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Functionals of infinitely divisible stochastic processes with exponential tails

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Abstract

We investigate the tail behavior of the distributions of subadditive functionals of the sample paths of infinitely divisible stochastic processes when the Lévy measure of the process has suitably defined exponentially decreasing tails. It is shown that the probability tails of such functionals are of the same order of magnitude as the tails of the same functionals with respect to the Lévy measure, and it turns out that the results of this kind cannot, in general, be improved. In certain situations we can further obtain both lower and upper bounds on the asymptotic ratio of the two tails. In the second part of the paper we consider the particular case of Lévy processes with exponentially decaying Lévy measures. Here we show that the tail of the maximum of the process is, up to a multiplicative constant, asymptotic to the tail of the Lévy measure. Most of the previously published work in the area considered heavier than exponential probability tails.

Keywords: Exponential distributions; Infinitely divisible processes; Tail behavior of the distributions of functionals of sample paths

1. Introduction and preliminaries

Let $X = \{X(t), t \in T\}$ be an infinitely divisible stochastic process, in the sense that all its finite-dimensional distributions are infinitely divisible. Following the lead of recent authors, we are interested in the tail behavior of the distributions of various functionals of the sample paths of X. Unlike much of the previous work in the area which dealt with suitably heavy tails (the major exception being the body of work on Gaussian processes), the functionals of the processes considered in the present paper

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will typically have probability tails that decrease exponentially fast. We now proceed with formal definitions.

We work with a general infinitely divisible process whose characteristic function is given in the form

$$E\exp\{\mathrm{i}\langle\boldsymbol{\beta},\boldsymbol{X}\rangle\} = \exp\left\{\mathrm{i}\langle\boldsymbol{\beta},\boldsymbol{b}\rangle - \boldsymbol{\beta}'\Sigma\boldsymbol{\beta}/2 + \int_{\mathbb{R}^T}[\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\beta},\boldsymbol{x}\rangle} - 1 - \mathrm{i}\langle\boldsymbol{\beta},\tau(\boldsymbol{x})\rangle]\nu(\mathrm{d}\boldsymbol{x})\right\}, \tag{1.1}$$

where $\boldsymbol{b}\in\mathbb{R}^{\mathrm{T}}$, Σ is the covariance matrix of the Gaussian part of $\boldsymbol{X}$ and ν is the Lévy measure of the Poisson part of $\boldsymbol{X}$. Here $\boldsymbol{\beta}\in\mathbb{R}^{(T)}$, the space of real functions $\boldsymbol{\beta}$ defined on T such that $\beta(t)=0$ for all but finitely many t's, $\langle\boldsymbol{\beta},\boldsymbol{x}\rangle = \sum_{t\in T}\beta(t)x(t)$, and $\tau(\boldsymbol{x})(t) = x(t)/(x^2(t)+1)$.

Let ϕ be a measurable subadditive function $\mathbb{R}^T\to(-\infty,\infty]$; i.e.

$$\phi(\boldsymbol{x}_1+\boldsymbol{x}_2)\leqslant\phi(\boldsymbol{x}_1)+\phi(\boldsymbol{x}_2)\quad\text{for all } \boldsymbol{x}_1,\boldsymbol{x}_2\in\mathbb{R}^T. \tag{1.2}$$

Such functions include suprema of the sample paths, oscillations, L^p-norms and many others (with measurability questions treated in a standard way). This framework has been considered by Rosinski and Samorodnitsky (1993), who have shown that

$$P(\phi(\boldsymbol{X})>\lambda)\sim H(\lambda)\quad\text{as } \lambda\to\infty, \tag{1.3}$$

where

$$H(\lambda) = \nu(\{x\in\mathbb{R}^T\colon \phi(\boldsymbol{x}))>\lambda\}) \tag{1.4}$$

as long as H is asymptotically equivalent to a tail of a *subexponential* probability distribution. We remind the reader that a distribution F on $[0,\infty)$ is subexponential if

$$l := \lim_{\lambda\to\infty}\frac{\overline{F*F}(\lambda)}{\bar{F}(\lambda)}\ \text{exists and is finite} \tag{1.5}$$

and $F\in\mathscr{L}(0)$, where

$$\mathscr{L}(\alpha) = \left\{F\colon \lim_{u\to\infty}\frac{\bar{F}(u+v)}{\bar{F}(u)} = \mathrm{e}^{-\alpha v},\ \text{any } v>0\right\}, \tag{1.6}$$

$\alpha\geqslant 0$. Here $\bar{F}(x) = 1-F(x)$. The name *subexponential* is due to the assumption $F\in\mathscr{L}(0)$, and in this paper we are interested in exponentially decreasing tails. We will therefore consider the case when H in (1.4) is asymptotically equivalent to a tail of a probability distribution in the *exponential* class $\mathscr{S}(\alpha)$, $\alpha>0$, defined as the class of distributions in $\mathscr{L}(\alpha)$, $\alpha>0$, satisfying (1.5). (It is a curiosity that the distribution F on $[0,\infty)$ with the density $f(x)=\alpha\mathrm{e}^{-\alpha x}$, $x>0$ *does not* belong to the exponential class $\mathscr{S}(\alpha)$.) These distributions were introduced by Chistyakov (1964) and Chover et al. (1973) and were studied by a number of authors. We refer the reader to Teugels (1975), Embrechts and Goldie (1982) and Cline (1986, 1987) for a detailed analysis of both subexponential and exponential classes of distributions. The question discussed in this paper is what version of (1.3) holds under the assumption of exponential tails of H.

Intuitively, recalling the effect of convolutions on subexponential and exponential tails (see Embrechts et al. (1979, Theorem 3), Embrechts and Goldie (1982, Theorem 4.2), one might guess that under the assumption of exponentiality the appropriate version of (1.3) is

$$P(\phi(X) > \lambda) \sim cH(\lambda) \quad \text{as } \lambda \to \infty \tag{1.7}$$

for some $c > 0$. What we discover in this paper is that, while (1.7) is false in general, it is "almost true". More precisely, the true statement is

$$0 < \liminf_{\lambda \to \infty} \frac{P(\phi(X) > \lambda)}{H(\lambda)} \leqslant \limsup_{\lambda \to \infty} \frac{P(\phi(X) > \lambda)}{H(\lambda)} < \infty. \tag{1.8}$$

This result is proved in the next section, and it requires a somewhat more involved argument than the corresponding subexponential result of Rosinski and Samorodnitsky (1993).

Before proceeding, let us collect some facts about distributions in exponential classes. First of all, in the remainder of this paper $\mathscr{S}(\alpha)$ refers to the collection of distributions on *the whole of* $\mathbb{R}$ which are in $\mathscr{L}(\alpha)$ and for which (1.5) holds. Although most of literature on exponential and subexponential classes treats only distributions concentrated on $[0, \infty)$, the extensions to the more general case are, as noted by Willekens (1986) (see also Bertoin and Doney, 1993), entirely straightforward. In particular, the law of X is in $\mathscr{S}(\alpha)$ if and only if the law of X_+ is.

Lemma 1.1 *Let $F \in \mathscr{S}(\alpha)$, $\alpha \geqslant 0$. Then*

(i) $m_F(\alpha) = \int_{-\infty}^{\infty} e^{\alpha x} F(dx) < \infty$ *and* $l = 2m_F(\alpha)$ *in* (1.5).

(ii) *If the limit* $c_i = \lim_{\lambda \to \infty} (\bar{G}_i(\lambda)/\bar{F}(\lambda))$ *exists and is finite for two distribution functions* G_1, G_2 *then*

$$\lim_{\lambda \to \infty} \frac{\overline{G_1 * G_2(\lambda)}}{\bar{F}(\lambda)} = c_1 m_{G_2}(\alpha) + c_2 m_{G_1}(\alpha).$$

Moreover, $G_i \in \mathscr{S}(\alpha)$ *if* $c_i > 0$.

(iii) *For every* $n \geqslant 1$, $\lim_{\lambda \to \infty} (\overline{F^{*n}}(\lambda)/\bar{F}(\lambda)) = n m_F(\alpha)^{n-1}$. *Furthermore, there is a* $K < \infty$ *such that for every* $n \geqslant 1$ *and* $\lambda > 0$

$$\overline{F^{*n}}(\lambda)/\bar{F}(\lambda) \leqslant K(1 + m_F(\alpha))^{n-1}.$$

(iv) *For a* $\mu > 0$ *let* $G(x) = e^{-\mu} \sum_{n=0}^{\infty} (\mu^n/n!) F^{*n}(x)$. *Then* $\lim_{\lambda \to \infty} (\bar{G}(\lambda)/\bar{F}(\lambda)) = \mu m_G(\alpha)$.

(v) *Let* $G \in \mathscr{L}(\alpha)$, *and* $\sup_{\lambda > 0} \bar{G}(\lambda)/\bar{F}(\lambda) < \infty$. *Then* $H = F * G$ *is in* $\mathscr{S}(\alpha)$ *and* $\bar{H}(\lambda) \sim m_G(\alpha)\bar{F}(\lambda) + m_F(\alpha)\bar{G}(\lambda)$ *as* $\lambda \to \infty$.

Proof. (i) This is an immediate extension of the corresponding result for distributions on $[0, \infty)$ due to Chover et al. (1973); see also Cline (1978, Theorem 2.9).

(ii) Again, this follows from the known result for the distributions on $[0, \infty)$ due to Embrechts and Goldie (1982); it is spelled out in Cline (1987, Corollary 2.10).

(iii) The first part is an immediate consequence of (ii). The second part is Lemma 2.6 of Embrechts and Goldie (1982). See also the proof of Lemma 1 of Chover et al. (1973).

(iv) This is Theorem 4.2(ii) of Embrechts and Goldie (1982).

(v) See Corollary 1 of Cline (1986). □

The main theorem establishing (1.8) is proved in the next section. We show further by example that (1.7) is false in general when $\alpha > 0$, and that, when (1.7) does hold, the asymptotic constant c is not determined by the function H. Finally, we provide bounds on the upper and lower limits in (1.8) under certain further assumptions on the process $\boldsymbol{X}$ and the functional ϕ.

In Section 3 we consider the important particular case of the maxima of Lévy processes with Lévy measures with exponential right tails. We prove that, in this case, the limiting relation (1.7) does hold, and we further provide bounds for the asymptotic constant c.

2. Tails of subadditive functionals

Our framework is similar to that of Rosinski and Samorodnitsky (1993). Specifically, to avoid measurability problems we will work in this section with processes defined on a countable set T. We assume that there is a lower-semicontinuous pseudonorm $q: \mathbb{R}^T \to [0, \infty]$ such that

$$|\phi(\boldsymbol{x})| \leqslant q(\boldsymbol{x}) \quad \text{for every } x \in \mathbb{R}^T. \tag{2.1}$$

(That is, $q(\boldsymbol{x}+\boldsymbol{y}) \leqslant q(\boldsymbol{x}) + q(\boldsymbol{y})$ for all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^T$, $q(\boldsymbol{0}) = 0$ and $q(\rho\boldsymbol{x}) \leqslant q(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^T$ and $|\rho| \leqslant 1$.) The following is our general theorem.

Theorem 2.1. *Let $\boldsymbol{X}$ be given by* (1.1), *ϕ and q be, correspondingly, a measurable subadditive function and a lower-semicontinuous seminorm related by* (2.1). *Assume that $P(q(\boldsymbol{X}) < \infty) = 1$ and that the distribution function $F(x) = 1 - \min(1, H(x))$ is in $\mathscr{S}(\alpha)$. Then* (1.8) *holds.*

We start with a lemma, which strengthens Lemma 2.2 of Rosinski and Samorodnitsky (1993).

Lemma 2.1. *Let $\boldsymbol{X}$ be an infinitely divisible process with characteristic function given by* (1.1). *Assume that $P(q(\boldsymbol{X}) < \infty) = 1$ and that $\nu(\{\boldsymbol{x} \in \mathbb{R}^T: q(\boldsymbol{x}) > r\}) = 0$ for some $r > \boldsymbol{0}$. Then $E\exp(\varepsilon q(\boldsymbol{X})) < \infty$ for every $\varepsilon > 0$.*

Remark. Lemma 2.2 of Rosinski and Samorodnitsky (1993) proved that $E\exp(\varepsilon q(\boldsymbol{X})) < \infty$ for some $\varepsilon > 0$. Note further that the result does not follow from the standard facts about the Banach space valued infinitely divisible random vectors (see e.g. de Acosta, 1980) because our q is not, in general, either homogeneous or continuous.

Proof. We begin as in Rosinski and Samorodnitsky (1993) by choosing an $\boldsymbol{a}: T \to (0, \infty)$ such that $\sum_{t \in T} |a(t)X(t)|^2 < \infty$ *a.s.* Let $\boldsymbol{X}'$ be an independent copy of $\boldsymbol{X}$, and consider an $l^2(T)$-valued process with stationary independent and symmetric increments $\{\boldsymbol{Z}(u),\ u \geqslant 0\}$ such that $\boldsymbol{Z}(1) \stackrel{\mathrm{d}}{=} \boldsymbol{a}(\boldsymbol{X} - \boldsymbol{X}')$. (We are talking about coordinate-wise products, of course.) Finally, let $p(\boldsymbol{h}) = q(\boldsymbol{a}^{-1}\boldsymbol{h})$, $\boldsymbol{h} \in l^2(T)$. Note that p is a lower-semicontinuous pseudonorm on $l^2(T)$, and that $P(p(\boldsymbol{Z}(1)) < \infty)) = 1$. If μ is the Lévy measure of $\boldsymbol{Z}(1)$, then it follows that $\mu(\{\boldsymbol{h} \in l^2(T){:}\ p(\boldsymbol{h}) > r\}) = 0$.

Since $E\exp(\varepsilon q(\boldsymbol{X} - \boldsymbol{X}')) = E\exp(\varepsilon p(\boldsymbol{Z}(1))$, a standard application of Fubini's theorem and subadditivity of q shows that it is enough to prove that for any $\varepsilon > 0$,

$$E\exp(\varepsilon p(\boldsymbol{Z}(1))) < \infty\,. \tag{2.2}$$

Observe that a standard argument (like the one used in Theorem 1.3.2 of Fernique, 1975) shows that (2.2) holds if $\boldsymbol{Z}(1)$ is Gaussian. Therefore, we use again subadditivity of p to note that it is enough to prove our statement in the case when $\boldsymbol{X}$, and thus $\boldsymbol{Z}$, have no Gaussian component.

For a $\delta > 0$ let μ^δ denote the restriction of μ to the set $\{\boldsymbol{h} \in l^2(T){:}\ \|\boldsymbol{h}\|_{l^2(T)} \leqslant \delta\}$, and let $\{\boldsymbol{Z}^\delta(u),\ u \geqslant 0\}$ be an $l^2(T)$-valued process with stationary independent and symmetric increments such that μ^δ is the Lévy measure of $\boldsymbol{Z}^\delta(1)$.

Choose a sequence $\delta_n \downarrow 0$. We put $\{\boldsymbol{Z}^{\delta_n}(1)\}_{n \geqslant 1}$ on the same probability space as follows. Let $\{\boldsymbol{U}_n\}_{n \geqslant 1}$ be a sequence of independent infinitely divisible random vectors in $l^2(T)$ such that the Lévy measure of $\boldsymbol{U}_n$ is the restriction of μ to the set $\{\boldsymbol{h} \in l^2(T){:}\ \delta_{n+1} < \|\boldsymbol{h}\|_{l^2(T)} \leqslant \delta_n\}$, $n \geqslant 1$. Then we set $\boldsymbol{Z}^{\delta_n}(1) = \sum_{i=n}^{\infty} \boldsymbol{U}_i$, $n \geqslant 1$. Now an immediate application of Kolmogorov's 0–1 law shows that there is a $\kappa \in [0, \infty]$ such that

$$\limsup_{n \to \infty} p(\boldsymbol{Z}^{\delta_n}(1)) = \kappa \quad \text{a.s.} \tag{2.3}$$

We claim that $\kappa < \infty$. Indeed, choose an $R > 0$ so large that

$$P(p(\boldsymbol{Z}(1)) > R/2) < 1/4.$$

Then by Lévy's inequality, for every $n, m \geqslant 1$,

$$P\left(\max_{n \leqslant k \leqslant n+m} p(\boldsymbol{Z}^{\delta_k}(1)) > R\right) \leqslant 2P(p(\boldsymbol{Z}(1)) > R/2),$$

and so

$$P\left(\sup_{k \geqslant n} p(\boldsymbol{Z}^{\delta_k}(1)) > R\right) \leqslant 2P(p(\boldsymbol{Z}(1)) > R/2) \leqslant 1/2$$

for every $n \geqslant 1$, showing that $\kappa \leqslant R$. An immediate conclusion from (2.3) is that for every $\gamma > \kappa$,

$$\lim_{n \to \infty} P(p(\boldsymbol{Z}^{\delta_n}(1)) > \gamma) = 0. \tag{2.4}$$

Now fix an $\varepsilon > 0$. Choose a $\gamma > \kappa$. By (2.4) we can make the probability in its left-hand side as small as we wish, and in particular there is an $n \geqslant 1$ such that

$$P(p(\boldsymbol{Z}^{\delta_n}(1)) > \gamma) < \tfrac{1}{4}\mathrm{e}^{-(8\gamma + 2r)}. \tag{2.5}$$

Obviously, $\boldsymbol{Z}(1) \stackrel{\mathrm{d}}{=} \boldsymbol{Z}^{\delta_n}(1) + \boldsymbol{W}$, where $\boldsymbol{Z}^{\delta_n}(1)$ and $\boldsymbol{W}$ are independent, and the latter random vector is infinitely divisible with Lévy measure μ_* equal to the restriction of μ to the set $\{\boldsymbol{h} \in l^2(T): \|\boldsymbol{h}\|_{l^2(T)} > \delta_n\}$. Note that μ_* is a finite measure. Let m_* be the total mass of μ_*. By the subadditivity of p,

$$E\exp(\varepsilon p(\boldsymbol{Z}(1))) \leqslant E\exp(\varepsilon p(\boldsymbol{Z}^{\delta_n}(1)))\, E\exp(\varepsilon p(\boldsymbol{W})).$$

Clearly,

$$\begin{aligned} E\exp(\varepsilon p(\boldsymbol{W})) &= E\exp\left(\varepsilon p\left(\sum_{j=1}^{N} \boldsymbol{Y}_j\right)\right) \leqslant E\exp\left(\varepsilon \sum_{j=1}^{N} p(\boldsymbol{Y}_j)\right) \\ &\leqslant E\exp(\varepsilon r N) < \infty, \end{aligned}$$

where N is a Poisson random variable with mean m_*, and $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots$ is an independent of N sequence of i.i.d. $l^2(T)$-valued random vectors with common distribution $(m_*)^{-1}\mu_*$. We have used the fact that μ does not charge vectors $\boldsymbol{h} \in l^2(T)$ with $p(\boldsymbol{h}) > r$. Our statement will therefore follow once we prove that

$$E\exp(\varepsilon p(\boldsymbol{Z}^{\delta_n}(1))) < \infty. \tag{2.6}$$

We now repeat the argument of the proof of Lemma 2.2 of Rosinski and Samorodnitsky (1993) (as applied to the process $\{\boldsymbol{Z}^{\delta_n}(u),\, u \geqslant 0\}$) to conclude that

$$E\exp(\varepsilon p(Z^{\delta_n}(1))) \leqslant \liminf_{\delta \to 0} M_\delta, \tag{2.7}$$

where for each $\delta > 0$, M_δ satisfies

$$M_\delta \leqslant 2M_\delta \exp(\varepsilon(8\gamma + 2r))P(p(\boldsymbol{Z}^{\delta_n}(1)) > \gamma) + \exp(8\varepsilon\gamma). \tag{2.8}$$

(This is just (2.9) and (2.10) of Rosinski and Samorodnitsky, 1993.) By the choice of n, we conclude from (2.5) and (2.8) that $M_\delta \leqslant 2\exp(8\varepsilon\gamma)$ for every $\delta > 0$, and so by (2.7)

$$E\exp(\varepsilon p(\boldsymbol{Z}^{\delta_n}(1))) \leqslant 2\exp(8\varepsilon\gamma) < \infty.$$

This completes the proof of the lemma. $\square$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Lemma 2.1 of Rosinski and Samorodnitsky (1993), there is an $r > 0$ such that $\nu(\{\boldsymbol{x} \in \mathbb{R}^T: q(\boldsymbol{x}) > r\}) < \infty$. Write

$$\boldsymbol{X} = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3, \tag{2.9}$$

where $\boldsymbol{X}_i$, $i = 1, 2, 3$ are independent infinitely divisible stochastic processes on T such that

$$E\exp\{i\langle \boldsymbol{\beta}, \boldsymbol{X}_j\rangle\} = \exp\left\{\int_{\mathbb{R}^T} [\mathrm{e}^{\mathrm{i}\langle \boldsymbol{\beta}, \boldsymbol{x}\rangle} - 1]\,\nu_j(\mathrm{d}\boldsymbol{x})\right\},$$

$j = 1, 2$, and

$$\begin{aligned} E\exp\{\mathrm{i}\langle \boldsymbol{\beta}, \boldsymbol{X}_3\rangle\} = \exp\Big\{\mathrm{i}\langle \boldsymbol{\beta}, \boldsymbol{b}_1\rangle - \boldsymbol{\beta}'\Sigma\boldsymbol{\beta}/2 + \int_{\mathbb{R}^T} [\mathrm{e}^{\mathrm{i}\langle \boldsymbol{\beta}, \boldsymbol{x}\rangle} - 1 \\ - \mathrm{i}\langle \boldsymbol{\beta}, \boldsymbol{\tau}(\boldsymbol{x})\rangle]\,\nu_3(\mathrm{d}\boldsymbol{x})\Big\}, \end{aligned}$$

where

$$\begin{aligned} \nu_1(A) &= \nu(A \cap \{x \in \mathbb{R}^T: \phi(x) > r\}), \\ \nu_2(A) &= \nu(A \cap \{x \in \mathbb{R}^T: \phi(x) \leqslant r, q(x) > r\}), \\ \nu_3(A) &= \nu(A \cap \{x \in \mathbb{R}^T: q(x) \leqslant r\}), \end{aligned} \tag{2.10}$$

and $b_1 \in \mathbb{R}^T$. Since ν_2 is a finite measure, the simple argument used in the proof of Lemma 2.1 based on the tail behavior of a Poisson random variable shows that for every $\varepsilon > 0$ (in particular, for an $\varepsilon > \alpha$)

$$\lim_{\lambda \to \infty} e^{\varepsilon\lambda} P(\phi(X_2) > \lambda) = 0, \tag{2.11}$$

and, further, by Lemma 2.1 we also have

$$\lim_{\lambda \to \infty} e^{\varepsilon\lambda} P(\phi(X_3) > \lambda) = 0 \tag{2.12}$$

for all $\varepsilon > 0$. We now consider X_1. Note that ν_1 is also a finite measure; let m be its total mass, and let $\{Y_n\}_{n \geqslant 1}$ be a sequence of i.i.d. stochastic processes on T with common law $m^{-1}\nu_1$. Clearly,

$$P(\phi(Y_1) > \lambda) = m^{-1} H(\lambda) \tag{2.13}$$

whenever $\lambda > r$, and so the distribution of $\phi(Y_1)$ is in $\mathscr{S}(\alpha)$, see Lemma 1.1(ii). Let N be a mean m Poisson random variable independent of the sequence $\{Y_n\}_{n \geqslant 1}$. Then by the subadditivity of ϕ,

$$\begin{aligned} P(\phi(X_1) > \lambda) &= P\left(\phi\left(\sum_{j=1}^{N} Y_j\right) > \lambda\right) \\ &\leqslant P\left(\sum_{j=1}^{N} \phi(Y_j) > \lambda\right) \sim mP(\phi(Y_1) > \lambda) E\exp\left(\alpha \sum_{j=1}^{N} \phi(Y_j)\right) \\ &= H(\lambda)\exp(m(E\exp(\alpha\phi(Y_1)) - 1)) \end{aligned} \tag{2.14}$$

as $\lambda \to \infty$ by (2.13) and Lemma 1.1(iv). Now the finiteness of the upper limit in (1.8) follows from the subadditivity upper bound

$$\phi(X) \leqslant \phi(X_1) + \phi(X_2) + \phi(X_3),$$

(2.11), (2.12), (2.14) and Lemma 1.1(ii).

The positivity of the lower limit in (1.8) is even simpler. First, by the subadditivity of ϕ and the first part of (2.14),

$$\begin{aligned} P(\phi(X_1) > \lambda) &\geqslant P\left(\phi(Y_1) - \sum_{j=2}^{N} \phi(-Y_j) > \lambda, N \geqslant 1\right) \\ &= (1 - e^{-m}) \int_{\mathbb{R}} P(\phi(Y_1) > \lambda + u) K(du), \end{aligned}$$

where K is the conditional distribution function of $\sum_{j=2}^{N} \phi(-Y_j)$ given $N \geqslant 1$. Using (2.13) and Fatou's lemma we conclude that

$$\liminf_{\lambda \to \infty} \frac{P(\phi(X_1) > \lambda)}{H(\lambda)} \geqslant m^{-1}(1 - e^{-m}) \int_{\mathbb{R}} e^{-\alpha u} K(du). \tag{2.15}$$

Now we use the subadditivity lower bound

$$\phi(X) \geqslant \phi(X_1) - \phi(-X_2) - \phi(-X_3)$$

to obtain

$$P(\phi(X) > \lambda) \geqslant \int_{\mathbb{R}} P(\phi(X_1) > \lambda + v) M(\mathrm{d}v),$$

where M is the law of $\phi(-X_2) + \phi(-X_3)$ (it is easily seen to be non-defective,) and so the positivity of the lower limit in (1.8) follows from (2.15) and Fatou's lemma. This completes the proof of the theorem. $\square$

Comparing the result of Theorem 2.1 with the corresponding statement in the subexponential case (Theorem 2.1 of Rosinski and Samorodnitsky, 1993) one cannot fail to observe that the latter has a much more definite conclusion than the former. The following example shows that this is the nature of the distinction between the exponential and subexponential cases; i.e. the limiting relation (1.7) does not hold in general.

Example 2.1. We start with the fact that any distribution F on $\mathbb{R}$ with $F(x) = 1 - x^{-p}L(x)\mathrm{e}^{-\alpha x}$ for $x > 0$, where $p > 1$ and L varies slowly at infinity is in $\mathscr{S}(\alpha)$ (see e.g. Cline, 1986, Theorem 4). It follows from this that, by choosing appropriately the slowly varying functions, one can construct two distributions, F_1 and F_2, both in $\mathscr{S}(\alpha)$, such that F_1 is a symmetric distribution, F_2 is concentrated on $[0, \infty)$, and

$$0 < \inf_{\lambda > 0} \bar{F}_1(\lambda)/\bar{F}_2(\lambda) \leqslant \sup_{\lambda > 0} \bar{F}_1(\lambda)/\bar{F}_2(\lambda) < \infty, \tag{2.16}$$

but

$$\lim_{\lambda \to \infty} \bar{F}_1(\lambda)/\bar{F}_2(\lambda) \text{ does not exist.} \tag{2.17}$$

Let $T = \{1, 2\}$, and let $X = \sum_{n=1}^{N} Y_j$, where N is a mean 1 Poisson random variable, independent of the sequence of i.i.d. random vectors in $\mathbb{R}^2$, $\{Y_j\}_{j \geqslant 1}$ such that

$$Y_1 \stackrel{\mathrm{d}}{=} (U, W), \text{ with } U \text{ and } W \text{ independent and } U \sim F_1,\ W \sim F_2.$$

Then, of course, X is an infinitely divisible stochastic process on T with $\nu = F_1 \times F_2$. Finally, let $\phi(x) = \phi(x_1, x_2) = |x_1| + |x_2|$.

Observe that by Lemma 1.1(v)

$$\begin{aligned} H(\lambda) &= P(\phi(Y_1) > \lambda) = P(|U| + W > \lambda) \\ &\sim 2m_{F_2}(\alpha)\bar{F}_1(\lambda) + 2m_{F_1}^{+}(\alpha)\bar{F}_2(\lambda), \end{aligned} \tag{2.18}$$

where

$$m_{F_1}^{+}(\alpha) = \int_0^\infty \mathrm{e}^{\alpha x} F_1(\mathrm{d}x).$$

In particular, it follows from (2.16) and (2.17) that

$$0 < \liminf_{\lambda \to \infty} \bar{F}_2(\lambda)/H(\lambda) := l < L := \limsup_{\lambda \to \infty} \bar{F}_2(\lambda)/H(\lambda) < \infty. \tag{2.19}$$

We claim that for every $k \geqslant 2$ there are *positive* numbers a_k and b_k such that

$$P\left(\phi\left(\sum_{n=1}^{k} Y_n\right) > \lambda\right) \sim a_k H(\lambda) + b_k \bar{F}_2(\lambda) \tag{2.20}$$

as $\lambda \to \infty$. To this end observe that

$$\phi\left(\sum_{n=1}^{k} Y_n\right) = \left|\sum_{n=1}^{k} U_n\right| + \sum_{n=1}^{k} W_n,$$

where $\{U_n\}_{n \geqslant 1}$ are i.i.d. with common law F_1, $\{W_n\}_{n \geqslant 1}$ are i.i.d. with common law F_2, and the two sequences are independent. Let $T_k = |\sum_{n=1}^{k} U_n|$, and observe that by Lemma 1.1(iii) we immediately have

$$P(T_k > \lambda) \sim 2k m_{F_1}(\alpha)^{k-1} \bar{F}_1(\lambda)$$

and

$$P\left(\sum_{n=1}^{k} W_n > \lambda\right) \sim k m_{F_2}(\alpha)^{k-1} \bar{F}_2(\lambda)$$

as $\lambda \to \infty$. Therefore, by (2.16) and Lemma 1.1(v) we conclude that, as $\lambda \to \infty$,

$$\begin{aligned} P\left(\phi\left(\sum_{n=1}^{k} Y_n\right) > \lambda\right) &\sim 2k m_{F_1}(\alpha)^{k-1} m_{F_2}(\alpha)^k \bar{F}_1(\lambda) + k m_{F_2}(\alpha)^{k-1} m_{T_k}(\alpha) \bar{F}_2(\lambda) \\ &\sim k(m_{F_1}(\alpha) m_{F_2}(\alpha))^{k-1} \left[H(\lambda) + \left(\frac{m_{T_k}(\alpha)}{m_{F_1}(\alpha)^{k-1}} - 2m_{F_1}^{+}(\alpha)\right) \bar{F}_2(\lambda)\right]. \end{aligned} \tag{2.21}$$

Now (2.20) follows from (2.21) and the simple fact that for every $k \geqslant 2$

$$m_{T_k}(\alpha) > 2m_{F_1}^{+}(\alpha) m_{F_1}(\alpha)^{k-1}.$$

We claim that

$$\liminf_{\lambda \to \infty} P(\phi(\boldsymbol{X}) > \lambda)/H(\lambda) = \mathrm{e}^{-1} \sum_{k=0}^{\infty} \frac{1}{k!}(a_k + l b_k) \tag{2.22}$$

and

$$\limsup_{\lambda \to \infty} P(\phi(\boldsymbol{X}) > \lambda)/H(\lambda) = \mathrm{e}^{-1} \sum_{k=0}^{\infty} \frac{1}{k!}(a_k + L b_k). \tag{2.23}$$

Clearly, the former is strictly smaller than the latter. This will obviously imply that the limit

$$\lim_{\lambda \to \infty} P(\phi(\boldsymbol{X}) > \lambda)/H(\lambda)$$

does not exist.

To prove (2.22) and (2.23) we use Lemma 1.1(iii) to conclude that

$$\begin{aligned}\liminf_{\lambda\to\infty} P(\phi(X)>\lambda)/H(\lambda) &= \liminf_{\lambda\to\infty} P\left(\phi\left(\sum_{n=1}^{N} Y_n\right)>\lambda\right)\Big/H(\lambda)\\ &= \liminf_{\lambda\to\infty} \mathrm{e}^{-1}\sum_{k=0}^{\infty}\frac{1}{k!}P\left(\phi\left(\sum_{n=1}^{k} Y_n\right)>\lambda\right)\Big/H(\lambda)\\ &= \lim_{K\to\infty}\liminf_{\lambda\to\infty} \mathrm{e}^{-1}\sum_{k=0}^{K}\frac{1}{k!}P\left(\phi\left(\sum_{n=1}^{k} Y_n\right)>\lambda\right)\Big/H(\lambda).\end{aligned}$$

Now, for each fixed $K \geqslant 1$ and $0<\varepsilon<1$ we have, for all $\lambda>0$ big enough,

$$\frac{\sum_{k=0}^{K}\frac{1}{k!}P(\phi(\sum_{n=1}^{k}Y_n)>\lambda)/H(\lambda)}{\sum_{k=0}^{K}\frac{1}{k!}(a_kH(\lambda)+b_k\bar{F}_2(\lambda))/H(\lambda)}\in(1-\varepsilon,1+\varepsilon),$$

from which (2.22) follows. We can prove (2.23) in the same manner. The fact that the sums in the right-hand sides of (2.22) and (2.23) converge follows from Theorem 2.1.

We remark that in the subexponential case ($\alpha=0$), $b_k=0$ for all k in (2.20), and so the limit $\lim_{\lambda\to\infty}P(\phi(X)>\lambda)/H(\lambda)$ exists in this case, as it should by the results of Rosiński and Samorodnitsky (1993).

The following example shows that even when the limiting relation (1.7) does hold, one cannot expect the limiting constant to be determined by the measure $1-H$.

Example 2.2. Take an arbitrary $F\in\mathscr{S}(\alpha)$, and let $\{X_i\}_{i\geqslant1}$ be a sequence of i.i.d. random variables with common distribution F. Let once again $T=\{1,2\}$, and let us define two infinitely divisible stochastic processes on T, $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$, as follows. For $m=1,2$ let $\boldsymbol{X}^{(m)}=\sum_{n=1}^{N}Y_i^{(m)}$, where N is a mean 1 Poisson random variable, independent of the sequence of i.i.d. random vectors in $\mathbb{R}^2$, $\{Y_i^{(m)}\}_{i\geqslant1}$ such that

$$Y_i^{(1)}=\begin{cases}(X_i,0) & \text{with probability } \frac{1}{2},\\ (0,X_i) & \text{with probability } \frac{1}{2},\end{cases}$$

and

$$Y_i^{(2)}=(X_i,X_i),$$

$i\geqslant1$.

Choose $\phi(x_1,x_2)=x_1\vee x_2$, and observe that

$$H^{(m)}(\lambda)=P(\phi(Y_1^{(m)})>\lambda)=\bar{F}(\lambda)$$

for both $m=1,2$. However, a trivial computation shows that, as $\lambda\to\infty$,

$$P(\phi(\boldsymbol{X}^{(1)})>\lambda)\sim\exp(-\tfrac{1}{2}(1-m_F(\alpha)))\bar{F}(\lambda),$$

while

$$P(\phi(\boldsymbol{X}^{(2)})>\lambda)\sim\exp(-(1-m_F(\alpha)))\bar{F}(\lambda),$$

and so the two constants are different unless $m_F(\alpha)=1$.

The last two examples notwithstanding, in certain situations one can estimate the lower and upper limits in (1.8). Suppose, for example, that the characteristic function of $\boldsymbol{X}$ is given in the form

$$E\exp\{\mathrm{i}\langle\boldsymbol{\beta},\boldsymbol{X}\rangle\} = \exp\left\{\int_{\mathbb{R}^T}(\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\beta},\boldsymbol{x}\rangle}-1)\nu(\mathrm{d}\boldsymbol{x})\right\}, \tag{2.24}$$

with ν such that the integral $\int_{(-1,1)^T}\boldsymbol{x}\nu(\mathrm{d}\boldsymbol{x})$ converges (coordinatewise). If ν is a finite measure, then the simple subadditivity argument used in (2.14) shows that

$$\begin{aligned}\limsup_{\lambda\to\infty}\frac{P(\phi(\boldsymbol{X})>\lambda)}{H(\lambda)} &\leqslant \exp\left\{\int_{\mathbb{R}}(\mathrm{e}^{\alpha x}-1)H(\mathrm{d}x)\right\}\\ &\leqslant \exp\left\{\int_0^\infty(\mathrm{e}^{\alpha x}-1)H(\mathrm{d}x)\right\}.\end{aligned} \tag{2.25}$$

An estimate for the lower bound can be obtained in a similar way. To this end, let

$$H_-(\lambda) = \nu(\{\boldsymbol{x}\in\mathbb{R}^T\colon \phi(-\boldsymbol{x})>\lambda\}). \tag{2.26}$$

Further, let $\phi_+ = \phi\vee 0$, and observe that ϕ_+ is subadditive if ϕ is. We can now estimate the asymptotics of $P(\phi(\boldsymbol{X})>\lambda)$ (which can be treated as $P(\phi(\boldsymbol{X}_1)>\lambda)$ in (2.14) if ν is a finite measure) as follows. Let $m=\nu(\mathbb{R}^T)$, and $\{\boldsymbol{Y}_j\}_{j\geqslant 1}$ be i.i.d., with common law $m^{-1}\nu$. For any $\lambda>0$ and $n\geqslant 1$,

$$P\left(\phi_+\left(\sum_{j=1}^n \boldsymbol{Y}_j\right)>\lambda\right)\geqslant P\left(\bigcup_{i=1}^n\left\{\phi_+(\boldsymbol{Y}_i)-\sum_{j\neq i}\phi_+(-\boldsymbol{Y}_j)>\lambda\right\}\right),$$

and so by Fatou's lemma

$$\begin{aligned}&\liminf_{\lambda\to\infty}P\left(\phi_+\left(\sum_{j=1}^n \boldsymbol{Y}_j\right)>\lambda\right)\Big/H(\lambda)\\ &\quad\geqslant\sum_{i=1}^n\liminf_{\lambda\to\infty}P\left(\phi_+(\boldsymbol{Y}_i)-\sum_{j\neq i}\phi_+(-\boldsymbol{Y}_j)>\lambda\right)\Big/H(\lambda)\\ &\quad\geqslant nm^{-1}(E\mathrm{e}^{-\alpha\phi_+(-\boldsymbol{Y}_1)})^{n-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\liminf_{\lambda\to\infty}\frac{P(\phi(\boldsymbol{X})>\lambda)}{H(\lambda)} &\geqslant \sum_{n=0}^\infty \mathrm{e}^{-m}\frac{m^n}{n!}\liminf_{\lambda\to\infty}P\left(\phi_+\left(\sum_{j=1}^n \boldsymbol{Y}_j\right)>\lambda\right)\Big/H(\lambda)\\ &\geqslant \sum_{n=1}^\infty \mathrm{e}^{-m}\frac{m^n}{n!}nm^{-1}(E\mathrm{e}^{-\alpha\varphi_+(-\boldsymbol{Y}_1)})^{n-1}\\ &= \exp(-m(1-E\mathrm{e}^{-\alpha\varphi_+(-\boldsymbol{Y}_1)}))\\ &= \exp\left\{\int_0^\infty(\mathrm{e}^{-\alpha x}-1)H_-(\mathrm{d}x)\right\}.\end{aligned} \tag{2.27}$$

The following proposition describes a situation in which the bounds (2.25) and (2.27) hold for infinitely divisible processes satisfying (2.24), even when the Lévy measure ν is infinite.

Proposition 2.1. *Let X be an infinitely divisible stochastic process given by* (2.24). *Under conditions of Theorem* 2.1 *assume, additionally, that*

$$\int_{\mathbb{R}^T} (1 \wedge q(\boldsymbol{x}))\nu(\mathrm{d}\boldsymbol{x}) < \infty. \tag{2.28}$$

Then

$$\exp\left\{\int_0^\infty (\mathrm{e}^{-\alpha x} - 1)H_-(\mathrm{d}x)\right\} \leqslant \liminf_{\lambda\to\infty} \frac{P(\phi(X) > \lambda)}{H(\lambda)}$$

$$\leqslant \limsup_{\lambda\to\infty} \frac{P(\phi(X) > \lambda)}{H(\lambda)} \leqslant \exp\left\{\int_0^\infty (\mathrm{e}^{\alpha x} - 1)H(\mathrm{d}x)\right\}. \tag{2.29}$$

Proof. We start with the obvious observation that (2.28) implies

$$\nu(\{\boldsymbol{x} \in \mathbb{R}^T\colon q(\boldsymbol{x}) > r\}) < \infty \tag{2.30}$$

for every $r > 0$. Because of this we can split the Lévy measure ν as in (2.10) for any $r > 0$, and the measures ν_1 and ν_2 will be finite. Fix now $r > 0$ and let X_i, $i = 1, 2, 3$ be independent infinitely divisible processes given by

$$E\exp\{\mathrm{i}\langle\boldsymbol{\beta}, X_i\rangle\} = \exp\left\{\int_{\mathbb{R}^T} [\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\beta},\boldsymbol{x}\rangle} - 1) >]\nu_i(\mathrm{d}\boldsymbol{x})\right\},$$

$i = 1, 2, 3$, and such that (2.9) holds. For the upper bound in (2.29) note that by the subadditivity of ϕ, (2.11), (2.12), Lemma 1.1(ii) and the fact that (2.25) holds for processes with a finite Lévy measure, we have

$$\limsup_{\lambda\to\infty} P(\phi(X) > \lambda)/H(\lambda)$$

$$\leqslant \exp\left\{\int_r^\infty (\mathrm{e}^{\alpha x} - 1)H(\mathrm{d}x)\right\} E\mathrm{e}^{\alpha\varphi(X_2)} E\mathrm{e}^{\alpha\varphi(X_3)}. \tag{2.31}$$

Therefore, to complete the proof of the upper bound in (2.29) it remains to show that

$$\limsup_{r\to 0} E\mathrm{e}^{\alpha\varphi(X_2)} \leqslant 1, \quad \limsup_{r\to 0} E\mathrm{e}^{\alpha\varphi(X_3)} \leqslant 1.$$

We start with X_2. Let m_r be the total mass of ν_2, and observe that by (2.28), $m_r = o(1/r)$. Therefore, letting N_r be a Poisson random variable with mean m_r, we conclude that

$$E\mathrm{e}^{\alpha\varphi(X_2)} \leqslant E\mathrm{e}^{\alpha r N_r} = \exp(m_r(\mathrm{e}^{\alpha r} - 1)) \to 1$$

as $r \to 0$. It remains to consider X_3, and here our claim will follow from (2.28) and the standard estimate

$$E\mathrm{e}^{\alpha\varphi(X_3)} \leqslant E\mathrm{e}^{\alpha q(X_3)} \leqslant \exp\left(\int_{q(x)\leqslant r} (\mathrm{e}^{\alpha q(x)} - 1)\nu(\mathrm{d}\boldsymbol{x})\right).$$

This proves the upper bound in (2.29). For the lower bound note that the argument similar to the one we used to establish (2.31) gives us

$$\liminf_{\lambda\to\infty} P(\phi(\boldsymbol{X}) > \lambda)/H(\lambda)$$
$$\geqslant \exp\left\{\int_0^\infty (\mathrm{e}^{-\alpha x} - 1)H_-^{(1,r)}(\mathrm{d}x)\right\} E\mathrm{e}^{-\alpha\varphi(-X_2)}E\mathrm{e}^{-\alpha\varphi(-X_3)}, \tag{2.32}$$

where

$$H_-^{(1,r)}(\lambda) = \nu(\{\boldsymbol{x}\in\mathbb{R}^T\colon \phi(\boldsymbol{x}) > r, \phi(-\boldsymbol{x}) > \lambda\}).$$

Define further

$$H_-^{(2,r)}(\lambda) = \nu(\{\boldsymbol{x}\in\mathbb{R}^T\colon \phi(\boldsymbol{x}) \leqslant r, q(\boldsymbol{x}) > r, \phi(-\boldsymbol{x}) > \lambda\}).$$

We then have

$$E\mathrm{e}^{-\alpha\varphi(-X_2)} \geqslant \exp\left\{\int_{\mathbb{R}} (\mathrm{e}^{-\alpha x} - 1)H_-^{(2,r)}(\mathrm{d}x)\right\}$$
$$\geqslant \exp\left\{\int_0^\infty (\mathrm{e}^{-\alpha x} - 1)H_-^{(2,r)}(\mathrm{d}x)\right\}.$$

Since $(H_-^{(1,r)} + H_-^{(2,r)})$ converges vaguely, as $r\to 0$, to $-H_-$, we do get our lower bound in (2.32) if we prove that

$$\liminf_{r\to 0} E\mathrm{e}^{-\alpha\varphi(-X_3)} \geqslant 1.$$

However, this follows from the corresponding statement for the upper bound and the inequality $E1/Z \geqslant 1/EZ$, $Z \geqslant 0$. Therefore, the proof of the proposition is now complete. □

Remark. One can extend the statement of Proposition 2.1 to, say, *symmetric* infinitely divisible processes satisfying (2.28), but for which the integral $\int_{[-1,1]^T} \boldsymbol{x}\nu(\mathrm{d}\boldsymbol{x})$ may diverge. We leave it to an interested reader to generalize this result further, by accomodating various possible shifts.

3. Maxima of Lévy processes

Let $\boldsymbol{X} = \{X(t),\ 0 \leqslant t \leqslant 1\}$ be a process with stationary independent increments (*Lévy process*) such that

$$E\exp(\mathrm{i}\theta X(t)) = \exp(t\psi(\theta)), \tag{3.1}$$

where

$$\psi(\theta) = \mathrm{i}b\theta - \sigma^2\theta^2/2 + \int_{-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x\mathbf{1}(|x| \leqslant 1))\rho(\mathrm{d}x), \tag{3.2}$$

with $b\in\mathbb{R}$, $\sigma \geqslant 0$ and ρ a Borel measure such that $\int_{-\infty}^{\infty}(1\wedge x^2)\rho(\mathrm{d}x) < \infty$.

We consider the tail of the supremum of X, $P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)$. It has been shown by Berman (1986) and Marcus (1987) that

$$\lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)}{\rho(\lambda, \infty)} = 1 \tag{3.3}$$

provided the right tail of the Lévy measure, $\rho(\lambda, \infty)$ was regularly varying at infinity (plus some extra conditions). Later Rosinski and Samorodnitsky (1993) showed that (3.3) holds under the assumption of subexponentiality of the right tail of the Lévy measure ρ. See also Willekens (1987). We are interested in the exponential case: assume that

$$\text{the distribution function } 1 - \min(\rho(x, \infty), 1) \text{ is in } \mathscr{S}(\alpha) \tag{3.4}$$

for an $\alpha > 0$. Clearly, this situation falls into the framework of Theorem 2.1, an application of which shows immediately that

$$0 < \liminf_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)}{\rho(\lambda, \infty)} \leqslant \liminf_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)}{\rho(\lambda, \infty)} < \infty.$$

It is the purpose of this section to demonstrate that in this important case the limit $\lim_{\lambda\to\infty} P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)/\rho(\lambda, \infty)$ does exist, and so the two tails are truly equivalent. For a related result in the context of a random walk drifting to $-\infty$ see Bertoin and Doney (1993).

We start with some notation. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables. Then $\{S_n = Y_1 + \cdots + Y_n, n \geqslant 0\}$ is a random walk, and we denote by $\{M_n = \max_{0\leqslant i\leqslant n} S_i,\ n \geqslant 0\}$ the corresponding ladder height process. Writing the maximum of $n+1$ numbers as the bigger of the first number and the maximum of the next n numbers, we obtain for every $n \geqslant 0$,

$$M_{n+1} \stackrel{\mathrm{d}}{=} \max(0, M_n + Y_{n+1}\},$$

and so if the common distribution F of $\{Y_j\}_{j\geqslant 1}$ is in $\mathscr{S}(\alpha)$, then by Lemma 1.1(ii) and an inductive argument we immediately conclude that

$$\lim_{\lambda\to\infty} \frac{P(M_n > \lambda)}{\bar{F}(\lambda)} = \sum_{i=1}^{n} m(\alpha)^{i-1} m_{M_{n-i}}(\alpha) \in (0, \infty), \tag{3.5}$$

where $m(\alpha) = E\mathrm{e}^{\alpha Y_1}$, and $m_{M_k}(\alpha) = E\mathrm{e}^{\alpha M_k}$, $k \geqslant 0$.

Suppose, for a moment, that our Lévy process is actually compound Poisson, with Lévy exponent ψ in (3.1) having the form

$$\psi(0) = \int_{-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}\theta x} - 1)\rho(\mathrm{d}x) \tag{3.6}$$

with ρ satisfying (3.4), and being a *finite* measure. That is,

$$\rho = \mu F, \tag{3.7}$$

where $\mu > 0$ and F is a distribution in $\mathscr{S}(\alpha)$. Then we let $Y_1, Y_2, \ldots$ be i.i.d. random variables with common law F, independent of a Poisson random variable N with mean μ. Then, clearly,

$$P\left(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda\right) = P(M_N > \lambda) = \sum_{n=0}^{\infty} \mathrm{e}^{-\mu}\frac{\mu^n}{n!} P(M_n > \lambda).$$

Observe further that by Lemma 1.1 (iii) we know that there is a $K < \infty$ such that for every $\lambda > 0$ and $n \geqslant 1$,

$$P(M_n > \lambda) \leqslant P\left(\sum_{i=1}^{n} (Y_i)_+ > \lambda\right) \leqslant K(1 + m_+(\alpha))^{n-1}\bar{F}(\lambda)$$

(where $m_+(\alpha) = E\mathrm{e}^{\alpha(Y_1)_+}$.) Therefore, by the Lebesgue dominated convergence theorem and (3.5) we have

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)}{\rho(\lambda, \infty)} &= \lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)}{\mu\bar{F}(\lambda)} \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{-\mu}\frac{\mu^{n-1}}{n!} \lim_{\lambda\to\infty} \frac{P(M_n > \lambda)}{\bar{F}(\lambda)} \\
&= \sum_{n=1}^{\infty} \mathrm{e}^{-\mu}\frac{\mu^{n-1}}{n!} \sum_{i=1}^{n} m(\alpha)^{i-1} m_{M_{n-i}}(\alpha) \in (0, \infty). \qquad (3.8)
\end{aligned}$$

This shows that the limit $\lim_{\lambda\to\infty} P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)/\rho(\lambda, \infty)$ exists when the Lévy process is compound Poisson. More importantly, it is also an important ingredient in the proof of the general case, stated in the following theorem.

Theorem 3.1. *Let X be a Lévy process with characteristic function given by* (3.1) *and* (3.2). *If the tail of ρ is equivalent to the tail of a distribution in $\mathscr{S}(\alpha)$* (i.e. (3.4) *holds,*) *then*

$$\lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)}{\rho(\lambda, \infty)} = c \qquad (3.9)$$

for some $c \in (0, \infty)$.

Proof. We prove the theorem by sequentially increasing the level of generality. On the first level (compound Poisson process) its statement follows from (3.8); we regard this situation as **Step 0** of the proof.

Step 1: Here we add the possibility of a drift. That is, suppose that the Lévy exponent ψ of the process has the form

$$\psi(\theta) = \mathrm{i}b\theta + \int_{-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}\theta x} - 1)\rho(\mathrm{d}x), \qquad (3.10)$$

with the Lévy measure ρ still of the form (3.7). Although the argument is somewhat different in the two cases, $b > 0$ and $b < 0$, the approach is the same, and we consider only the (marginally more complicated) case $b > 0$.

Fix an $\varepsilon > 0$ small enough so that $b - \varepsilon > 0$, and let $K > 0$ be a large positive number to be specified later. Consider two Lévy processes, $X_+ = \{X_+(t), 0 \leqslant t \leqslant 1\}$

and $\boldsymbol{X}_- = \{X_-(t), 0 \leqslant t \leqslant 1\}$ defined by their corresponding Lévy exponents

$$\psi_+(\theta) = K(\mathrm{e}^{\mathrm{i}\theta(b+\varepsilon)/K} - 1) + \int_{-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}\theta x} - 1)\rho(\mathrm{d}x), \tag{3.11}$$

and

$$\psi_-(\theta) = K(\mathrm{e}^{\mathrm{i}\theta(b-\varepsilon)/K} - 1) + \int_{-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}\theta x} - 1)\rho(\mathrm{d}x). \tag{3.12}$$

Observe that both $\boldsymbol{X}_+$ and $\boldsymbol{X}_-$ are compound Poisson processes, with Lévy measures

$$\rho_+ = \rho + K\delta_{\{(b+\varepsilon)/K\}}$$

and

$$\rho_- = \rho + K\delta_{\{(b-\varepsilon)/K\}}$$

correspondingly. In particular, $\rho_+(\lambda, \infty) \sim \rho_-(\lambda, \infty) \sim \rho(\lambda, \infty)$ as $\lambda \to \infty$, and since the statement of the theorem has been proved for such processes, we conclude that the limits

$$L_+(K,\varepsilon) = \lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X_+(t) > \lambda)}{\rho(\lambda, \infty)}$$

and

$$L_-(K,\varepsilon) = \lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X_-(t) > \lambda)}{\rho(\lambda, \infty)}$$

exist and are in $(0, \infty)$.

Recall that $Y_1, Y_2, \ldots$ are i.i.d. random variables with common law F, and let $\Gamma_1, \Gamma_2, \ldots$ and $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots$ be the sequences of arrival times of two independent Poisson processes with rates μ and K accordingly, independent of the sequence $Y_1, Y_2, \ldots$ as well. Let $\hat{\Gamma}_j = \Gamma_j \wedge 1$, $j \geqslant 1$. Then for every $\lambda > 0$,

$$P\left(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda\right) = P\left(\max_{i:\Gamma_i\leqslant 1}\left(\sum_{j=1}^{i} Y_j + b\hat{\Gamma}_{i+1}\right) > \lambda\right). \tag{3.13}$$

Furthermore,

$$P\left(\sup_{0\leqslant t\leqslant 1} X_+(t) > \lambda\right) = P\left(\max_{i:\Gamma_i\leqslant 1}\left(\sum_{j=1}^{i} Y_j + \frac{b+\varepsilon}{K} R_i\right) > \lambda\right), \tag{3.14}$$

where R_i = number of j: $\tilde{\Gamma}_j \leqslant \hat{\Gamma}_{i+1}$, $i \geqslant 0$.

Choose any $\delta \in (0, 1)$. We have

$$\begin{aligned}
&P\left(\sup_{0\leqslant t\leqslant 1} X_+(t) > \lambda\right) \\
&\quad\geqslant P\left(\max_{i:\Gamma_i\leqslant 1}\left(\sum_{j=1}^{i} Y_j + \frac{b+\varepsilon}{K} R_i\right) > \lambda, \frac{b+\varepsilon}{K} R_i \geqslant b\hat{\Gamma}_{i+1} \forall i: \Gamma_i \leqslant 1\right) \\
&\quad\geqslant P\left(\max_{i:\Gamma_i\leqslant 1}\left(\sum_{j=1}^{i} Y_j + b\hat{\Gamma}_{i+1}\right) > \lambda, \frac{b+\varepsilon}{K} R_i \geqslant b\hat{\Gamma}_{i+1} \forall i: \Gamma_i \leqslant 1\right)
\end{aligned}$$

$$= P\left(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda\right) - P\left(\max_{i:\Gamma_i\leqslant 1}\left(\sum_{j=1}^{i} Y_j + b\hat{\Gamma}_{i+1}\right) > \lambda,\right.$$

$$\left.\frac{b+\varepsilon}{K} R_i < b\hat{\Gamma}_{i+1} \text{ for some } i\text{: } \Gamma_i \leqslant 1\right)$$

$$:= P\left(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda\right) - Q(\lambda). \tag{3.15}$$

Now, for every $N \in \{1,2, \dots\}$

$$Q(\lambda) \leqslant P\left(\max_{i:\Gamma_i\leqslant 1}\left(\sum_{j=1}^{i} Y_j\right) > \lambda - b,\ \frac{b+\varepsilon}{K} R_i < b\hat{\Gamma}_{i+1} \text{ for some } i\text{: } \Gamma_i \leqslant 1\right)$$

$$\leqslant P\left(\max_{i:\Gamma_i\leqslant 1}\left(\sum_{j=1}^{i} Y_j\right) > \lambda - b,\ \Gamma_N \leqslant 1\right)$$

$$+ P\left(\max_{i\leqslant N}\left(\sum_{j=1}^{i} Y_j\right) > \lambda - b\right) P\left(\frac{b+\varepsilon}{K} R_i < b\hat{\Gamma}_{i+1} \text{ for some } i \leqslant N\right)$$

$$:= Q_1(\lambda) + Q_2(\lambda). \tag{3.16}$$

Observe that by (3.5) and Lemma 1.1(iii) we have

$$\lim_{\lambda\to\infty} \frac{Q_1(\lambda)}{\rho(\lambda, \infty)} = e^{\alpha b} \sum_{n=N}^{\infty} e^{-\mu} \frac{\mu^{n-1}}{n!} \sum_{i=1}^{n} m(\alpha)^{i-1} m_{M_{n-i}}(\alpha) \leqslant \delta/2 \tag{3.17}$$

if N is large enough.

In the following k will stand for a finite positive constant that is allowed to change from line to line. With this in mind we use Lemma 1.1(iv) to conclude that

$$\limsup_{\lambda\to\infty} \frac{Q_2(\lambda)}{\rho(\lambda, \infty)} \leqslant kP\left(\frac{b+\varepsilon}{K} R_i < b\hat{\Gamma}_{i+1} \text{ for some } i \leqslant N\right)$$

$$\leqslant k \sum_{n=1}^{N} P\left(\frac{b+\varepsilon}{K} R_i < b\hat{\Gamma}_{i+1}\right) \leqslant \delta/2 \tag{3.18}$$

if $K > K_0 = K_0(\varepsilon, \delta)$ because

$$\frac{R_i}{K\hat{\Gamma}_{i+1}} \Rightarrow \delta_{\{1\}}$$

(the point mass at 1) at $K \to \infty$. Therefore, for such K,

$$\limsup_{\lambda\to\infty} \frac{Q(\lambda)}{\rho(\lambda, \infty)} \leqslant \delta,$$

and we conclude by (3.15) that for every $K > K_0(\varepsilon, \delta)$

$$\limsup_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)}{\rho(\lambda, \infty)} \leqslant \lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X_+(t) > \lambda)}{\rho(\lambda, \infty)}$$

$$+ \delta = L_+(K, \varepsilon) + \delta. \tag{3.19}$$

We now turn to lower bounds. As in (3.14) we have

$$P\Big(\sup_{0\leqslant t\leqslant 1} X_-(t)>\lambda\Big)=P\Big(\max_{i:\Gamma_i\leqslant 1}\Big(\sum_{j=1}^{i} Y_j+\frac{b-\varepsilon}{K}R_i\Big)>\lambda\Big),\tag{3.20}$$

and proceeding in a similar fashion to the above arguments we can write for every $\delta\in(0,1)$:

$$\begin{aligned}
&P\Big(\sup_{0\leqslant t\leqslant 1} X_-(t)>\lambda\Big)\\
&=P\Big(\max_{i:\Gamma_i\leqslant 1}\Big(\sum_{j=1}^{i} Y_j-\frac{b-\varepsilon}{K}R_i\Big)>\lambda,\ \frac{b-\varepsilon}{K}R_i\leqslant b\hat{\Gamma}_{i+1}\forall i\colon \Gamma_i\leqslant 1\Big)\\
&\quad+P\Big(\max_{i:\Gamma_i\leqslant 1}\Big(\sum_{j=1}^{i} Y_j-\frac{b-\varepsilon}{K}R_i\Big)>\lambda,\\
&\qquad\qquad \frac{b-\varepsilon}{K}R_i>b\hat{\Gamma}_{i+1}\text{ for some } i\colon \Gamma_i\leqslant 1\Big)\\
&\leqslant P\Big(\sup_{0\leqslant t\leqslant 1} X(t)>\lambda\Big)\\
&\quad+P\Big(\sum_{j=1}^{L}(Y_j)_+ +\frac{b-\varepsilon}{K}\tilde{L}>\lambda,\ \frac{b-\varepsilon}{K}R_i>b\hat{\Gamma}_{i+1}\text{ for some } i\leqslant L\Big)\\
&:=P\Big(\sup_{0\leqslant t\leqslant 1} X(t)>\lambda\Big)+Q_3(\lambda).
\end{aligned}\tag{3.21}$$

Here $L=\max\{i\colon \Gamma_i\leqslant 1\}$ and $\tilde{L}=\max\{i\colon \tilde{\Gamma}_i\leqslant 1\}$.

We continue in a similar manner. For every $N\in\{1,2,\dots\}$ we have

$$\begin{aligned}
Q_3(\lambda)&\leqslant P\Big(\sum_{j=1}^{L}(Y_j)_+ +\frac{b-\varepsilon}{K}\tilde{L}>\lambda,\ L\geqslant N\Big)\\
&\quad+P\Big(\sum_{j=1}^{N}(Y_j)_+ +\frac{b-\varepsilon}{K}\tilde{L}>\lambda,\ \frac{b-\varepsilon}{K}R_i>b\hat{\Gamma}_{i+1}\text{ for some } i\leqslant N\Big)\\
&:=Q_4(\lambda)+Q_5(\lambda).
\end{aligned}\tag{3.22}$$

Observe that $Ee^{\alpha(b-\varepsilon)/K\tilde{L}}$ is bounded from above uniformly over $0<\varepsilon<b$ and $K>1$. Therefore, by Lemma 1.1(ii) and (iii), we conclude that

$$\lim_{\lambda\to\infty}\frac{Q_4(\lambda)}{\rho(\lambda,\infty)}=\sum_{n=N}^{\infty} e^{-\mu}\frac{\mu^{n-1}}{n!}nm_+(\alpha)^{n-1}Ee^{\alpha(b-\varepsilon)/K\tilde{L}}\leqslant\delta/2\tag{3.23}$$

if N is large enough. Furthermore, for every $\tilde{N}\in\{1,2,\dots\}$ we have

$$\begin{aligned}
Q_5(\lambda)&\leqslant P\Big(\sum_{j=1}^{N}(Y_j)_+ +\frac{b-\varepsilon}{K}\tilde{L}>\lambda,\ L>\tilde{N}\Big)\\
&\quad+P\Big(\sum_{j=1}^{N}(Y_j)_+ +\frac{b-\varepsilon}{K}\tilde{N}>\lambda,\ \frac{b-\varepsilon}{K}R_i>b\hat{\Gamma}_{i+1}\text{ for some } i\leqslant N\Big)\\
&:=Q_6(\lambda)+Q_7(\lambda).
\end{aligned}$$

Now, by Lemma 1.1(ii) and (iii) we immediately conclude that

$$\lim_{\lambda\to\infty}\frac{Q_6(\lambda)}{\rho(\lambda,\infty)}=\mu^{-1}Nm_+(\alpha)^{N-1}E(\mathrm{e}^{\alpha\tilde{L}(b-\varepsilon)/K}1(\tilde{L}>\tilde{N}))\leqslant\delta/4 \tag{3.24}$$

if $\tilde{N}$ is large comparatively to N. Finally, letting k being a constant that depends on the choice of N and $\tilde{N}$, we obtain, similarly to (3.18)

$$\begin{aligned}\limsup_{\lambda\to\infty}\frac{Q_7(\lambda)}{\rho(\lambda,\infty)}&\leqslant kP\left(\frac{b-\varepsilon}{K}R_i>b\hat{\Gamma}_{i+1}\text{ for some }i\leqslant N\right)\\&\leqslant k\sum_{n=1}^{N}P\left(\frac{b-\varepsilon}{K}R_i>b\hat{\Gamma}_{i+1}\right)\leqslant\delta/4\end{aligned} \tag{3.25}$$

if $K>K_1=K_1(\varepsilon,\delta)$. We conclude by (3.21)–(3.25) that for every $K>K_1(\varepsilon,\delta)$,

$$\begin{aligned}\liminf_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}X(t)>\lambda)}{\rho(\lambda,\infty)}&\geqslant\lim_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}X_-(t)>\lambda)}{\rho(\lambda,\infty)}-\delta\\&=L_-(K,\varepsilon)-\delta.\end{aligned} \tag{3.26}$$

It remains to compare $L_+(K,\varepsilon)$ and $L_-(K,\varepsilon)$. For any $\gamma>0$ by (3.14) and (3.20) we have

$$\begin{aligned}&P\left(\sup_{0\leqslant t\leqslant 1}X_+(t)>\lambda\right)\\&\quad\leqslant P\left(\sup_{0\leqslant t\leqslant 1}X_-(t)>\lambda-\gamma\right)+P\left(\sup_{0\leqslant t\leqslant 1}X_+(t)>\lambda,\frac{2\varepsilon}{K}\tilde{L}>\gamma\right)\\&\quad\leqslant P\left(\sup_{0\leqslant t\leqslant 1}X_-(t)>\lambda-\gamma\right)+P\left(\sum_{j=1}^{L}(Y_j)_++\frac{b+\varepsilon}{K}\tilde{L}>\lambda,\frac{2\varepsilon}{K}\tilde{L}>\gamma\right),\end{aligned}$$

and so by Lemma 1.1(ii) and (iv) we conclude that

$$L_+(K,\varepsilon)\leqslant\mathrm{e}^{\alpha\gamma}L_-(K,\varepsilon)+\mu^{-1}E\mathrm{e}^{\alpha\sum_{j=1}^{L}(Y_j)_+}E\left(\mathrm{e}^{\alpha(b+\varepsilon)/K\tilde{L}}1\left(\frac{2\varepsilon}{K}\tilde{L}>\gamma\right)\right). \tag{3.27}$$

But

$$\begin{aligned}&E\left(\mathrm{e}^{2\alpha(b+\varepsilon)/K\tilde{L}}1\left(\frac{2\varepsilon}{K}\tilde{L}>\gamma\right)\right)\\&\quad\leqslant(E\mathrm{e}^{2\alpha(b+\varepsilon)/K\tilde{L}})^{1/2}(P(\tilde{L}/K>\gamma/2\varepsilon))^{1/2}\\&\quad\leqslant\exp\{\tfrac{1}{2}K(\mathrm{e}^{2\alpha(b+\varepsilon)/K}-1)\}\left(\frac{2\varepsilon}{\gamma}\right)^{1/2}\leqslant\mathrm{e}^{k\alpha(b+\varepsilon)}\left(\frac{2\varepsilon}{\gamma}\right)^{1/2}\end{aligned}$$

for an absolute finite constant k as long as $K>1$ and $\varepsilon<b$ (say). Observe, further, that for such K and ε the limit $L_-(K,\varepsilon)$ is unformly bounded from above. Therefore, choosing γ in (3.27) small and then choosing ε small, we may achieve

$$L_+(K,\varepsilon)-L_-(K,\varepsilon)\leqslant\delta$$

and so by (3.19) and (3.26) we have

$$\limsup_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}X(t)>\lambda)}{\rho(\lambda,\infty)}-\liminf_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}X(t)>\lambda)}{\rho(\lambda,\infty)}\leqslant 3\delta,$$

and since $\delta>0$ is arbitrarily small, the statement of the theorem has now been proved for Lévy processes of the form (3.10).

Step 2: Here we prove the theorem for general Lévy processes without a Brownian component. That is, we assume now that the Lévy exponent of X has the form

$$\psi(\theta)=\mathrm{i}b\theta+\int_{-\infty}^{\infty}(\mathrm{e}^{\mathrm{i}\theta x}-1-\mathrm{i}\theta x\mathbf{1}(|x|\leqslant 1))\,\rho(\mathrm{d}x) \tag{3.28}$$

without any additional assumptions on the Lévy measure ρ.

Fix an $\varepsilon>0$, and let X_1 and X_2 be two independent Lévy motions, with Lévy exponents

$$\psi_1(\theta)=\mathrm{i}b\theta+\int_{|x|>\varepsilon}(\mathrm{e}^{\mathrm{i}\theta x}-1-\mathrm{i}\theta x\mathbf{1}(|x|\leqslant 1))\,\rho(\mathrm{d}x)$$

and

$$\psi_2(\theta)=\int_{|x|\leqslant\varepsilon}(\mathrm{e}^{\mathrm{i}\theta x}-1-\mathrm{i}\theta x\mathbf{1}(|x|\leqslant 1))\,\rho(\mathrm{d}x)$$

correspondingly, such that $X=X_1+X_2$. Observe first that X_1 is a Lévy process of the type (3.10), and for such processes the theorem has already been proved. Therefore,

$$\lim_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}X_1(t)>\lambda)}{\rho(\lambda,\infty)}=L(\varepsilon)\in(0,\infty).$$

Note that the support of the Lévy measure of X_2 is bounded, and therefore the probability tail $P(\sup_{0\leqslant t\leqslant 1}X_2(t)>\lambda)$ decreases faster than $\mathrm{e}^{-\beta x}$ for any $\beta>0$. In particular,

$$\lim_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}X_2(t)>\lambda)}{\rho(\lambda,\infty)}=0 \tag{3.29}$$

for every $\varepsilon>0$ and

$$E\exp\left\{\alpha\sup_{0\leqslant t\leqslant 1}|X_2(t)|\right\}\to 1 \tag{3.30}$$

as $\varepsilon\to 0$. Since

$$P\left(\sup_{0\leqslant t\leqslant 1}X_1(t)-\sup_{0\leqslant t\leqslant 1}|X_2(t)|>\lambda\right)\leqslant P\left(\sup_{0\leqslant t\leqslant 1}X(t)>\lambda\right)$$

$$\leqslant P\left(\sup_{0\leqslant t\leqslant 1}X_1(t)+\sup_{0\leqslant t\leqslant 1}|X_2(t)|>\lambda\right)$$

we conclude by (3.29) and Lemma 1.1(ii) that

$$Ee^{-\alpha\sup_{0\leqslant t\leqslant 1}|X_2(t)|}L(\varepsilon)$$

$$\leqslant \liminf_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}X(t)>\lambda)}{\rho(\lambda,\infty)}$$

$$\leqslant \limsup_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}X(t)>\lambda)}{\rho(\lambda,\infty)}\leqslant Ee^{\alpha\sup_{0\leqslant t\leqslant 1}|X_2(t)|}L(\varepsilon)$$

and an immediate application of (3.30) shows that the lower and the upper limits are, in fact, equal. This proves the statement of the theorem for Lévy processes without a Brownian component.

Step 3: Finally, we add a possible Brownian component. That is, the Lévy exponent ψ is given now in its most general form (3.2). Again, the idea is to use a Poisson approximation to the Brownian component. For a $K>0$ consider a Lévy process $\tilde{X}=\{\tilde{X}(t),\ 0\leqslant t\leqslant 1\}$ with Lévy exponent

$$\tilde{\psi}(\theta)=\mathrm{i}b\theta+\sqrt{K}(\sqrt{K}(\mathrm{e}^{\mathrm{i}\theta\sigma/\sqrt{K}}-1)-\mathrm{i}\theta\sigma)$$
$$+\int_{-\infty}^{\infty}(\mathrm{e}^{\mathrm{i}\theta x}-1-\mathrm{i}\theta x\mathbf{1}(|x|\leqslant 1))\rho(\mathrm{d}x). \tag{3.31}$$

Observe that we may write

$$\boldsymbol{X}\stackrel{\mathrm{d}}{=}\boldsymbol{X}_0+\boldsymbol{B},\qquad \tilde{\boldsymbol{X}}\stackrel{\mathrm{d}}{=}\boldsymbol{X}_0+\boldsymbol{Z}_K, \tag{3.32}$$

where $\boldsymbol{X}_0$ is a Lévy process with Lévy exponent given by (3.28), $\boldsymbol{B}$ is an independent of $\boldsymbol{X}_0$ symmetric Brownian motion with variance σ^2, and $\boldsymbol{Z}_K$ is an independent of $\boldsymbol{X}_0$ Lévy process with Lévy exponent

$$\psi(\theta)=\sqrt{K}(\sqrt{K}(\mathrm{e}^{\mathrm{i}\theta\sigma/\sqrt{K}}-1)-\mathrm{i}\theta\sigma).$$

Both $\tilde{\boldsymbol{X}}$ and $\boldsymbol{X}_0$ are Lévy processes of the kinds already considered, so the limits

$$\tilde{L}=\lim_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}\tilde{X}(t)>\lambda)}{\rho(\lambda,\infty)}$$

and

$$L_0=\lim_{\lambda\to\infty}\frac{P(\sup_{0\leqslant t\leqslant 1}X_0(t)>\lambda)}{\rho(\lambda,\infty)}$$

exist and are in $(0,\infty)$.

Clearly, $\boldsymbol{Z}_K\Rightarrow\boldsymbol{B}$ weakly in $D[0,1]$, equipped with Skorohod's topology J_1, as $K\to\infty$. Let now $K\to\infty$ through the positive integers. We put everything on the same probability space in the following way. By a standard embedding theorem (see e.g. Theorem IV.3.13, p. 71 of Pollard, 1984) there is a probability space $(\Omega_1,\mathscr{F}_1,P_1)$ on which we can define the processes $(\boldsymbol{Z}_K\}_{K\geqslant 1}$ and $\boldsymbol{B}$ such that $\boldsymbol{Z}_K\to\boldsymbol{B}$ a.s. in $D[0,1]$ as $K\to\infty$. Let further $\boldsymbol{X}_0$ be defined on a different probability space $(\Omega_2,\mathscr{F}_2,P_2)$. Let $(\Omega,\mathscr{F},P)$ be the product probability space.

Let $D_K = \sup_{0 \leqslant t \leqslant 1} |Z_K(t) - B(t)|$. Then $D_K \to 0$ a.s. as $K \to \infty$. We have by (3.32) for any $\gamma > 0$,

$$P\left(\sup_{0 \leqslant t \leqslant 1} X(t) > \lambda\right) \geqslant P\left(\sup_{0 \leqslant t \leqslant 1} \tilde{X}(t) > \lambda + \gamma, D_K \leqslant \gamma\right)$$

$$= P\left(\sup_{0 \leqslant t \leqslant 1} \tilde{X}(t) > \lambda + \gamma\right) - P\left(\sup_{0 \leqslant t \leqslant 1} \tilde{X}(t) > \lambda + \gamma, D_K > \gamma\right).$$

Now,

$$\limsup_{\lambda \to \infty} \frac{P(\sup_{0 \leqslant t \leqslant 1} \tilde{X}(t) > \lambda + \gamma, D_K > \gamma)}{\rho(\lambda, \infty)}$$

$$\leqslant \limsup_{\lambda \to \infty} \frac{P(\sup_{0 \leqslant t \leqslant 1} X_0(t) + \sup_{0 \leqslant t \leqslant 1} Z_K(t) > \lambda + \gamma, D_K > \gamma)}{\rho(\lambda, \infty)}$$

$$= \mathrm{e}^{-\gamma\alpha} L_0 E(\mathrm{e}^{\alpha \sup_{0 \leqslant t \leqslant 1} Z_K(t)} \mathbf{1}(D_K > \gamma)).$$

Using sequentially the Cauchy–Schwartz inequality and then a maximal inequality for submartingales we conclude that for all $K \geqslant 1$

$$E(\mathrm{e}^{\alpha \sup_{0 \leqslant t \leqslant 1} Z_K(t)} \mathbf{1}(D_K > \gamma)) \leqslant (E\mathrm{e}^{2\alpha \sup_{0 \leqslant t \leqslant 1} Z_K(t)})^{1/2} (P(D_K > \gamma))^{1/2}$$

$$\leqslant k(P(D_K > \gamma))^{1/2},$$

where k is an absolute finite constant. Observe that this argument also shows that the limit $\tilde{L}$, regarded as a function of K, is uniformly bounded from above for $K \geqslant 1$. Therefore, for any fixed $\delta \in (0, 1)$ we can choose first γ small and then K so large that

$$\liminf_{\lambda \to \infty} \frac{P(\sup_{0 \leqslant t \leqslant 1} X(t) > \lambda)}{\rho(\lambda, \infty)} \geqslant \tilde{L} - \delta. \tag{3.33}$$

Similarly, for every $\gamma > 0$

$$P\left(\sup_{0 \leqslant t \leqslant 1} X(t) > \lambda\right) \leqslant P\left(\sup_{0 \leqslant t \leqslant 1} \tilde{X}(t) > \lambda - \gamma\right)$$

$$+ P\left(\sup_{0 \leqslant t \leqslant 1} X(t) > \lambda, D_K > \gamma\right).$$

Arguing as above we conclude that

$$\limsup_{\lambda \to \infty} \frac{P(\sup_{0 \leqslant t \leqslant 1} X(t) > \lambda, D_K > \gamma)}{\rho(\lambda, \infty)}$$

$$\leqslant \limsup_{\lambda \to \infty} \frac{P(\sup_{0 \leqslant t \leqslant 1} X_0(t) + \sup_{0 \leqslant t \leqslant 1} B(t) > \lambda, D_K > \gamma)}{\rho(\lambda, \infty)}$$

$$= L_0 E(\mathrm{e}^{\alpha \sup_{0 \leqslant t \leqslant 1} B(t)} \mathbf{1}(D_K > \gamma)),$$

and, for a fixed $\delta \in (0,1)$ we take a sufficiently small γ and then a sufficiently large K to obtain

$$\limsup_{\lambda \to \infty} \frac{P(\sup_{0 \leqslant t \leqslant 1} X(t) > \lambda)}{\rho(\lambda, \infty)} \leqslant \tilde{L} + \delta. \tag{3.34}$$

Since δ can be taken as close to 0 as we wish, the proof of the theorem is now completed in the full generality by comparing (3.33) and (3.34). $\square$

We conclude this paper with a discussion of the value of the limit c in (3.9). If the Lévy exponent of the Lévy process ψ has the form (3.6) then one may use the general bounds of Proposition 2.1 in our particular case (note that (2.28) holds automatically in this case). However, we can get better bounds than those given by the general result, and these bounds are contained in the following proposition.

Proposition 3.1. *Under conditions of Theorem* 3.1 *we have*

$$c \geqslant \exp\left\{\alpha b + \alpha^2\sigma^2/2 + \int_{-\infty}^{\infty} (\mathrm{e}^{\alpha x} - 1 - \alpha x \mathbf{1}(|x| \leqslant 1))\rho(\mathrm{d}x)\right\}. \tag{3.35}$$

Furthermore, if the Lévy exponent ψ of the process is given in the form (3.6), *then*

$$c \leqslant \exp\left\{\int_0^{\infty} (\mathrm{e}^{\alpha x} - 1)\rho(\mathrm{d}x)\right\} \frac{1 - \exp\{-\int_{-\infty}^0 (1 - \mathrm{e}^{\alpha x})\rho(\mathrm{d}x)\}}{\int_{-\infty}^0 (1 - \mathrm{e}^{\alpha x})\rho(\mathrm{d}x)}. \tag{3.36}$$

Proof. Clearly, $P(\sup_{0 \leqslant t \leqslant 1} X(t) > \lambda) \geqslant P(X(1) > \lambda)$. Now (3.35) follows from the following simple generalization of Lemma 1.1(iv): for every $t > 0$,

$$\lim_{\lambda \to \infty} \frac{P(X(t) > \lambda)}{\rho(\lambda, \infty)} = tE\mathrm{e}^{\alpha X(t)}$$

$$= t\exp\left\{\alpha bt + \alpha^2\sigma^2 t/2 + t\int_{-\infty}^{\infty} (\mathrm{e}^{\alpha x} - 1 - \alpha x \mathbf{1}(|x| \leqslant 1))\rho(\mathrm{d}x)\right\} \tag{3.37}$$

Relation (3.37) has been undoubtedly known to (among other people) Embrechts and Goldie, who included in their paper (1982) only the compound Poisson case (probably because other parts of their result are not as easy to extend to the case of infinite Lévy measure). We add for completeness that one can easily derive (3.37) from Lemma 1.1(iv) by the usual argument consisting of representing $X(t)$ as a sum of two independent infinitely divisible random variables by splitting ρ into two parts, that around the origin, and that away from the origin.

We apply the same idea to prove (3.36). To this end, fix an $\varepsilon > 0$ and, as in the proof of Theorem 3.1, consider two independent Lévy processes, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ satisfying $\boldsymbol{X} = \boldsymbol{X}_1 + \boldsymbol{X}_2$, with Lévy exponents

$$\psi_1(\theta) = \int_{|x| > \varepsilon} (\mathrm{e}^{\mathrm{i}\theta x} - 1)\rho(\mathrm{d}x)$$

and

$$\psi_2(\theta) = \int_{|x| \leqslant \varepsilon} (e^{i\theta x} - 1)\rho(dx)$$

correspondingly. By (3.8) we conclude that

$$\lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X_1(t) > \lambda)}{\rho(\lambda, \infty)} = \sum_{n=1}^{\infty} e^{-\mu} \frac{\mu^{n-1}}{n!} \sum_{i=1}^{n} m(\alpha)^{i-1} m_{M_{n-1}}(\alpha), \tag{3.38}$$

where $\mu = \rho\{x: |x| > \varepsilon\}$, and $m(\alpha)$ and $m_{M_k}(\alpha)$ correspond to a random walk with the step distribution $F_\varepsilon(A) = \rho(A \cap \{x: |x| > \varepsilon\})/\mu$. Observe that for any $k \geqslant 0$

$$m_{M_k}(\alpha) \leqslant m_+(\alpha)^k = \left(\int_{-\infty}^{\infty} (1 \vee e^{\alpha x}) F_\varepsilon(dx)\right)^k. \tag{3.39}$$

Substituting (3.39) into (3.38) and simplifying we obtain

$$\begin{aligned} &\lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X_1(t) > \lambda)}{\rho(\lambda, \infty)} \\ &\quad \leqslant \exp\left\{\int_0^{\infty} (e^{\alpha x} - 1)\rho(dx)\right\} \frac{1 - \exp\{-\int_{-\infty}^{-\varepsilon}(1 - e^{\alpha x})\rho(dx)\}}{\int_{-\infty}^{-\varepsilon}(1 - e^{\alpha x})\rho(dx)}. \\ &\quad := l(\varepsilon). \end{aligned}$$

The probability tail of $\sup_{0\leqslant t\leqslant 1} X_2(t)$ is lighter than that of $\sup_{0\leqslant t\leqslant 1} X_1(t)$, and so we conclude by Lemma 1.1(ii) that

$$\lim_{\lambda\to\infty} \frac{P(\sup_{0\leqslant t\leqslant 1} X(t) > \lambda)}{\rho(\lambda, \infty)} \leqslant l(\varepsilon) E \exp\left\{\alpha \sup_{0\leqslant t\leqslant 1} X_2(t)\right\}, \tag{3.40}$$

and now (3.36) follows from the obvious fact that the right-hand side of (3.40) converges to the right-hand side of the former when $\varepsilon \to 0$. □

Remark. Of course, one can use (3.36) and subadditivity to derive an upper bound on c when a drift and/or Brownian component is present. Furthermore, one can get tighter than (3.35) lower bounds on c by minorizing stochastically $\sup_{0\leqslant t\leqslant 1} X(t)$ by the maximum of the process observed at the point i/n, $i = 0, 1, \ldots, n$ for some $n > 1$ and then appealing to (3.8). The resulting bounds are somewhat less transparent than (3.35), and so are not presented here.

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