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Multivariate subexponential distributions

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We present a formulation of subexponential and exponential tail behavior for multivariate distributions. The definitions are necessarily in terms of vague convergence of Radon measures rather than of ratios of distribution tails. With the proper setting, we show that if all one dimensional marginals of a d-dimensional distribution are subexponential, then the distribution is multivariate subexponential. Known results for univariate subexponential distributions are extended to the multivariate setting. Point process arguments are used for the proofs.

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subexponentiality * convolution tails * point processes * vague convergence

1. Introduction

A one-dimensional subexponential distribution is defined by the property that the distribution tail is asymptotically equivalent to the tails of the convolution powers of the distribution. The class of one dimensional subexponential distributions has proven useful in a variety of contexts where the subexponential property provides a necessary and sufficient condition for some sort of tail equivalence. (See, for example, the surveys by Embrechts, 1985; Bingham, Goldie and Teugels, 1989, pp. 429–432, and the references therein.) Tail equivalence is a useful property because for instance when a distribution is in a domain of attraction (either in the sense of extreme values or of partial sums of i.i.d. random variables) any tail equivalent distribution will also be in the domain of attraction and the normalizing constants will be the same (cf. Resnick, 1971). Our goal is to see what sensible generalizations of these concepts are possible in higher dimensions.

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In one dimension, the definitions are as follows: a distribution function F on $\mathbb{R}$ is in the class $\mathscr{L}(\alpha)$ for $\alpha \geq 0$ if its tail $\bar{F} = 1 - F$ satisfies

$$\lim_{x\to\infty} \frac{\bar{F}(x-y)}{\bar{F}(x)} = e^{\alpha y}, \quad y \in \mathbb{R}, \tag{1.1}$$

and the distribution F is in the class $\mathscr{S}(\alpha)$ if $F \in \mathscr{L}(\alpha)$ and

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\bar{F}(x)} = D < \infty. \tag{1.2}$$

The constant D is known to equal $2\int e^{\alpha x} F(dx)$, which was proved for the case $F(0) = 0$ by Chover, Ney and Wainger (1973) and by Cline (1987) and extended to the case that F concentrates on $\mathbb{R}$ by Willekens (1986). When $F(0) = 0$ and $D = 2$, (1.2) implies (1.1) with $\alpha = 0$ (Chistyakov, 1964) and in this case the class $\mathscr{S}(0)$ has been called the *subexponential class*. For our purposes, it is not natural to restrict distributions to $[0, \infty)$. Examples include the log normal, generalized inverse Gaussian, Pareto and distributions with tails of the form $kx^{\gamma} e^{-x^p}$, $0 < p < 1$.

For d-dimensions ($d \geq 1$) we propose the following definitions. Let

$$E = [-\infty, \infty]^d \setminus \{-\boldsymbol{\infty}\}$$

be the compactified Euclidean space punctured by the removal of the bottom point. Relatively compact sets are thus those which are bounded away from $-\boldsymbol{\infty}$. Let ν be a Radon measure on E such that:

(a) $\nu \not\equiv 0$.

(b) Each one dimensional marginal $(1 \leq i \leq d)$,

$$\nu_i(\cdot) := \nu([-\infty, \infty]^{i-1} \times (\cdot) \times [-\infty, \infty]^{d-i})$$

(where we interpret $[-\infty, \infty]^0 \times A = A$) has the property $\nu_i((x, \infty]) > 0$, for all $x \in \mathbb{R}$.

Also let $\boldsymbol{b}(t) = (b_1(t), \ldots, b_d(t))$ be a function satisfying $b_i(t) \to \infty$ as $t \to \infty$ for $i = 1, \ldots, d$. For a distribution F on $\mathbb{R}^d$ we say $F \in \mathscr{L}(\nu; \boldsymbol{b})$ if, as $t \to \infty$,

$$tF(\boldsymbol{b}(t) + \cdot) \xrightarrow{v} \nu \tag{1.3}$$

where '$\xrightarrow{v}$' denotes vague convergence of measures on E and ν satisfies (a) and (b) above. We say the distribution F is in the class $\mathscr{S}(\nu; b)$ if $F \in \mathscr{L}(\nu; \boldsymbol{b})$ and

$$tF * F(\boldsymbol{b}(t) + \cdot) \xrightarrow{v} \nu^{(2)} \tag{1.4}$$

for some Radon measure $\nu^{(2)}$. (This will entail $\nu^{(2)}$ satisfies (a) and (b) above and thus that $F * F \in \mathscr{L}(\nu^{(2)}; \boldsymbol{b})$.) In Section 2 we show that $\nu^{(2)} = 2\nu * F$.

The formulation of the multivariate subexponential property in terms of vague convergence of measures rather than convergence of distribution functions is advantageous because, first of all, multivariate distribution functions are much more awkward to deal with than are one dimensional distribution functions and, secondly, vague convergence of measures allows access to point process techniques for proving vague and weak convergence (Resnick, 1987).

Pick $x>-\infty$ and continuous functions with compact supports on E and which approximate the indicator of a set of the form $(x,\infty]\times[-\infty,\infty]^{d-1}$. By inserting these functions into the vague convergence given in (1.3) and (1.4) we may deduce marginal vague convergence. Thus, at continuity points of the limit

$$\lim_{t\to\infty} t\bar{F}_i(x+b_i(t))=\bar{\nu}_i(x):=\nu_i((x,\infty]) \tag{1.3'}$$

and

$$\lim_{t\to\infty} t\overline{F_i*F_i}(x+b_i(t))=\bar{\nu}_i^{(2)}(x):=\nu_i^{(2)}((x,\infty]), \tag{1.4'}$$

where F_i and ν_i are the ith one-dimensional marginals of F and ν, respectively. From Bingham, Goldie and Teugels (1989, Theorem 1.10.3; cf. also de Haan, 1970; Feller, 1970; Resnick, 1987, Proposition 0.4) we have that (1.3') implies $\bar{F}_i(\log x)$ is regularly varying with some index $-\alpha_i$, $\alpha_i\geqslant 0$. That is, (1.3') implies, for $x\in\mathbb{R}$,

$$\lim_{t\to\infty}\frac{\bar{F}_i(t+x)}{\bar{F}_i(t)}=\mathrm{e}^{-\alpha_i x} \tag{1.5}$$

so that F_i satisfies (1.1) and $F_i\in\mathscr{L}(\alpha_i)$. Also, we have for X_1, X_2 independent with distribution F_i that

$$\begin{aligned}\overline{F_i*F_i}(x)&\geqslant P[X_1+X_2\geqslant x,\ X_1\leqslant\tfrac{1}{2}x]+P[X_1+X_2\geqslant x,\ X_2\leqslant\tfrac{1}{2}x]\\&=2\int_{-\infty}^{x/2}\bar{F}_i(x-s)F_i(\mathrm{d}s)\end{aligned}$$

and therefore

$$t\overline{F_i*F_i}(x+b_i(t))\geqslant 2\int_{-\infty}^{(x+b_i(t))/2} t\bar{F}_i(x-s+b_i(t))F_i(\mathrm{d}s).$$

If (1.4') holds, then by Fatou's lemma,

$$\lim_{t\to\infty} t\overline{F_i*F_i}(x+b_i(t))\geqslant 2\int_{-\infty}^{\infty}\bar{\nu}_i(x-s)F_i(\mathrm{d}s)=2\overline{\nu_i*F_i}(x),$$

where $\overline{\nu_i*F_i}(x):=\nu_i*F_i((x,\infty])$. Since we assumed $\bar{\nu}_i(x)>0$ for $x\in\mathbb{R}$ and since $\lim_{t\to\infty}t\overline{F_i*F_i}(x+b_i(t))=\bar{\nu}_i^{(2)}(x)$ at points of continuity, we conclude $\bar{\nu}_i^{(2)}(x)>0$. From (1.4') we get therefore that $\overline{F_i*F_i}(\log x)$ is regularly varying, and since $b_i(t)$ is the same in both (1.3') and (1.4') we conclude that for some constant $D>0$,

$$\overline{F_i*F_i}(t)\sim D\bar{F}_i(t)$$

and hence that (1.2) holds. Thus we infer the important fact that (1.3) and (1.4) imply that each marginal distribution F_i satisfies (1.1) and (1.2) for some $\alpha_i\geqslant 0$. We will write $F_i\in\mathscr{L}(\alpha_i)$, and $F_i\in\mathscr{S}(\alpha_i)$ for the marginal properties. The purpose of Section 2 is to prove that the converse is true in the sense that if (1.3) holds and each marginal F_i satisfies (1.2), then (1.4) holds.

We now present a slight elaboration of the previous discussion, showing that our formulation subsumes the univariate definitions. We make use of results to be proven in Section 4.

Proposition 1.1. (i) *$F \in \mathscr{L}(\nu; \boldsymbol{b})$ for some $\boldsymbol{b}$ implies $F_i \in \mathscr{L}(\alpha_i)$ for each i and the latter is true if and only if* (1.3′) *holds.*

(ii) *$F \in \mathscr{L}(\nu; \boldsymbol{b})$ and $G \in \mathscr{L}(\mu; \boldsymbol{b})$ for some ν, μ and $\boldsymbol{b}$ implies*

$$\lim_{t\to\infty} \frac{\bar{F}_i(t)}{\bar{G}_i(t)} = \frac{\bar{\nu}_i(0)}{\bar{\mu}_i(0)} \quad \textit{for each } i.$$

(iii) *$F \in \mathscr{S}(\nu; \boldsymbol{b})$ for some $\boldsymbol{b}$ implies $F_i \in \mathscr{S}(\alpha_i)$ for each i and the latter is true if and only if both* (1.3′) *and* (1.4′) *hold.*

Proof. (i) We have already pointed out that (1.3) implies (1.3′) and that (1.3′) implies $F_i \in \mathscr{L}(\alpha_i)$. Conversely, if $F_i \in \mathscr{L}(\alpha_i)$ we let $g(t) = 1/\bar{F}_i(t)$. The function g is right-continuous and its left-continuous version $g^-(t) = \sup_{x<t} g(x)$ satisfies

$$1 = \sup_{\varepsilon>0} \lim_{t\to\infty} \frac{\bar{F}_i(t)}{\bar{F}_i(t-\varepsilon)} \leq \lim_{t\to\infty} \frac{g^-(t)}{g(t)} \leq 1.$$

Let $g^{\leftarrow}(t) = \inf\{x: g(x) \geq t\}$. By Lemma 4.1(iii),

$$\lim_{t\to\infty} t\bar{F}_i(g^{\leftarrow}(t)+u) = \lim_{t\to\infty} \frac{\bar{F}_i(g^{\leftarrow}(t)+u)}{\bar{F}_i(g^{\leftarrow}(t))} \frac{t}{g(g^{\leftarrow}(t))} = e^{-\alpha_i u}.$$

So $F_i \in \mathscr{L}(\alpha_i)$ implies (1.3′) with $b_i = g^{\leftarrow}$.

(ii) For each i,

$$\lim_{t\to\infty} t\bar{F}_i(b_i(t)) = \bar{\nu}_i(0)$$

and

$$\lim_{t\to\infty} t\bar{G}_i(b_i(t)) = \bar{\mu}_i(0).$$

The result follows as a consequence of Lemma 4.1(iv).

(iii) Suppose $F \in \mathscr{S}(\nu; \boldsymbol{b})$. This means $F \in \mathscr{L}(\nu; \boldsymbol{b})$ and $F * F \in \mathscr{L}(\nu^{(2)}; \boldsymbol{b})$. By (ii),

$$\lim_{t\to\infty} \frac{\overline{F_i * F_i}(t)}{\bar{F}_i(t)} = \frac{\bar{\nu}_i^{(2)}(0)}{\bar{\nu}_i(0)}. \tag{1.6}$$

and $F_i \in \mathscr{S}(\alpha_i)$.

Next, suppose (1.3′) and (1.4′) hold. By (i) this is true only if $F_i \in \mathscr{L}(\alpha_i)$ and $F_i * F_i \in \mathscr{L}(\alpha_i)$. Since the same norming sequence $b_i(t)$ is used, Lemma 4.1(iv) gives the further implication that (1.6) holds. That is, $F_i \in \mathscr{S}(\alpha_i)$.

Finally, suppose $F_i \in \mathscr{S}(\alpha_i)$. Then $F_i \in \mathscr{L}(\alpha_i)$ and choosing g and b_i as in part (i), (1.3′) holds. Also, (1.4′) holds as an immediate consequence of (1.2) and (1.3′). □

In case $\alpha_i > 0$ for each i, ν can be considered as the exponent measure of a multivariate max-stable distribution. De Haan and Resnick (1977), Cline (1988) and Omey (1989) provide characterizations of such measures.

When $\alpha_i = 0$, for all $i = 1, \dots, d$, the form of the limit measure ν in (1.3) is distinctive. From (1.5) there exists $c_i > 0$ such that for all $x \in \mathbb{R}$,

$$\nu_i((x, \infty]) = \lim_{t\to\infty} t\bar{F}_i(x + b_i(t)) = c_i.$$

Thus $\nu_i(\mathbb{R}) = 0$ and

$$\nu\left(\bigcup_{i=1}^{d} [-\infty, \infty]^{i-1} \times \mathbb{R} \times [-\infty, \infty]^{d-i}\right) = 0$$

so that ν concentrates on

$$\Xi := E \Big\backslash \left(\bigcup_{i=1}^{d} [-\infty, \infty]^{i-1} \times \mathbb{R} \times [-\infty, \infty]^{d-i}\right) = \{-\infty, \infty\}^d \backslash \{-\boldsymbol{\infty}\}.$$

That is, ν concentrates on the $2^d - 1$ points whose coordinates are $\pm\infty$ but not all of whose coordinates are $-\infty$. Thus ν is of the form

$$\nu^{(0)} := \sum_{\sigma \in \Xi} w_{\sigma} \varepsilon_{\sigma}. \tag{1.7}$$

where

$$w_{\sigma} = \nu(A_{\sigma_1} \times \cdots \times A_{\sigma_d}) = \lim_{t\to\infty} tF(A_{\sigma_1} \times \cdots \times A_{\sigma_d} + \boldsymbol{b}(t))$$

and

$$A_{\sigma} = \begin{cases} (1, \infty], & \text{if } \sigma = \infty, \\ [-\infty, 1], & \text{if } \sigma = -\infty. \end{cases}$$

If in addition (1.4) holds, then the limit measure $\nu^{(2)}$ in (1.4) (we will show) is equal to $2\nu * F = 2\nu$. We emphasize (1.7) is for the case $\alpha_i = 0$ for all $i = 1, \dots, d$.

In case some, but not all, of the α_i's are zero, the limit measure ν cannot in general be expressed as a mixture between the two types (see Section 3).

Additional special cases of interest are when (1.3) holds with F being a product measure and when (1.3) holds with F concentrating on $\{\boldsymbol{x}: x^{(1)} = \cdots = x^{(d)}\}$. These cases are taken up in Section 3.

In Section 2 we prove that if $F \in \mathscr{L}(\nu)$ with marginal properties (1.4′), then $F \in \mathscr{S}(\nu)$ and that (1.4) holds. The limit measure $\nu^{(2)}$ in (1.4) is shown to satisfy

$$\nu^{(2)} = 2\nu * F.$$

The mode of proof uses a point process transform technique which equates tail properties of measures with weak convergence of a sequence of induced point processes to limiting Poisson processes. In Section 2 we also show that for multivariate subexponential distributions, domains of attraction are preserved by taking convolution powers.

In Section 3 we consider a variety of applications, extensions and examples. We show that if F is a multivariate distribution which is regularly varying at $\boldsymbol{\infty}$ (Resnick, 1987; Omey, 1989) then $F \in \mathscr{S}(\nu^{(0)}; \boldsymbol{b})$ where $\nu^{(0)}$ is specified in (1.7). Also, we prove that, if $F \in \mathscr{L}(\nu; \boldsymbol{b})$ and $F^{*n} \in \mathscr{S}(\nu^{(n)}; \boldsymbol{b})$ for some n and $\nu^{(n)}$, then $F \in \mathscr{S}(\nu)$. Finally we consider compound distributions and some specific examples. Section 4 presents a discussion of the normalizing function $\boldsymbol{b}(t)$ and proves lemmas used elsewhere in the paper.

2. Marginal and global properties; domains of attraction

In this section we show that an $\mathscr{L}(\nu; \boldsymbol{b})$ distribution F whose one dimensional marginals F_i are in $\mathscr{S}(\alpha_i)$, $1 \leqslant i \leqslant d$, is also in $\mathscr{S}(\nu; \boldsymbol{b})$ and that the limit measure $\nu^{(2)}$ in (1.4) is $2\nu * F$.

First a word about notation. Operations on vectors are to be interpreted component-wise. Thus if $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,d}) \in \mathbb{R}^d$, $n = 1, 2$, we have

$$\boldsymbol{x}_1^a = ((x_{1,1})^a, \ldots, (x_{1,d})^a),$$

$$\boldsymbol{x}_1 + \boldsymbol{x}_2 = (x_{1,1} + x_{2,1}, \ldots, x_{1,d} + x_{2,d}),$$

$$\boldsymbol{x}_1 \vee \boldsymbol{x}_2 = (x_{1,1} \vee x_{2,1}, \ldots, x_{1,d} \vee x_{2,d}),$$

$$\boldsymbol{x}_1\boldsymbol{x}_2 = (x_{1,1}x_{2,1}, \ldots, x_{1,d}x_{2,d})$$

and

$$\frac{\boldsymbol{x}_1}{\boldsymbol{x}_2} = \left(\frac{x_{1,1}}{x_{2,1}}, \ldots, \frac{x_{1,d}}{x_{2,d}}\right).$$

Similarly, $\boldsymbol{x}_1 \leqslant \boldsymbol{x}_2$ means $x_{1,i} \leqslant x_{2,i}$, $i = 1, \ldots, d$, and if $\boldsymbol{x}_1 \leqslant \boldsymbol{x}_2$ we write $[\boldsymbol{x}_1, \boldsymbol{x}_2] = \{\boldsymbol{x}: \boldsymbol{x}_1 \leqslant \boldsymbol{x} \leqslant \boldsymbol{x}_2\}$. We write $-\boldsymbol{\infty} = (-\infty, \ldots, -\infty)$ and $\boldsymbol{\infty} = (\infty, \ldots, \infty)$.

We proceed by means of a point process transform technique (Resnick, 1987, 1986; see also Davis and Resnick, 1985a,b, 1986, 1988; Kallenberg, 1983). Suppose E' is a LCCB space (i.e., locally compact with a countable basis). We set $M_p(E')$ equal to the space of point measures on E' and metrize $M_p(E')$ by the vague metric (denoted ρ). A point measure on E' is a Radon measure on E' of the form $\sum_i \varepsilon_{x_i}$ where $x_i \in E'$ and for a Borel subset $B \subset E'$ we have $\varepsilon_x(B) = 1$ if $x \in B$ and $\varepsilon_x(B) = 0$ otherwise. A Poisson process on E' with mean measure μ will be denoted PRM(μ); i.e., a Poisson random measure with mean measure μ. Recall from Section 1 that we are primarily interested in the LCCB space $E = [-\infty, \infty]^d \backslash \{-\boldsymbol{\infty}\}$. Lebesgue measure on $[0, \infty)$ will be denoted with λ.

Proposition 2.1. *Let F and G be probability measures on $\mathbb{R}^d$ and let ν and μ be Radon measures on E satisfying* (a) *and* (b) *of the definition of the multivariate class $\mathscr{L}$ given in Section* 1. *Suppose $\boldsymbol{a}(t) \in \mathbb{R}_+^d$, $\boldsymbol{b}(t) \in \mathbb{R}^d$ are functions such that $\boldsymbol{b}(t) \to \boldsymbol{\infty}$ and $\boldsymbol{a}(t) \to \boldsymbol{\gamma}^{-1} \in (0, \infty]^d$ and suppose*

$$tF(\boldsymbol{a}(t) \cdot + \boldsymbol{b}(t)) \xrightarrow{v} \nu, \tag{2.1}$$

$$tG(\boldsymbol{a}(t) \cdot + \boldsymbol{b}(t)) \xrightarrow{v} \mu, \tag{2.2}$$

on E. Suppose also, for $i=1,\ldots,d$ that the marginals of F and G satisfy: $F_i \in \mathcal{S}(\alpha_i)$, $G_i \in \mathcal{S}(\alpha_i)$. Let $\{\boldsymbol{X}_k, k \geq 1\}$ be i.i.d. random vectors with distribution F and let $\{\boldsymbol{Y}_k, k \geq 1\}$ be i.i.d. random vectors with distribution G and independent of $\{\boldsymbol{X}_k\}$. Then as $t \to \infty$,

$$\sum_{k=1}^{\infty} \varepsilon_{(k/t,(\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t))} \Rightarrow \mathrm{PRM}(\lambda \times (\nu * G(\boldsymbol{\gamma}^{-1}\cdot) + \mu * F(\boldsymbol{\gamma}^{-1}\cdot))) \tag{2.3}$$

in $M_{\mathrm{p}}([0,\infty) \times E)$ and so equivalently (Resnick, 1987, Proposition 3.21)

$$tF * G(\boldsymbol{a}(t)\cdot + \boldsymbol{b}(t)) \xrightarrow{\mathrm{v}} \nu * G(\boldsymbol{\gamma}^{-1}\cdot) + \mu * F(\boldsymbol{\gamma}^{-1}\cdot). \tag{2.4}$$

Remark. The following are the cases of interest:

(a) $F \in \mathcal{L}(\nu; \boldsymbol{b})$ and $G \in \mathcal{L}(\mu; \boldsymbol{b})$. Then $\boldsymbol{a}(t) \equiv \boldsymbol{1}$ and $\boldsymbol{\gamma} = \boldsymbol{1}$.

(b) F and G are regularly varying so that $\boldsymbol{a}(t) = \boldsymbol{b}(t)$.

(c) F and G are in a type III multivariate domain of attraction and each marginal F_i and G_i is in the univariate domain of attraction of $\Lambda(x) := \exp(-\mathrm{e}^{-x})$. $x \in \mathbb{R}$. Then assuming also that $F_i(x) < 1$, $G_i(x) < 1$ for all $x \in \mathbb{R}$, we have $F_i \in D(\Lambda) \cap \mathcal{S}(\alpha_i)$, $G_i \in D(\Lambda) \cap \mathcal{S}(\alpha_i)$ and $\boldsymbol{a}(t) \to \boldsymbol{\gamma}^{-1} = \boldsymbol{\alpha}^{-1}$.

Proof of Proposition 2.1. We proceed in a series of steps which are somewhat analogous to those in Goldie and Resnick (1988). Recall first that (2.1) and (2.2) are equivalent respectively to

$$\sum_{k=1}^{\infty} \varepsilon_{(k/t,(\boldsymbol{X}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t))} \Rightarrow \sum_k \varepsilon_{(t_k^{(1)},j_k^{(1)})} := \mathrm{PRM}(\lambda \times \nu) \tag{2.5}$$

and

$$\sum_{k=1}^{\infty} \varepsilon_{(k/t,(\boldsymbol{Y}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t))} \Rightarrow \sum_k \varepsilon_{(t_k^{(2)},j_k^{(2)})} := \mathrm{PRM}(\lambda \times \mu) \tag{2.6}$$

in $M_{\mathrm{p}}([0,\infty) \times E)$ (Resnick, 1987, p. 154).

For what follows we need the following variant of Proposition 3.21 in Resnick (1987).

Lemma 2.2. *Suppose E_1 and E_2 are* LCCB *spaces and for each n, $\{Z_{nk}, W_{nk}, k \geq 1\}$ are i.i.d. random elements of $E_1 \times E_2$ defined on the same probability space. The following statements* (a), (b) *and* (c) *are equivalent:*

(a) *For all compact A and B,*

$$nP[Z_{n1} \in A, W_{n1} \in B] \to 0,$$

$$nP[Z_{n1} \in \cdot\,] \xrightarrow{\mathrm{v}} \mu_1,$$

$$nP[W_{n1} \in \cdot\,] \xrightarrow{\mathrm{v}} \mu_2. \tag{2.7}$$

(b) *In $M_{\mathrm{p}}(E_1) \times M_{\mathrm{p}}(E_2)$,*

$$\left(\sum_{k=1}^{n} \varepsilon_{Z_{nk}}, \sum_{k=1}^{n} \varepsilon_{W_{nk}}\right) \Rightarrow \left(\sum_k \varepsilon_{Z_k}, \sum_k \varepsilon_{W_k}\right) \tag{2.8}$$

where the limits are independent Poisson random measures with mean measures μ_1 and μ_2, respectively.

(c) *In $M_p([0,\infty)\times E_1)\times M_p([0,\infty)\times E_2)$,*

$$\left(\sum_{k=1}^{n}\varepsilon_{(k/n,Z_{nk})},\sum_{k=1}^{n}\varepsilon_{(k/n,W_{nk})}\right)\Rightarrow\left(\sum_k\varepsilon_{(t_k,Z_k)},\sum_k\varepsilon_{(s_k,W_k)}\right),\tag{2.9}$$

where the limits are independent Poisson random measures with mean measures $\lambda\times\mu_1$ and $\lambda\times\mu_2$, respectively.

Proof. Let $f_1\in C_K(E_1)$, $f_2\in C_K(E_2)$. Taking joint Laplace functionals at (f_1,f_2) shows that (2.8) is equivalent to

$$\lim_{n\to\infty}(E\,\mathrm{e}^{-(f_1(Z_{n1})+f_2(W_{n1}))})^n$$
$$=\exp\left[-\int_{E_1}(1-\mathrm{e}^{-f_1})\,\mathrm{d}\mu_1+\int_{E_2}(1-\mathrm{e}^{-f_2})\,\mathrm{d}\mu_2\right].\tag{2.10}$$

The left side of (2.10) is rewritten as

$$\lim_{n\to\infty}\left(1-\frac{nE[1-\mathrm{e}^{-f_1(Z_{n1})-f_2(W_{n1})}]}{n}\right)^n$$

and so (2.10) is equivalent to

$$\lim_{n\to\infty}\iint_{E_1\times E_2}(1-\mathrm{e}^{-f_1(z)-f_2(w)})nP[Z_{n1}\in\mathrm{d}z,\ W_{n1}\in\mathrm{d}w]$$
$$=\int_{E_1}(1-\mathrm{e}^{-f_1})\,\mathrm{d}\mu_1+\int_{E_2}(1-\mathrm{e}^{-f_2})\,\mathrm{d}\mu_2.\tag{2.11}$$

Let A_i be the support of f_i, $i=1,2$. Decompose the left side of (2.11) as

$$\iint_{E_1\times E_2}=\iint_{A_1\times A_2^c}+\iint_{A_1^c\times A_2}+\iint_{A_1\times A_2}+\iint_{A_1^c\times A_2^c}$$
$$=\mathrm{I}_n+\mathrm{II}_n+\mathrm{III}_n+\mathrm{IV}_n.$$

Suppose (a) holds. Then

$$I_n=\int_{A_1}(1-\mathrm{e}^{-f_1(z)})nP[Z_{n1}\in\mathrm{d}z]$$
$$-\int_{A_1\times A_2}(1-\mathrm{e}^{-f_1(z)})nP[Z_{n1}\in\mathrm{d}z,\ W_{n1}\in\mathrm{d}w].$$

Since $1-\exp\{-f_1\}\le 1$, the second term is bounded above by $nP[Z_{n1}\in A_1,\ W_{n1}\in A_2]$ whose limit is zero. So $\mathrm{I}_n\to\mu_1(f_1)$. Similarly, $\mathrm{II}_n\to\mu_2(f_2)$, $\mathrm{III}_n\to 0$ and $\mathrm{IV}_n=0$. This verifies (2.11).

Conversely, suppose (2.11) holds so that

$$\lim_{n\to\infty}[nE[1-\mathrm{e}^{-f_1(Z_{n1})-f_2(W_{n1})}]$$
$$-nE[1-\mathrm{e}^{-f_1(Z_{n1})}]-nE[1-\mathrm{e}^{-f_2(W_{n1})}]]=0.\tag{2.12}$$

Then applying (2.12),

$$\begin{aligned}\lim_{n\to\infty} nE[(1-e^{-f_1(Z_{n1})})(1-e^{-f_2(W_{n1})})]\\ &= nE[1-e^{-f_1(Z_{n1})}-e^{-f_2(W_{n1})}+e^{-f_1(Z_{n1})-f_2(W_{n1})}]\\ &=0,\end{aligned}$$

for any $f_i\in C_K(E_i)$, $i=1,2$ and this is equivalent to (a).

We have verified, therefore, that (a) and (b) are equivalent. The rest of the proof is similar to the proof of Resnick (1987, Proposition 3.21). □

Proof of Proposition 2.1 (continued). We now apply Lemma 2.2 with

$$Z_{tk}=\left(\frac{\boldsymbol{X}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)},\frac{\boldsymbol{Y}_k}{\boldsymbol{a}(t)}\right),\qquad W_{tk}=\left(\frac{\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)},\frac{\boldsymbol{X}_k}{\boldsymbol{a}(t)}\right),$$

where both Z_{tk} and W_{tk} live in $E\times[-\infty,\infty]^d$. For A_1, A_2 compact in E, A_3, A_4 arbitrary,

$$\begin{aligned}\lim_{t\to\infty} tP\left[\left(\frac{\boldsymbol{X}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)},\frac{\boldsymbol{Y}_k}{\boldsymbol{a}(t)}\right)\in A_1\times A_3,\left(\frac{\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)},\frac{\boldsymbol{X}_k}{\boldsymbol{a}(t)}\right)\in A_2\times A_4\right]\\ &\le \lim_{t\to\infty} tP\left[\frac{\boldsymbol{X}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\in A_1,\frac{\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\in A_2\right]\\ &=\lim_{t\to\infty}\nu(A_1)\mu(A_2)/t=0.\end{aligned}$$

So (2.7) is satisfied. Furthermore since $\boldsymbol{X}_k$ and $\boldsymbol{Y}_k$ are independent and since $\boldsymbol{a}(t)\to\boldsymbol{\gamma}^{-1}$ it is clear

$$tP\left[\left(\frac{\boldsymbol{X}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)},\frac{\boldsymbol{Y}_k}{\boldsymbol{a}(t)}\right)\in\cdot\right]\xrightarrow{v}\nu\times G(\boldsymbol{\gamma}^{-1}\cdot)$$

and

$$tP\left[\left(\frac{\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)},\frac{\boldsymbol{X}_k}{\boldsymbol{a}(t)}\right)\in\cdot\right]\xrightarrow{v}\mu\times F(\boldsymbol{\gamma}^{-1}\cdot).$$

So it follows from Lemma 2.2 that as $t\to\infty$,

$$\begin{aligned}\left(\sum_{k=1}^{\infty}\varepsilon_{(k/t,(\boldsymbol{X}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t),\boldsymbol{Y}_k/\boldsymbol{a}(t))},\sum_{k=1}^{\infty}\varepsilon_{(k/t,(\boldsymbol{Y}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t),\boldsymbol{X}_k/\boldsymbol{a}(t))}\right)\\ \Rightarrow\left(\sum_k\varepsilon_{(t_k,j_k^{(1)},\boldsymbol{\gamma}\boldsymbol{Y}_k')},\sum_k\varepsilon_{(s_k,j_k^{(2)},\boldsymbol{\gamma}\boldsymbol{X}_k')}\right)\end{aligned}\tag{2.13}$$

in $(M_p([0,\infty)\times E\times[-\infty,\infty]^d))^2$ where the limit consists of two independent Poisson processes with mean measures $\lambda\times\nu\times G(\boldsymbol{\gamma}^{-1}\cdot)$ and $\lambda\times\mu\times F(\boldsymbol{\gamma}^{-1}\cdot)$, respectively. Because addition is vaguely continuous we get from (2.13),

$$\begin{aligned}\sum_{k=1}^{\infty}[\varepsilon_{(k/t,(\boldsymbol{X}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t),\boldsymbol{Y}_k/\boldsymbol{a}(t))}+\varepsilon_{(k/t,(\boldsymbol{Y}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t),\boldsymbol{X}_k/\boldsymbol{a}(t))}]\\ \Rightarrow\sum_k[\varepsilon_{(t_k,j_k^{(1)},\boldsymbol{\gamma}\boldsymbol{Y}_k')}+\varepsilon_{(s_k,j_k^{(2)},\boldsymbol{\gamma}\boldsymbol{X}_k')}]\end{aligned}\tag{2.14}$$

in $M_p([\infty,\infty)\times E\times[-\infty,\infty]^d)$ where the limit is Poisson with mean measure

$$\lambda\times\nu\times G(\gamma^{-1}\cdot)+\lambda\times\mu\times F(\gamma^{-1}\cdot)=\lambda\times(\nu\times G(\gamma^{-1}\cdot)+\mu\times F(\gamma^{-1}\cdot)).$$

Now let $1_k^{(1)}$, $1_k^{(2)}$ be the indicators of the events

$$\left[\frac{\boldsymbol{X}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant-\theta\mathbf{1}\right]^{\mathrm{c}},\qquad\left[\frac{\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant-\theta\mathbf{1}\right]^{\mathrm{c}},$$

respectively, and restrict the state space in (2.14) to the compact set $[0,T]\times[-\infty,-\theta\mathbf{1}]^{\mathrm{c}}\times[-\infty,\infty]^d$. With the state space so restricted we may add the $\boldsymbol{X}$ and $\boldsymbol{Y}$ components in (2.14) to get (via Proposition 3.18 of Resnick, 1987)

$$\begin{aligned}N_{t,\theta}&=\sum_{k\leqslant tT}\varepsilon_{(k/t,(\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t))}1_k^{(1)}+\sum_{k\leqslant tT}\varepsilon_{(k/t,(\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t))}1_k^{(2)}\\&=\sum_{k\leqslant tT}\varepsilon_{(k/t,(\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t))}(1_k^{(1)}+1_k^{(2)})\qquad(2.15)\\&\Rightarrow\sum_k\varepsilon_{(t_k,j_k^{(1)}+\gamma\boldsymbol{Y}_k')}1_{[[j_k^{(1)}\leqslant-\theta\mathbf{1}]^{\mathrm{c}}\cap[t_k\leqslant T]]}\\&\quad+\sum_k\varepsilon_{(s_k,j_k^{(2)}+\gamma\boldsymbol{X}_k')}1_{[[j_k^{(2)}\leqslant-\theta\mathbf{1}]^{\mathrm{c}}\cap[s_k\leqslant T]]}.\end{aligned}$$

Call this latter limit N_θ. As $\theta\to\infty$ we have, almost surely,

$$N_\theta\to\sum_k\varepsilon_{(t_k,j_k^{(1)}+\gamma\boldsymbol{Y}_k')}1_{[t_k\leqslant T]}+\sum_k\varepsilon_{(s_k,j_k^{(2)}+\gamma\boldsymbol{X}_k')}1_{[s_k\leqslant T]}$$

and so by Billingsley (1968, Theorem 4.2) we need to show for $\delta>0$,

$$\lim_{\theta\to\infty}\limsup_{t\to\infty}P[\rho(N_{t,\theta},N_t)>\delta]=0\qquad(2.16)$$

where ρ is the vague metric and

$$N_t=\sum_{k\leqslant tT}\varepsilon_{(k/t,(\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t))/\boldsymbol{a}(t))}.$$

For (2.16) it is enough to prove that for any $h\in C_{\mathrm{K}}(E)$ and any $\eta>0$,

$$\begin{aligned}\lim_{\theta\to\infty}\limsup_{t\to\infty}P\Bigg[\Bigg|&\sum_{k\leqslant tT}h\left(\frac{\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\right)(1_k^{(1)}+1_k^{(2)})\\&-\sum_{k\leqslant tT}h\left(\frac{\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\right)\Bigg|>\eta\Bigg]=0.\qquad(2.17)\end{aligned}$$

The difference referred to in (2.17) is bounded by

$$\sum_{k\leqslant tT}h\left(\frac{\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\right)|1-1_k^{(1)}-1_k^{(2)}|$$

and so the probability in (2.17) is bounded by

$$P\left[\bigcup_{k\leqslant tT}\left\{h\left(\frac{\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\right)>0, |1-1_k^{(1)}-1_k^{(2)}|\neq 0\right\}\right].$$

Suppose the support of h is contained in $[-\boldsymbol{\infty}, -M\boldsymbol{1}]^{\mathrm{c}}$. Then the previous probability is bounded by

$$tTP\left\{\left[h\left(\frac{\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\right)>0\right]\cap([1_k^{(1)}=0=1_k^{(2)}]\cup[1_k^{(1)}=1=1_k^{(2)}])\right\}$$

$$\leqslant tTP\left\{\left[\frac{\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant -M\boldsymbol{1}\right]^{\mathrm{c}}\cap\left[\frac{\boldsymbol{X}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant -\theta\boldsymbol{1}\right]\cap\left[\frac{\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant -\theta\boldsymbol{1}\right]\right\}$$

$$+\,tTP\left\{\left[\frac{\boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant -M\boldsymbol{1}\right]^{\mathrm{c}}\cap\left[\frac{\boldsymbol{X}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant -\theta\boldsymbol{1}\right]^{\mathrm{c}}\cap\left[\frac{\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant -\theta\boldsymbol{1}\right]^{\mathrm{c}}\right\}$$

$$=\mathrm{I}_{t,\theta}+\mathrm{II}_{t,\theta}.$$

Now

$$\limsup_{t\to\infty}\mathrm{II}_{t,\theta}\leqslant\limsup_{t\to\infty} tTP\left\{\left[\frac{\boldsymbol{X}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant -\theta\boldsymbol{1}\right]^{\mathrm{c}}\cap\left[\frac{\boldsymbol{Y}_k-\boldsymbol{b}(t)}{\boldsymbol{a}(t)}\leqslant -\theta\boldsymbol{1}\right]^{\mathrm{c}}\right\}=0,$$

because of (2.1), (2.2) and the independence of $\boldsymbol{X}_k$ and $\boldsymbol{Y}_k$.

As for $\mathrm{I}_{t,\theta}$, it is dominated:

$$\mathrm{I}_{t,\theta}\leqslant tT\sum_{i=1}^{d}P\left[\frac{X_{k,i}+Y_{k,i}-b_i(t)}{a_i(t)}>-M,\right.$$

$$\left.\frac{X_{k,i}-b_i(t)}{a_i(t)}\leqslant -\theta, \frac{Y_{k,i}-b_i(t)}{a_i(t)}\leqslant -\theta\right]. \tag{2.18}$$

Once $b_i(t)>2\theta-M$, the ith term within this sum is bounded by

$$P\left[\frac{X_{k,i}+Y_{k,i}-b_i(t)}{a_i(t)}>-M, \left(\frac{X_{k,i}}{a_i(t)}\wedge\frac{Y_{k,i}}{a_i(t)}\right)>\theta-M\right]$$

$$=P\left[\frac{X_{k,i}+Y_{k,i}-b_i(t)}{a_i(t)}>-M\right]$$

$$-P\left[\frac{X_{k,i}+Y_{k,i}-b_i(t)}{a_i(t)}>-M, \frac{X_{k,i}}{a_i(t)}\leqslant\theta-M\right]$$

$$-P\left[\frac{X_{k,i}+Y_{k,i}-b_i(t)}{a_i(t)}>-M, \frac{Y_{k,i}}{a_i(t)}\leqslant\theta-M\right]. \tag{2.19}$$

By Lemma 4.1(iv) and by the exponential tails of ν_i and μ_i,

$$\lim_{t\to\infty}\frac{\bar{G}_i(t)}{\bar{F}_i(t)}=\frac{\bar{\mu}_i(0)}{\bar{\nu}_i(0)}=\frac{\bar{\mu}_i(-M)}{\bar{\nu}_i(-M)},$$

where the limit is actually independent of M. Using a standard result (cf. Embrechts and Goldie, 1982), the first term on the right in (2.19) satisfies

$$\begin{aligned}
&\lim_{t\to\infty} tP\left[\frac{X_{k,i}+Y_{k,i}-b_i(t)}{a_i(t)} > -M\right] \\
&\quad = \lim_{t\to\infty} \frac{\overline{F_i * G_i}(b_i(t)-a_i(t)M)}{\bar{F}_i(b_i(t)-a_i(t)M)}\, t\bar{F}_i(b_i(t)-a_i(t)M) \\
&\quad = \left(\int_{-\infty}^{\infty} e^{\alpha_i y} G_i(dy) + \frac{\bar{\mu}_i(-M)}{\bar{\nu}_i(-M)} \int_{-\infty}^{\infty} e^{\alpha_i x} F_i(dx)\right) \bar{\nu}_i(-M) \\
&\quad = \bar{\nu}_i(-M) \int_{-\infty}^{\infty} e^{\alpha_i y} G_i(dy) + \bar{\mu}_i(-M) \int_{-\infty}^{\infty} e^{\alpha_i x} F_i(dx).
\end{aligned} \tag{2.20}$$

By Fatou's lemma and the fact that F_i, $G_i \in \mathscr{L}(\alpha_i)$, the two terms subtracted in (2.19) satisfy

$$\begin{aligned}
&\liminf_{t\to\infty} tP\left[\frac{X_{k,i}+Y_{k,i}-b_i(t)}{a_i(t)} > -M, \frac{X_{k,i}}{a_i(t)} \leqslant \theta - M\right] \\
&\qquad + tP\left[\frac{X_{k,i}+Y_{k,i}-b_i(t)}{a_i(t)} > -M, \frac{Y_{k,i}}{a_i(t)} \leqslant \theta - M\right] \\
&\quad = \liminf_{t\to\infty} \int_{-\infty}^{a_i(t)(\theta-M)} t\bar{F}_i(b_i(t)-a_i(t)M-y) G_i(dy) \\
&\qquad + \int_{-\infty}^{a_i(t)(\theta-M)} t\bar{G}_i(b_i(t)-a_i(t)M-x) F_i(dx) \\
&\quad \geqslant \bar{\nu}_i(-M) \int_{-\infty}^{\gamma_i^{-1}(\theta-M)} e^{\alpha_i y} G_i(dy) \\
&\qquad + \bar{\mu}_i(-M) \int_{-\infty}^{\gamma_i^{-1}(\theta-M)} e^{\alpha_i x} F_i(dx).
\end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21) into (2.18) and (2.19) and letting $\theta\to\infty$ yields the desired result that

$$\lim_{\theta\to\infty} \limsup_{t\to\infty} \mathrm{I}_{t,\theta} = 0. \qquad \square$$

Corollary 2.3. *Suppose $F \in \mathscr{L}(\nu; \boldsymbol{b}), G \in \mathscr{L}(\mu; \boldsymbol{b})$. If also $F_i \in \mathscr{S}(\alpha_i)$, $G_i \in \mathscr{S}(\alpha_i)$, $i = 1, \ldots, d$, then as $t\to\infty$,*

$$\sum_{k=1}^{\infty} \varepsilon_{(k/t, \boldsymbol{X}_k+\boldsymbol{Y}_k-\boldsymbol{b}(t))} \Rightarrow \mathrm{PRM}(\lambda \times (\nu * G + \mu * F))$$

so that

$$tF * G(\cdot + \boldsymbol{b}(t)) \xrightarrow{v} \nu * G + \mu * F.$$

Proof. Set $\boldsymbol{a}(t) \equiv \mathbf{1}$ in Proposition 2.1. □

Corollary 2.4. *Suppose* $F \in \mathscr{L}(\nu; \boldsymbol{b})$ *and the marginals of* F *satisfy* $F_i \in \mathscr{S}(\alpha_i)$, $i = 1, \ldots, d$. *Suppose also that* $\{\boldsymbol{X}_k\}$ *and* $\{\boldsymbol{Y}_k\}$ *are independent i.i.d. sequences with common distribution* F. *Then*

$$\sum_{k=1}^{\infty} \varepsilon_{(k/t, \boldsymbol{X}_k + \boldsymbol{Y}_k - \boldsymbol{b}(t))} \Rightarrow \mathrm{PRM}(\lambda \times 2\nu * F)$$

so that

$$tF * F(\cdot + \boldsymbol{b}(t)) \xrightarrow{v} 2\nu * F.$$

Thus $F \in \mathscr{S}(\nu, \boldsymbol{b})$ *and* $\nu^{(2)} = 2\nu * F$.

Proof. Set $F = G$, $\boldsymbol{a}(t) \equiv \mathbf{1}$ and apply Proposition 2.1. □

The next result shows how domains of attraction in extreme value theory may be preserved by convolution.

Corollary 2.5. *Suppose F and G are in a domain of attraction of a multivariate extreme value distribution* (*Resnick*, 1987) *and that* (2.1) *and* (2.2) *hold with* ν, μ *the exponent measures of multivariate extreme value distributions with marginals of the type* $\Lambda(x) = \exp\{-\mathrm{e}^{-x}\}$, $x \in \mathbb{R}$. *Suppose further that* $F_i \in \mathscr{S}(\alpha_i)$, $G_i \in \mathscr{S}(\alpha_i)$, $i = 1, \ldots, d$. *Then as* $t \to \infty$, (2.3) *and* (2.4) *hold so that* $F * G$ *is in the domain of attraction of the multivariate extreme value distribution*

$$\exp\{-\nu * G(\boldsymbol{\alpha}^{-1}[-\infty, \boldsymbol{x}]^{\mathrm{c}}) - \mu * F(\boldsymbol{\alpha}^{-1}[-\infty, \boldsymbol{x}]^{\mathrm{c}})\}$$

which also has marginals of the type $\Lambda(x)$.

Proof. Apply Theorem 2.1 with $\boldsymbol{\gamma} = \boldsymbol{\alpha}$. That the marginals of the limit distributions are of type $\Lambda(x)$ follows from Resnick (1987, Proposition 1.19). □

3. Examples and extensions

In this section we provide several examples in addition to the Type III domain of attraction example given above. We also extend known results about the univariate classes $\mathscr{S}(\alpha_i)$ to the multivariate setting.

A simple example is the case where the components of $\boldsymbol{X}$ are independent or, more generally, for $1 \leq i \neq j \leq d$,

$$\lim_{t \to \infty} tP[X_i > x_i + b_i(t), X_j > x_j + b_j(t)] = 0.$$

One may easily show that $F_i \in \mathscr{L}(\alpha_i)$ for each i implies $F \in \mathscr{L}(\nu; \boldsymbol{b})$ where

$$\bar{\nu}(\boldsymbol{u}) := \nu([-\infty, \boldsymbol{u}]^{\mathrm{c}}) = \sum_{i=1}^{d} \mathrm{e}^{-\alpha_i u_i}.$$

Note that ν therefore concentrates on the axes, $\Omega_i = \{\boldsymbol{u}: u_i > -\infty,\ u_j = -\infty,\ j \neq i\}$, $i = 1, \ldots, d$.

At the other extreme, suppose $X_1 = \cdots = X_d$ almost surely. Then $F_1 \in \mathscr{L}(\alpha_1)$ implies $F \in \mathscr{L}(\nu; \boldsymbol{b})$ where ν has all its mass on the ray $\{r\mathbf{1}: r \in (-\infty, \infty]\}$ and

$$\bar{\nu}(\boldsymbol{u}) = \mathrm{e}^{-\alpha_1(\bigwedge_i u_i)}.$$

We next present an example which is a mixture of the cases $\alpha_i = 0$ and $\alpha_i > 0$ and also a mixture of independence and total dependence. Let X_1 and X_1' be independent exponential(1) random variables and let $X_2 = \mathrm{e}^{X_1}$ with probability δ and $X_2 = \mathrm{e}^{X_1'}$ with probability $1 - \delta$. A little computation shows the joint distribution of (X_1, X_2) is in the class $\mathscr{L}(\nu;\ (\log t, t))$ with limiting measure concentrating on $(-\infty, \infty) \times \{-\infty\} \cup [-\infty, \infty) \times \{\infty\}$ and so that

$$\begin{aligned}\nu((x_1, \infty) \times \{-\infty\}) &= \lim_{t \to \infty} tP[(X_1 - \log t, X_2 - t) \in (x_1, \infty) \times [-\infty, x_2]] \\ &= \mathrm{e}^{-x_1} - \delta(1 \wedge \mathrm{e}^{-x_1})\end{aligned}$$

and

$$\begin{aligned}\nu((x_1, \infty) \times \{\infty\}) &= \lim_{t \to \infty} tP[(X_1 - \log t, X_2 - t) \in (x_1, \infty) \times (x_2, \infty]] \\ &= \delta(1 \wedge \mathrm{e}^{-x_1}) + (1 - \delta) 1_{\{-\infty\}}(x_1).\end{aligned}$$

This example shows that when some but not all of the exponents α_i are zero, ν is not generally expressed as a mixture of the two pure types of exponential measures.

Regular variation

A distribution F on $\mathbb{R}_+^d$ has regularly varying tails on the cone $[0, \infty]^d \backslash \{\mathbf{0}\}$ with limit measure μ (written $F \in \mathscr{R}(\mu; \boldsymbol{b})$) if there exists a sequence $\boldsymbol{b}(t)$ and a Radon measure μ on $[0, \infty]^d \backslash \{\mathbf{0}\}$, satisfying (a) and (b) of the introduction, such that

$$tF(\boldsymbol{b}(t)\ \cdot) \xrightarrow{\mathrm{v}} \mu(\ \cdot\ ) \tag{3.1}$$

on $[0, \infty)^d \backslash \{\mathbf{0}\}$. (cf. de Haan and Resnick, 1977; Resnick, 1987; Omey 1989.)

We extend the well known univariate result that regular variation implies subexponentiality.

Proposition 3.1. *Suppose $F \in \mathscr{R}(\mu; \boldsymbol{b})$. Define $A_{+\infty} = (1, \infty)$, $A_{-\infty} = (-\infty, 1]$ and let $\nu^{(0)}$ to be the discrete measure on E, having all its mass on $\Xi := \{-\infty, \infty\}^d \backslash \{-\boldsymbol{\infty}\}$ given in* (1.7) *and with weights*

$$\nu^{(0)}(\{\boldsymbol{\sigma}\}) = \mu(A_{\sigma_1} \times \cdots \times A_{\sigma_d}), \quad \boldsymbol{\sigma} \in \Xi.$$

Then $F \in \mathscr{S}(\nu^{(0)}; \boldsymbol{b})$ and $F * F \in \mathscr{R}(2\mu; \boldsymbol{b}) \cap \mathscr{S}(2\nu^{(0)}; \boldsymbol{b})$.

Proof. Regular variation of F implies regular variation for the probability tails of all subvectors. (To see that i dimensional marginals are regularly varying, take a sequence of functions which are continuous with support compact in E and which approximate the indicator function of $(x_1,\infty]\times\cdots\times(x_i,\infty]\times[-\infty,\infty]^{d-i}$ and insert these into the definition of multivariate regular variation.) Thus it suffices for what follows to take $\boldsymbol{u}$ to be a finite vector. From (3.1) and the locally uniform convergence implicit in regular variation,

$$\lim_{t\to\infty} t\bar{F}(\boldsymbol{u}+\boldsymbol{b}(t))=\lim_{t\to\infty} t\bar{F}(\boldsymbol{b}(t)(\boldsymbol{1}+(u_i/b_i(t))_i))$$
$$=\bar{\mu}(\boldsymbol{1})=\sum_{\sigma\in\Xi}\nu^{(0)}(\boldsymbol{\sigma}).$$

This being so for all subvectors as well, it follows that $F\in\mathscr{L}(\nu^{(0)};\boldsymbol{b})$.

The argument of Resnick (1986, Proposition 4.1) is easily modified to allow for vector scaling. Furthermore it is valid even if some $\alpha_i=0$. See Resnick (1986, Section 5). Hence, $F*F\in\mathscr{R}(2\mu;\boldsymbol{b})$ and

$$\lim_{t\to\infty} t\overline{F*F}(\boldsymbol{b}(t)\boldsymbol{u})=2\bar{\mu}(\boldsymbol{u}).$$

Applying this to the above argument, $F*F\in\mathscr{L}(2\nu^{(0)};\boldsymbol{b})$ and hence $F\in\mathscr{S}(\nu^{(0)};\boldsymbol{b})$. And applying the last conclusion to $F*F$, we also have $F*F\in\mathscr{S}(2\nu^{(0)};\boldsymbol{b})$. $\square$

Multivariate stable laws and Type I max-stable laws are examples of such distributions.

There exist, however, class $\mathscr{L}$ distributions which do not have multivariate regularly varying tails even though the marginal tails are regularly varying. As an example of this, consider F such that for $x_i\geq 0$,

$$P[X_1>x_1,X_2>x_2]=\frac{1+\gamma\sin(\log r(\boldsymbol{x}))\sin(\pi\phi(\boldsymbol{x}))}{r(\boldsymbol{x})},\tag{3.2}$$

where $r(\boldsymbol{x})=1+x_1+x_2$, $\phi(\boldsymbol{x})=(x_1-x_2)/r(\boldsymbol{x})$ and $0<|\gamma|\leq\frac{1}{12}$. The tails are asymptotically Pareto:

$$P[X_i>x]\sim x^{-1},\quad x\to\infty.$$

One may easily show that

$$\lim_{t\to\infty} tP[X_i>x_i+t]=1,\quad i=1,2,$$

and

$$\lim_{t\to\infty} tP[X_1>x_1+t,X_2>x_2+t]=\tfrac{1}{2}.$$

But $tP[X_1>t,X_2>ct]$ does not generally converge. Since the marginals are subexponential, this distribution is in fact multivariate subexponential.

To show that marginal membership in $\mathscr{L}(\alpha_i)$ does not imply membership in $\mathscr{L}(\nu;\boldsymbol{b})$, consider the related example

$$P[X_1>x_1,X_2>x_2]=\frac{1+\gamma\sin(\log r(\boldsymbol{x}))\cos(\frac{1}{2}\pi\phi(\boldsymbol{x}))}{r(\boldsymbol{x})}.$$

Again, the marginals are Pareto. If this distribution is to be in $\mathscr{L}(\nu; \boldsymbol{b})$ then we could choose $\boldsymbol{b}(t) = (c_1 t, c_2 t)$ for some $c_i > 0$. But in this case, $tP[X_1 > c_1 t, X_2 > c_2 t]$ converges only if $\cos(\frac{1}{2}\pi\phi(c_1 t, c_2 t)) \to 0$, that is, only if either $c_1 = 0$ or $c_2 = 0$.

Higher order convolutions

Much of the past effort on the univariate class $\mathscr{S}(\alpha)$ has been directed at the behavior of F^{*n} and of the distribution of a randomly stopped sum (cf. Chover, Ney and Wainger, 1973a; Embrechts and Goldie, 1982; Embrechts, Goldie and Veraverbeke, 1979; Cline, 1987). These results may now be extended to the multivariate case.

Proposition 3.2. *Suppose $F \in \mathscr{L}(\nu; \boldsymbol{b})$. The following are equivalent:*

(i) $F \in \mathscr{S}(\nu; \boldsymbol{b})$.

(ii) $F^{*n} \in \mathscr{L}(n\nu * F^{*(n-1)}; \boldsymbol{b})$ *for some* $n \geq 2$.

(iii) $F^{*n} \in \mathscr{S}(\nu^{(n)}; \boldsymbol{a})$ *for some* $n \geq 2$, *some* $\nu^{(n)}$, *some* $\boldsymbol{a}$.

When these hold, they hold for all n and with $\boldsymbol{a} = \boldsymbol{b}$, $\nu^{(n)} = n\nu * F^{*(n-1)}$.

Proof. (i)$\Rightarrow$(ii). For each n, (ii) follows by application of Corollary 2.3 and induction.

(i)$\Rightarrow$(iii). Since by the above $F^{*n} \in \mathscr{L}(\nu^{(n)}; \boldsymbol{b})$ and $F^{*2n} \in \mathscr{L}(\nu^{(2n)}; \boldsymbol{b})$ for some $\nu^{(n)}$ and $\nu^{(2n)}$, then $F^{*n} \in \mathscr{S}(\nu^{(n)}; \boldsymbol{b})$.

(ii)$\Rightarrow$(i). Since $F \in \mathscr{L}(\nu; \boldsymbol{b})$ is assumed, we only need to show that $F_i \in \mathscr{S}(\alpha_i)$ for each i. We have

$$\lim_{t\to\infty} t\bar{F}_i(b_i(t)) = \bar{\nu}_i(0)$$

and

$$\lim_{t\to\infty} t\overline{F^{*n}}(b_i(t)) = \overline{n\nu_i * F_i^{*n-1}}(0) = n\left(\int_{-\infty}^{\infty} e^{\alpha_i x} F_i(dx)\right)^{n-1} \bar{\nu}_i(0).$$

By Proposition 1.1(ii),

$$\lim_{t\to\infty} \frac{\overline{F^{*n}}(t)}{\bar{F}_i(t)} = n\left(\int_{-\infty}^{\infty} e^{\alpha_i x} F_i(dx)\right)^{n-1}.$$

According to the univariate theory (e.g., Cline, 1987, Corollary 2.11, which is valid even if $F_i(0) > 0$), this is sufficient to conclude $F_i \in \mathscr{S}(\alpha_i)$.

(iii)$\Rightarrow$(i). Again, it suffices to show $F_i \in \mathscr{S}(\alpha_i)$ for each i. According to Proposition 1.1(iii), $F_i^{*n} \in \mathscr{S}(\alpha_i)$. This implies $F_i \in \mathscr{S}(\alpha_i)$ (cf. Cline, 1987, Corollary 2.11). □

Note that it is necessary to specify the norming sequence $\boldsymbol{b}$ in (ii) of Proposition 3.2. For example, the gamma(1) and gamma(2) distributions are each in the class $\mathscr{L}$ but with non-equivalent norming sequences. Hence neither is in the class $\mathscr{S}$.

For the following discussion on compound distributions, let $\{\lambda_n\}$ be a probability measure on $\{1, 2, \ldots\}$ and define $\lambda(z) = \sum_{n=1}^{\infty} \lambda_n z^n$ for real z and $H = \sum_{n=1}^{\infty} \lambda_n F^{*n}$, $H' = \sum_{n=1}^{\infty} n\lambda_n F^{*n}$ for the measure F.

Proposition 3.3. *Let* $F \in \mathscr{L}(\nu; \boldsymbol{b})$ *such that* $\bar{F}_i(0)=1$ *for each* i, *and let* $m_i = \int_0^\infty e^{\alpha_i x} F_i(dx)$. Suppose $\lambda(m_i+\varepsilon)<\infty$ *for some* $\varepsilon>0$ *and for each* i. *Then the following are equivalent*:

(i) $F \in \mathscr{S}(\nu; \boldsymbol{b})$.

(ii) $H \in \mathscr{L}(\nu * H'; \boldsymbol{b})$.

(iii) $H \in \mathscr{S}(\nu_H; \boldsymbol{a})$ *for some* ν_H *and some* $\boldsymbol{a}$; *and one of the following holds*:

(a) $\limsup_{t\to\infty} t\bar{H}_i(b_i(t))<\infty$ *for each* i.

(b) $\lambda(q+\varepsilon)<\infty$, *where* $q=\sup_i \lambda(m_i)$.

(c) $\limsup_{n\to\infty}(\lambda_{n+1}/\lambda_n)<\inf_i(1/m_i)$.

Proof. (i)$\Rightarrow$(ii). Let $\boldsymbol{u}\in(-\infty,\infty]^d$. For each marginal, Cline (1987, Theorem 2.13) gives

$$\begin{aligned}
\lim_{t\to\infty} t\bar{H}_i(b_i(t)+u_i) &= \lim_{t\to\infty}\frac{\bar{H}_i(t+u_i)}{\bar{F}_i(t)}\bar{\nu}_i(0)\\
&= e^{-\alpha_i u_i}\left(\sum_{n=1}^{\infty}\lambda_n n m_i^{n-1}\right)\bar{\nu}_i(0)\\
&= e^{-\alpha_i u_i}\sum_{n=1}^{\infty} n\lambda_n \overline{\nu_i * F_i^{*n-1}}(0)\\
&= \sum_{n=1}^{\infty}\lim_{t\to\infty}\lambda_n \overline{F_i^{*n}}(u_i+b_i(t)).
\end{aligned}$$

Also,

$$\overline{F^{*n}}(\boldsymbol{u}+\boldsymbol{b}(t)) \leq \sum_{i=1}^{d}\overline{F_i^{*n}}(u_i+b_i(t)).$$

Thus, by Proposition 3.2 and dominated convergence,

$$\begin{aligned}
\lim_{t\to\infty} t\bar{H}(\boldsymbol{u}+\boldsymbol{b}(t)) &= \sum_{n=1}^{\infty}\lim_{t\to\infty}\lambda_n\overline{F^{*n}}(\boldsymbol{u}+\boldsymbol{b}(t))\\
&= \sum_{n=1}^{\infty} n\lambda_n\overline{\nu * F^{*n-1}}(\boldsymbol{u})\\
&= \overline{\nu * H'}(\boldsymbol{u}),
\end{aligned}$$

whenever $\boldsymbol{u}\in[-\infty,\infty]^d$, $u_i>-\infty$ for each i. That is, $H\in\mathscr{L}(\nu * H'; \boldsymbol{b})$.

(i)$\Rightarrow$(iii). This follows by applying the above implication to each of H and $H * H = \lambda^2(F)$.

(ii)$\Rightarrow$(i). As in the corresponding argument for Proposition 3.2, (ii) implies

$$\lim_{t\to\infty}\frac{\bar{H}_i(t)}{\bar{F}_i(t)} = \lambda'(m_i)\nu_i(0)$$

which is sufficient (Cline, 1987, Theorem 2.13) for $F_i\in\mathscr{S}(\alpha_i)$, every i, and hence for $F\in\mathscr{S}(\nu; \boldsymbol{b})$.

(iii)⇒(i). The assumption (iii) implies $H_i \in \mathscr{S}(\alpha_i)$. With one of the additional assumptions (a), (b) or (c), it follows that $F_i \in \mathscr{S}(\alpha_i)$ for each i (cf. Cline, 1987, Theorem 2.13 and Corollary 2.14). Thus $F \in \mathscr{S}(\nu; \boldsymbol{b})$. □

Other examples

Suppose $\boldsymbol{Y} \sim G \in \mathscr{R}(\mu'; \boldsymbol{b}')$. Then componentwise transformations will give rise to a variety of examples.

For one such example, suppose $\bar{G}_i \in \mathrm{RV}_{-\beta_i}$ with $\beta_i > 0$ and $Y_i > 0$ a.s. and let F be the distribution of

$$\boldsymbol{X} = (c_i \log Y_i)_i, \quad c_i > 0.$$

Here $G \in \mathscr{R}(\mu'; \boldsymbol{b}')$ is equivalent to $F \in \mathscr{L}(\mu; \boldsymbol{b})$, for some μ and $\boldsymbol{b}$, and μ has exponents $\alpha_i = \beta_i / c_i > 0$. Thus F is in $\mathscr{S}(\nu; \boldsymbol{b})$ if and only if every $F_i \in \mathscr{S}(\alpha_i)$. Cline (1986) gives examples both of $F_i \in \mathscr{S}(\alpha_i)$ and of $F_i \notin \mathscr{S}(\alpha_i)$. For instance, if $\mathrm{e}^{\alpha_i t} \bar{F}_i(t) \in \mathrm{RV}_{\gamma_i}$ for some real γ_i, then $F \in \mathscr{S}(\mu; \boldsymbol{b})$ if and only if

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha_i x} F_i(\mathrm{d}x) < \infty \quad \text{for every } i.$$

As another example, again suppose $\boldsymbol{Y} \sim G \in \mathscr{R}(\mu'; \boldsymbol{b}')$ and let $F_i \in \mathscr{S}(0) \cap \mathscr{D}(\Lambda)$. (Cf. Goldie and Resnick (1988) for sufficient conditions.) Define

$$X_i = F_i^{\leftarrow}(G_i(Y_i)).$$

By de Haan and Resnick (1977, Theorem 4) F is in the (extreme value) domain of attraction of a max-stable law with double exponential (Λ) marginals. Since the marginals of F are also subexponential then F is multivariate subexponential.

A specific two-dimensional example of this, with lognormal marginals, is for $x_i \geq 0$,

$$F(x_1, x_2) = 1 - \psi_1(x_1) - \psi_2(x_2) + \frac{\psi_1(x_1)\psi_2(x_2)}{\psi_1(x_1) + \psi_2(x_2)},$$

where $\psi_i(x) = 1 - \Phi((\log x - \mu_i)/\sigma_i))$ and Φ is the standard normal distribution.

4. Norming sequences

In this section we examine the norming function $\boldsymbol{b}(t)$ appearing in definitions (1.3) and (1.4) with the intent of describing equivalent versions. When we say two norming functions $\boldsymbol{b}_1(t)$, $\boldsymbol{b}_2(t)$ are equivalent for given distribution F, we mean that either could be used in the definitions of the classes given in (1.3) and (1.4).

Let g be a nondecreasing function and define its left- and right-continuous versions:

$$g^-(t) = \sup_{x<t} g(x) \quad \text{and} \quad g^+(t) = \inf_{x>t} g(x).$$

We will use the left-continuous version of the inverse:

$$g^{\leftarrow}(t) = \inf\{x: g(x) \geq t\}.$$

Note that $g(x) \geq t$ implies $x \geq g^{\leftarrow}(t)$ and $g(x) < t$ implies $x \leq g^{\leftarrow}(t)$. We also observe that $(g^{\leftarrow})^{\leftarrow} = g^{-}$. Following Geluk and de Haan (1987, p. 32), two functions $h_1, h_2 : [0, \infty) \mapsto \mathbb{R}$ are *inversely asymptotic* if for every $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon)$ such that for $t \geq t_0$,

$$h_2((1-\varepsilon)t) \leq h_1(t) \leq h_2((1+\varepsilon)t). \tag{4.1}$$

If h_1 and h_2 are non-decreasing then the relation inversely asymptotic means $h_1^{\leftarrow} \sim h_2^{\leftarrow}$; i.e., the inverses are asymptotically equivalent. We start with a lemma about inverse asymptotic equivalence.

Lemma 4.1. (i) *Suppose g is nondecreasing. Then $g(b(t)) \sim t$ if and only if both $g(g^{\leftarrow}(t)) \sim t$ and $g^{\leftarrow}$ and b are inversely asymptotic.*

(ii) *Suppose g and b are nondecreasing. If $g(b(t)) \sim t$ then $b^{\leftarrow}(t) \sim g(t)$.*

(iii) *Suppose g is nondecreasing. Then $g(g^{\leftarrow}(t)) \sim t$ if and only if $g^{+}(t) \sim g^{-}(t)$.*

(iv) *Suppose g_1, g_2 are each nondecreasing. If $g_1(b(t)) \sim g_2(b(t)) \sim t$, then $g_1 \sim g_2$.*

Proof. (i) b_1 and b_2 be two functions such that $g(b_1(t)) \sim t$ and b_1 and b_2 are inversely asymptotic so that for each $\varepsilon > 0$,

$$b_1((1-\varepsilon)t) \leq b_2(t) \leq b_1((1+\varepsilon)t)$$

whenever t is large enough. Then for some t_1 and all $t \geq t_1$,

$$(1-\varepsilon)^2 t \leq g(b_1((1-\varepsilon)t)) \leq g(b_2(t)) \leq g(b_1((1+\varepsilon)t)) \leq (1+\varepsilon)^2 t.$$

Hence $g(b_2(t)) \sim t$.

The 'if' part is satisfied, therefore, by choosing $b_1 = g^{\leftarrow}$ and $b_2 = b$.

On the other hand suppose $g(b(t)) \sim t$. Given $\varepsilon > 0$ and large enough t,

$$g(b((1-\varepsilon)t)) < t < g(b((1+\varepsilon)t)),$$

which implies

$$b((1-\varepsilon)t) \leq g^{\leftarrow}(t) \leq b((1+\varepsilon)t).$$

This implies (4.1) for some t_0 and the "only if" part follows with $b_1 = b$ and $b_2 = g^{\leftarrow}$.

(ii) One may easily show that for any $\varepsilon > 0$,

$$b((1-\varepsilon)b^{\leftarrow}(t)) \leq t \leq b((1+\varepsilon)b^{\leftarrow}(t)),$$

so that

$$\begin{aligned} 1-\varepsilon &= \lim_{t\to\infty} \frac{g(b((1-\varepsilon)b^{\leftarrow}(t)))}{b^{\leftarrow}(t)} \\ &\leq \liminf_{t\to\infty} \frac{g(t)}{b^{\leftarrow}(t)} \\ &\leq \limsup_{t\to\infty} \frac{g(t)}{b^{\leftarrow}(t)} \\ &\leq \lim_{t\to\infty} \frac{g(b((1+\varepsilon)b^{\leftarrow}(t)))}{b^{\leftarrow}(t)} = 1+\varepsilon. \end{aligned}$$

Thus $g \sim b^{\leftarrow}$.

(iii) If $g(g^{\leftarrow}(t)) \sim t$ then, by (ii), $g^- = (g^{\leftarrow})^{\leftarrow} \sim g$. However, the argument for (ii) holds equally well if we replace $b^{\leftarrow}$ with the right-continuous version of the inverse, $b_r^{\leftarrow}$. Since $g^+ = (g_r^{\leftarrow})_r^{\leftarrow}$, we also conclude $g \sim g^+$ and hence $g^+ \sim g^-$.

Conversely, note that $g^-(g^{\leftarrow}(t)) \leq t \leq g^+(g^{\leftarrow}(t))$. Thus $g^+ \sim g^-$ implies

$$\lim_{t\to\infty} \frac{g^-(g^{\leftarrow}(t))}{t} = \lim_{t\to\infty} \frac{g^+(g^{\leftarrow}(t))}{t} = 1$$

and, since $g^- \leq g \leq g^+$, we have that $g(g^{\leftarrow}(t)) \sim t$.

(iv) By (i), g_1 and g_2 are each inversely asymptotic to b and thus to each other. Therefore, $g_1(g_2^{\leftarrow}(t)) \sim t$. By (ii) and then (iii) $g_1 \sim (g_2^{\leftarrow})^{\leftarrow} = g_2^- \sim g_2$. □

The condition $g(g^{\leftarrow}(t)) \sim t$ does not generally hold. It does hold for $g = r \circ s$ where $r \in \mathrm{RV}_\gamma$ and s is a continuous 1–1 function. In particular, it holds for $g = 1/\bar{F}_i$, when $F_i \in \mathscr{L}(\alpha_i)$. Let

$$q_i(t) = \bar{F}_i^{\leftarrow}(1/t) = \left(\frac{1}{1-F_i}\right)^{\leftarrow}(t).$$

An immediate consequence of Lemma 4.1 is that $F_i \in \mathscr{L}(\alpha_i)$ if and only if $t\bar{F}_i(q_i(t)+u) \to \mathrm{e}^{-\alpha_i u}$ (see Proposition 1.1).

Proposition 4.2. *Suppose $F \in \mathscr{L}(\nu; \boldsymbol{b})$. Then $F \in \mathscr{L}(\nu; \boldsymbol{a})$ if and only if $\boldsymbol{a}$ is such that $a_i(t)$ and $b_i(t)$ are inversely asymptotic for each $i = 1, \ldots, d$. In particular, we may take $a_i(t) = q_i(t/\bar{\nu}_i(0))$.*

Proof. Suppose a_i and b_i are inversely asymptotic for each $i = 1, \ldots, d$. Then for every $\varepsilon > 0$ there is t_0 such that for all i and all $t \geq t_0$,

$$a_i((1-\varepsilon)t) \leq b_i(t) \leq a_i((1+\varepsilon)t). \tag{4.2}$$

Thus for $t \geq t_0$ and any $\boldsymbol{u} \in (-\infty, \infty]^d$,

$$\bar{F}(\boldsymbol{u} + \boldsymbol{b}((1+\varepsilon)t)) \leq \bar{F}(\boldsymbol{u} + \boldsymbol{a}(t)) \leq \bar{F}(\boldsymbol{u} + \boldsymbol{b}((1-\varepsilon)t)).$$

Hence

$$\frac{1}{1+\varepsilon}\,\bar{\nu}(\boldsymbol{u}) \leq \liminf_{t\to\infty} t\bar{F}(\boldsymbol{u} + \boldsymbol{a}(t))$$

$$\leq \limsup_{t\to\infty} t\bar{F}(\boldsymbol{u} + \boldsymbol{a}(t)) \leq \frac{1}{1-\varepsilon}\,\bar{\nu}(\boldsymbol{u}).$$

This shows $F \in \mathscr{L}(\nu; \boldsymbol{a})$.

In particular, let $a_i(t) = q_i(t/\bar{\nu}_i(0))$. Since $\lim_{t\to\infty} t\bar{F}_i(b_i(t)) = \bar{\nu}_i(0)$ then for large enough t,

$$\frac{\bar{\nu}_i(0)}{(1+\varepsilon)t} < \bar{F}_i(b_i(t)) < \frac{\bar{\nu}_i(0)}{(1-\varepsilon)t}.$$

Hence $a_i((1-\varepsilon)t) \leq b_i(t) \leq a_i((1+\varepsilon)t)$, which is (4.2).

On the other hand, if $F \in \mathscr{L}(\nu; \boldsymbol{a})$ and we let $g(t) = \bar{\nu}_i(0)/\bar{F}_i(t)$ then

$$\lim_{t\to\infty} t^{-1} g(b_i(t)) = \lim_{t\to\infty} t^{-1} g(a_i(t)) = 1.$$

By Lemma 4.1(i), it must be a_i and b_i are each inversely asymptotic to $g^{\leftarrow}$ and hence to each other. □

Given that $F \in \mathscr{L}(\nu; \boldsymbol{b})$, Proposition 4.2 characterizes the possible norming sequences $\boldsymbol{a}$ such that $F \in \mathscr{L}(\nu; \boldsymbol{a})$. However, we would like to characterize the sequences $\boldsymbol{a}$ such that $F \in \mathscr{L}(\mu, \boldsymbol{a})$ for some μ. At least, we want to know when $b_i(t)$ may be replaced with $q_i(c_i t)$ for an arbitrary positive c_i.

For those marginals whose characteristic exponent α_i is positive, a characterization follows from a multivariate convergence of types result.

Proposition 4.3. *Assume $F \in \mathscr{L}(\nu; \boldsymbol{b})$ and let $\boldsymbol{a}$ be such that $a_i(t) = b_i(t)$ whenever $\alpha_i = 0$. Then $F \in \mathscr{L}(\mu; \boldsymbol{a})$ for some μ if and only if for each i,*

$$d_i = \lim_{t\to\infty} (b_i(t) - a_i(t)) \quad \textit{exists finite.} \tag{4.3}$$

In this case $\mu(\cdot) = \nu(\cdot - \boldsymbol{d})$.

Furthermore, (4.3) *is satisfied when $a_i(t) = q_i(c_i t)$ for $\alpha_i > 0$ and $c_i > 0$.*

Proof. Let F_1 be the marginal distribution for the subvector of those X_i's having $\alpha_i > 0$. Then $F_1 \in \mathscr{L}(\nu_1; \boldsymbol{b}_1)$ for corresponding choices of ν_1 and $\boldsymbol{b}_1$. This is equivalent to saying $F_1^*(\boldsymbol{x}) = F_1((\log x_1, \ldots, \log x_d)) \in \mathscr{R}(\nu_1; \boldsymbol{b}_1)$ (see Section 3 or de Haan and Resnick, 1977). That is, F_1^* is in a Type I multivariate extreme value domain of attraction. By the convergence of types theorem, therefore, we have $F_1 \in \mathscr{L}(\mu_1; \boldsymbol{a}_1)$ if and only if for each i the ratio $\mathrm{e}^{b_{1i}(t)}/\mathrm{e}^{a_{1i}(t)}$converges to a finite positive constant, that is, if and only if

$$d_{1i} = \lim_{t\to\infty} (b_{1i}(t) - a_{1i}(t)) \text{ exists.} \tag{4.4}$$

If, therefore, $F \in \mathscr{L}(\mu, \boldsymbol{a})$, then (4.3) follows from (4.4) for each i such that $\alpha_i > 0$ and it follows by assumption for each i such that $\alpha_i = 0$.

Conversely, if (4.3) holds then by the uniform marginal convergence

$$\lim_{t\to\infty} |t\bar{F}(\boldsymbol{u} + \boldsymbol{a}(t)) - t\bar{F}(\boldsymbol{u} + \boldsymbol{b}(t) - \boldsymbol{d})|$$

$$\leq \lim_{t\to\infty} \sum_{i=1}^{d} |t\bar{F}_i(u_i + a_i(t)) - t\bar{F}_i(u_i + b_i(t) - d_i)| = 0.$$

Hence $F \in \mathscr{L}(\nu(\cdot - \boldsymbol{d}); \boldsymbol{a})$.

Finally, we know from Proposition 4.2 that $F \in \mathscr{L}(\nu; \boldsymbol{b})$ with $b_i(t) = q_i(t/\bar{\nu}_i(0))$. But for i such that $\alpha_i > 0$, we have for any $c > 0$,

$$\lim_{t \to \alpha} (q_i(ct) - q_i(t)) = \frac{\log c}{\alpha_i}. \tag{4.5}$$

(This is again due to the fact that $\bar{F}_i(\log x) \in \mathrm{RV}_{-\alpha_i}$.) Thus for each such i, $b_i(t)$ may be replaced with $q_i(c_i t)$. $\square$

Indeed, if every $\alpha_i > 0$, then $F \in \mathscr{L}(\nu(\cdot - \boldsymbol{d}); \boldsymbol{q})$ where $d_i = (\log \bar{\nu}_i(0))/\alpha_i$.

In case the α_i's are zero, one may use a similar argument (convergence of types) when F is in a multivariate extreme value domain of attraction (see de Haan and Resnick, 1977). A general approach which encompasses all of these is as follows. Let $\boldsymbol{M}_n = (\bigvee_{k=1}^n X_{ki})_i$ for an i.i.d. sequence $\{\boldsymbol{X}_n\}$.

Proposition 4.4. *Suppose there exists a sequence of vector-valued functions* $\boldsymbol{g}_n(\boldsymbol{x}) = (g_{ni}(x_i))_i$ *such that* $P[\boldsymbol{M}_n \leqslant \boldsymbol{g}_n(\boldsymbol{x})]$ *converges to a probability distribution* $H(\boldsymbol{x})$ *with* exponential(1) *marginals. Then for each* $\boldsymbol{c} \in [0, \infty)^d \backslash \{\boldsymbol{0}\}$,

$$\lim_{t} t\bar{F}(\boldsymbol{q}(\boldsymbol{c}t)) = -\log H((c_1^{-1}, \ldots, c_d^{-1})).$$

Proof. From the assumption we immediately have that for each i,

$$\lim_{n \to \infty} n\bar{F}_i(g_{ni}(x_i)) = \lim_{n \to \infty} -\log P[M_{ni} \leqslant g_{ni}(x_i)] = x_i.$$

By Lemma 4.1(i), it follows that

$$\lim_{n \to \infty} n\bar{F}_i(q_i(n/x_i)) = x_i.$$

Thus,

$$\begin{aligned} &\lim_{n \to \infty} n|\bar{F}(\boldsymbol{g}_n(\boldsymbol{x})) - \bar{F}(\boldsymbol{q}((n/x_i)_i))| \\ &\quad \leqslant \lim_{n \to \infty} \sum_{i=1}^{d} n|\bar{F}_i(g_{ni}(x_1)) - \bar{F}_i(q_i(n/x_i))| \\ &\quad = 0. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{n \to \infty} n\bar{F}(\boldsymbol{q}((n/x_i)_i)) &= \lim_{n \to \infty} -\log P[\boldsymbol{M}_n \leqslant \boldsymbol{g}_n(\boldsymbol{x})] \\ &= -\log H(\boldsymbol{x}), \end{aligned}$$

which is equivalent to the assertion. $\square$

Therefore if $F \in \mathscr{L}(\nu; \boldsymbol{b})$ with $\boldsymbol{\alpha} = \boldsymbol{0}$ and F also satisfies the conditions of Proposition 4.4, then for $\boldsymbol{x}$ finite,

$$\lim_{t \to \infty} t\bar{F}(\boldsymbol{x} + \boldsymbol{q}(t)) = -\log H(\boldsymbol{1}).$$

When one or more components of x are equal to $+\infty$, and we define $e_i = \infty 1_{\{\infty\}}(x_i) + 1_{(-\infty,\infty)}(x_i)$, then

$$\lim_{t\to\infty} t\bar{F}(x + q(t)) = -\log H(e).$$

Furthermore, suppose F satisfies the conditions of Proposition 4.4 and has marginal equivalency, i.e., for each i,

$$c_i = \lim_{u\to\infty} \frac{\bar{F}_i(u)}{\bar{F}_1(u)} \text{ exists finite, positive.}$$

Then it is clear (by Lemma 4.1 and the argument of Proposition 4.4) that $q_i(t)$ may be replaced with $q_1(c_i t)$ and

$$\lim_{t\to\infty} t\bar{F}(q_1(t)\mathbf{1}) = -\log \bar{H}(c).$$

On the other hand, the condition in Proposition 4.4 need not be satisfied, as the example in (3.2) shows.

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