

# A law of large numbers for upcrossing measures

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## Abstract

We present a mathematical treatment of the so called RFC-counting which is applied to functions from subsets of  $\mathbb{R}$  to  $\mathbb{R}$  and which essentially counts upcrossings for each pair of levels. In mechanical engineering it is applied to stress or strain histories to assess their potential fatigue damage. We associate three measures on  $\mathbb{R}^2$  with RFC-counting and study their properties. Using the subadditive ergodic theorem of Kingman (1975) we prove a law of large numbers for these measures when they are applied to the paths of a stationary process. We compute the limit  $\tilde{\mu}$  explicitly e.g. for one-dimensional stationary diffusion processes.  $\tilde{\mu}$  may be compared with the spectral measure.

**Keywords:** Upcrossing measure; Rainflow counting; Fatigue analysis; Stationary process; Excursion; Law of large numbers; Random measure; Vague convergence

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## 1. Introduction

In this article we define the notion of an upcrossing measure and establish a law of large numbers for the upcrossing measure of the paths of a stationary real-valued process.

If  $f: [0, T] \rightarrow \mathbb{R}$  is continuous and  $x < y$  then let  $u(f, x, y)$  be the number of upcrossings of  $f$  from  $x$  to  $y$  on the interval  $[0, T]$ . We will show that the collection  $\mathcal{U} = \{u(f, x, y); x < y\}$  can be represented as a discrete measure  $\nu$  on  $\Delta = \{(x, y) \in \mathbb{R}^2: x < y\}$  via  $\nu_{[0, T]}(f, (-\infty, x] \times [y, \infty)) = u(f, x, y)$ , so  $\nu$  is something like a discrete density of the “distribution function”  $u$ . This representation allows us to introduce a topology on  $\mathcal{U}$  in a natural way – namely the vague topology on the set  $\mathcal{M}$  of Radon measures on  $\Delta$ .

If  $f$  is the realization of a stationary process on  $[0, \infty)$ , then it is natural to ask whether  $T^{-1}\nu_{[0, T]}(f, \cdot)$  converges almost surely in the vague topology to some Radon measure  $\tilde{\mu}$  as  $T \rightarrow \infty$ . We will show that this is true under slight integrability conditions in Chapter 3 and that  $\tilde{\mu}$  is deterministic in case the process is ergodic. Note that the almost sure convergence of the average number of upcrossings per unit time to some limit for fixed  $x < y$  follows easily from the subadditive ergodic theorem of Kingman (1975) but our theorem provides more information. The examples in

Chapter 4 show that in some cases the limit  $\tilde{\mu}$  can be computed explicitly. In the ergodic case the limit  $\tilde{\mu}$  may be contrasted with the spectral measure  $m$  provided both exist. While  $m$  behaves nicely under linear systems (which  $\tilde{\mu}$  does not),  $\tilde{\mu}$  behaves nicely under nonlinear transformation of the state and under (random) time change (which  $m$  does not). In a number of cases – such as fatigue analysis which we describe below –  $\tilde{\mu}$  seems to be more relevant than  $m$ . This suggests a time series analysis based on estimates of  $\tilde{\mu}$  rather than or in addition to periodogram analysis in certain cases.

We believe that upcrossing measures are of some intrinsic mathematical interest but they also have important applications in fatigue analysis which was our motivation for studying them. Let  $f(t)$ ,  $t \geq 0$  be a real-valued stress- or strain function which is applied to some mechanical unit (part of a car, airplane, machine etc.). It has been verified experimentally that in many situations the number of upcrossings between all pairs of levels is a “sufficient statistic” to a considerable degree of accuracy: if two different stress (or strain-) functions  $f$  and  $g$  on  $[0, T]$  have the same number of upcrossings between any pair of levels then the number of replications of  $f$  and  $g$  needed to lead to failure of a given unit are approximately the same.

In fatigue applications a sample of the strain function  $f$  is obtained by measurement (in a car on the street for example). If  $f$  is modelled as a realization of a stationary ergodic process, then its scaled upcrossing measure or some smoothed version of it can be considered to be an estimate of  $\tilde{\mu}$  – in fact a reasonably good one since the samples are usually rather long. Some companies then generate artificial (very long) strain functions from the estimate of  $\tilde{\mu}$  and apply them to the unit on a test stand. To justify this method mathematically some kind of LLN for upcrossing measures is needed.

Instead of considering upcrossing-measures engineers have looked at the closely related “rainflow” (RFC) or “range-pair-range” statistic which was introduced independently by Matsuishi and Endo (1968) and de Jonge (1982). The relationship between (up)crossings and RFC have been worked out independently by Rychlik (1992c) (see also Frendahl and Rychlik (1993)) and the author (see Beste et al. (1992, and Ref. [6] therein)).

Let us explain the idea of rainflow counting. In Fig. 1  $\varepsilon(t)$  and  $\sigma(t)$  represent an example of a strain/stress relationship. Stress and strain are not functions of one another but exhibit a hysteresis phenomenon. In Fig. 1 there are three closed hysteresis loops. One can associate with  $f(t) = \varepsilon(t)$ ,  $0 \leq t \leq T$  a measure  $\mu$  on  $\Delta$  which has mass  $n \in \mathbb{N}_0$  at  $(x, y)$  if there are  $n$  closed loops with lower strain level  $x$  and upper strain level  $y$ . In the example  $\mu$  has three atoms of unit mass each. Note that in order to determine  $\mu$  knowledge of the function  $\varepsilon(t)$  is sufficient. The precise relationship between  $\sigma(t)$  and  $\varepsilon(t)$  is immaterial for this. This measure  $\mu$  – or rather its discretization to a matrix – is known as rainflow statistic or rainflow matrix. We will term  $\mu$  “rainflow measure” or “oscillation measure” of  $f$ .

The relation between  $\mu$  and  $\nu$  is rather simple: every closed loop whose strain range contains  $[x, y]$  contributes exactly one upcrossing of the strain function  $\varepsilon$  from  $x$  to  $y$ . To get *all* upcrossings one has to add the number of upcrossings of the “residual” strain function which by definition is the function of  $t$  which is left over when all closed

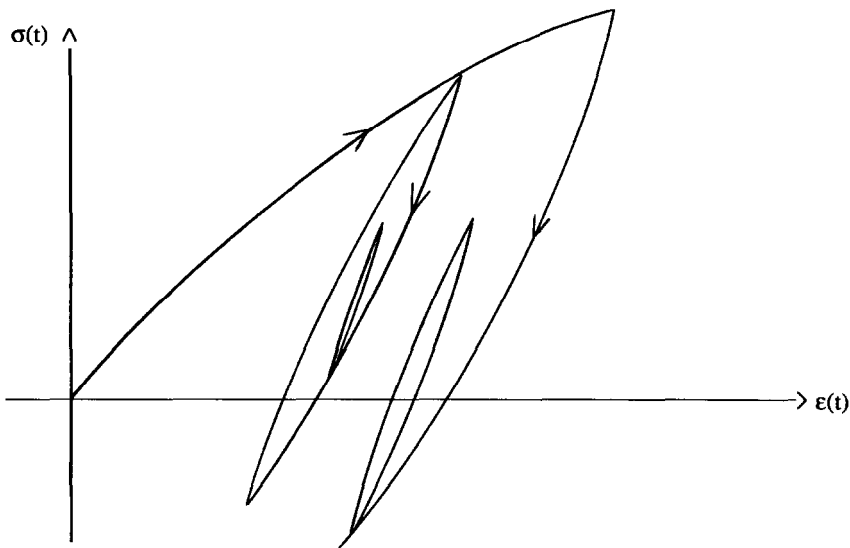


Fig. 1.

loops have been deleted. In particular for any given  $f(t)$ ,  $0 \leq t \leq T$   $\mu(f, A) \leq \nu(f, A)$  for every  $A \subseteq \Delta$ .

One may expect that  $\mu$  and  $\nu$  applied to the paths of a stationary process exhibit the same asymptotics as  $t \rightarrow \infty$  due to the fact the residual will typically be negligible compared to the closed loops for large  $t$ . We will show in Chapter 3 that this is true. What is the significance of this result? Engineers are primarily interested in a LLN for  $\mu$ . Rychlik (1987, 1988, 1992) has done some work on this but his formulae for the limit  $\tilde{\mu}$  are rather nonexplicit and restricted to special cases even in cases where an explicit computation is possible. Knowing that the LLN gives the same limit  $\tilde{\mu}$  for both  $\mu$  and  $\nu$  however allows us to compute  $\tilde{\mu}((-\infty, x] \times [y, \infty))$  as the limiting average number of upcrossings of the process from  $x$  to  $y$  – a quantity which can be computed easily and explicitly in some cases – basically whenever one has a formula for expected first passage times e.g. for diffusions. Of course this requires in addition a proof of the fact that  $T^{-1}\nu_{[0, T]}(f, x, y)$  converges to  $\tilde{\mu}((-\infty, x] \times [y, \infty))$  almost surely.

Beside the fatigue interpretation the measure  $\mu$  also helps to establish the LLN for  $\nu$ . Loosely speaking  $\mu$  has some nice superadditivity properties which  $\nu$  is lacking and which allow for the effective application of the subadditive ergodic theorem of Kingman (1975).

Apart from the literature quoted so far Dowling (1972) studies the rainflow statistic and emphasises its reliability in fatigue analysis. Lindgren and Rootzén (1987) contains a more mathematical treatment mostly based on Rychlik's work. Krüger et al. (1985) contains an on-line algorithm designed by the author for generating random sequences from  $\mu$  (which can be applied to units on a test stand). We point out that level crossing counting, which is popular in fatigue analysis (e.g. Holm and de Maré (1985)), is a function of  $\nu$ . Level crossing counting cannot be applied to nonsmooth

functions but for smooth functions it may be used for conservative estimates of fatigue life. In addition explicit formulae for the asymptotics are known in a number of cases for which  $\tilde{\mu}$  is not known explicitly.

Results on distributional properties of  $\nu$  or  $\mu$  applied to the paths of Brownian excursion or Brownian motion upto a fixed or stopping time will appear in a forthcoming paper (recall that upcrossings, local time and excursion theory are intimately related). Let us point out that the tree structure of excursions as defined in Le Gall (1991, p. 1404 ff) (which is used in connection with the explicit construction of superprocesses) can be visualized in the stress-strain diagram (Fig. 1). In Fig. 1 the residual (the “root”) contains two closed loops (“offspring”) one of which contains a further loop.

The paper is organized as follows: we construct  $\nu$  in 2.2 via a (deterministic) point process  $\rho$  on  $\Delta \times$  time interval which marks the time when an additional upcrossing of range  $(x, y) \in \Delta$  has been completed. We believe that it is worth treating the case where  $f$  is just regular and not necessarily continuous. The price we have to pay is a slightly complicated definition of  $\rho$ . The remainder of Chapter 2 contains the definition of  $\mu$  and properties of  $\mu$  and  $\nu$  needed in Chapter 3. Chapter 2 is completely nonstochastic. In Chapter 3 we establish the LLN mentioned before and in Chapter 4 we provide five examples.

For definition and elementary properties of vague convergence of Radon measures on locally compact second countable Hausdorff spaces we refer the reader to Bauer (1968), Kallenberg (1976) or Karr (1986, Appendix).

## 2. Definition and properties of upcrossing- and oscillation measures

In this chapter we assume that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a regular function i.e.  $f$  has finite right and left limits at every  $t \in \mathbb{R}$ . In particular  $f$  is bounded on bounded intervals. We start by defining an “upcrossing measure”  $\nu_f$  associated with  $f$  restricted to a bounded interval  $I$  of  $\mathbb{R}$ .

### Notation

$$R = \{f: \mathbb{R} \rightarrow \mathbb{R}, f \text{ is regular}\},$$

$$\Delta = \{(x, y) \in \mathbb{R}^2: x < y\} \text{ equipped with the topology of } \mathbb{R}^2,$$

$$\mathcal{B}(\Delta) = \text{Borel sets of } \Delta,$$

$$\mathcal{B}_c(\Delta) = \text{relatively compact Borel sets of } \Delta,$$

$$E_{xy} = (-\infty, x] \times [y, \infty) \text{ for } (x, y) \in \Delta,$$

$$\mathcal{M} = \text{Radon measures on } \Delta \text{ equipped with the vague topology,}$$

$$I, J = \text{intervals of } \mathbb{R} \text{ (bounded or unbounded, open, closed or halfopen).}$$

$\mathcal{J}(\mathbb{R}) = \bigcup_{s \in \mathbb{R}} \{s^-, s, s^+\}$  equipped with the total ordering defined by

$$s^- < s < s^+ < t^- \text{ for } s, t \in \mathbb{R}, s < t.$$

$$\mathcal{J}(J) = \{t \in \mathcal{J}(\mathbb{R}) : \forall s, u \in J : s \leq t \leq u\} \cup \{\inf(J)^+\}$$

$$\cup \{\sup(J)^-\}, \text{ where } \{s^+\} = \{s^-\} := \emptyset \text{ if } s \notin \mathbb{R}.$$

Using  $\mathcal{J}(J)$  rather than  $J$  as the index set of a regular function will turn out to be very handy in the following. We will only use the order structure of  $\mathcal{J}(J)$  but no topology on  $\mathcal{J}(J)$ . If  $f \in R$ , then  $f$  can be naturally extended to  $\mathcal{J}(\mathbb{R})$  by defining  $f(s^+) = \lim_{t \downarrow s} f(t)$  and  $f(s^-) = \lim_{t \uparrow s} f(t)$  for  $s \in \mathbb{R}$ . The extension will again be denoted by  $f$ .

**Definition 2.1.** For  $(x, y) \in \Delta$  and  $f: J \rightarrow \mathbb{R}$

$$\tilde{u}_J(f, x, y) := \sup \{k \in \mathbb{N} : \forall s_1 < t_1 < s_2 \dots < s_k < t_k; s_1, t_k \in J :$$

$$f(s_i) \leq x, f(t_i) \geq y \text{ for all } i \in \{1, \dots, k\}\}$$

is called the number of strict upcrossings of  $f$  on  $J$  from  $x$  to  $y$ .

$$u_J(f, x, y) := \lim_{\varepsilon \downarrow 0} \tilde{u}_J(f, x + \varepsilon, y - \varepsilon)$$

is called the number of (almost) upcrossings of  $f$  on  $J$  from  $x$  to  $y$ .

**Remark.** The reason for the slight deviation from the usual definition is that (almost) upcrossings will play a more important role than strict upcrossings in the following. Of course both notions coincide if  $f$  is continuous and  $J$  compact. Further (almost) upcrossings become strict ones if one allows  $s_i, t_i$  to be in  $\mathcal{J}(J)$  in the definition of  $\tilde{u}$  and if  $J$  is bounded and  $f$  is regular.

Next we define the upcrossing measure  $\nu_I(f, \cdot)$  in a constructive way rather than using Caratheodory's extension theorem. We first define a discrete measure  $\rho$  on  $I \times \Delta$  and then let  $\nu_I(f, \cdot)$  be its marginal on  $\Delta$ . Roughly speaking  $\rho$  has an atom at  $(t, x, y)$  in case an upcrossing of range  $(x, y)$  is completed at  $t$ . Especially in the stochastic case the point process  $\rho_I$  is of independent interest and will be used in the next chapter. Our approach is very similar but not equivalent (even in the continuous case) to that of Rychlik (1993, p. 378).

**Definition 2.2.** Let  $f \in R$ ,  $(x, y) \in \Delta$ ,  $I$  a bounded interval and  $t \in \mathcal{J}(I)$ . Define

$$s = s_I(t, f) := \begin{cases} \sup \{r \in \mathcal{J}(I), r < t, f(r) > y\} & \text{if such an } r \text{ exists,} \\ \inf \{\mathcal{J}(I)\} & \text{otherwise,} \end{cases}$$

$$u = u_I(t, f) := \begin{cases} \inf \{r \in \mathcal{J}(I), r > t, f(r) \geq y\} & \text{if such an } r \text{ exists,} \\ \infty & \text{otherwise,} \end{cases}$$

$$m_1 := \inf_{s \leq r \leq t} f(r),$$

$$m_2 := \begin{cases} \inf_{t \leq r \leq u} f(r) & \text{if } u \neq \infty, \\ -\infty & \text{if } u = \infty \text{ (where } r \in \mathcal{J}(I)), \end{cases}$$

$$\rho_I(f, t, \{(x, y)\}) := \begin{cases} 1 & \text{if } x = m_1 \vee m_2 \text{ and } y = f(t), \\ 0 & \text{otherwise,} \end{cases}$$

$$v_I(f, \{(x, y)\}) := \sum_{t \in \mathcal{J}(I)} \rho_I(f, t, \{(x, y)\}),$$

$$v_I(f, A) := \sum_{(x, y) \in A} v_I(f, \{(x, y)\}) \text{ for } A \subseteq \Delta.$$

$v_I(f, \cdot)$  is called “upcrossing measure of  $f$  (restricted to  $I$ )”.

**Remark.**  $v_I(f, \cdot)$  can be regarded as a measure either on  $\mathcal{B}(\Delta)$  or  $2^{\Delta}$ . Obviously it takes values in  $\mathbb{N}_0 \cup \{\infty\}$ . The reason for calling  $v_I$  “upcrossing measure” will become apparent in the following proposition. The readers who are eager to see an example may peep ahead to 4.5.

**Proposition 2.3.** *Let  $f \in R$  and  $I$  be a bounded interval. Then  $v_I(f, \cdot)$  – regarded as a measure on  $\mathcal{B}(\Delta)$  – is the only measure satisfying  $v_I(f, E_{xy}) = u_I(f, x, y)$  for all  $(x, y) \in \Delta$ .*

**Proof.** Uniqueness is immediate since  $\mathcal{E} := \{E_{xy}, (x, y) \in \Delta\}$  generates  $\mathcal{B}(\Delta)$ ,  $\mathcal{E}$  is closed under finite intersections,  $u_I(f, x, y) < \infty$  for  $(x, y) \in \Delta$  and  $\Delta$  is the union of countably many members of  $\mathcal{E}$ .

It remains to prove  $v_I(f, E_{xy}) = u_I(f, x, y)$  for all  $(x, y) \in \Delta$ . Fix  $(x, y) \in \Delta$  and let  $m = u_I(f, x, y)$  ( $< \infty$ ). The assertion is clearly true for  $m = 0$ , so we assume  $m \geq 1$ .

Define by induction

$$S_1 = \inf\{t \in \mathcal{J}(I): f(t) \leq x\}$$

$$T_p = \inf\{t \in \mathcal{J}(I): t > S_p, f(t) \geq y\}, 1 \leq p \leq m$$

$$S_p = \inf\{t \in \mathcal{J}(I): t > T_{p-1}, f(t) \leq x\}, 2 \leq p \leq m$$

$$S_{m+1} = \sup \mathcal{J}(I)$$

$$M_p = \max_{r \in [S_p, S_{p+1}]} f(r)$$

$$t_p = \max\{t \in [S_p, S_{p+1}]: f(t) = M_p\} \in \mathcal{J}(I), 1 \leq p \leq m.$$

Then – by Definition 2.2 –  $\rho_I(f, t_p, \{(v, M_p)\}) = 1$  for precisely one  $v \leq x$  while for all  $t \notin \{t_1, \dots, t_m\}$ ,  $v \leq x$  and  $w \geq y$  we have  $\rho_I(f, t, \{(v, w)\}) = 0$  proving the assertion.  $\square$

It is obvious that for fixed  $f \in R$  and bounded interval  $I$   $v_I(f, \cdot)$  takes values in  $\mathbb{N}_0 \cup \{\infty\}$ ,  $v_I(f, K) < \infty$  for compact sets  $K \subseteq \Delta$  (so  $v_I(f, \cdot)$  is a Radon measure on

$(\Delta, \mathcal{B}(\Delta))$  and that  $v_t(f, \cdot)$  has bounded support in the Euclidean metric of  $\mathbb{R}^2$ . It is not hard to see that these three properties characterize upcrossing measures i.e. for a measure  $v$  with these properties there exists  $f \in C[0, 1]$  say such that  $v(\cdot) = v_{[0, 1]}(f, \cdot)$ . We will not need this fact however. We also mention that one has obvious transformation formulae for  $v_t$  w.r.t. a nondecreasing, continuous time-change and w.r.t. a non-decreasing and continuous transformation of the state space  $\mathbb{R}$ .

The following obvious super- and subadditivity properties will be of fundamental importance for us in Chapter 3.

**Properties 2.4** (Super- and Subadditivity). *Let  $I, I_1, I_2$  be bounded intervals such that  $I = I_1 \cup I_2$  and  $I_1 \cap I_2 = \emptyset, f \in \mathbb{R}$  and  $(x, y) \in \Delta$ .*

(a)  $v_I(f, E_{xy}) \geq v_{I_1}(f, E_{xy}) + v_{I_2}(f, E_{xy})$  and  $v_I(f, E_{xy}) \leq v_{I_1}(f, E_{xy}) + v_{I_2}(f, E_{xy}) + 1$ . The same inequalities hold if  $E_{xy}$  is replaced by  $E_{xy}^{(1)} = \{(v, w) \in \Delta : v < x < y \leq w\}$  or  $E_{xy}^{(2)} = \{(v, w) \in \Delta : v \leq x < y < w\}$  or  $E_{xy}^{(3)} = \{(v, w) \in \Delta : v < x < y < w\}$ .

(b) Let  $K_1, K_2$  be (bounded) intervals such that  $A = K_1 \times K_2 \in \mathcal{B}_c(\Delta)$ . Then  $v_{I_1}(f, A) + v_{I_2}(f, A) - 2 \leq v_I(f, A) \leq v_{I_1}(f, A) + v_{I_2}(f, A) + 2$ .

**Proof.** (a) is obvious, (b) follows from (a).  $\square$

Up to now we only considered bounded intervals of  $\mathbb{R}$  as the domain of definition for  $\rho$  and  $v$ . To avoid technical complications we only generalize  $\rho$  to unbounded intervals since this way we will get a stationary point process in the next chapter under stationarity conditions on the corresponding stochastic process. Proposition 2.6 will be needed in the next chapter. It states in what sense  $v_t$  and  $\sigma_t$  – which will be introduced in 2.7 – are close.

**Definition and Lemma 2.5.** *Let  $J$  be an interval (possibly unbounded),  $f \in \mathbb{R}$  and  $t \in \mathcal{J}(J)$ . Assume there exists a bounded interval  $I \subset J$  such that for each bounded interval  $I \subseteq K \subseteq J$  there exists some  $x_K < f(t)$  such that  $\rho_K(f, t, \{(x_K, f(t))\}) = 1$ . Then  $x = \lim_{\mathcal{J}(K) \uparrow \mathcal{J}(J)} x_K \in [-\infty, f(t)[$  exists. Define  $\rho_J(f, t, \{(x, f(t))\}) = 1$  (and  $= 0$  otherwise). Then  $\rho_J$  agrees with the former definition if  $J$  is bounded.*

**Proof.** Definition 2.2 shows that  $x_K$  decreases when  $K$  increases as long as  $u = \infty$  or as long as  $u \neq \infty$ . Since  $u_L(t) \neq \infty$  implies  $u_K(t) \neq \infty$  for bounded  $K \supseteq L$ , the existence of the limit  $x \in [-\infty, f(t)[$  follows. The last assertion holds trivially.  $\square$

**Remark.** Note that  $\rho_J(f, \cdot)$  is a discrete measure on  $\mathcal{J}(J) \times \bar{\Delta}$ , where  $\bar{\Delta} = \Delta \cup \{(-\infty, y) : y \in \mathbb{R}\}$ . Observe that  $\sum_{t \in \mathcal{J}(J)} \rho_J(f, t, \bar{\Delta} \setminus \Delta) \leq 1$ , so restricting to  $\Delta$  we lose at most one atom.

**Proposition 2.6.** *Assume  $f \in \mathbb{R}, I$  bounded,  $J \supseteq I$  an interval,  $(x, y) \in \Delta$  and define  $E_{xy}^{(i)}, i = 1, 2, 3$  as in 2.4(a),  $E_{xy}^{(4)} = E_{xy}$ .*

(a)  $v_I(f, E_{xy}^{(i)}) - 2 \leq \sum_{t \in \mathcal{J}(I)} \rho_J(f, t, E_{xy}^{(i)}) \leq v_I(f, E_{xy}^{(i)}) + 1, i = 1, 2, 3, 4$ .

(b)  $v_I(f, K) - 6 \leq \sum_{t \in \mathcal{J}(I)} \rho_J(f, t, K) \leq v_I(f, K) + 6$  for every  $K = K_1 \times K_2 \in \mathcal{B}_c(\Delta)$ ,

where  $K_1, K_2$  are (bounded) intervals.

$$(c) \quad v_I(f, \Delta) - 2 \leq \sum_{t \in \mathcal{J}(I)} \rho_J(f, t, \Delta) \leq v_I(f, \Delta) + 1.$$

**Proof.** (b) follows immediately from (a).

(a) Assume  $t_1, t_2 \in \mathcal{J}(I), t_1 < t_2$  satisfy  $\rho_J(f, t_j, E_{xy}) = 1$  for  $j = 1, 2$ . Then  $x \geq \inf_{t_1 \leq r \leq t_2} f(r)$  and therefore  $\rho_I(f, t_2, E_{xy}) = 1$  i.e. we have shown the right inequality for  $i = 4$ .

Conversely assume  $t_1, t_2 \in \mathcal{J}(I), t_1 < t_2$  satisfy  $\rho_I(f, t_j, E_{xy}) = 1$  for  $j = 1, 2$ . Again  $x \geq \inf_{t_1 \leq r \leq t_2} f(r)$  and therefore  $\rho_J(f, t_1, E_{xy}) = 1$  or  $\rho_J(f, t_1, \bar{\Delta} \setminus \Delta) = 1$ . Recalling the previous remark we have shown the left inequality for  $i = 4$ .

The inequalities for  $i = 1, 2, 3$  follow from the continuity of the corresponding measures.

(c) If  $\rho_I(f, t, \Delta) = 1$  for  $t \in \mathcal{J}(I)$  and  $t \neq \sup \mathcal{J}(I)$  then  $\rho_J(f, t, \bar{\Delta}) = 1$  which—together with the previous remark—implies the first inequality. Conversely, if  $\rho_J(f, t, \Delta) = 1$  for  $t \in \mathcal{J}(I)$  and  $f$  is not constant on  $[\inf \mathcal{J}(I), t]$ , then  $\rho_I(f, t, \Delta) = 1$ .  $\square$

### 2.1. Oscillation measures

Now we give a rigorous definition of the oscillation- or rainflow measure  $\mu$  which has been mentioned in the introduction. Recall that  $\mu_I(f, \{(x, y)\})$  is supposed to represent the number of “closed loops” of strain-range  $[x, y]$  in the stress–strain diagram. To arrive at a mathematical definition first note that a closed loop never disappears or changes in case the (strain-) function  $f(t), 0 \leq t \leq T$  is extended to domain  $\mathbb{R}$  in an arbitrary way. Furthermore  $f$  can be extended in such a way to a function from  $\mathbb{R}$  to  $\mathbb{R}$  that the number of loops of range  $[x, y]$  on  $[0, T]$  equals  $v_K(f, \{(x, y)\})$  for some  $K \supseteq I$ . Therefore we define  $\mu_I(f, \{(x, y)\})$  as the infimum of the mass  $v_K(g, \{(x, y)\})$  over all extensions of  $f|_I$  to larger intervals  $K$ . This way we avoid any reference to stress–strain relationship. Apart from the obvious results 2.8(a) and (b) the superadditivity property 2.8(e) will be of fundamental importance for us. Note that 2.8(e) is rather obvious heuristically using the “closed loop” interpretation. It is true—but we will neither need nor prove—that our definition coincides with the result of rainflow-counting algorithms in the discrete-time case.

**Definition 2.7.** Let  $I$  be a bounded interval,  $f \in R$ .

$$\mathcal{S} = \mathcal{S}_I(f) := \{g \in R: g|_I = f|_I\}$$

$$\mu_I(f, \{(x, y)\}) := \inf_{K \supseteq I, g \in \mathcal{S}} v_K(g, \{(x, y)\}), \quad K \text{ bounded interval, } (x, y) \in \Delta$$

$$\mu_I(f, A) := \sum_{(x, y) \in A} \mu_I(f, \{(x, y)\}) \quad \text{for } A \subseteq \Delta.$$

The measure  $\mu_I(f, \cdot)$  on  $(\Delta, \mathcal{B}(\Delta))$  is called “oscillation (or rainflow-) measure of  $f$  on  $I$ ”.

$$r_I(f, \cdot) := v_I(f, \cdot) - \mu_I(f, \cdot)$$



is called “residual upcrossing measure of  $f$  on  $I$ ”. Further we define the measure  $\sigma$  on  $\Delta$  by

$$\sigma_I(f, \cdot) := \sum_{t \in \mathcal{J}(I)} \rho_R(f, t, \cdot)|_{\Delta}.$$

**Remark.** The fact that  $r_I(f, \cdot)$  is a measure i.e.  $v_I(f, A) - \mu_I(f, A) \geq 0$  for all  $A \in \mathcal{B}(\Delta)$  follows from the next proposition. It is *not* true that  $\mu_I(f, A) = \inf_{K \supseteq I, g \in \mathcal{S}} v_K(g, A)$  for all  $A \in \mathcal{B}(\Delta)$ ! Take e.g.  $A = E_{0,1}$ ,  $I = [0, 1]$ ,  $f(x) = x$ ; then the left side is zero and the right side is one.

Given  $f, I$  and  $(x, y)$  let us specify some  $K$  and  $g$  which minimize the right hand side of the equation defining  $\mu$ . Let  $K$  be any bounded interval strictly extending  $I$  on both sides. Define  $g = f$  on  $I$  and  $g = y + 1$  to the right of  $I$  iff the last visit of  $f$  to  $(-\infty, x] \cup [y, \infty)$  in  $I$  was in  $[y, \infty)$  and  $g = x - 1$  otherwise and analogously to the left of  $I$ . We write  $g^f$  for this function (adding arguments  $I, (x, y)$  in case of ambiguity). The fact that  $K, g^f$  are minimizers is straightforward to check.

**Proposition 2.8.** Assume  $f \in R$ ,  $I$  a bounded interval,  $(x, y) \in \Delta$ ,  $A \in \mathcal{B}(\Delta)$ .

- (a)  $\mu_I(f, A) \leq \mu_K(f, A)$ ,  $I \subseteq K$  a bounded interval,
- (b)  $\mu_I(f, A) \leq v_I(f, A)$ ,
- (c)  $\sigma_I(g^f, \{(x, y)\}) \leq \sigma_I(f, \{(x, y)\})$ ,
- (d)  $\mu_I(f, A) \leq \sigma_I(f, A)$
- (e) If  $I = I_1 \cup I_2$ ,  $I_1 \cap I_2 = \emptyset$ , then  $\sigma_I(f, A) = \sigma_{I_1}(f, A) + \sigma_{I_2}(f, A)$ ,  $\mu_I(f, A) \geq \mu_{I_1}(f, A) + \mu_{I_2}(f, A)$ .

**Proof.** (a), (b) and the first half of (e) are obvious. (c) is straightforward to check. It suffices to show (d) and the second half of (e) for  $A = \{(x, y)\}$ . Let  $K$  be a bounded interval strictly extending  $I$  on both sides.

- (d)  $\mu_I(f, \{(x, y)\}) = v_K(g^f(I), \{(x, y)\}) = \sigma_I(g^f(I), \{(x, y)\}) \leq \sigma_I(f, \{(x, y)\})$  by (c).
- (e)  $\mu_I(f, \{(x, y)\}) = \sigma_I(g^f(I), \{(x, y)\}) \geq \mu_{I_1}(g^f(I), \{(x, y)\}) + \mu_{I_2}(g^f(I), \{(x, y)\})$   
 $= \mu_{I_1}(f, \{(x, y)\}) + \mu_{I_2}(f, \{(x, y)\})$ ,

where the inequality follows from the first half of (e) and (d).  $\square$

**Remarks.** It is a simple consequence of the definition of  $\mu$  and of  $g^f$  that  $\mu$  is invariant under time-reversal of  $f$  and also that working with downcrossings rather than upcrossings the resulting oscillation measure is identical with  $\mu$ .

Summarizing  $\mu$  has nice symmetry and superadditivity properties while  $v$  has nice “almost” additivity properties on rectangles and  $f \rightarrow v_I(f, \cdot)$  has nice continuity properties (we did not elaborate on the latter since we will not need them here but they are important when processes are approximated).  $\sigma$  is even additive but does not share any of the other mentioned properties.  $\mu_I(f, \cdot)$  and  $v_I(f, \cdot)$  are functions of  $f$  on  $I$  which  $\sigma_I(f, \cdot)$  is not.

### 3. Law of large numbers

Before we state the main result – Theorem 3.1 – we need some notation.

$$\mathcal{K} = \{[x_1, x_2] \times [y_1, y_2]: -\infty < x_1 < x_2 < y_1 < y_2 < \infty\}$$

$$\mathcal{K}_q = \{[x_1, x_2] \times [y_1, y_2]: x_1 \leq x_2 < y_1 \leq y_2; x_1, x_2, y_1, y_2 \in \mathbb{Q}\}.$$

**Theorem 3.1.** *Let  $X(t), t \in \mathbb{R}$  be a stationary process with cadlag paths and  $\mathbb{E}v_{[0,1]}(K) < \infty$  for  $K \in \mathcal{K}$ . Let  $\mathcal{I}$  be the invariant  $\sigma$ -algebra on  $(\Omega, \mathcal{F}, \mathbb{P})$  generated by the stationary  $\mathcal{M}$ -valued process  $\sigma_{[n, n+1]}(X, \cdot), n \in \mathbb{N}$ .*

*Then there exists a  $\mathcal{I}$ -measurable  $\mathcal{M}$ -valued  $\tilde{\mu}$  such that for every  $A \in \mathcal{B}(\Delta)$*

$$\begin{aligned} \tilde{\mu}(A) &= \mathbb{E}(\sigma_{[0,1]}(X, A) | \mathcal{I}) \text{ a.s. and } \lim_{t \rightarrow \infty} \frac{1}{t} v_{[0,t]}(X, \cdot) = \lim_{t \rightarrow \infty} \frac{1}{t} \mu_{[0,t]}(X, \cdot) \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \sigma_{[0,t]}(X, \cdot) = \tilde{\mu}(\cdot) \end{aligned}$$

*vaguely almost surely. If  $X$  is ergodic, then  $\tilde{\mu}$  is deterministic. Further for all  $A \in \mathcal{B}(\Delta)$*

*$\lim_{t \rightarrow \infty} \frac{1}{t} \mu_{[0,t]}(X, A) = \lim_{t \rightarrow \infty} \frac{1}{t} \sigma_{[0,t]}(X, A) = \tilde{\mu}(A)$  a.s. and for all  $A \in \mathcal{B}(\Delta)$  such that  $\mathbb{E}\tilde{\mu}(A) < \infty$  the convergence is also in  $L^1$ . Further*

$$\lim_{t \rightarrow \infty} \frac{1}{t} v_{[0,t]}(X, A) = \tilde{\mu}(A) \text{ a.s. for all } A \in \mathcal{B}_c(\Delta), A = E_{xy}((x, y) \in \Delta) \text{ and } A = \Delta \text{ and}$$

*the convergence is in  $L^1$  if  $A \in \mathcal{B}_c(\Delta)$  (for additional information see 3.7 and 3.3).*

We will prove Theorem 3.1 via a number of lemmas and propositions. In Lemma 3.2 we settle some measurability questions, Lemma 3.3 is an important intermediate step from which most assertions of 3.1 concerning  $\mu$  and  $\sigma$  can be deduced. The following propositions basically establish that the scaled vague limits for  $\sigma$ ,  $\mu$  and  $v$  all coincide. To see that the limits for  $\sigma$  and  $v$  coincide (Proposition 3.5) we need 2.6(b) which shows that  $v$  and  $\sigma$  are sufficiently close. In Proposition 3.5 we establish the coincidence of the limits of  $\mu$  and  $v$  by proving, that the scaled residual upcrossing measure is asymptotically negligible.

**Lemma 3.2.** *Let  $D$  be the set of cadlag functions from  $\mathbb{R}$  to  $\mathbb{R}$  equipped with the  $\sigma$ -algebra  $\mathcal{D}$  generated by the projections. Let  $I$  be a bounded nonempty interval. Then the maps*

$$X \mapsto v_I(X, \cdot), \quad X \mapsto \sigma_I(X, \cdot) \quad \text{and} \quad X \mapsto \mu_I(X, \cdot)$$

*are measurable from  $(D, \mathcal{D})$  to  $\mathcal{M}$  equipped with its Borel- $\sigma$ -algebra. Further for each  $A \in \mathcal{B}(\Delta)$  the maps*

$$X \mapsto v_I(X, A), \quad X \mapsto \sigma_I(X, A) \quad \text{and} \quad X \mapsto \mu_I(X, A)$$

*are measurable from  $(D, \mathcal{D})$  to  $\mathbb{R} \cup \{\infty\}$ .*

**Proof.** It suffices to show the last three assertions for all  $A = E_{xy}, (x, y) \in \Delta$ , since the result then follows via the Dynkin class theorem (also called monotone class theorem) for general  $A \in \mathcal{B}(\Delta)$ . The first three assertions then follow e.g. from Proposition A.5 in Karr (1986).

Fix  $(x, y) \in \Delta$ . Assume first that  $I$  is compact. As is well-known first contact times of right continuous processes of closed sets are measurable:

define  $T_x: D \rightarrow I \cup \partial$  by  $T_x(X) = \inf\{t \in I: X_{t-} \leq x \text{ or } X_t \leq x\}$  if the set is nonempty and  $T_x(X) = \partial$  otherwise. Then for  $t \in I$  we have

$$\{T_x(X) \leq t\} = \bigcap_{n \in \mathbb{N}} \bigcup_{q \in \mathbb{Q}_t} \left\{ X_q < x + \frac{1}{n} \right\} \cup \{X_t \leq x\} \in \mathcal{D},$$

where  $\mathbb{Q}_t = \mathbb{Q} \cap I \cap ] - \infty, t]$ . Analogously the first contact time of  $[y, \infty[$  is measurable. This implies measurability of  $v_I(E_{xy})$  for compact  $I$  and – by monotonicity – for all bounded  $I$ .

It is easy to see from the proof of Proposition 2.6 that  $X \mapsto v_I(X, E_{xy}) - \sigma_I(X, E_{xy}) \in \{-1, 0, 1, 2\}$  is measurable.

$\mu_I(X, E_{xy})$  can be expressed as a measurable function of infima and suprema between successive contact times of  $] - \infty, x]$  and  $[y, \infty[$  and is therefore measurable. We omit details.  $\square$

The following Lemma will be applied to both  $\lambda_{st}(\omega, \cdot) = \mu_{[s, t]}(X(\omega), \cdot)$  and  $\lambda_{st}(\omega, \cdot) = \sigma_{[s, t]}(X(\omega), \cdot)$  in case  $X$  is a cadlag stationary process.

**Lemma 3.3.** *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $\tilde{\Delta} := \{(s, t): 0 \leq s < t < \infty\}$  and assume  $\lambda: \Omega \times \tilde{\Delta} \rightarrow \mathcal{M}$  satisfies*

$$\omega \mapsto \lambda_{st}(\omega, A) \text{ is measurable for } A \in \mathcal{B}(\Delta), (s, t) \in \tilde{\Delta}, \quad (3.1)$$

$$t \mapsto \lambda_{0t}(\omega, A) \text{ is nondecreasing for } A \in \mathcal{B}(\Delta), \omega \in \Omega \quad (3.2)$$

$$\lambda_{lm}(\omega, A) \geq \lambda_{lm}(\omega, A) + \lambda_{mn}(\omega, A) \text{ for } A \in \mathcal{B}(\Delta), 0 \leq l < m < n, l, m, n \in \mathbb{N}_0, \omega \in \Omega \quad (3.3)$$

$$\mathcal{L}(\lambda_{n, m+n}; n \in \mathbb{N}_0, m \in \mathbb{N}) = \mathcal{L}(\lambda_{n+1, m+n+1}; n \in \mathbb{N}_0, m \in \mathbb{N}) \quad (3.4)$$

and

$$\sup_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E} \lambda_{0n}(K) < \infty \text{ for } K \in \mathcal{K}. \quad (3.5)$$

Then there exists a random element  $\tilde{\lambda}$  of  $\mathcal{M}$  such that

$$\frac{1}{t} \lambda_{0t} \rightarrow \tilde{\lambda} \text{ vaguely as } t \rightarrow \infty \text{ a.s.} \quad (3.6)$$

$$\frac{1}{t} \lambda_{0t}(A) \rightarrow \tilde{\lambda}(A) \text{ a.s. as } t \rightarrow \infty \text{ for } A \in \mathcal{B}(\Delta) \quad (3.7)$$

$$\mathbb{E} \tilde{\lambda}(A) < \infty \text{ whenever } A \in \mathcal{B}_c(\Delta). \quad (3.8)$$

If  $A \in \mathcal{B}(\Delta)$  and  $\mathbb{E} \tilde{\lambda}(A) < \infty$ , then

$$\frac{1}{t} \lambda_{0t}(A) \rightarrow \tilde{\lambda}(A) \quad \text{as } t \rightarrow \infty \text{ also in } L_1. \quad (3.9)$$

Further there exists a  $\mathbb{P}$ -nullset  $N$  such that

$$\frac{1}{t} \lambda_{0t}(\omega, K) \xrightarrow[t \rightarrow \infty]{} \tilde{\lambda}(\omega, K) \quad \text{for all } K \in \mathcal{K} \text{ and all } \omega \notin N. \quad (3.10)$$

**Remark.** In general  $\mathcal{K}$  cannot be replaced by  $\mathcal{B}(\Delta)$  in (3.10).

**Proof.** We assume w.l.o.g. that  $(\Omega, \mathcal{F}, \mathbb{P})$  is complete. Define  $\lambda_t := \lambda_{0t}$ ,  $t > 0$ . For  $A \in \mathcal{B}(\Delta)$  we can use Kingman's subadditive ergodic theorem (Kingman (1975, p. 169 and p. 187 for the extended version)) to get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \lambda_n(A) = \kappa_A \quad \text{a.s.}$$

for some (possibly extended) random variable  $\kappa_A$ . Due to (3.2) we also have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \lambda_t(A) = \kappa_A \quad \text{a.s.} \quad (3.11)$$

The convergence in (3.11) is also in  $L_1$  whenever  $\sup_{n \in \mathbb{N}} (1/n) \mathbb{E} \lambda_n(A) < \infty$ , otherwise  $\mathbb{E} \kappa_A = \infty$ .

Let  $M$  be the set of  $\omega \in \Omega$  such that  $(1/t) \lambda_t(\omega, K) \rightarrow \kappa_K(\omega) \in \mathbb{R}$  for all  $K \in \mathcal{K}_q$ . Clearly  $\mathbb{P}(M) = 1$ . Fix  $\omega \in M$ . Then the collection  $\{(1/t) \lambda_t(\omega, \cdot), t \geq 1\}$  is bounded in the sense that  $\sup_{t \geq 1} (1/t) \lambda_t(\omega, K) < \infty$  for every compact  $K \subset \Delta$  which implies that  $\{(1/t) \lambda_t(\omega), t \geq 1\}$  is vaguely relatively compact. Suppose  $\lambda$  and  $\tilde{\lambda}$  are different vague limit points as  $t \rightarrow \infty$ . Then there exists some  $K \in \mathcal{K}_q$  such that  $\lambda(K) < \tilde{\lambda}(K)$  (say). Further – since  $\lambda$  is regular – we find an open  $G \subset \Delta$  and  $\tilde{K} \in \mathcal{K}_q$  such that  $K \subset G \subset \tilde{K}$  and  $\lambda(\tilde{K}) < \tilde{\lambda}(K)$ . Then

$$\tilde{\lambda}(G) \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \lambda_t(G) \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \lambda_t(\tilde{K}) = \liminf_{t \rightarrow \infty} \frac{1}{t} \lambda_t(\tilde{K}) \leq \lambda(\tilde{K}) < \tilde{\lambda}(K) \leq \lambda(G),$$

which is a contradiction. Hence there exists a random element  $\tilde{\lambda}$  of  $\mathcal{M}$  such that  $(1/t) \lambda_t(\omega, \cdot) \xrightarrow{v} \tilde{\lambda}(\omega, \cdot)$  as  $t \rightarrow \infty$  for every  $\omega \in M$  (measurability of  $\tilde{\lambda}$  follows since  $\mathcal{M}$  is metrizable). So we have shown (3.6).

(3.8) follows easily: if  $A \in \mathcal{B}_c(\Delta)$  and  $A \subset G$  where  $G$  is open and relatively compact, then  $E \tilde{\lambda}(A) \leq \mathbb{E} \tilde{\lambda}(G) \leq \mathbb{E} \kappa_G < \infty$ . Define

$$\chi(A) := \mathbb{E} \kappa_A, \quad A \in \mathcal{B}(\Delta) \quad \text{and}$$

$$\chi_n(A) := \frac{1}{n} \mathbb{E} \lambda_n(A), \quad A \in \mathcal{B}(\Delta), n \in \mathbb{N}.$$

Due to (3.11) and the sentence following it, we have

$$\lim_{n \rightarrow \infty} \chi_n(A) = \chi(A), \quad A \in \mathcal{B}(\Delta)$$

(since  $\sup(1/n) \mathbb{E}\lambda_n(A) = \lim(1/n) \mathbb{E}\lambda_n(A)$ ). The convergence theorem of Nikodym (1933) implies that  $\chi$  is a measure on  $(\Delta, \mathcal{B}(\Delta))$ . (Nikodym considered finite measures, but the extension to  $\sigma$ -finite measures is obvious).  $\chi$  is Radon and so is  $\tilde{\chi}(\cdot) := \mathbb{E}\tilde{\lambda}(\cdot)$  (by (3.8)). Since  $\kappa_K \leq \tilde{\lambda}(K)$  a.s. for  $K$  compact and  $\kappa_G \geq \tilde{\lambda}(G)$  a.s. for  $G$  open we have  $\chi(K) \leq \tilde{\chi}(K)$  for  $K$  compact and  $\chi(G) \geq \tilde{\chi}(G)$  for  $G$  open, so  $\chi = \tilde{\chi}$  by regularity. For every compact  $K$  we therefore have  $\tilde{\lambda}(K) = \kappa_K$  a.s.

Let us consider the special case where the process in (3.4) is ergodic for the moment. Then  $\kappa_A = \chi(A)$  a.s. for every (fixed)  $A \in \mathcal{B}(\Delta)$  and since  $\kappa_K = \chi(K) = \tilde{\lambda}(K)$  a.s. for all  $K \in \mathcal{K}_q$  we have  $\kappa_A = \tilde{\lambda}(A)$  a.s. for every (fixed)  $A \in \mathcal{B}(\Delta)$  showing (3.7) and (3.9). In the general case let  $\hat{\mathbb{P}}$  be the law of  $\lambda(\omega) = (\lambda_{mn}(\omega))_{0 \leq m < n} (m, n \in \mathbb{N}_0)$  on the restriction to the subset  $S$  which satisfies (3.3) (with the restriction of the product- $\sigma$ -algebra of  $\mathcal{M}$ ). Define the shift  $(\tau(\lambda))_{mn} = \lambda_{m+1, n+1}$  on  $S$   $\hat{\mathbb{P}}$  is invariant with respect to  $\tau$  by (3.4) and – since  $S$  is standard – we have a representation

$$\hat{\mathbb{P}}(\cdot) = \int_{\mathcal{H}} \hat{\mathbb{P}}_x(\cdot) dp(x) \quad (3.12)$$

for some  $\mathcal{H}$ , a probability measure  $p$  and ergodic probability measures  $\hat{\mathbb{P}}_x$  on  $S$  (Parthasarathy (1977, p. 256)). For  $A \in \mathcal{B}(\Delta)$  we have shown that  $\hat{\mathbb{P}}_x(\lambda \in S: \kappa_A = \tilde{\lambda}(A)) = 1$  for every  $x \in \mathcal{H}$ , so (3.12) implies (3.7) and (3.9) in the general case.

Again we show (3.10) in the ergodic case first. Then  $\tilde{\lambda}$  is deterministic.

Obviously the sets  $\{(x, y) \in \Delta: \tilde{\lambda}(\{(x, y)\}) > 0\}$ ,  $\{(q_1, q_2, y) \in (\Delta \cap \mathbb{Q}^2) \times \mathbb{R}: q_2 < y \text{ and } \tilde{\lambda}([q_1, q_2] \times \{y\}) > 0\}$  and  $\{(x, q_1, q_2) \in \mathbb{R} \times (\Delta \cap \mathbb{Q}^2): x < q_1 \text{ and } \tilde{\lambda}(\{x\} \times [q_1, q_2]) > 0\}$  are countable, so we find a null set  $N \subset \Omega$  such that  $(1/t) \lambda_t(\omega, A) \rightarrow \tilde{\lambda}(\omega, A)$  for all  $\omega \notin N$  and all  $A \subseteq \Omega$  as above. It follows that for the same nullset  $N$  we have (3.10).

Using (3.12) we get (3.10) in the general case.  $\square$

**Proposition 3.4.** *Let  $X(t), t \in \mathbb{R}$  be a stationary real-valued process with cadlag paths and assume  $\mathbb{E}v_{[0,1]}^{\mu}(K) < \infty$  for  $K \in \mathcal{K}$ . Then both*

$$\lambda_{st}(\omega, \cdot) = \mu_{[s,t]}(X(\omega), \cdot) \quad \text{and} \quad \lambda_{st}(\omega, \cdot) = \sigma_{[s,t]}(X(\omega), \cdot)$$

*satisfy the assumptions of Lemma 3.3.*

**Proof.** (3.1) follows from 3.2. (3.2) is obvious, (3.3) was established in 2.8(e). (3.4) is clear. (3.5) follows for  $\sigma$  from 2.6(b) and for  $\mu$  as follows:

$$\mathbb{E}\mu_{[0,n]}(K) \leq \mathbb{E}v_{[0,n]}(K) \leq n\mathbb{E}v_{[0,1]}(K) + 2(n-1)$$

by 2.4(b), so

$$\sup_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E}\mu_{[0,n]}(K) \leq \mathbb{E}v_{[0,1]}(K) + 2. \quad \square$$

**Notation.** Under the assumptions of 3.4 we denote the limiting measures  $\tilde{\lambda}$  in 3.3 corresponding to  $\mu$  and  $\sigma$  by  $\tilde{\mu}$  and  $\tilde{\sigma}$  respectively. It is clear that in (3.6), (3.7), (3.9) and (3.10) we can write  $\mu_{[0,t]}$  resp.  $\sigma_{[0,t]}$  instead of  $\mu_{[0,t]}$  resp.  $\sigma_{[0,t]}$ .

Our next aim is to show that  $\tilde{\mu} = \tilde{\sigma}$  and that  $(1/t) v_{[0,t]}(X(\omega), \cdot)$  also converges to  $\tilde{\mu}$ .

**Proposition 3.5.** *Under the assumptions of 3.4*

$$\frac{1}{t} v_{[0,t]}(X(\omega), \cdot) \rightarrow \tilde{\sigma}(\omega, \cdot) \text{ vaguely as } t \rightarrow \infty \text{ a.s.}$$

Further there exists a nullset  $N$  such that

$$\frac{1}{t} v_{[0,t]}(X(\omega), K) \rightarrow \tilde{\sigma}(\omega, K) \text{ for all } \omega \notin N \text{ and all } K \in \mathcal{K}.$$

**Proof.** By 2.6(b) and 3.4 the family  $\{(1/t) v_{[0,t]}(X(\omega)), t \geq 1\}$  is almost surely vaguely relatively compact.

If  $\tilde{v}(\omega, \cdot)$  is a vague limit point 2.6(b) and 3.4 imply that

$$\tilde{v}(\omega, K) \geq \lim_{t \rightarrow \infty} \frac{1}{t} v_{[0,t]}(X(\omega), K) = \tilde{\sigma}(\omega, K) \text{ a.s. for } K \in \mathcal{K}.$$

Showing that  $\tilde{v}(\omega, K) = \tilde{\sigma}(\omega, K)$  for all  $K \in \mathcal{K}_q$  by contradiction as in the proof of 3.3 we see that  $\tilde{v} = \tilde{\sigma}$  almost surely. The last assertion follows from 3.4 and 2.6(b).  $\square$

**Proposition 3.6.** *Under the assumptions of 3.4  $\tilde{\mu} = \tilde{\sigma}$  almost surely.*

**Proof.** It suffices to show

$$\lim_{t \rightarrow \infty} \frac{1}{t} r_{[0,t]}(X(\omega), K) = 0 \text{ a.s. for all } K \in \mathcal{K}_q,$$

since we already know from 2.8(b), 3.4 and 3.5 that

$$0 \leq \frac{1}{t} r_{[0,t]}(X(\omega), \cdot) = \frac{1}{t} (v_{[0,t]}(X(\omega), \cdot) - \mu_{[0,t]}(X(\omega), \cdot)) \rightarrow \tilde{\sigma} - \tilde{\mu} =: \tilde{r}$$

vaguely almost surely as  $t \rightarrow \infty$ . By 3.4 and 3.5 we further know that there is a nullset  $N$  such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} r_{[0,t]}(X(\omega), K) = \tilde{r}(\omega, K) \text{ for all } \omega \notin N^c \text{ and all } K \in \mathcal{K}. \quad (3.13)$$

Define  $M(\omega) = \sup_{t \in \mathbb{R}} X_t(\omega)$  and  $m(\omega) = 1$  if  $M(\omega) = \bigcup_{n=1}^{\infty} \sup_{t \in [-n, n]} X_t(\omega)$  and  $m(\omega) = 0$  otherwise. Further let  $A(\omega) := [M(\omega), \infty[$  if  $m(\omega) = 0$  and  $]M(\omega), \infty[$  if  $m(\omega) = 1$ . Fix  $x_1 < x_2 < y_1 < y_2$ . Then

$$r_{[0,t]}(X(\omega), [x_1, x_2] \times ([y_1, y_2] \cap A(\omega))) = 0$$

since  $v_{[0,t]}$  of the same set is zero. We show that

$$\lim_{t \rightarrow \infty} \frac{1}{t} r_{[0,t]}(X(\omega), [x_1, x_2] \times ([y_1, y_2] \cap A^c(\omega))) = 0 \text{ a.s.} \quad (3.14)$$

Define  $T_1^y(\omega) := \inf\{t \in \mathcal{I}([0, \infty[): X(t, \omega) \geq y\}$ ,  $T_n^y(\omega) := \inf\{t \in \mathcal{I}([0, \infty[), t \geq T_{n-1}^y(\omega) + 1: X(t, \omega) \geq y\}$ ,  $n \geq 2$ . By Birkhoff's ergodic theorem there exists a null set  $\tilde{N} \supseteq N$  such that  $T_n^y(\omega) < \infty$  for all  $n \in \mathbb{N}$  and all  $y \in \mathbb{R}$  on  $\{M(\omega) > y\} \cap \tilde{N}^c$  and on  $\{M(\omega) = y\} \cap \{m(\omega) = 1\} \cap \tilde{N}^c$ . So  $r_{[0, T_n^y]}(X(\omega), [x_1, x_2] \times [y_1, y_2 \wedge y]) \leq r_{[0, T_n^y]}(X(\omega), [x_1, x_2] \times [y_1, y_2 \wedge y]) + 1$  for all  $n$  and  $\omega, y$  as above. Since  $T_n^y \rightarrow \infty$  as  $n \rightarrow \infty$  we have

$$\liminf_{t \rightarrow \infty} \frac{1}{t} r_{[0,t]}(X(\omega), [x_1, x_2] \times [y_1, y_2 \wedge y]) = 0$$

for  $\omega, y$  as above. The  $\liminf$  is a  $\lim$  by (3.13). In case  $m(\omega) = 1$  – setting  $y = M(\omega)$  – we get (3.14). Otherwise by (3.13)

$$\tilde{r}(\omega, [x_1, x_2] \times ([y_1, y_2] \cap A^c(\omega))) = \lim_{y \uparrow M(\omega)} \tilde{r}(\omega, [x_1, x_2] \times [y_1, y_2 \wedge y]) = 0.$$

Since  $[x_1, x_2] \times ([y_1, y_2] \cap A^c(\omega))$  is the difference of two sets in  $\mathcal{K}$  we get (3.14) also in case  $m(\omega) = 0$ .  $\square$

**Proposition 3.7.** *Let  $X(t)$ ,  $t \in \mathbb{R}$  be stationary with cadlag paths and  $\mathbb{E}v_{[0,1]}(K) < \infty$  for  $K \in \mathcal{K}$ . Then*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} v_{[0,t]}(A) \geq \tilde{\mu}(A) \text{ a.s., } A \in \mathcal{B}(\Delta) \quad (3.15)$$

and

$$\frac{1}{t} v_{[0,t]}(A) \rightarrow \tilde{\mu}(A) \text{ a.s., } t \rightarrow \infty \quad (3.16)$$

for  $A \in \mathcal{B}_c(\Delta)$ , for  $A = E_{xy}((x, y) \in \Delta)$  and for  $A = \Delta$ . For every  $A \in \mathcal{B}(\Delta)$  for which there exists  $A \subseteq B \in \mathcal{B}(\Delta)$  such that  $(1/t) v_{[0,t]}(B) \rightarrow \tilde{\mu}(B)$  a.s. (3.16) holds on the set  $\{\tilde{\mu}(B) < \infty\}$ .

Convergence in (3.16) holds in  $L_1$  for  $A \in \mathcal{B}_c(\Delta)$ . If  $L_1$ -convergence holds for  $B \in \mathcal{B}(\Delta)$ , then also for  $A \subseteq B$ ,  $A \in \mathcal{B}(\Delta)$ . If  $\mathbb{E}\tilde{\mu}(E_{xy}) < \infty$  resp.  $\mathbb{E}\tilde{\mu}(\Delta) < \infty$ , then  $L_1$ -convergence holds for  $A = E_{xy}$  resp.  $A = \Delta$ .

**Proof.** (3.15) is clear by 3.4 and  $v_{[0,t]}(A) \geq \mu_{[0,t]}(A)$ . Almost sure and  $L_1$ -convergence for  $A \in \mathcal{K}$ ,  $A = E_{xy}$  and  $A = \Delta$  follow from 2.6, 3.4, 3.5 and 3.6.

Assume  $(1/t) v_{[0,t]}(B) \rightarrow \tilde{\mu}(B)$  a.s. and  $A \subseteq B$ ,  $A, B \in \mathcal{B}(\Delta)$ . Then  $\tilde{\mu}(B) = \lim(1/t) v_{[0,t]}(B) \geq \liminf(1/t) v_{[0,t]}(A) + \liminf(1/t) v_{[0,t]}(B \setminus A) \geq \tilde{\mu}(A) + \tilde{\mu}(B \setminus A) = \tilde{\mu}(B)$  implies

$$\lim_{t \rightarrow \infty} \frac{1}{t} v_{[0,t]}(A) = \tilde{\mu}(A) \text{ a.s. on } \{\tilde{\mu}(B) < \infty\}.$$

This also implies a.s. convergence for  $A \in \mathcal{B}_c(\Delta)$  since  $A$  can be covered by finitely many  $K_i \in \mathcal{K}$  and  $\tilde{\mu}(K) < \infty$  a.s. for  $K \in \mathcal{K}$ .

Now assume  $L_1$ -convergence holds for  $B \in \mathcal{B}(\Delta)$  and let  $A \subseteq B$ ,  $A, B \in \mathcal{B}(\Delta)$ . Recall  $0 \leq r_{[0, t]} = v_{[0, t]} - \mu_{[0, t]}$ . Then

$$\begin{aligned} \mathbb{E} \left| \frac{1}{t} r_{[0, t]}(A) \right| &\leq \mathbb{E} \left| \frac{1}{t} r_{[0, t]}(B) \right| \leq \mathbb{E} \left| \frac{1}{t} v_{[0, t]}(B) - \tilde{\mu}(B) \right| \\ &+ \mathbb{E} \left| \frac{1}{t} \mu_{[0, t]}(B) - \tilde{\mu}(B) \right| \rightarrow 0 \quad \text{by 3.4.} \end{aligned}$$

Therefore

$$\mathbb{E} \left| \frac{1}{t} v_{[0, t]}(A) - \tilde{\mu}(A) \right| \leq \mathbb{E} \left| \frac{1}{t} r_{[0, t]}(A) \right| + \mathbb{E} \left| \frac{1}{t} \mu_{[0, t]}(A) - \tilde{\mu}(A) \right| \rightarrow 0.$$

$L_1$ -convergence for  $A \in \mathcal{B}_c(\Delta)$  follows from this as a.s. convergence above.  $\square$

Now we are able to finish the proof of Theorem 3.1.

**Proof of Theorem 3.1.**  $\tilde{\mu}(A) = \mathbb{E}(\sigma_{[0, 1]}(X, A) | \mathcal{J})$  a.s. follows from Birkhoff's ergodic theorem (in the generalized sense if  $\mathbb{E}\tilde{\mu}(A) = \infty$ ). By changing the version of  $\tilde{\mu}$  if necessary we can assume that  $\tilde{\mu}(K)$  is  $\mathcal{J}$ -measurable for all  $K \in \mathcal{K}_q$ . This implies that  $\tilde{\mu}(A)$  is  $\mathcal{J}$ -measurable for all open  $A$  and – by regularity – for all  $A \in \mathcal{B}(\Delta)$  and hence  $\tilde{\mu}$  is  $\mathcal{J}$ -measurable. If  $X$  is ergodic, then  $\mathcal{J}$  is trivial, so  $\tilde{\mu}$  is deterministic. All other statements have been proved in 3.3–3.7.  $\square$

**Remark.** If  $X(n)$ ,  $n \in \mathbb{N}_0$  is real-valued stationary, we can extend it to a stationary cadlag process  $Y$  on  $\mathbb{R}$  on a possibly enlarged probability space by first extending it to  $\mathbb{Z}$  and then defining a random variable  $\varphi$  which is uniformly distributed on  $]0, 1]$  and independent of  $X(n)$ ,  $n \in \mathbb{Z}$  and defining

$$Y(t) := X(n), \quad \text{for } t \in [n - 1 + \varphi(\omega), n + \varphi(\omega)[.$$

Then  $v_{[0, n]}(X, \cdot) = v_{[0, n]}(Y, \cdot)$  and  $\mu_{[0, n]}(X, \cdot) = \mu_{[0, n]}(Y, \cdot)$ . The integrability condition in 3.1 is automatically satisfied. Obviously  $\tilde{\mu}(A) \leq \frac{1}{2}$  for all  $\omega$ , so 3.7 implies that for every  $A \in \mathcal{B}(\Delta)$  we have  $(1/n)v_{[0, n]}(X, A) \rightarrow \tilde{\mu}(A)$  a.s.

Proposition 3.7 leaves open the question whether for every  $A \in \mathcal{B}(\Delta)$   $(1/t)v_{[0, t]}(X, A)$  converges to  $\tilde{\mu}(A)$  a.s. The following example shows that this need not be the case. It also provides a counterexample to other conjectures one might propose.

**Example 3.8.** We give an example of a stationary ergodic process  $X(t)$ ,  $t \in \mathbb{R}$  and  $A \in \mathcal{B}(\Delta)$  such that  $\lim(1/t)v_{[0, t]}(X, A) = \infty$  a.s. but  $\tilde{\mu}(A) = 0$ .

For  $m \in \mathbb{N}$ , let  $g_m: [0, 1[ \rightarrow \mathbb{Z}$  be the function which takes the values  $0, 1, -1, 3, -3, \dots, 2m+1, -2m, 2m, -2m+2, 2m-2, \dots, -2, 2, 0$  in that order on  $[0, 1[$  – each on a right-open left-closed interval of equal length  $1/(4m+3)$ ,  $m \in \mathbb{N}$ . Let



$Y_n, n \in \mathbb{Z}$  be i.i.d and  $\mathbb{P}(Y_0 = g_{2^m}) = 2^{-m}, m \in \mathbb{N}$ . Let  $\varphi$  be uniform on  $[0, 1[$  and independent of the  $Y_i, i \in \mathbb{N}_0$ . Define

$$X(t) = Y_{[t+\varphi]}(t + \varphi - [t + \varphi]), \quad t \in \mathbb{R}([ ]: \text{Gauss-brackets}) .$$

Clearly  $X$  is ergodic and  $\mathbb{E}v_{[0,1]}(X, K) < \infty$  for  $K \in \mathcal{K}$ . Define  $A = \{(x, y) \in \Delta: x, y \in \mathbb{Z} \text{ and } y - x \text{ is even}\}$ . By inspection  $\mu_{[0,n]}(X, A) = 0$  for all  $n \in \mathbb{N}$ , so  $\tilde{\mu}(A) = 0$ , but  $v_{[0,n]}(X, A) \geq 2 \cdot 2^{m^2}$  if  $m = \max\{k: Y_i = g_{2^k} \text{ for some } i \in \{1, \dots, n-1\}\}$ . An application of the Borel–Cantelli Lemma shows that for  $\alpha > 0$   $\mathbb{P}(\{v_{[0,n]}(X, A) < \alpha n \text{ i.o.}\}) = 0$ , so  $\lim n^{-1}v_{[0,n]}(X, A) = \infty$  a.s.

A slight modification of the example also shows that  $\mathbb{E}v_{[0,1]}(X, B) < \infty$  does *not* imply  $\mathbb{E}\tilde{\mu}(B) < \infty$ : define  $X$  as above but let it be zero for a unit between successive  $Y_i$ 's. If  $B = \{(x, y) \in \Delta: x, y \in \mathbb{Z} \text{ and } y - x \text{ is odd}\}$ , then  $\mathbb{E}v_{[0,1]}(X, B) \leq 1$  but  $\tilde{\mu}(B) = \infty$  (but  $\mu_{[0,1]} \equiv 0$ ). The last example shows further that  $\mathbb{E}v_{[0,1]}(X, B) < \infty$  does *not* imply  $\mathbb{E}v_{[0,n]}(X, B) < \infty$  for all  $n \in \mathbb{N}$ : take the same  $B$  and  $n = 3$ .

#### 4. Examples

The first three examples concern the explicit computation of the limit  $\tilde{\mu}$  in Theorem 3.1. In the ergodic case all we have to do is to compute  $\tilde{\mu}(E_{xy}) = \lim_{n \rightarrow \infty} (1/n) \mathbb{E}v_{[0,n]}(X, E_{xy})$  for all  $(x, y)$ . If  $\tilde{\mu}(E_{xy}) < \infty$  for all  $(x, y) \in \Delta$  then we know  $\tilde{\mu}$ . So we have to calculate the mean number of upcrossings for any pair  $(x, y) \in \Delta$ .

**Example 4.1.** Let  $X_1, X_2, \dots$  be i.i.d. real-valued with continuous distribution function. Without loss of generality we assume that  $X_1$  is uniformly distributed on  $[0, 1]$  (otherwise perform an appropriate transformation). Define  $\tau^y := \inf\{n \in \mathbb{N}: X_n \geq y\}$  and  $\tau_x := \inf\{n \in \mathbb{N}: X_n \leq x\}$ . Then for  $0 < x < y < 1$   $\mathbb{E}\tau^y = 1/(1 - y)$  and  $\mathbb{E}\tau_x = 1/x$ , so

$$\tilde{\mu}(E_{xy}) = \left( \frac{1}{x} + \frac{1}{1 - y} \right)^{-1}$$

and  $\tilde{\mu}$  has a density w.r.t. Lebesgue measure on  $\Delta$  given by

$$\frac{d\tilde{\mu}}{d\lambda}(x, y) = 2x(1 - y)(x + 1 - y)^{-3} .$$

Note that  $\tilde{\mu}(\Delta) = 1/3$ , which is not surprising since the probability that  $X_m(m \geq 2)$  is a local maximum of the sequence  $X_1, X_2, \dots$  is  $1/3$ .

**Example 4.2.** Let  $X(t), t \in \mathbb{R}$  be a one-dimensional diffusion process (on  $\mathbb{R}$ ) with (strictly increasing, continuous) scale function  $p(x), x \in \mathbb{R}$  ( $p(0) = 0$ ) and speed measure  $m(dx)$  on  $\mathbb{R}$  (see Mandl (1968) for definitions and properties). Assume  $m(\mathbb{R}) < \infty, p(\infty) = \infty$  and  $p(-\infty) = -\infty$ , so  $X(t), t \in \mathbb{R}$  can be chosen to be stationary. From explicit formulas for the expected first passage times (Mandl (1968, p. 91))

and the strong Markov property we get

$$\tilde{\mu}(E_{xy}) = \frac{1}{(p(y) - p(x))m(\mathbb{R})}.$$

Note that  $p$  can be identified from  $\tilde{\mu}$  up to a factor which is arbitrary anyway (sometimes the factor is fixed by requiring  $p'(0) = 0$  in case  $p$  is  $C^1$ ).

For the special case of the stationary Ornstein–Uhlenbeck process  $X$  solving

$$dX(t) = -\alpha X(t) dt + \sigma dW(t), \quad \alpha, \sigma > 0,$$

we get

$$\tilde{\mu}(E_{xy}) = \frac{\sigma\sqrt{\alpha}}{2\sqrt{\pi}} \left( \int_x^y \exp\left(-\frac{\alpha}{\sigma^2} z^2\right) dz \right)^{-1}.$$

Let us show that for any  $x_0 \in \mathbb{R}$ , the diffusion  $X$  starting in  $x_0 \in \mathbb{R}$  at time zero satisfies  $(1/t)\mu_{[0,t]}(X, \cdot) \rightarrow \tilde{\mu}(\cdot)$  vaguely almost surely and the same for  $v$ .  $\mathcal{L}(X(1))$  is equivalent to  $m$ , so the laws of  $(X_{1+t})_{t \geq 0}$  and of the stationary process are equivalent and we therefore have  $(1/t)\mu_{[1,t]}(X) \rightarrow \tilde{\mu}$  and  $(1/t)v_{[1,t]}(X) \rightarrow \tilde{\mu}$  vaguely almost surely and we have almost sure convergence on  $\mathcal{B}_c(\mathcal{A})$ . Now, for  $K \in \mathcal{K}$  – omitting the argument  $X$  – we have

$$\mu_{[1,t]}(K) \leq \mu_{[0,t]}(K) \leq v_{[0,t]}(K) \leq v_{[0,1]}(K) + 2 + v_{[1,t]}(K)$$

implying

$$\lim_{t \rightarrow \infty} \frac{1}{t} v_{[0,t]}(K) = \lim_{t \rightarrow \infty} \frac{1}{t} \mu_{[0,t]}(K) = \tilde{\mu}(K) \quad \text{a.s.}$$

So

$$\frac{1}{t} (\mu_{[0,t]} - \mu_{[1,t]}) \xrightarrow{v} 0 \quad \text{a.s.} \quad \text{and} \quad \frac{1}{t} (v_{[0,t]} - \mu_{[1,t]}) \xrightarrow{v} 0 \quad \text{a.s.}$$

and therefore  $\lim(1/t) \mu_{[0,t]} = \lim(1/t) v_{[0,t]} = \tilde{\mu}$  a.s. and a.s. convergence holds on  $\mathcal{B}_c(\mathcal{A})$ .

**Example 4.3.** Let  $E$  be a nonempty countable set and  $(Y_n)_{n \in \mathbb{Z}}$  an irreducible stationary Markov chain with values in  $E$ , transition probabilities  $(p_{ij})$  and invariant probability measure  $(\pi_i)$ . Let  $f: E \rightarrow \mathbb{R}$  satisfy  $|f^{-1}(K)| < \infty$  for every bounded  $K \subset \mathbb{R}$  and define  $X_n := f(Y_n)$ ,  $n \in \mathbb{Z}$ . We wish to calculate  $\tilde{\mu}$  associated with  $(X_n)$ . Fix  $(x, y) \in \mathcal{A}$ . To compute  $\tilde{\mu}(E_{xy})$  we may assume  $y \leq \sup_{e \in E} f(e)$  and  $x \geq \inf_{e \in E} f(e)$ . Then  $\tilde{\mu}(E_{xy})$  equals the probability that  $X_0 \leq x$  and there exists  $k > 0$  such that  $X_k \geq y$  and  $x < X_m < y$  for all  $0 < m < k$ . If we define  $A = \{e \in E: f(e) \leq x\}$ ,  $B = \{e \in E: x < f(e) < y\}$ ,  $C = \{e \in E: f(e) \geq y\}$  and  $e_j^{xy} = \mathbb{P}_j(Y_n \text{ exits from } B \text{ into } C)$ ,  $j \in B$ , then – since  $B$  is finite –  $e_j^{xy}$  can be computed by solving a finite linear system only involving  $p_{ij}$  for  $i \in B$ . Therefore

$$\tilde{\mu}(E_{xy}) = \sum_{i \in A} \pi_i \left( \sum_{j \in C} p_{ij} + \sum_{j \in B} p_{ij} e_j^{xy} \right).$$

**Example 4.4.** Let  $\bar{\mu}$  be the measure on  $\mathcal{A}$  with density  $(y-x)^{-3}$  (i.e.  $\bar{\mu}(E_{xy}) = \frac{1}{2}(y-x)^{-1}$ ),  $B$  standard Brownian motion and  $M$  a continuous local martingale satisfying  $M_0 = 0$  and  $[M]_\infty = \infty$  a.s. (for definitions and properties of continuous local martingales, local time etc. see Von Weizsäcker and Winkler, 1990; or Revuz and Yor, 1991). Further let  $\mathcal{C}_t^M(r)$ ,  $t \geq 0$  be local time at  $r \in \mathbb{R}$  of  $M$  and  $Z$  the modulus of an  $\mathcal{N}(0, 1)$ -variable. We show that

$$(i) \quad t^{-1/2} \mu_{[0,t]}(B) \xrightarrow{\mathcal{L}} Z \cdot \bar{\mu} \stackrel{\mathcal{L}}{=} \mathcal{C}_1^B(0) \cdot \bar{\mu},$$

$$(ii) \quad [M]_t^{-1/2} \mu_{[0,t]}(M) \xrightarrow{\mathcal{L}} Z \cdot \bar{\mu},$$

$$(iii) \quad (\mathcal{C}_t^M(0))^{-1} \mu_{[0,t]}(M) \xrightarrow{\text{a.s.}} \bar{\mu},$$

and the same holds for  $\mu$  replaced by  $\nu$ .

We prove (i) not directly but by time-changing  $B$  such that 4.2 can be applied. Let  $\sigma \in C^\infty(\mathbb{R}, \mathbb{R})$  satisfy  $\sigma \geq 1$  and  $\int_{-\infty}^{\infty} \sigma^{-2}(x) dx = 1$ . Define  $T_t$ ,  $t \geq 0$  by  $t = \int_0^{T_t} \sigma^{-2}(B_s) ds$  and  $Y(t) := B(T_t)$ . Then  $Y$  is a diffusion with (finite) speed measure  $m(dx) = 2\sigma^{-2}(x) dx$  and scale function  $p(x) = x$ . It is easy to see via scaling properties of Brownian motion (and well-known) that

$$r^{-1/2} \int_0^r \sigma^{-2}(B_s) ds = r^{-1/2} \int_{-\infty}^{\infty} \sigma^{-2}(u) \mathcal{C}_r^B(u) du \xrightarrow[r \rightarrow \infty]{\mathcal{L}} Z \stackrel{\mathcal{L}}{=} \mathcal{C}_1^B(0).$$

Together with 4.2 this implies

$$T_t^{-1/2} \mu_{[0, T_t]}(B) = (T_t^{-1/2} \cdot t)(t^{-1} \mu_{[0,t]}(Y)) \xrightarrow[t \rightarrow \infty]{\mathcal{L}} Z \cdot \mu \stackrel{\mathcal{L}}{=} \mathcal{C}_1^B(0) \cdot \bar{\mu}. \quad (4.1)$$

Hence (i) follows (observe that  $T_t \uparrow \infty$  as  $t \uparrow \infty$  a.s.).

To show (ii) define  $S_r := \inf \{t \geq 0: [M]_t > r\}$ . Then  $W_r := M_{S_r}$  is a Brownian motion. Therefore  $W_{[M]_t} = M_t$  which – together with (i) – implies (ii).

To show (iii) for  $M = B$  first observe that the ergodic theorem for additive functionals (Revuz and Yor (1991, p. 397)) implies that  $\mathcal{C}_t^B(0)^{-1} \cdot \int_0^t \sigma^{-2}(B_s) ds$  converges to 1 almost surely as  $t \rightarrow \infty$ . So a computation analogous to (4.1) shows

$$\lim_{t \rightarrow \infty} (\mathcal{C}_{T_t}^B(0))^{-1} \mu_{[0, T_t]}(B) = \lim_{t \rightarrow \infty} t^{-1} \mu_{[0,t]}(B) = \bar{\mu},$$

where the limit is almost sure. The result for general  $M$  follows by time-change.

**Example 4.5.** The following example has nothing to do with Chapter 3 and only serves to show that upcrossing measures may occasionally be useful to obtain explicit formulas. We omit detailed proofs.

Let  $f: [0, t] \rightarrow ]0, \infty[$  be a regular function bounded away from zero which represents the amount of DM one gets for 1 US\$ at  $t$ . Suppose an investor has 1 DM at  $s = 0$  and his aim is to own as many DM as possible at  $t$  solely by trading between DM and \$ (without loans!). We assume that  $f$  is known to the investor beforehand and

that he has to pay  $c$  units transaction costs per unit traded ( $0 \leq c$ ). It is not hard to see that his final wealth  $w_c$  when trading optimally is

$$\begin{aligned} w_c &= \exp \left( \int_A \log^+ \left( \frac{y}{x} (1+c)^{-2} \right) dv_{[0,t]}(f, (x, y)) \right) \\ &= \exp \left( \int_A (y - x - K)^+ dv_{[0,t]}(\log f, (x, y)) \right), \end{aligned}$$

where  $K = 2 \log(1+c)$ . Note that  $w_c < \infty$  for  $c > 0$ ,  $w_c$  is nonincreasing and  $w_\infty = 1$ .  $w_0 < \infty$  iff the total variation of  $f$  (or  $\log f$ ) is finite. It is clear that the optimal trading strategy is to sell all \$ at  $s$  whenever  $\rho_{[0,t]}(\log f, s, E_{xy}) = 1$ , where  $y = f(s)$  and  $x = y - K$ . Note that we tacitly assumed that trading is permitted at any  $s \in \mathcal{J}([0, t])$ .

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