 (see also Billingsley [7], Theorem 8.2) hold. For short, we omit time parameter and write Z_τ^u instead of $Z_\tau^{1,u}$. We split the check of the above mentioned conditions into two separate items: in *(i)*, we check convergence of finite dimensional distributions; in *(ii)*, we verify regularity of the paths.

(i) Fix $0 < \tau_1 \le \tau_2 < \infty$; to avoid inconsistencies, put $0 \cdot L(0) \overset{\text{def}}{=} 0$. Then $\mathrm{E}\, Z_{\tau_1}^u = \mathrm{E}\, Z_{\tau_2}^u = 0$ and

$$\frac{u^{2\kappa}L^2(u)}{\sigma^2}\operatorname{Cov}(Z_{\tau_1}^u, Z_{\tau_2}^u) = \frac{1}{\sigma^2}\mathrm{E}(X_1 - X_{1+\tau_1 u})(X_1 - X_{1+\tau_2 u}) =$$

$$1 - l(\tau_1 u) - l(\tau_2 u) + (1+\tau_1 u)^{2\gamma} l\left(\frac{(\tau_2-\tau_1)u}{1+\tau_1 u}\right) =$$

$$-\frac{1}{2}\Big[(1+\tau_1 u)^{2\gamma} + (1+\tau_2 u)^{2\gamma} - u^{2\kappa}\left(\tau_1^{2\kappa}L^2(\tau_1 u) + \tau_2^{2\kappa}L^2(\tau_2 u)\right) -$$

$$(1+\tau_1 u)^{2\gamma}\left(1+\left(\frac{1+\tau_2 u}{1+\tau_1 u}\right)^{2\gamma}-\left(\frac{(\tau_2-\tau_1)u}{1+\tau_1 u}\right)^{2\kappa}L^2\left(\frac{(\tau_2-\tau_1)u}{1+\tau_1 u}\right)\right)\Bigg]=$$
$$\frac{u^{2\kappa}}{2}\left[\tau_1^{2\kappa}L^2(\tau_1 u)+\tau_2^{2\kappa}L^2(\tau_2 u)-(\tau_2-\tau_1)^{2\kappa}(1+\tau_1 u)^{2(\gamma-\kappa)}L^2\left(\frac{(\tau_2-\tau_1)u}{1+\tau_1 u}\right)\right].$$

Since L varies slowly at 0, Bingham et al. [8], Theorem 1.2.1 implies that

$$\frac{L(\tau_i u)}{L(u)}\xrightarrow[u\to 0+0]{}1,\, i=1,2,\quad \text{and that}\quad \frac{L\left(\frac{(\tau_2-\tau_1)u}{1+\tau_1 u}\right)}{L(u)}\xrightarrow[u\to 0+0]{}1.$$

Thus,

$$\mathrm{Cov}(Z^u_{\tau_1},Z^u_{\tau_2})\xrightarrow[u\to 0+0]{}\frac{\sigma^2}{2}\left(\tau_1^{2\kappa}+\tau_2^{2\kappa}-(\tau_2-\tau_1)^{2\kappa}\right)$$

or equivalently, $Z^u \xrightarrow{\text{fdd}} \sigma B^\kappa$, $u\to 0+0$.

(ii) Fix $\epsilon,\delta\in(0,\infty)$ and $0\le a<b<\infty$, and any $(u_n)_{n\ge 1}: u_n \xrightarrow[n\to\infty]{} 0+0$. For the sake of clarity, we divide a verification of regularity of paths into several steps.

Step 1. Let $0\le c<d<\infty$ be fixed. Then (because of a.s. continuity of $\tau\longmapsto Z^{u_n}_\tau$)

$$\left\{\eta>0\,\middle|\,\exists s\in[c,d]:\sup_{c\le\tau<s}\left|Z^{u_n}_\tau-Z^{u_n}_c\right|<\eta,\ \sup_{s\le\tau\le d}\left|Z^{u_n}_\tau-Z^{u_n}_d\right|<\eta\right\}\supset$$
$$\left\{\eta>0\,\middle|\,\sup_{c\le\tau\le d}\left|Z^{u_n}_\tau-Z^{u_n}_c\right|<\eta/2\right\}\cup\left\{\eta>0\,\middle|\,\sup_{c\le\tau\le d}\left|Z^{u_n}_\tau-Z^{u_n}_d\right|<\eta/2\right\}\ a.s.$$

Consequently,

$$\mathrm{P}\left(\Delta(Z^{u_n},[c,d])>\eta\right)\le \mathrm{P}\left(\sup_{c\le\tau\le d}\left|Z^{u_n}_\tau-Z^{u_n}_c\right|\ge\frac{\eta}{2}\right)$$
$$+\mathrm{P}\left(\sup_{c\le\tau\le d}\left|Z^{u_n}_\tau-Z^{u_n}_d\right|\ge\frac{\eta}{2}\right)\le$$
$$2\left[\mathrm{P}\left(\sup_{c\le\tau\le d}(Z^{u_n}_\tau-Z^{u_n}_c)\ge\frac{\eta}{2}\right)+\mathrm{P}\left(\sup_{c\le\tau\le d}(Z^{u_n}_\tau-Z^{u_n}_d)\ge\frac{\eta}{2}\right)\right]$$

with the last being true because of the fact that Z^{u_n} is centred. Thus, in order to bound left hand side, it suffices to bound each probability on the right hand side. We give a detailed implementation for the first one. The other one is handled in the same way.

Step 2. Let $0\le c<d<\infty$ be from the *Step 1*. Assume

$$d-c<4^{-\frac{1}{\kappa}}. \tag{3.1}$$

Define a process $(\xi^{n,c,d}_t)_{t\in[c,d]}$ by $\xi^{n,c,d}_t=Z^{u_n}_t-Z^{u_n}_c$, $t\in[c,d]$. Then $(\xi^{n,c,d}_t)$ is centred and continuous. Moreover, for $t\in[c,d]$,

$$u_n^{2\kappa}L^2(u_n)\,\mathrm{E}(\xi^{n,c,d}_t)^2=\mathrm{E}(X_{1+u_nt}-X_{1+u_nc})^2=\sigma^2(1+u_nc)^{2(\gamma-\kappa)}((t-c)u_n)^{2\kappa}$$
$$\times L^2\left(\frac{(t-c)u_n}{1+u_nc}\right).$$

Since $t-c \in [0,1)$, and $u_n \to 0+0$, Bingham et al. [8], Theorem 1.5.6 implies an existence of n_0 such that, for $\forall n \geq n_0$, it holds

$$\left(\frac{L\left(\frac{(t-c)u_n}{1+u_n c}\right)}{L(u)}\right)^2 \leq 2\sigma^{-2}(t-c)^{-\kappa} \vee (t-c)^{\kappa} \leq \frac{2\sigma^{-2}}{(t-c)^{\kappa}}. \tag{3.2}$$

We can also assume that n_0 is chosen so that $(1+u_n c)^{2(\gamma-\kappa)} \leq 2$ for $\forall n \geq n_0$. Then by (3.1),

$$\sigma^2_{[c,d]} \stackrel{\text{def}}{=} \sup_{t\in[c,d]} \mathrm{E}(\xi_t^{n,c,d})^2 \leq 4(d-c)^{\kappa} < 1. \tag{3.3}$$

Next, note that, for $c \leq s < t \leq d$, it holds $\xi_t^{n,c,d} - \xi_s^{n,c,d} = Z_t^{u_n} - Z_s^{u_n} = \xi_t^{n,s,d}$. Hence, by the above, the canonical metric of $\xi^{n,c,d}$ may be bounded as follows:

$$\rho^2_{\xi^{n,c,d}}(s,t) \leq 4|t-s|^{\kappa}.$$

Consequently, the smallest number of balls having $\tilde{\epsilon} \in (0,1]$ radius with respect to $\rho_{\xi^{n,c,d}}$ and covering $[c,d]$ satisfies

$$N([c,d], \rho_{\xi^{n,c,d}}, \tilde{\epsilon}) \leq \left(\frac{A}{\tilde{\epsilon}}\right)^{\frac{2}{\kappa}},$$

where $A \geq 2(d-c)^{\frac{\kappa}{2}}$ can be chosen arbitrary. Let $\epsilon_0 = \sigma_{[c,d]}$. Then application of Adler and Taylor [1], Theorem 4.1.2 yields

$$\mathrm{P}\left(\sup_{t\in[c,d]} \xi^{n,c,d} \geq \eta\right) \leq \left(\frac{\kappa}{2}\frac{(KA\eta)^2}{\sigma^4_{[c,d]}}\right)^{1/\kappa} \bar{\Phi}\left(\frac{\eta}{\sigma_{[c,d]}}\right)$$

for any $\eta \geq \sigma_{[c,d]}\left(1+\sqrt{\frac{2}{\kappa}}\right)$. In particular, setting $\eta = \theta_\kappa\sqrt{\sigma_{[c,d]}}$, $\theta_\kappa = \left(1+\sqrt{\frac{2}{\kappa}}\right)$ and taking into account the bound (3.3), one has

$$\mathrm{P}\left(\sup_{t\in[c,d]} \xi^{n,c,d} \geq \theta_\kappa\sqrt{\sigma_{[c,d]}}\right) \leq \frac{\tilde{K}}{\sigma^{\frac{3}{\kappa}}_{[c,d]}} \bar{\Phi}\left(\frac{\theta_\kappa}{\sqrt{\sigma_{[c,d]}}}\right) \tag{3.4}$$

for $n \geq n_0$ and $\tilde{K} = (\frac{\kappa}{2}(KA\theta_\kappa)^2)^{\frac{1}{\kappa}}$. Finishing this step, we note the following.

- The constant $\tilde{K}$ on the right hand side of (3.4) may be regarded as universal, provided one neglects an obvious dependence on κ; it suffices to assume (3.1). Then, taking $A \geq 1 > \sigma_{[c,d]}$, we have that the constraint imposed on A by the Adler and Taylor [1], Theorem 4.1.2 also holds.
- $\sigma^2_{[c,d]} = O((d-c)^{\kappa}) = o(1)$, $(d-c) \to 0+0$.
- Because of (3.2) and condition $(1+u_n c)^{2(\gamma-\kappa)} \leq 2$, $n \geq n_0$, we have that n_0 assuring (3.4) depends on c and $d-c$. It is an increasing function of both. However, assumption (3.1) discards the dependence on difference $d-c$.

Step 3. Let integer $k \geq 0$, $m \geq 1$ be such that $[k, k+m] \supset [a,b]$, and k is the biggest whereas m is the smallest among all having this property. Partition each $[j, j+1]$, $j = k, \ldots, k+m-1$, into equal intervals $[t_l^j, t_{l+1}^j]$, $l = 0, \ldots, q-1$, so that $4(t_{l+1}^j - t_l^j)^{\kappa} < 1 \wedge (\delta\theta_\kappa^{-1})^2$. Then, for $\forall j, l$,

$$t_{l+1}^j - t_l^j < 4^{-\frac{1}{\kappa}} \quad \text{and} \quad \theta_\kappa\sqrt{\sigma_{[t_l^j, t_{l+1}^j]}} < \delta.$$

Therefore, an application of the results obtained in the previous steps (from line to line varying constant value is denoted by the same letter K because its magnitude does not affect the limit) yields

$$\mathrm{P}\left(\max_{j,l} \Delta(Z^{u_n}, [t_l^j, t_{l+1}^j] > \delta)\right) \le K \cdot q \cdot m \max_{j,l} \sigma_{[t_l^j, t_{l+1}^j]}^{-\frac{3}{\kappa}} \bar{\Phi}\left(\frac{\theta_\kappa}{\sqrt{\sigma_{[t_l^j, t_{l+1}^j]}}}\right) \le$$

$$K(b-a) \max_{j,l} \sigma_{[t_l^j, t_{l+1}^j]}^{-\frac{5}{\kappa}} \bar{\Phi}\left(\frac{\theta_\kappa}{\sqrt{\sigma_{[t_l^j, t_{l+1}^j]}}}\right)$$

since $q^{-1} = t_{l+1}^j - t_l^j$ for all j, l, and m is proportional to $(b-a)$. It is clear that $x^{\frac{5}{\kappa}} \bar{\Phi}(\theta_\kappa \sqrt{x}) \xrightarrow[x\to\infty]{} 0$. Hence, if there is a need, one can increase the value of q up to the smallest integer for which the right hand side does not exceed ϵ. Then it remains to pass to the upper limit as $n \to \infty$. □

Proof of Theorem 2.2. *Step 1.* Fix $t \in (0, \infty)$, and define a random process $(Z_\tau^t)_{\tau>0}$ by

$$Z_\tau^t = X_t - X_{t+\frac{1}{\tau}}, \tau > 0.$$

Let $y \to \infty$. Put $u = y^{-1}$, $f_t(y) = a_t(u)$. Then $\left(\frac{Z_{\tau y}^t}{f_t(y)}\right)_{\tau>0} \xrightarrow{\text{fdd}} \left(Y_{\frac{1}{\tau}}^t\right)_{\tau>0}$. Therefore, Bingham et al. [8], Theorems 8.5.1–8.5.2 imply that $\left(Y_{\frac{1}{\tau}}^t\right)_{\tau>0}$ is self-similar with some index $\kappa_t \in \mathbb{R}$, and f_t is regularly varying with κ_t. Consequently, $\left(Y_\tau^t\right)_{\tau>0}$ is self-similar with index $-\kappa_t$, and a_t is regularly varying at 0 with $-\kappa_t$. Since

$$Y_0^t \overset{d}{=} \lim_{u\to 0+0} \frac{X_t - X_{t+0\cdot u}}{a_t(u)} \equiv 0,$$

$\left(Y_\tau^t\right)_{\tau\ge 0}$ is also self-similar with index $-\kappa_t$. Moreover, assumption $a_t(u) \xrightarrow[u\to 0+0]{} 0+0$ implies that $\kappa_t \le 0$. In fact, one must necessary have $\kappa_t < 0$. Indeed, if it were true that $\kappa_t = 0$, then, by Bingham et al. [8], Theorems 8.5.1–8.5.2, it were true that $Y_\tau^t \overset{d}{=} Y_1^t + b \ln \tau$, $\tau > 0$. Since the limit of the Gaussian process is Gaussian, fdd convergence yields convergence of the first two moments. Thus, for any $\tau > 0$,

$$\mathrm{E}\left(\frac{X_t - X_{t+\tau u}}{a_t(u)}\right) \xrightarrow[u\to 0+0]{} \mathrm{E}\, Y_\tau^t = 0 \Rightarrow 0 = \mathrm{E}\, Y_\tau^t = \mathrm{E}\, Y_1^t + b \ln \tau = b \ln \tau \Rightarrow b$$

$$= 0 \Rightarrow Y_\tau^t \overset{d}{=} Y_1^t.$$

By assumption, Y_1^t is non-degenerate. On the other hand, continuity of the paths on the right yields

$$\mathrm{P}\left(Y_1^t = 0\right) = \mathrm{P}\left(\lim_{\tau\to 0+0} Y_\tau^t = 0\right) = 1.$$

Obtained contradiction excludes the case $\kappa_t = 0$. Also note that, in Bingham et al. [8], Theorem 8.5.1, constant c is equal to 0 because of the same condition $Y_0^t = 0$.

Step 2. Fix $t \in (0, \infty)$. By results of the *Step 1*,

$$\frac{X_t - X_{t+tu}}{a_t(u)} \xrightarrow{d} Y_t^t \Rightarrow \mathrm{D}\left(\frac{X_t - X_{t+tu}}{a_t(u)}\right) \to \mathrm{D}(Y_t^t) = t^{-2\kappa_t}\, \mathrm{D}(Y_1^t).$$

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On the other hand, by self-similarity of X,

$$\mathrm{D}\left(\frac{X_t - X_{t+tu}}{a_t(u)}\right) = t^{2\gamma}\frac{a_1^2(u)}{a_t^2(u)}\mathrm{D}\left(\frac{X_1 - X_{1+u}}{a_1(u)}\right) \sim t^{2\gamma}\frac{a_1^2(u)}{a_t^2(u)}\mathrm{D}(Y_1^1).$$

Thus,

$$a_t(u) \sim t^{\gamma+\kappa_t}a_1(u)\sqrt{\frac{\mathrm{D}(Y_1^1)}{\mathrm{D}(Y_1^t)}} = t^{\gamma+\kappa_t}u^{-\kappa_1}L_1(u)\sqrt{\frac{\mathrm{D}(Y_1^1)}{\mathrm{D}(Y_1^t)}},$$

where L_1 is slowly varying at 0 by the *Step 1*. Hence, for $\forall t,\ \kappa_t = \kappa_1 \overset{\text{def}}{=} -\kappa$.

Step 3.

$$\begin{aligned}\mathrm{D}(X_1 - X_{1+u}) &= \sigma^2(1 + (1+u)^{2\gamma} - 2l(u)) \Rightarrow l(u)\\ &= \frac{1}{2}\left(1 + (1+u)^{2\gamma} - \frac{\mathrm{D}(X_1 - X_{1+u})}{\sigma^2}\right).\end{aligned}$$

By all above, $\mathrm{D}(X_1 - X_{1+u}) \sim a_1^2(u) \sim u^{2\kappa}L_1^2(u)$. Therefore, $L(u) \overset{\text{def}}{=} \sqrt{\frac{\mathrm{D}(X_1 - X_{1+u})}{\sigma^2 u^{2\kappa}}}$ varies slowly at 0.

Step 4. It remains to prove the last claim. Fix $t \in (0,\infty)$. Note that $c_t Y^t \in \mathrm{Tan}(X,t)$ for any $r_n \downarrow 0$ and $q_n \overset{c_t \neq 0}{=} c_t^{-1}a_t(r_n)$. Take arbitrary $\tilde{Y} \in \mathrm{Tan}(X,t)$. If $r_n \downarrow 0, q_n \downarrow 0$ are such that $\left(\frac{X_{t+\tau r_n} - X_t}{q_n}\right)_{\tau\geq 0} \xrightarrow{d} \tilde{Y}$, then Gaussianity yields $\mathrm{D}\left(\frac{X_t - X_{t+tr_n}}{q_n}\right) \xrightarrow[n\to\infty]{} \mathrm{D}\left(\tilde{Y}_t\right)$. If $\mathrm{D}\left(\tilde{Y}_t\right) > 0$, then

$$t^{2\gamma}\frac{a_1^2(r_n)}{q_n^2}\mathrm{D}(Y_1^1) \sim t^{2\gamma}\frac{a_1^2(r_n)}{q_n^2}\mathrm{D}\left(\frac{X_1 - X_{1+r_n}}{a_1(r_n)}\right) = \mathrm{D}\left(\frac{X_t - X_{t+tr_n}}{q_n}\right) \to \mathrm{D}\left(\tilde{Y}_t\right) > 0 \Rightarrow$$

$$q_n \sim t^\gamma\sqrt{\frac{\mathrm{D}(Y_1^1)}{\mathrm{D}\left(\tilde{Y}_t\right)}}a_1(r_n).$$

Consequently, $\tilde{Y}$ is a constant multiple of Y^t. If $\mathrm{D}\left(\tilde{Y}_t\right) = 0$, then, by Falconer [16], Proposition 3.3 and self-similarity of X, $\mathrm{D}\left(\tilde{Y}_\tau\right) = 0$ for all $\tau > 0$. Thus, $\tilde{Y}$ is zero multiple of Y^t. Summing up, Y^t is a unique tangent process of X at t. By Falconer [16], Corollary 4.3, the set of such $t \in (0,\infty)$ for which Y^t is not a scalar multiple of B^κ has the Lebesgue measure 0. More than that, self-similarity of X implies that this set is empty. Indeed, fix arbitrary t_0 having property $Y^{t_0} \overset{d}{=} c_{t_0}B^\kappa$, and take any $t_1 \in (0,\infty)\setminus\{t_0\}$. Then the claim follows by noting that

$$\left(\frac{X_{t_1} - X_{t_1+\tau u}}{a_{t_1}(u)}\right)_{\tau\geq 0} \overset{d}{=} \left(\frac{t_1}{t_0}\right)^\gamma\left(\frac{X_{t_0} - X_{t_0+\tau\frac{ut_0}{t_1}}}{a_{t_1}(u)}\right)_{\tau\geq 0} =$$

$$c_{t_0,t_1}(u)\left(\frac{X_{t_0} - X_{t_0+\tau\frac{ut_0}{t_1}}}{a_{t_0}(\frac{t_0 u}{t_1})}\right)_{\tau\geq 0} \xrightarrow{d} \tilde{c}_{t_0,t_1}c_{t_0}B^\kappa,\ u \to 0+0,$$

where $c_{t_0,t_1}(u) \xrightarrow[u\to 0+0]{} \tilde{c}_{t_0,t_1} \in (0,\infty)$. □

Proof of Theorem 2.3. Define a process $(Z_t^T)_{t\in[0,1]}$ by $Z_t^T = X_{T_0+tT}, t \in [0,1]$. Below, we show that, under assumptions made above, Theorems 4.1–4.2 of Bardet and Surgailis [3] apply to Z^T with $H(t) \equiv \kappa$ and $c(t) = (\varepsilon^\gamma \varrho^\kappa \sigma L(0)(1+\varrho t)^{\gamma-\kappa})^2$. Note that in some expressions time argument of Z^T falls into the range of its domain only asymptotically. If this is the case, we do not comment. However, one should keep in mind that the mentioned expressions are well defined, provided n is large enough. For short, we assume that $T = 1$ and denote Z^1 by Z. The case of $T \neq 1$ reduces to this one because of self-similarity. We also make use of notion $\bar{\varepsilon} = 1 - \varepsilon$. Finally, for reproducibility, we label conditions of Bardet and Surgailis [3] exactly so as it is done in the original source.

(A.1)′ Fix $k \in \{1, 2, \dots\}$ and $t \in (0, 1)$. Then

$$\begin{aligned} &\mathrm{E}\left(Z_{\frac{[nt]+k}{n}} - Z_{\frac{[nt]}{n}}\right)^2 = \\ &\mathrm{E}\left(Z_{t-\frac{\{nt\}-k}{n}} - Z_{t-\frac{\{nt\}}{n}}\right)^2 = \varepsilon^{2\gamma}\,\mathrm{E}\left(X_{1+\varrho\left(t-\frac{\{nt\}-k}{n}\right)} - X_{1+\varrho\left(t-\frac{\{nt\}}{n}\right)}\right)^2 = \\ &\left[n\left(1+t-\frac{\{nt\}}{n}\right) = \frac{1}{u_n^t}\right] = \varepsilon^{2\gamma}(nu_n^t)^{-2\gamma}\,\mathrm{E}(X_{1+\varrho k u_n^t} - X_1)^2 = \\ &\varepsilon^{2\gamma}\varrho^{2\kappa}\sigma^2\left(\frac{k}{n}\right)^{2\kappa}(nu_n^t)^{2(\kappa-\gamma)}L^2(\varrho k u_n^t) \end{aligned} \tag{3.5}$$

and $(nu_n^t)^{-1} = 1 + \varrho\left(t - \frac{\{nt\}}{n}\right) = 1 + \varrho t + O\left(\frac{1}{n}\right)$. Since k is fixed,

$$\begin{aligned} &\frac{\mathrm{E}\left(Z_{\frac{[nt]+k}{n}} - Z_{\frac{[nt]}{n}}\right)^2}{\left(\frac{k}{n}\right)^{2\kappa}} = \varepsilon^{2\gamma}\varrho^{2\kappa}\sigma^2\left(1+\varrho t + O\left(\frac{1}{n}\right)\right)^{2(\gamma-\kappa)}\left(L^2(0) + o(\sqrt{ku_n^t})\right) = \\ &\varepsilon^{2\gamma}\varrho^{2\kappa}\sigma^2(1+\varrho t)^{2(\gamma-\kappa)}\left(1+O\left(\frac{1}{n}\right)\right)\left(L^2(0)+o\left(\frac{1}{\sqrt{n}}\right)\right) = \\ &\varepsilon^{2\gamma}\varrho^{2\kappa}\sigma^2L^2(0)(1+\varrho t)^{2(\gamma-\kappa)} + o\left(\frac{1}{\sqrt{n}}\right) = c(t) + o\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

Thus, the first two conditions of Bardet and Surgailis [3], (A.1)′ hold. The third one follows easily by noting that $t \mapsto c(t)$ is continuously differentiable on $[0, 1]$.

$(\mathrm{A.2})_p$ for $p = 2$. Let $j \in \{0, 1, \dots, n-2\}$, $u_{n,j} \overset{\mathrm{def}}{=} \frac{\varrho}{n+\varrho j}$. Then $u_{n,j+m} = \frac{u_{n,j}}{1+mu_{n,j}}$, $m \geq 0$. Thus, for $k \geq 1$,

$$\begin{aligned} &\sigma^{-2}\,\mathrm{E}\left(\Delta Z_{\frac{j}{n},\frac{1}{n}}\Delta Z_{\frac{j+k}{n},\frac{1}{n}}\right) = \\ &\varepsilon^{2\gamma}\sigma^{-2}\,\mathrm{E}\left(X_{1+\varrho\frac{j+1}{n}} - X_{1+\varrho\frac{j}{n}}\right)\left(X_{1+\varrho\frac{j+k+1}{n}} - X_{1+\varrho\frac{j+k}{n}}\right) = \\ &\varepsilon^{2\gamma}\left[\left(1+\varrho\frac{j+1}{n}\right)^{2\gamma}\left(l\left(\frac{\varrho\frac{k}{n}}{1+\varrho\frac{j+1}{n}}\right) - l\left(\frac{\varrho\frac{k-1}{n}}{1+\varrho\frac{j+1}{n}}\right)\right) - \right. \\ &\quad\left.\left(1+\varrho\frac{j}{n}\right)^{2\gamma}\left(l\left(\frac{\varrho\frac{k+1}{n}}{1+\varrho\frac{j}{n}}\right) - l\left(\frac{\varrho\frac{k}{n}}{1+\varrho\frac{j}{n}}\right)\right)\right] = \\ &(\bar{\varepsilon})^{2\gamma}n^{-2\gamma}\left(u_{n,j+1}^{-2\gamma}(l(ku_{n,j+1}) - l((k-1)u_{n,j+1}))\right. \\ &\quad\left. - u_{n,j}^{-2\gamma}(l((k+1)u_{n,j}) - l(ku_{n,j}))\right) = \\ &(\bar{\varepsilon})^{2\gamma}\frac{(nu_{n,j})^{-2\gamma}}{2}\left[(1+u_{n,j})^{2\gamma}\left(\frac{(1+(k+1)u_{n,j})^{2\gamma} - (1+ku_{n,j})^{2\gamma}}{(1+u_{n,j})^{2\gamma}}\right.\right. \end{aligned}$$

$$-\Delta p_{(k-1)\frac{u_{n,j}}{1+u_{n,j}},\frac{u_{n,j}}{1+u_{n,j}}}\Big) -$$
$$\Big((1+(k+1)u_{n,j})^{2\gamma} - (1+ku_{n,j})^{2\gamma} - \Delta p_{ku_{n,j},u_{n,j}}\Big)\Big] =$$
$$(\bar{\varepsilon})^{2\gamma}\frac{(nu_{n,j})^{-2\gamma}}{2}\left(\Delta p_{ku_{n,j},u_{n,j}} - (1+u_{n,j})^{2\gamma}\Delta p_{(k-1)\frac{u_{n,j}}{1+u_{n,j}},\frac{u_{n,j}}{1+u_{n,j}}}\right). \tag{3.6}$$

Let u_n^t, $t \in (0,1)$, be as in *(A1)*. Then $u_n^{\frac{j}{n}} = \varrho^{-1}u_{n,j}$. Therefore, taking $k=1$ in (3.5)–(3.6), we have

$$\left(\frac{\bar{\varepsilon}^\gamma}{\sigma}\right)^2 \mathrm{D}\left(\Delta Z_{\frac{j}{n},\frac{1}{n}}\right) = u_{n,j}^{2\kappa}(nu_{n,j})^{-2\gamma}L^2(u_{n,j});$$

$$\left(\frac{\bar{\varepsilon}^\gamma}{\sigma}\right)^2 \mathrm{D}\left(\Delta^{(2)} Z_{\frac{j}{n},\frac{1}{n}}\right) = \left(\frac{\bar{\varepsilon}^\gamma}{\sigma}\right)^2\Big(\mathrm{D}\left(\Delta Z_{\frac{j+1}{n},\frac{1}{n}}\right) + \mathrm{D}\left(\Delta Z_{\frac{j}{n},\frac{1}{n}}\right)$$
$$-2\,\mathrm{E}\left(\Delta Z_{\frac{j+1}{n},\frac{1}{n}}\Delta Z_{\frac{j}{n},\frac{1}{n}}\right)\Big) =$$
$$\left(\frac{u_{n,j}}{1+u_{n,j}}\right)^{2\kappa}\left(\frac{nu_{n,j}}{1+u_{n,j}}\right)^{-2\gamma}L^2\left(\frac{u_{n,j}}{1+u_{n,j}}\right) -$$
$$2\frac{(nu_{n,j})^{-2\gamma}}{2}\left(\Delta p_{u_{n,j},u_{n,j}} - (1+u_{n,j})^{2\gamma}\Delta p_{0,\frac{u_{n,j}}{1+u_{n,j}}}\right)$$
$$+(u_{n,j})^{2\kappa}(nu_{n,j})^{-2\gamma}L^2(u_{n,j}) =$$
$$(u_{n,j})^{2\kappa}(nu_{n,j})^{-2\gamma}\Bigg(2(1+u_{n,j})^{2(\gamma-\kappa)}L^2\left(\frac{u_{n,j}}{1+u_{n,j}}\right)$$
$$+2L^2(u_{n,j}) + 2^{2\kappa}L^2(2u_{n,j})\Bigg) \overset{n\to\infty}{\sim}$$
$$L^2(0)(4+4^\kappa)(u_{n,j})^{2\kappa}(nu_{n,j})^{-2\gamma}.$$

Consequently, for $k \in \{2,\dots,n\} : k+j \le n$,

$$\left(\frac{\bar{\varepsilon}^\gamma}{\sigma}\right)^2 \mathrm{E}\left(\Delta^{(2)} Z_{\frac{j}{n},\frac{1}{n}}\Delta^{(2)} Z_{\frac{j+k}{n},\frac{1}{n}}\right) = \left(\frac{\bar{\varepsilon}^\gamma}{\sigma}\right)^2 \mathrm{E}\left(\Delta Z_{\frac{j+1}{n},\frac{1}{n}} - \Delta Z_{\frac{j}{n},\frac{1}{n}}\right)\left(\Delta Z_{\frac{j+k+1}{n},\frac{1}{n}} - \Delta Z_{\frac{j+k}{n},\frac{1}{n}}\right) =$$
$$\frac{(nu_{n,j+1})^{-2\gamma}}{2}\Big[\Delta p_{ku_{n,j+1},u_{n,j+1}} - (1+u_{n,j+1})^{2\gamma}\Delta p_{(k-1)\frac{u_{n,j+1}}{1+u_{n,j+1}},\frac{u_{n,j+1}}{1+u_{n,j+1}}} -$$
$$\left(\Delta p_{(k-1)u_{n,j+1},u_{n,j+1}} - (1+u_{n,j+1})^{2\gamma}\Delta p_{(k-2)\frac{u_{n,j+1}}{1+u_{n,j+1}},\frac{u_{n,j+1}}{1+u_{n,j+1}}}\right)\Big] -$$
$$\frac{(nu_{n,j})^{-2\gamma}}{2}\Big[\Delta p_{(k+1)u_{n,j},u_{n,j}} - (1+u_{n,j})^{2\gamma}\Delta p_{k\frac{u_{n,j}}{1+u_{n,j}},\frac{u_{n,j}}{1+u_{n,j}}} -$$
$$\left(\Delta p_{ku_{n,j},u_{n,j}} - (1+u_{n,j})^{2\gamma}\Delta p_{(k-1)\frac{u_{n,j}}{1+u_{n,j}},\frac{u_{n,j}}{1+u_{n,j}}}\right)\Big] = -\frac{(nu_{n,j})^{-2\gamma}}{2}\Big[\Delta^{(2)}p_{ku_{n,j},u_{n,j}} -$$
$$2(1+u_{n,j})^{2\gamma}\Delta^{(2)}p_{(k-1)\frac{u_{n,j}}{1+u_{n,j}},\frac{u_{n,j}}{1+u_{n,j}}} + (1+2u_{n,j})^{2\gamma}\Delta^{(2)}p_{(k-2)\frac{u_{n,j}}{1+2u_{n,j}},\frac{u_{n,j}}{1+2u_{n,j}}}\Big] \overset{n\to\infty}{\sim}$$
$$K\cdot\frac{\Delta^{(2)}p_{ku_{n,j},u_{n,j}} - 2(1+u_{n,j})^{2\gamma}\Delta^{(2)}p_{(k-1)\frac{u_{n,j}}{1+u_{n,j}},\frac{u_{n,j}}{1+u_{n,j}}} + (1+2u_{n,j})^{2\gamma}\Delta^{(2)}p_{(k-2)\frac{u_{n,j}}{1+2u_{n,j}},\frac{u_{n,j}}{1+2u_{n,j}}}}{\sqrt[-1/2]{\mathrm{D}\left(\Delta^{(2)}Z_{\frac{j}{n},\frac{1}{n}}\right)\mathrm{D}\left(\Delta^{(2)}Z_{\frac{j+k}{n},\frac{1}{n}}\right)}(u_{n,j})^{2\kappa}},$$

where $K = K(\varepsilon,\kappa,j,k,n)$ is uniformly bounded for all j,k,n. □

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Proof of Corollary 2.1. To prove the corollary, apply the Delta method. □

In order to prove Proposition 2.1, we need the following lemma. We believe that a similar proposition was proved elsewhere. However, we could not find a corresponding reference. Therefore, we provide a proof for completeness.

Lemma 3.1. *Let $x \in (0,4)$ be fixed, and let $g^x : [0,\infty) \to \mathbb{R}$ be defined by*

$$g^x(y) = (y+2)^x - 2(y+1)^x + y^x.$$

Then there exists $C_x \in (0,\infty)$ such that $|\Delta^{(2)} g^x_{y,1}| \le \frac{C_x}{y^{4-x}}$ for $\forall y > 0$.

Proof. The claim follows immediately by noting that

$$\Delta^{(2)} g^x_{y,1} = D_x \int_0^1 \int_0^1 \int_0^1 \int_0^1 (y + v_1 + v_2 + v_3 + v_4)^{x-4} \mathrm{d}v_1 \mathrm{d}v_2 \mathrm{d}v_3 \mathrm{d}v_4,$$

where $D_x = x(x-1)(x-2)(x-3)$. □

Proof of Proposition 2.1. Since the proof of proposition reduces to a careful application of the Taylor's formula and Lemma 3.1, we give a detailed exposition for the sfBm. In case of other families, it is a slightly reworked repetition of the latter. For short, we omit a subscript denoting that the quantities under consideration correspond to the sfBm, i.e., we write $R, l, \ldots,$ instead of $R_{S^H}, l_{S^H}, \ldots$. We also assume that $H \in (0,1)$ is fixed and put $\binom{x}{0} \overset{\text{def}}{=} 1$; $\binom{x}{j} \overset{\text{def}}{=} \frac{x(x-1)\cdots(x-j+1)}{j!}$, $j \ge 1$.

Step 1. Let $t, s, h > 0$, and let $u \in [0,1)$. Then

$$\begin{aligned}
&R(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left(|s+t|^{2H} + |t-s|^{2H}\right) \Rightarrow \\
&\quad R(t,t+h) = t^{2H} + (t+h)^{2H} - \frac{1}{2}\left((2t+h)^{2H} + h^{2H}\right) = \\
&\quad t^{2H}\left(1 + \left(1+\frac{h}{t}\right)^{2H} - \frac{1}{2}\left(\left(2+\frac{h}{t}\right)^{2H} + \left(\frac{h}{t}\right)^{2H}\right)\right) \Rightarrow \\
&\quad l(u) = \frac{1}{2-2^{2H-1}}\left(1 + (1+u)^{2H} - \frac{1}{2}\left((2+u)^{2H} + u^{2H}\right)\right) \Rightarrow \\
&\quad l(u) - \frac{1}{2}\left(1 + (1+u)^{2H}\right) = -\frac{1}{2}u^{2\kappa}L^2(u) = \\
&\quad \frac{2^{2H-1}}{2-2^{2H-1}}\left(\frac{1+(1+u)^{2H}}{2} - \left(1+\frac{u}{2}\right)^{2H} - \left(\frac{u}{2}\right)^{2H}\right) = \\
&\quad \frac{2^{2H-1}}{2-2^{2H-1}}\left(\sum_{k=1}^{\infty}\binom{2H}{k}u^k\left(\frac{1}{2}-\frac{1}{2^k}\right) - \left(\frac{u}{2}\right)^{2H}\right) = -\frac{u^{2H}}{4-4^H}\left(1 + O\left(u^{2(1-H)}\right)\right),
\end{aligned}$$

where the last two equalities are due to Taylor's expansion in the neighbourhood of 0. Hence, under assumptions made, *(L1)–(L2)* hold.

Step 2. By *Step 1*,

$$p(u) = u^{2\kappa}L^2(u) = \frac{2^{2H}}{2-2^{2H-1}}\left(\left(\frac{u}{2}\right)^{2H} + \left(1+\frac{u}{2}\right)^{2H} - \frac{1+(1+u)^{2H}}{2}\right).$$

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Since constant does not affect the order of differences Δp, $\Delta^{(2)} p$, it suffices to show that condition *(L3)* of Theorem 2.3 applies to $\tilde{p} \overset{\text{def}}{=} \left(\frac{2^{2H}}{2-2^{2H-1}}\right)^{-1} p$. Let g^x be the same as in Lemma 3.1. Then

$$\Delta \tilde{p}_{ku,u} = \left(\frac{u}{2}\right)^{2H} \left((k+1)^{2H} - k^{2H}\right) + \left(\left(1+(k+1)\frac{u}{2}\right)^{2H} - \left(1+k\frac{u}{2}\right)^{2H}\right) - \frac{1}{2}\left((1+(k+1)u)^{2H} - (1+ku)^{2H}\right) \Rightarrow$$

$$\Delta^{(2)} \tilde{p}_{ku,u} = \left(\frac{u}{2}\right)^{2H} g^{2H}(k) + \tilde{g}_k\left(\frac{u}{2}\right) - \frac{1}{2}\tilde{g}_k(u),$$

where $\tilde{g}_k(u) = (1+(k+2)u)^{2H} - 2(1+(k+1)u)^{2H} + (1+ku)^{2H}$. Next, note that

$$(1+ju)^{2H}\tilde{g}_{k-j}\left(\frac{u}{1+ju}\right) = (1+ju)^{2H}\Bigg[\left(1+(k-j+2)\frac{u}{1+ju}\right)^{2H} - 2\left(1+(k-j+1)\frac{u}{1+ju}\right)^{2H} + \left(1+(k-j)\frac{u}{1+ju}\right)^{2H}\Bigg] = \tilde{g}_k(u),$$

and that

$$(1+ju)^{2H}\tilde{g}_{k-j}\left(\frac{u}{2(1+ju)}\right) = (1+ju)^{2H}\Bigg[\left(1+(k-j+2)\frac{u}{2(1+ju)}\right)^{2H} - 2\left(1+(k-j+1)\frac{u}{2(1+ju)}\right)^{2H} + \left(1+(k-j)\frac{u}{2(1+ju)}\right)^{2H}\Bigg] = \tilde{g}_{k+j}\left(\frac{u}{2}\right).$$

Therefore, for $u > 0$,

$$\Delta^{(2)}\tilde{p}_{ku,u} - 2(1+u)^{2H}\Delta^{(2)}\tilde{p}_{(k-1)\frac{u}{1+u},\frac{u}{1+u}} + (1+2u)^{2H}\Delta^{(2)}\tilde{p}_{(k-2)\frac{u}{1+2u},\frac{u}{1+2u}} = \left(\frac{u}{2}\right)^{2H}\Delta^{(2)}g^{2H}_{k-2,1} + \tilde{g}_k\left(\frac{u}{2}\right) - 2\tilde{g}_{k+1}\left(\frac{u}{2}\right) + \tilde{g}_{k+2}\left(\frac{u}{2}\right) = \left(\frac{u}{2}\right)^{2H} \times \left(\Delta^{(2)}g^{2H}_{k-2,1} + \Delta^{(2)}g^{2H}_{k+\frac{2}{u},1}\right).$$

By Lemma 3.1, $\Delta^{(2)}g^{2H}_{k-2,1} + \Delta^{(2)}g^{2H}_{k+\frac{2}{u},1} = O\left(\frac{1}{k^{4-2H}}\right)$. □

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