

Synchronization of firing times in a stochastic neural network model with excitatory connections

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We investigate a finite, stochastic, completely connected neural network model with excitatory couplings. The dynamics of the moments of firing in the net is described by a Markov chain. We derive exponential bounds for its transition probabilities. Moreover, the exponential fast synchronization of the moments of firing is proved. The results are illustrated by computer simulations.

Ornstein–Uhlenbeck process * Markov chains * neural network * synchronization * excitatory connections

1. Introduction

The neural model we consider here is some finite system of interacting Markov processes. This model was developed by Kryukov, G. Borisyuk, R. Borisyuk, Kirillov and Kovalenko (1990) under the assumption that the electrical activity (the membrane potential) of a single independent neuron can be described by some Markov process as was proposed by Kryukov (1976). The neural model of Kryukov et al. (1990) takes into account the following facts well-known from physiology: the spiking nature of the neuronal activity, the exponential decay of the membrane potential in the absence of afferent spikes, and the relative refractory period. Simulation modelling of the neural net of Kryukov et al. (1990) exhibits a number of interesting critical cooperative effects. The properties of the neural model found by Kryukov et al. (1990) are used to build a number of specific models of different structures of the central nervous system. Nevertheless the neural modelling needs a more detailed mathematical description. Our aim in the present paper is to analyze the corresponding stochastic system. Tuckwell (1989) presents an account of the theory of stochastic processes that arises in connection with a nervous system. Here we study a large-

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time behaviour of the neural model in the case of excitatory connections between neurons, viz. the synchronization of firing times.

Consider an N -particle system. Each particle (or neuron) is numerated by a number $i \in \{1, \dots, N\}$ and is described by a membrane potential $x_i(t)$ and a threshold function $y_i(t)$, $t \geq 0$. The i th neuron of the net fires and sends an impulse to other neurons when its membrane potential $x_i(t)$ reaches a threshold $y_i(t)$. The threshold $y_i(t)$ decays exponentially in t to y_i^∞ if there are no impulses in the net.

There are two kinds of inputs to the i th neuron. The first one is 'noise' represented by an Ornstein–Uhlenbeck process $\eta_i(t)$. Another one is a sum of the impulses from the other neurons. We assume that the processes $\eta_i(t)$, $i \in \{1, \dots, N\}$ are independent for different i . Justifications for the use of the Ornstein–Uhlenbeck process in modelling neural activity are given by Tuckwell (1989), and Ricciardi and Sacerdote (1979). The impulse from the j th neuron changes the membrane potential of the i th neuron jumpwise by a value a_{ij} and decays exponentially in time, where a_{ij} , $i \neq j$, $i, j = 1, \dots, N$, are coupling constants. A connection (interaction function) is called excitatory if it is positive, and inhibitory if it is negative. We consider here the case of excitatory connections only. This type of connections is discussed by Kryukov et al. (1990).

After firing of the i th neuron, the membrane potential $x_i(t)$ and the threshold $y_i(t)$ jump to values a_{ii} and $y_i(0)$, respectively, and the process of the membrane potential accumulation is repeated till the next firing.

It is clear that when no connections are present we get a system of N independent renewal processes.

We obtain exponential bounds for the transition probabilities of the Markov chain associated with the moments of firing. Moreover the exponential fast synchronization of the firing of all neurons is proved under the condition of positive coupling constants.

Similar Markov models of a neural network, but with constant inhibitory connections, are studied by Cottrell (1992), and Fricker, Robert, Saada and Tibi (1991). The case of inhibitory connections shows different limit behaviours of the models depending on coupling parameters as investigated by Cottrell (1992) and Fricker et al. (1991). Let us note that the connections between neurons in our model are positive and depend on time. Fricker et al. (1991) assumed that the distribution of the intervals between two consecutive firings for a single independent neuron is exponential. In our case an exponential bound for the tail of the corresponding distribution follows from the description of the model.

2. Model

We define a system of N right-continuous processes $S_i(t)$, $t \geq 0$, $1 \leq i \leq N$, with zero initial state, i.e. $S_i(0) = 0$ for $1 \leq i \leq N$. The process $S(t)$ has state space $[0; \infty)^N$ and is defined on some probability space (Ω, Σ, P) . In terms of our neural model the process $S_i(t)$ describes 'an age' of an impulse (or the time elapsed since the last firing until the moment t) of the i th neuron, $i \in \{1, \dots, N\}$, $t \geq 0$. We call any moment T such that $S_i(T) = 0$ for some

$i \in \{1, \dots, N\}$ a moment of firing of the i th neuron. Let $\eta_i(t)$, $t \geq 0$, $1 \leq i \leq N$, be Ornstein–Uhlenbeck processes satisfying the following system of Itô equations:

$$d\eta_i(t) = -\alpha_i \eta_i(t) dt + \sigma_i dW_i(t), \quad (1)$$

with initial conditions

$$\eta_i(0) = 0,$$

where $W_i(t)$, $i \in \{1, \dots, N\}$, are standard Wiener processes on (Ω, Σ, P) , independent for different i and $\alpha_i, \sigma_i > 0$. We define also the right-continuous processes

$$y_i(t) = y_i^\infty + y_i e^{-\alpha_i S_i(t)}$$

and

$$x_i(t) = \eta_i(S_i(t)) + [a_{ii} + a_i^0 \delta(t - S_i(t))] e^{-\alpha_i S_i(t)} + \sum_{j: S_j(t) < S_i(t)} a_{ij} e^{-\alpha_i S_j(t)} \quad (2)$$

to be the threshold function and the membrane potential of the i th neuron, respectively, where $y_i > 0$, $y_i^\infty \geq 0$, $y_i^\infty + y_i > a_{ii}$, $a_{ij} > 0$, $i \neq j$, $i, j = 1, \dots, N$. The random vector $a^0 = (a_1^0, \dots, a_N^0)$ is arbitrarily distributed on a finite domain $(b_1; y_1 - a_{11}) \times \dots \times (b_N; y_N - a_{NN})$, $b_i > -\infty$, $i = 1, \dots, N$, and

$$\delta(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we do not require the condition $a_{ii} > 0$, $i = 1, \dots, N$, which makes our model general. We put the initial state of $S(T)$ equal to zero, and, correspondingly,

$$y_i(0) = y_i^\infty + y_i,$$

and

$$x_i(0) = a_{ii} + a_i^0,$$

$i = 1, \dots, N$. Thus the initial state of the process $x(t)$ is actually given by the random vector a^0 , which we consider to be an initial condition for our system. Note that the first two components of the i th membrane potential depend on the age $S_i(t)$ of the impulse of the i th neuron only (see the right-hand side of (2)). The last term in the right side of (2) represents the interactions in the net. It is a sum over all j such that the impulse of the j th neuron at the moment t is ‘younger’ than $S_i(t)$. The values of impulses decay exponentially in time. It is clear that the process $x(t)$ is non-Markov.

The dynamics of the system $\{S(t), x(t), y(t)\}$ is the following. Every i th coordinate $S_i(t)$ increases in time with rate one from zero until the first time T_1 that $x_{i1}(t)$ reaches the boundary $y_{i1}(t)$ (which decays exponentially) for some $i_1 \in \{1, \dots, N\}$. Note that for $0 \leq t < T_1$, $i = 1, \dots, N$,

$$x_i(t) = \eta_i(S_i(t)) + (a_{ii} + a_i^0) e^{-\alpha_i S_i(t)},$$

i.e., there is no interaction term until the time $t = T_1$. Then the coordinate $S_{i1}(t)$ jumps in the zero state, i.e., $S_{i1}(T_1) = 0$. Simultaneously, the process $x_{i1}(t)$ jumps in the state $x_{i1}(t) = \eta_{i1}(S_{i1}(t)) + a_{i1i1}e^{-\alpha_{i1}S_{i1}(t)}$ and sends an impulse to the other processes. The latter means that every process $x_j(t)$, $j \in \{1, \dots, N\}$, which does not reach the corresponding boundary $y_j(t)$ receives an increment $a_{ji1}e^{-\alpha_j S_{i1}(t)}$. In particular, a trajectory of $x_j(t)$ has a gap equal to a_{ji1} at time T_1 . From this state $\{S(T_1), x(T_1), y(T_1)\}$ the procedure is repeated with some modification in the interaction. More precisely, if $x_i(t)$ reaches the threshold $y_i(t)$ at the moments $t_1, t_2 > 0$ consecutively, and $x_j(t)$ does not reach the threshold $y_j(t)$ during time interval $[t_1; t_2]$ for some $i, j \in \{1, \dots, N\}$, then $x_j(t)$ receives an increment $a_{ji}(e^{-\alpha_j S_{i1}(t)} - e^{-\alpha_j(t_2 - t_1)})$ at moment t_2 .

Our model does not take into account the relative refractory period, i.e. a time interval after a firing of a neuron, during which the neuron is unable to receive any impulse. But in order to exclude the situation when the i th neuron receives an impulse at the rest state a_{ii} (immediately after its firing) we assume the following. Notice that at the moment when the i th membrane potential reaches its threshold at least for one $i \in \{1, \dots, N\}$, the j th potential can step over the corresponding j th threshold because of the increment induced by the

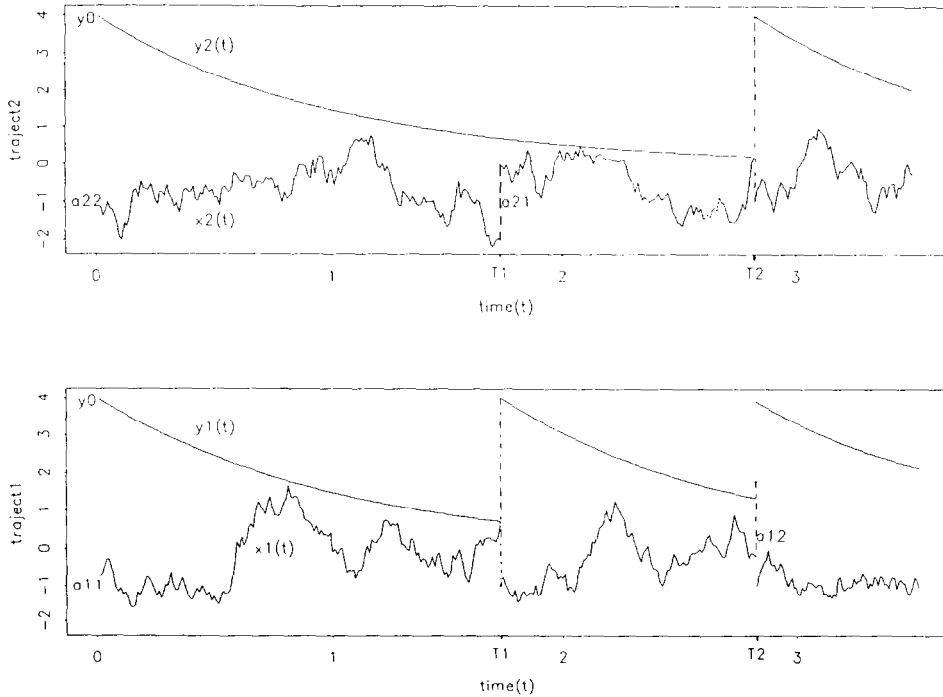


Fig. 1. Simulated trajectories of two membrane potentials. $N = 2$, $a_{11} = a_{22} = -1$, $a_{12} = a_{21} = 2$, $y_1(0) = y_2(0) = 4$, $\sigma_1 = \sigma_2 = 2$, $\alpha_1 = \alpha_2 = 1$, $a^0 = 0$. Neuron 1 fires at time T_1 , gives rise to excitation a_{21} in membrane potential $x_2(t)$, and threshold function $y_1(t)$ returns to the initial state. Neuron 2 fires at time T_2 , causes excitation a_{12} in membrane potential $x_1(t)$, and threshold function $y_2(t)$ returns to the initial state. Since the trajectory $x_1(t)$ at time T_2 steps over the threshold function $y_1(t)$ due to the excitation a_{12} , we have full synchronization of the firing times at the moment T_2 .

impulse from the firing neuron. We suppose that at this case the j th neuron does not send an impulse back to the firing i th neuron.

Simulated trajectories of the membrane potentials are given in Figure 1 in the case $N=2$. Here, T_1 is the firing time of the first neuron and T_2 is the firing time of the second neuron, which turns out to be the first moment of synchronization. Notice the excitation a_{21} at time T_1 of the second membrane potential due to the firing of the first neuron, and similarly the excitation a_{12} at time T_2 of the first membrane potential due to the firing of the second neuron.

3. Results

Let us fix $T>0$ arbitrarily. Define a sequence of vectors $\{\tau_{nT}\}_{n=0}^{\infty}$, where $\tau_t = (\tau_t^1, \dots, \tau_t^N)$, and

$$\tau_t^i = \inf\{\tau > 0: S_i(t + \tau) = 0\},$$

i.e., τ_t^i is the first moment of firing of the i th neuron since time t . Notice that $\tau_t^i < \infty$ a.s., for any $i=1, \dots, N$, $t>0$ (see the upperbound (7) below). It is easy to see that the discrete time process $\{\tau_t\}$, $t=T, 2T, \dots$, is a Markov chain on $[0; \infty)^N$ with initial state τ_0 . The distribution of τ_0 is defined by system (2).

First of all we consider the transition probabilities of the Markov chain $\{\tau_{nT}\}$.

Theorem 1. *Let $N < \infty$, $0 < T < \infty$ and suppose all coupling constants a_{ij} , $i \neq j$, $i, j \in \{1, \dots, N\}$ are positive. Then for every $M \subseteq \{1, \dots, N\}$, $v = (v^1, \dots, v^N) \in \mathbb{R}_+^N$, and $u = (u^1, \dots, u^N) \in \mathbb{R}_+^N$, where $u^i \in [0; T]$ if and only if $i \in M$, one has*

$$\begin{aligned} & \Pr\{\tau_{(n+1)T}^i > v^i, i=1, \dots, N \mid \tau_{nT}^i = u^i, i=1, \dots, N\} \\ & \leq A e^{-\alpha \sum_{i \in M'} v^i} \quad \text{if } v^i \leq u^i - T, i \notin M, \end{aligned} \quad (3)$$

and

$$\Pr\{\tau_{(n+1)T}^i > v^i, i=1, \dots, N \mid \tau_{nT} = u\} = 0 \quad \text{otherwise},$$

for every $n > 0$ and some constants $A, \alpha > 0$, where α is independent of a^0 and T .

The proof of Theorem 1 is given in Section 4.

The exponential bound for the transitional distribution function obtained in Theorem 1 allows us to derive some remarkable properties of the system (2) with excitatory connections only. Consider the vector d_n of the $N-1$ gaps d_n^i between first firings of different neurons since time nT :

$$d_n := (|\tau_{nT}^1 - \tau_{nT}^2|, |\tau_{nT}^2 - \tau_{nT}^3|, \dots, |\tau_{nT}^{N-1} - \tau_{nT}^N|).$$

It is clear that the sequence $\{d_n\}$ is a Markov chain also. The following theorem establishes the large-time limit behaviour of $\{d_n\}$, in particular, its recurrence.

Theorem 2. *Let*

$$\lambda = \min\{n > 0: d_n^i = 0, i = 1, \dots, N-1\}.$$

Then

$$\Pr\{\lambda > t, d_t^i > u^i, i = 1, \dots, N-1\} \leq \Gamma e^{-\gamma t} e^{-\alpha \sum_{i=1}^{N-1} u^i} \quad (4)$$

for every $t > 0, u = (u^1, \dots, u^{N-1}) \in \mathbb{R}_+^{N-1}$ where $\Gamma, \gamma, \alpha > 0$ are some constants and α, γ are independent of a^0 .

Theorem 2 is proved in Section 5.

Since the (random) moment λ represents the first moment of synchronization of firings, the result of Theorem 2 shows an exponential fast synchronization (in time) of the moments of firing in the neural net. Let us define the sequence of moments of synchronization:

$$\{\lambda_k\}_{k=1}^\infty,$$

where

$$\lambda_1 = \lambda, \quad \lambda_{k+1} = \min\{\lambda > \lambda_k: d_\lambda^i = 0, i = 1, \dots, N-1\}.$$

Clearly, the random variables

$$\lambda_{k+1} - \lambda_k, \quad k = 1, 2, \dots,$$

are independent, identically distributed for different k . Moreover, the exponential bound for their distribution is given by the following:

Corollary. *For any $t > 0, k = 1, 2, \dots$,*

$$\Pr\{\lambda_{k+1} - \lambda_k > t\} \leq \Gamma e^{-\gamma t}.$$

Proof. It follows from (2) that after the first firing the membrane potential of the i th neuron does not depend on a_i^0 for any i . This means that after the moment of full synchronization the processes of the membrane potentials are described by system (2) with $a^0 = (0, \dots, 0)$. Hence for every $t > 0$

$$\Pr\{\lambda_{k+1} - \lambda_k > t\} = \Pr\{\lambda > t | a^0 = (0, \dots, 0)\} \leq \Gamma e^{-\gamma t},$$

where the inequality follows from (4). \square

4. Proof of Theorem 1

We consider the transition probability function

$$\Pr\{\tau_{(n+1)T} \in A | \tau_{nT} = u\}$$

for $n \geq 0, A \in \mathcal{B}(\mathbb{R}_+^N)$. Evidently, this function is independent of n . Without loss of generality we set $n = 0$.

Consider first the case $M = \{1, \dots, N\}$. Let

$$u \in \{u = (u^1, \dots, u^N) \in \mathbb{R}_+^N : u^i < T, i = 1, \dots, N\}.$$

Hence, $S(T)$ is distributed on $\mathcal{D} = [0; T - u^1] \times \dots \times [0; T - u^N]$. For arbitrarily fixed $i \in \{1, \dots, N\}$, we consider, under the condition $S_i(T) = \hat{S}_i(T)$, $0 \leq \hat{S}_i(T) \leq T - u^i$, the stochastic process

$$x_i(T - S_i(T) + t), \quad t \in [0; S_i(T) + \tau_T^i],$$

which describes the dynamics of the membrane potential of the i th neuron in the time interval between the last moment of firing before time T and the first moment of firing after time T . We get from formula (2) the following expression:

$$\begin{aligned} & x_i(T - \hat{S}_i(T) + t) \\ &= \eta_i(t) + a_{ii} e^{-\alpha_i t} + \sum_{j: S_j(T - \hat{S}_i(T) + t) \leq t} a_{ij} \exp(-\alpha_i S_j(T - \hat{S}_i(T) + t)). \end{aligned}$$

Consequently,

$$\hat{S}_i(T) + \tau_T^i = \inf\{t > \hat{S}_i(T) : \eta_i(t) \geq Y_i(t) \mid \eta_i(t) < Y_i(t), 0 \leq t \leq \hat{S}_i(T)\},$$

where

$$\begin{aligned} Y_i(t) &= y_i^\infty + (y_i - a_{ii}) e^{-\alpha_i t} \\ &\quad - \sum_{j: S_j(T - \hat{S}_i(T) + t) \leq t} a_{ij} \exp(-\alpha_i S_j(T - \hat{S}_i(T) + t)). \end{aligned} \quad (5)$$

Let

$$\theta = \exp(2\alpha_i t) - 1, \quad t \geq 0.$$

Since $\eta_i(t)$ is the Ornstein–Uhlenbeck process defined by (1), we have

$$\eta_i(t) \sqrt{\theta + 1} \stackrel{d}{=} \frac{\sigma_i}{\sqrt{2\alpha_i}} W_i(\theta), \quad \theta \geq 0,$$

where ‘ $\stackrel{d}{=}$ ’ means equality in distribution, and $W_i(\theta)$, $i = 1, \dots, N$ are Wiener processes, independent for different i .

Hence, we get the following:

$$\begin{aligned} & \Pr\{\tau_T^i > v^i, i = 1, \dots, N \mid \tau_0 = u\} \\ & \leq \sup_{S \in \mathcal{D}} \Pr\{W_i(\theta) < \tilde{Y}_i(\theta), \theta < e^{2\alpha_i(S_i + v^i)} - 1, i = 1, \dots, N \mid S(T) = S, \tau_0 = u\}, \end{aligned} \quad (6)$$

where

$$\tilde{Y}_i(\theta) \equiv \frac{\sqrt{2\alpha_i}}{\sigma_i} \sqrt{\theta + 1} Y_i\left(\frac{1}{2\alpha_i} \ln(\theta + 1)\right)$$

for all $\theta \geq 0$.

Note that from (5),

$$\tilde{Y}_i(\theta) \leq \frac{\sqrt{2\alpha_i}}{\sigma_i} ((y_i - a_{ii}) + y_i^\infty \sqrt{\theta + 1})$$

under the condition of positive connections. Therefore from (6) one can derive the bound

$$\begin{aligned} & \Pr\{\tau_T^i > v^i, i = 1, \dots, N \mid \tau_0 = u\} \\ & \leq \sup_{S \in \mathcal{S}} \Pr\left\{ \left\{ W_i(\theta) < \frac{\sqrt{2\alpha_i}}{\sigma_i} ((y_i - a_{ii}) + y_i^\infty \sqrt{\theta + 1}), \right. \right. \\ & \quad \left. \left. 0 < \theta \leq \exp\{2\alpha_i(S_i + v^i)\} - 1, i = 1, \dots, N \mid S(T) = S, \tau_0 = u \right\} \right\}. \end{aligned} \quad (7)$$

We use the well-known formula (see, for example Itô and McKean, 1974) for the distribution of the first passage time through a constant threshold $d > 0$,

$$\Pr\{W(\tau) < d \mid \forall \tau \leq t\} = \frac{2}{\sqrt{2\pi t}} \int_0^d e^{-u^2/(2t)} du$$

in the case $y_i^\infty = 0$, and the result by Sato (1977),

$$\Pr\{W(\tau) < c\sqrt{\tau + 1} \mid \forall \tau \leq t\} \sim qt^{-p(c)},$$

where $q > 0$, and $0 < p(c) < \frac{1}{2}$ for all $0 < c < \infty$, in the case $y_i^\infty > 0$, to obtain from (7),

$$\Pr\{\tau_T^i > v^i, i = 1, \dots, N \mid \tau_0 = u\} \leq A e^{\varepsilon T N} e^{-\alpha \sum_{i=1}^N v^i}, \quad (8)$$

for all $v \in \mathbb{R}_+^N$, $u \in [0; T]^N$, and some constants $A, \alpha > \varepsilon > 0$ independent of a^0 and T , which proves (3) in the case $M = \{1, \dots, N\}$. Note, that α can be chosen arbitrarily such that $0 < \alpha < \min_i \alpha_i$ if $y_i^\infty = 0$, and $0 < \alpha < 2p((\sqrt{2\alpha_i}/\sigma_i)y_i^\infty) \min_i \alpha_i$ if $y_i^\infty > 0$.

It is clear that in the case $M \subset \{1, \dots, N\}$ the bound (3) is obtained analogously to (8). Theorem 1 is proved. \square

Let us note that the exponential bound

$$\Pr\{\tau_0^i > u^i, i = 1, \dots, N\} \leq A' e^{-\alpha' \sum_{i=1}^N u^i}, \quad (9)$$

$u \in \mathbb{R}_+^N$, for some constants $A', \alpha' > 0$, follows immediately from the proof of Theorem 1.

5. Proof of Theorem 2

First we note that the probability of a joint firing of all the neurons in the system $\{S_i(t), x_i(t), y_i(t), i = 1, \dots, N\}$ is positive under some conditions. More precisely, we will prove that for every fixed $0 < T < \infty$ there exists $c > 0$ such that

$$\Pr\{\tau_{(n+1)T}^1 = \dots = \tau_{(n+1)T}^N \mid \tau_{nT} = u\} \geq c, \quad (10)$$

whenever $u \in [0; T)^N$. We rewrite this probability in integral form:

$$\begin{aligned} & \Pr\{\tau_{(n+1)T}^1 = \dots = \tau_{(n+1)T}^N | \tau_{nT} = u\} \\ &= \int_0^\infty \Pr\{\tau_{(n+1)T}^i \in dv, i = 1, \dots, N | \tau_{nT} = u\} . \end{aligned}$$

Consider an event $\{\tau_{(n+1)T}^i \in dv, i = 1, \dots, N | \tau_{nT} = u\}$ for $v > 0$ fixed arbitrarily. Let us suppose that for some $i \in \{1, \dots, N\}$ the continuous trajectory $x_i(t)$, $t > (n+1)T$, hits the threshold $y_i(t)$ at moment $t_i = (n+1)T + v$ for the first time among all trajectories $x_j(t)$, $j = 1, \dots, N$. This means that the corresponding Wiener process $W_i(\theta)$, $\theta > 0$ hits the threshold $\tilde{Y}_i(\theta)$ at the moment $\theta_i = e^{2\alpha_i(S_i((n+1)T) + v)} - 1$. Note that threshold $\tilde{Y}_j(\theta)$ of the j th neuron has a discontinuity at the moment of firing of the i th neuron, and more strongly, for every $j \neq i, j \in \{1, \dots, N\}$,

$$\tilde{Y}_j(\theta_j^-) - \tilde{Y}_j(\theta_j) > \frac{\sqrt{2\alpha_j}}{\sigma_j} a_{ji}(1 - e^{-\alpha_j(t_i - t_i^0)}) > 0 ,$$

where $\theta_j = e^{2\alpha_j(S_j((n+1)T) + v)} - 1$, and $t_i^0 = \sup\{t < t_i; S_i(t) = 0\}$, i.e., t_i^0 represents the last moment of firing of the i th neuron before time t_i . It follows from the conditions on the parameters of our model that $\Pr\{t_i - t_i^0 \geq \Delta\} \geq p$ for some $\Delta, p > 0$, for every $t_i > T$, $i = 1, \dots, N$. Consequently, all neurons can fire simultaneously, not only in the case that each of the corresponding Wiener processes hits the threshold at the same moment, but also in the case that at least one of them hits the threshold, while every other one may take a value from some finite non-zero segment. For every neuron this segment has the size of the gap in its threshold induced by the impulse from the firing neuron. If $W_j(\theta)$ takes a value from this segment at time $\theta = \theta_j^-$, then at the moment θ_j the process $W_j(\theta)$ is at or over the corresponding threshold, which has a jump-wise decay at point θ_j . Formally, for $u \in [0; T)^N$ we have

$$\begin{aligned} & \Pr\{\tau_{(n+1)T}^i \in dv, i = 1, \dots, N | \tau_{nT} = u\} \\ & \geq \inf_{S \in \mathcal{D}} \sum_{i=1}^N \Pr\{W_j(e^{2\alpha_j(S_j + v)} - 1) \in [\tilde{Y}_j((e^{2\alpha_j(S_j + v)} - 1)^-); \\ & \quad \tilde{Y}_j(e^{2\alpha_j(S_j + v)} - 1)] , \\ & \quad W_j(\theta_j) < \tilde{Y}_j(\theta_j), \theta_j < \exp(2\alpha_j(S_j + v)) - 1, i \neq j \in \{1, \dots, N\} , \\ & \quad W_i(\theta) < \tilde{Y}_i(\theta), \theta < \theta_i, W_i(\theta_i) = \tilde{Y}_i(\theta_i), \\ & \quad \theta_i \in 2\alpha_i e^{2\alpha_i v} dv | \tau_{nT} = u\} \\ & \geq g(v) dv \end{aligned}$$

where $g(v)$ is some positive function on $(0; \infty)$. Hence (10) is proved.

Making use of (3) and (8) we obtain

$$\begin{aligned} & \Pr\{\tau_{nT}^i > T, n = k, \dots, k+l-1 \mid \tau_{(k-1)T}^i < T\} \\ &= \Pr\{\tau_{kT}^i > lT \mid \tau_{(k-1)T}^i < T\} \leq A e^{-\alpha l T} \end{aligned} \quad (11)$$

and

$$\begin{aligned} & \Pr\{\tau_{nT}^i \geq T, n = k_1, \dots, k_l, \tau_{nT}^i < T, n \in \{0, \dots, t-1\} \setminus \{k_1, \dots, k_l\}, \tau_{iT}^i > v^i\} \\ & \leq A^{l+1} e^{-l\alpha T} e^{-\alpha v^i} \end{aligned}$$

for some constants $A, \alpha > 0$ independent of T , and every $i \in \{1, \dots, N\}$, $k, l > 0$, $0 \leq k_1 < \dots < k_l < t$.

The last bound, (11) and Theorem 1 allow us to derive the following:

$$\begin{aligned} & \Pr\{\tau_{nT} \notin [0; T]^N, n \leq t-1, \tau_{iT}^i > v^i, i = 1, \dots, N \mid \tau_0 = u_0\} \\ & \leq B_1 B^i e^{-\alpha T} \exp\left\{\alpha \sum_{i=1}^N (u_0^i - v^i)\right\} \end{aligned} \quad (12)$$

for every $u_0, v \in \mathbb{R}_+^N$, $t \in \{1, 2, \dots\}$, and some $B_1 = B_1(N, T) > 0$, $B = B(N) > 0$.

Let us define $T_1 := mT$, where $m = \min\{k > 0: e^{k\alpha T} > B\}$. Consider the subsequence $\{\tau_{nT_1}\}_{n=1}^\infty$. From (12) follows the existence of some constant $\gamma_1 > 0$ such that

$$\begin{aligned} & \Pr\{\tau_{nT_1} \notin [0; T_1]^N, n \leq t-1, \tau_{iT_1}^i > v^i, i = 1, \dots, N \mid \tau_0 = u_0\} \\ & \leq A_1 e^{-\gamma_1 t} \exp\left\{\alpha \sum_{i=1}^N (u_0^i - v^i)\right\}, \end{aligned}$$

which implies

$$\begin{aligned} & \Pr\{\tau_{nT} \notin [0; T_1]^N, n \leq t-1, \tau_{iT}^i > v^i, i = 1, \dots, N \mid \tau_0 = u_0\} \\ & \leq \Pr\{\tau_{nT_1} \notin [0; T_1]^N, n \leq [(t-1)/m], \\ & \quad \tau_{(\lfloor (t-1)/m \rfloor + 1)T_1}^i > v^i - mT, i = 1, \dots, N \mid \tau_0 = u_0\} \\ & \leq A_2 e^{-\gamma t} \exp\left\{\alpha \sum_{i=1}^N (u_0^i - v^i)\right\} \end{aligned} \quad (13)$$

for every $u_0, v \in \mathbb{R}_+^N$ and some constants $A_2, A_1, \gamma > 0$.

To prove (4) we use an idea of Malyshev, Podorol'skii and Turova (1989). Let $t \in \{1, 2, \dots\}$ be fixed arbitrarily and define the set

$$D := \{u = (u^1, \dots, u^N): u^1 = \dots = u^N\}.$$

We define the following probabilities for every $0 < k \leq t$, $u_0 \in \mathbb{R}_+^N$, and $0 < t_1 < \dots < t_k \leq t$:

$$\tilde{p}(t; u_0, G; t_1, \dots, t_k) = \Pr\{\tau_{iT} \in G_t, i = 1, \dots, t-1; \tau_{iT} \in G \setminus D \mid \tau_0 = u_0\},$$

where $G \in \mathcal{B}(\mathbb{R}_+^N)$ and

$$G_l = \begin{cases} \{[0; T_1]^N \setminus D & \text{if } l \in \{t_1, \dots, t_k\}, \\ \mathbb{R}_+^N \setminus ([0; T_1]^N \cup D) & \text{otherwise.} \end{cases}$$

Note that $t_l \equiv t$. Let us put

$$\bar{p}(0; u_0, [0; T_1]^N, 0) = 1,$$

for $u_0 \in [0; T_1]^N \setminus D$. We define also

$$\bar{p}(t; u_0, G) = \sum_{k=1}^t \sum_{0 < t_1 < \dots < t_k \leq t} \bar{p}(t; u_0, G; t_1, \dots, t_k).$$

It is clear that

$$\bar{p}(t; u_0, G) = \Pr\{\lambda > t, \tau_{iT} \in G \setminus D \mid \tau_0 = u_0\}.$$

We derive from (13)

$$\bar{p}(t; u_0, [0; T_1]^N, t) \leq A_3 e^{-\alpha_1 t} \quad (14)$$

for some constants $A_3, \alpha_1 > 0$, if $u_0 \in [0; T_1]^N$.

Since (10) holds we get

$$\begin{aligned} & \Pr\{\text{the process } \{\tau_{nT}\}_{n=1}^\infty \text{ hits the set } [0; T_1]^N, \text{ but } \tau_{nT} \notin D, n > 0 \mid \tau_0 = u_0\} \\ &= \sum_{n=1}^\infty \bar{p}(n; u_0, [0; T_1]^N, n) \\ &\geq \Pr\{\tau_{nT_1} \notin D, n > 0 \mid \tau_0 \in [0; T_1]^N \setminus D\} = 1 - \varepsilon_1 \end{aligned} \quad (15)$$

for some $0 < \varepsilon_1 < 1$. From (14) and (15) follows the existence of constants $z_0 > 1, \varepsilon_2 > 0$ which satisfy

$$\sum_{n=1}^\infty z_0'' \bar{p}(n; u_0, [0; T_1]^N, n) < 1 - \varepsilon_2 \quad (16)$$

if $u_0 \in [0; T_1]^N$. We shall now prove that

$$\bar{p}(t; u_0, [0; T_1]^N) < B_2 e^{-\beta t} \quad (17)$$

for some $B_2, \beta > 0$, if $u_0 \in [0; T_1]^N$. Indeed, we derive

$$\begin{aligned} & \sum_{t=1}^\infty z_0^t \bar{p}(t; u_0, [0; T_1]^N) \\ &= \sum_{t=1}^\infty z_0^t \sum_{k=1}^t \sum_{0 < t_1 < \dots < t_k \leq t} \bar{p}(t; u_0, [0; T_1]^N; t_1, \dots, t_k) \\ &\leq \sum_{t=1}^\infty \sum_{k=1}^t \sum_{0 < t_1 < \dots < t_k \leq t} z_0^t \bar{p}(t_1; u_0, [0; T_1]^N; t_1) \\ &\quad \times z_0^{t-t_1} \max_{v \in [0; T_1]^N \setminus D} \bar{p}(t-t_1; v, [0; T_1]^N; t_2-t_1, \dots, t_k-t_1) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{t=1}^{\infty} \sum_{k=1}^t \sum_{0 < t_1 < \dots < t_k \leq t} z_0^t \bar{p}(t_1; u_0, [0; T_1]^N; t_1) \\
&\quad \times z_0^{t_2-t_1} \max_{v \in [0; T_1]^N \setminus D} \bar{p}(t_2 - t_1; v, [0; T_1]^N; t_2 - t_1) \\
&\quad \times z_0^{t-t_2} \max_{v \in [0; T_1]^N \setminus D} \bar{p}(t - t_2; v, [0; T_1]^N; t_3 - t_2, \dots, t_k - t_2),
\end{aligned}$$

where the first inequality follows from the Markov property of the process $\{\tau_{nT}\}$. By repeating the same procedure $k-1$ times for every term in the last series one obtains, taking into account (16),

$$\sum_{t=1}^{\infty} z_0^t \bar{p}(t; u_0, [0; T_1]^N) \leq \sum_{t=1}^{\infty} \left(\sum_{s=1}^{\infty} z_0^s \max_{v \in [0; T_1]^N} \bar{p}(s; v, [0; T_1]^N; s) \right)^t < \infty,$$

which proves (17).

If $u_0 \notin [0; T_1]^N$ we obtain from (17) and (13) that

$$\bar{p}(t; u_0, [0; T_1]^N) < B_3 e^{-\beta_1 t} e^{\alpha \sum_{i=1}^N u_0^i} \quad (18)$$

for some $B_3 > 0$ and $\beta_1 = \min\{\beta, \gamma\}$.

Taking into account that at least $\tau_{iT}^i > u^i$ or $\tau_{iT}^{i+1} > u^i$ if $d_i^i > u^i$, and making use of (9), we derive from (13) and (18),

$$\begin{aligned}
&\Pr\{\lambda > t, d_i^i > u^i, i = 1, \dots, N-1\} \\
&= \sum_{n=1}^{t-1} \Pr\{\lambda > t, d_i^i > u^i, i = 1, \dots, N-1; \tau_{(n-1)T} \in [0; T_1]^N, \\
&\quad \tau_{kT} \notin [0; T_1]^N, k = n, \dots, t-1\} \\
&\quad + \Pr\{\lambda > t, d_i^i > u^i, i = 1, \dots, N-1; \tau_{kT} \notin [0; T_1]^N, k = 0, \dots, t-1\} \\
&\leq C \sum_{n=0}^t e^{-\beta_1 n} e^{-\gamma(t-n)} e^{-\alpha \sum_{i=1}^{N-1} u^i} \leq F e^{-\gamma' t} e^{-\alpha \sum_{i=1}^{N-1} u^i},
\end{aligned}$$

for some constants $C, F, \gamma' > 0$. This completes the proof of Theorem 2. \square

6. Concluding remarks

The main result of this paper establishes the exponentially fast full synchronization of the moments of firing in a completely connected neural network model when all connections are positive. We point out that our result concerns the synchronization when the time parameter goes to infinity. We illustrate our theoretical results with some simulations as well, where the coupling constants have been chosen so as to realize full synchronization

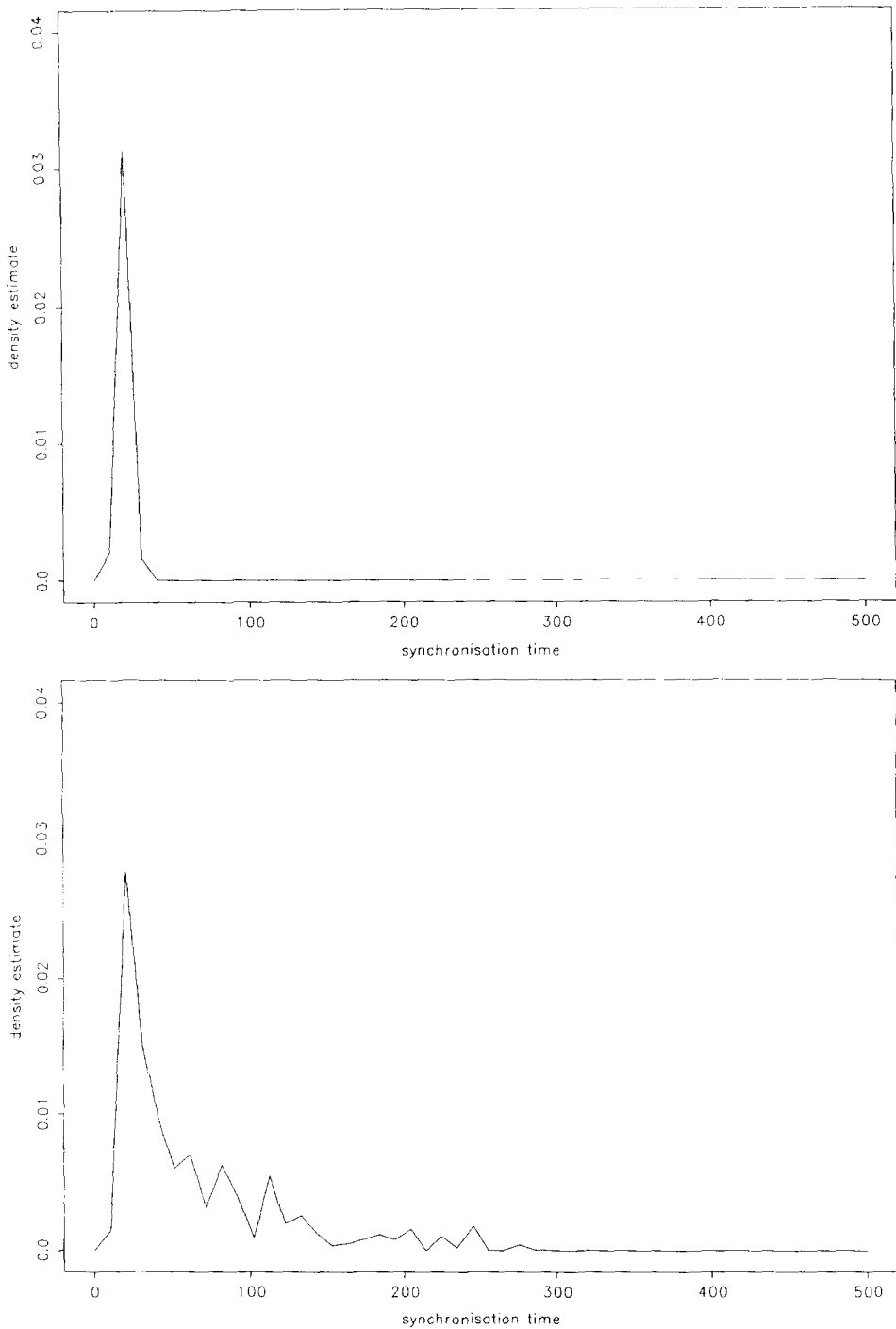


Fig. 2. Density estimates of intervals between moments of full synchronization based on 10000 simulations. $N=10$, $a_{ii}=1$, $y_i(0)=4$, $\sigma_i=2$, $\alpha_i=1$, $a^0=0$, $i=1, \dots, 10$. (a) $a_{ij}=1$, $i \neq j$. (b) $a_{ij}=0.5$, $i \neq j$.

in reasonable time. We have observed the moments of full synchronization in a net consisting of 10 neurons. On the basis of the observations we estimated the density of the distribution of the intervals between moments of full synchronization for different choices of coupling constants. The plots for our estimates are given in Figure 2(a) and Figure 2(b) for $a_{ij} = 1$ and $a_{ij} = 0.5$, $i \neq j$, respectively. It is easy to discern the exponential decay of that density. Our simulations show that the model is very sensitive to the values of the coupling constants. It is clear from formula (4) that for the right-hand side to tend to zero one needs t to be at least of order γ^{-1} . But from the proof of Theorem 2 it follows that $\gamma \rightarrow 0$ when all $a_{ij} \rightarrow 0$, $i \neq j$. This is why there could be a very low probability of full synchronization during some finite 'real' time interval in the case of sufficiently small coupling constants. Comparing the pictures in Figure 2(a) and Figure 2(b), one can see that the decay of the estimated density is slower in the case of smaller coupling constants (Figure 2(b)), i.e., the probability that there is no synchronization during a given time interval is greater in this case. In order to obtain our estimate we used a density estimator with a Gaussian kernel. Note that one can obtain synchronization of firing of a given set of neurons within a given time interval by a suitable choice of coupling constants.

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