



# Reflecting Brownian snake and a Neumann–Dirichlet problem

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## Abstract

The paper deals with a path-valued Markov process: the reflecting Brownian snake. It is a particular case of the path-valued process previously introduced by Le Gall. Here the spatial motion is a reflecting Brownian motion in a domain  $D$  of  $\mathbb{R}^d$ . Using this probabilistic tool, we construct an explicit function  $v$  solution of an integral equation which is, under some hypotheses on the regularity of  $v$ , equivalent to a semi-linear partial differential equation in  $D$  with some mixed Neumann–Dirichlet conditions on the boundary. When the hypotheses on  $v$  are not satisfied, we prove that  $v$  is still solution of a weak formulation of the Neumann–Dirichlet problem. © 2000 Elsevier Science B.V. All rights reserved.

**Keywords:** Brownian snake; Reflecting Brownian motion; Semi-linear partial differential equations; Neumann problem

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## 1. Introduction

The connections between superprocesses and partial differential equations have been pointed out by Dynkin (1991). Indeed, by considering the exit measure of a superdiffusion, Dynkin gave a probabilistic representation of nonnegative solutions of a semilinear Dirichlet problem. This representation was first used to study probabilistic properties of superprocesses (see for instance Dawson et al., 1989) but, more recently (see the works of Dynkin and Kuznetsov, 1995; Le Gall (1997), etc.) this probabilistic tool made possible to prove new analytic results and is still the object of active research.

The aim of this paper is to give a probabilistic approach to the analogous Neumann problem. We are interested in the nonnegative solutions of the partial differential equation  $\Delta u = 4u^2$  in a smooth domain  $D$  of  $\mathbb{R}^d$ . The main tool of the study is the path-valued Markov process (called Brownian snake) introduced by Le Gall (1993). This process is a powerful tool in the study of path properties of superprocesses and

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also for the probabilistic representation of solutions of the Dirichlet problem associated to the partial differential equation.

The Neumann problem for the equation  $\Delta u = 0$  in a domain has been studied by Brosamler (1976) via reflecting Brownian motion in the domain. So, it seems natural to consider a Brownian snake with a reflecting Brownian motion as spatial motion to tackle the Neumann problem. We call this process the reflecting Brownian snake. The reflecting Brownian snake is a Markov process  $(W_s, \zeta_s)$  which takes values in the set of stopped paths in  $\mathbb{R}^d$ . Heuristically, one can think of  $W_s$  as a reflecting Brownian path in the considered domain, stopped at the random time  $\zeta_s$ . This lifetime evolves according to a one-dimensional reflecting Brownian motion. When  $\zeta_s$  decreases, the path  $W_s$  is erased from its final point, and when  $\zeta_s$  increases, the path  $W_s$  is extended independently of the past. A more rigorous description is given in Section 3. We will also introduce the excursion measure  $\mathbb{N}_x$  of this process which plays a key role here. It is an infinite measure which represents the “law” of the Brownian snake when considering for the lifetime process only a Brownian excursion instead of a whole reflecting Brownian motion.

For technical reasons which will be explained later, we must stop the trajectories. Here, they will be stopped when they reach a set on the boundary. In return, we only get a mixed Neumann–Dirichlet problem: we have a Dirichlet condition on the set where the paths are stopped.

Let us denote by  $\partial D$  the topological boundary of  $D$ . Let  $F$  be a closed subset of  $\partial D$  and let  $\tau(w)$  be the hitting time of  $F$  of the path  $w$ . We now stop the paths of the process  $(W_s)$  at time  $\tau(W_s)$  and construct two measures on the boundary  $X_F^D$  and  $\tilde{X}_F^D$ . The measure  $X_F^D$  charges the endpoints of the paths of the snake that touch  $F$  and therefore has support included in  $F$ . It is the analogous of the exit measure of Le Gall (1994b). The measure  $\tilde{X}_F^D$  charges all the reflecting points of the paths of the snake before their absorption on  $F$ . Its support is included in  $F^c = \partial D \setminus F$ .

Let  $f$  be a bounded continuous function on  $F$  and  $g$  be a bounded continuous function on  $F^c$ . The Neumann–Dirichlet problem considered is the following:

$$\begin{aligned} \Delta u &= 4u^2 \quad \text{on } D, \\ u &= f \quad \text{on } \mathring{F}, \\ \frac{\partial u}{\partial n} &= 2g \quad \text{on } F^c, \end{aligned} \tag{1}$$

where  $n$  is the outward unit normal vector and  $\mathring{F}$  represents the interior of  $F$  (viewed as a subset of the topological space  $\partial D$ ). We consider the function  $v(x) = \mathbb{N}_x[1 - \exp - (\langle X_F^D, f \rangle + \langle \tilde{X}_F^D, g \rangle)]$  with  $\langle \mu, \varphi \rangle$  representing the integral of  $\varphi$  with respect to the measure  $\mu$ . We shall prove in Section 4 that function  $v$  is solution of an integral equation. Owing to this equation, we prove that, if  $f$  and  $g$  are continuous and if  $v$  is of class  $\mathcal{C}^1$  on the closure of  $D$ ,  $v$  is a nonnegative solution of problem (1). Finally, we shall prove that, even if  $v$  is not  $\mathcal{C}^1$ , it is still a weak solution of the problem provided that  $f$  is Hölder.

We shall recall in Section 2 the definition of the reflecting Brownian motion in terms of transition density kernels. For this definition to apply, we need to consider only

sufficiently smooth domains, namely of class  $\mathcal{C}^3$ . We will then construct a local time on the boundary and recall some fundamental properties of reflecting Brownian motion. We also give alternative definitions of this process. For more results on reflecting Brownian motion see Hsu (1984). For a general definition of diffusions with reflection in a domain, see Stroock and Varadhan (1971). In Section 3, we shall briefly present the construction of the reflecting Brownian snake (see Le Gall, 1993 and Le Gall, 1994b for a comprehensive presentation of the Brownian snake) and we shall also define the two measures on the boundary as Revuz measures of some additive functionals of the reflecting Brownian snake. We finally prove the main results in Section 4.

**Notations.** We denote by  $\mathbb{R}_+$  the real halfline  $[0, +\infty)$  and by  $\mathbb{R}_+^*$  the interval  $(0, +\infty)$ . The indicator function of a set  $A$  will be denoted by  $\mathbb{1}_A$ . We denote by  $C$  a generic constant which may change from line to line.

## 2. The reflecting Brownian motion

### 2.1. Definition and first properties

Let  $D$  be a bounded domain of  $\mathbb{R}^d$  with  $\mathcal{C}^3$  boundary. We can construct the standard reflecting Brownian motion (with normal reflection) in  $D$ . It is a  $\bar{D}$ -valued diffusion process  $(X_t)_{t \geq 0}$  whose transition density kernels satisfy

$$\forall (t, x, y) \in \mathbb{R}_+^* \times D \times D \quad \frac{\partial}{\partial t} p(t, x, y) = \frac{1}{2} \Delta_x p(t, x, y),$$

$$\forall (x, y) \in \bar{D} \times \bar{D} \quad \lim_{t \rightarrow 0} p(t, x, y) = \delta_y(x),$$

$$\forall (t, x, y) \in \mathbb{R}_+^* \times \partial D \times \bar{D} \quad \frac{\partial}{\partial n_x} p(t, x, y) = 0,$$

where  $n_x$  is the outward unit normal vector at  $x \in \partial D$ . For existence and uniqueness of such a process, see e.g. Itô (1957) or Sato and Ueno (1965). Let us recall that  $p$  also satisfies the following properties

- The function  $p: \mathbb{R}_+^* \times \bar{D} \times \bar{D} \rightarrow \mathbb{R}_+^*$  is continuous.
- For every fixed  $(x, y) \in \bar{D} \times \bar{D}$ , the function  $t \mapsto p(t, x, y)$  is  $\mathcal{C}^1$ .
- For every fixed  $(t, y) \in \mathbb{R}_+ \times \bar{D}$ , the function  $x \mapsto p(t, x, y)$  is in  $\mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$ .
- $p$  is symmetric in  $x$  and  $y$ .

We denote by  $P_x$  its law when it starts at point  $x \in \bar{D}$  and  $E_x$  the expectation relative to  $P_x$ .

**Proposition 2.1** (Chung and Hsu, 1984). *Let  $\tilde{p}(t, x, y)$  be the transition density function of the standard Brownian motion in  $\mathbb{R}^d$ .*

*Then,  $p(t, x, y)$  can be written in the form*

$$p(t, x, y) = p_0(t, x, y) + p_1(t, x, y)$$

with

(a) There exist positive constants  $C_1, C_2$  and  $a$  such that

$$C_2 \tilde{p}(t, x, y) \leq p_0(t, x, y) \leq C_1 \tilde{p}(at, x, y).$$

(b)  $p_1(t, x, y) = \int_0^t \int_D \int_D p_0(t-u, x, z) q(u, z, y) \, dz \, du$  with  $\sup_{y \in \mathbb{R}^d} \int_D |q(t, x, y)| \, dx \leq C/\sqrt{t}$ .

**Proposition 2.2.** For every measurable positive functions  $f$  and  $g$  on  $D$  and for every  $t \geq 0$ , we have,

$$\int_D E_x[f(X_t)] g(x) \, dx = \int_D f(x) E_x[g(X_t)] \, dx.$$

**Proof.** This is a consequence of the fact that the function  $p(t, \cdot, \cdot)$  is symmetric.  $\square$

**Proposition 2.3.** If  $\varphi \in L^1(\tilde{D})$ , then for every  $t > 0$ ,  $x \mapsto E_x[\varphi(X_t)]$  is continuous on  $\tilde{D}$ .

## 2.2. Local time on the boundary

For  $\varepsilon > 0$ , we set

$$D_\varepsilon = \{x \in \tilde{D} \mid d(x, \partial D) \leq \varepsilon\}.$$

Then, we define the local time of  $X$  on  $\partial D$  by

$$\forall t \geq 0, \quad \ell_t^D = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{D_\varepsilon}(X_s) \, ds,$$

where the limit exists in  $L^2$  and a.s. (Sato and Tanaka, 1962).

**Proposition 2.4** (Sato and Ueno, 1965). The process  $\ell^D$  is a continuous additive functional of the reflecting Brownian motion  $X$  which increases a.s. only on the set of times  $t$  such that  $X_t \in \partial D$ .

**Proposition 2.5.** Let  $\varphi$  be a bounded measurable function on  $\tilde{D}$ . Then, for all  $t \geq 0$  and all  $x \in \tilde{D}$ ,

$$E_x \left[ \int_0^t \varphi(X_s) \, d\ell_s^D \right] = \int_0^t \, ds \int_{\partial D} \varphi(y) p(s, x, y) \sigma(dy),$$

where  $\sigma$  is the area measure on  $\partial D$ .

**Proposition 2.6** (Hsu, 1984). For every integer  $n$ , there exists a constant  $K_n$  such that, for all  $t \geq 0$ ,

$$\sup_{x \in \tilde{D}} E_x[(\ell_t^D)^n] \leq K_n t^{n/2}.$$

We presented the reflecting Brownian motion via its transition densities but this definition only applies for sufficiently smooth domains. We will restrict ourselves here to that case but, we can give another definition of this process via a stochastic differential equation involving the boundary local time.

**Proposition 2.7** (see Chung and Hsu, 1984 or Lions and Sznitman, 1984). *The reflecting Brownian motion  $X$  is solution of the Skorokhod problem*

$$dX_t = dB_t - \frac{1}{2}n(X_t) d\ell_t^D.$$

This proposition can be viewed as an alternative definition of the reflecting Brownian motion. It is stronger than the definition via transition densities as it also apply for less smooth domains (see for instance Bass, 1996 and the references therein).

We also give another characterisation of the reflecting Brownian motion in terms of a martingale problem (see El Karoui and Chaleyat-Maurel, 1978 for a general presentation of this approach).

**Proposition 2.8.** *For every function  $\varphi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$ ,*

$$\varphi(X_t) - \varphi(X_0) - \frac{1}{2} \int_0^t \Delta \varphi(X_s) ds + \frac{1}{2} \int_0^t \frac{\partial \varphi}{\partial n}(X_s) d\ell_s^D$$

*is a  $P^x$ -martingale for all  $x \in \bar{D}$ .*

**Proof.** This is a consequence of Proposition 2.7 and the Itô formula.  $\square$

### 3. Reflecting Brownian snake and measures on the boundary

#### 3.1. The reflecting Brownian snake

As  $X$  is a Markov process in  $\mathbb{R}^d$ , we can construct as in Le Gall (1993) the Brownian snake with spatial motion  $X$  (in Le Gall, 1993, we need an hypothesis on the continuity of the process that has been removed in Bertoin et al. (1997) by changing the topological structure on  $\mathcal{W}$ ). More precisely, we define  $\mathcal{W}$  as the set of killed paths in  $\mathbb{R}^d$ . A killed path is a continuous function  $w: [0, \zeta] \rightarrow \mathbb{R}^d$ , where  $\zeta \in [0, +\infty)$  is called the lifetime of  $w$ . In the following, we will denote  $\hat{w} = w(\zeta)$ . Notice that every point  $x$  in  $\mathbb{R}^d$  can be viewed as a killed path of  $\mathcal{W}$  with lifetime 0. We endowed  $\mathcal{W}$  with the following distance:

$$d(w, w') = \|w(0) - w'(0)\| + |\zeta - \zeta'| + \int_0^{\zeta \wedge \zeta'} (d_u(w^{(u)}, w'^{(u)}) \wedge 1) du,$$

where  $w^{(u)}$  is the restriction of  $w$  to  $[0, u]$ , and  $d_u$  denotes the Skorokhod distance on the set of the càdlàg functions on  $[0, u]$ . It is easy to check that  $(\mathcal{W}, d)$  is a Polish space.

We now fix  $x \in \mathbb{R}^d$  and denote by  $\mathcal{W}_x$  the set of killed paths starting at  $x$ . We also denote by  $\mathcal{W}_x^D$  the set of paths of  $\mathcal{W}_x$  that stay in  $\bar{D}$ . The reflecting Brownian snake with starting point  $x$  is a continuous strong Markov process with values in  $\mathcal{W}_x^D$ . We denote by  $(W_s, s \geq 0)$  this process and we set  $\zeta_s$  the lifetime of  $W_s$ . We denote by  $\mathbb{P}_w$  the law of  $(W_s)$  starting at  $w \in \mathcal{W}_x^D$ . It is characterised by the following properties:

- (i) The law of the lifetime process  $(\zeta_s)$  under  $\mathbb{P}_w$  is that of a one-dimensional reflecting Brownian motion starting from  $\zeta_w$ .

(ii) Under  $\mathbb{P}_w$ , given the process  $(\zeta_s)$ , the process  $(W_s)$  is still an (inhomogeneous) Markov process whose transition kernels are described by

$$\text{Let } s < s' \text{ and let } m_{s,s'} = \inf_{u \in [s,s']} \zeta_u.$$

The paths  $W_s$  and  $W_{s'}$  coincide until time  $m_{s,s'}$  and then, conditionally on  $W_s(m_{s,s'})$ , the process  $(W_{s'}(t+m_{s,s'}), t \geq 0)$  is a reflecting Brownian motion in  $D$  starting from  $W_s(m_{s,s'})$  independent from  $(W_{s'}(t), 0 \leq t \leq m_{s,s'})$ .

We refer to Le Gall (1993) and Le Gall (1994b) for more precise construction and properties of this process.

We denote by  $x$  the trivial path with lifetime 0 reduced to point  $x$ . As 0 is a regular point for the process  $(\zeta_s)$ , it is clear that  $x$  is a regular point for the process  $(W_s)$ . Consequently, we can define the excursion measure of  $(W_s)$  out of  $x$ , denoted by  $\mathbb{N}_x$ .

This measure is an infinite measure which is characterized in the same way as  $\mathbb{P}_w$  by (i)' Under  $\mathbb{N}_x$ , the “law” of the lifetime process  $(\zeta_s)$  is Itô measure of positive

Brownian excursions with the normalisation:

$$\mathbb{N}_x \left( \sup_{s \geq 0} \zeta_s > \varepsilon \right) = \frac{1}{2\varepsilon}.$$

(ii) still holds.

We will also need another probability measure  $\mathbb{P}_x^*$ . This is the distribution of the process  $(W_s)$  starting from  $w \in \mathcal{W}_x^D$  and stopped when the lifetime process first reaches 0. We can describe the probability  $\mathbb{P}_w^*$  in the following way: we set  $\sigma = \inf\{s \geq 0, \zeta_s = 0\}$  and  $\zeta_s^* = \inf_{(0 \leq u \leq s)} \zeta_u$ . We denote by  $(\alpha_i, \beta_i)$ ,  $i \in I$  the excursion interval of  $\zeta - \zeta^*$  away from 0 before time  $\sigma$  and for every  $i \in I$  and  $s \geq 0$ , we set

$$W_s^i(t) = W_{(\alpha_i+s) \wedge \beta_i}(\zeta_{\alpha_i} + t), \quad 0 \leq t \leq \zeta_{(\alpha_i+s) \wedge \beta_i} - \zeta_{\alpha_i}.$$

**Proposition 3.1** (Le Gall, 1994b). *Under  $\mathbb{P}_w^*$ , the random measure*

$$\sum_{i \in I} \delta_{(\zeta_{\alpha_i}, W^i)}$$

*is a Poisson point measure on  $[0, \zeta(w)] \times \mathcal{C}(\mathbb{R}_+, \mathcal{W})$  with intensity*

$$2dt \mathbb{N}_{w(t)}(dk).$$

### 3.2. The measures on the boundary

We fix  $x \in \bar{D}$ . Let  $F$  be a closed subset of  $\partial D$ . We write  $\mathring{F}$  for the interior of  $F$  (with respect to the topology on  $\partial D$ ) and  $\partial F = F \setminus \mathring{F}$ . We always suppose that  $\mathring{F} \neq \emptyset$  and that  $\sigma(\partial F) = 0$ . If  $w \in \mathcal{W}_x^D$ , we set

$$\tau(w) = \inf\{t \in [0, \zeta] \mid w(t) \in F\}.$$

**Lemma 3.2.** *For every  $x \in \bar{D}$ ,  $P_x(\tau < +\infty) = 1$ .*

**Proof.** The polar sets for the reflecting Brownian motion in  $D$  and the standard Brownian motion are the same since  $\partial D$  is smooth.  $\square$

**Lemma 3.3.** *The application  $x \mapsto P_x(\tau \leq t)$  is lower semi-continuous for every  $t > 0$ .*

**Proof.** Let  $0 < s < t$ . Then,

$$x \mapsto P_x(\exists u \in [s, t], X_u \in F) = \int p(s, x, y) P_y(\tau \leq t - s) dy$$

is continuous using the regularity of  $p$ . Moreover, it converges to  $P_x(\tau < t)$  as  $s$  tends to 0 and the convergence is nondecreasing.  $\square$

**Lemma 3.4.** *We have  $\sup_{x \in \bar{D}} E_x[\tau] < \infty$ .*

**Proof.** Let  $\delta = \inf_{x \in \bar{D}} P_x(\tau \leq 1)$ . Because of the preceding lemma and because of  $\mathbb{P}_x(\tau \leq 1) > 0$  for all  $x \in \bar{D}$ ,  $\delta > 0$ . Applying the strong Markov property, we have for every  $x \in \bar{D}$ ,

$$\begin{aligned} P_x(\tau > n) &= E_x[\mathbb{1}_{\{\tau > n-1\}} P_{X_{n-1}}(\tau > 1)] \\ &\leq (1 - \delta) P_x(\tau > n - 1). \end{aligned}$$

So  $\sup_{x \in \bar{D}} P_x(\tau > n) \leq (1 - \delta)^n$ .

Then,

$$\begin{aligned} E_x[\tau] &= \int_0^{+\infty} P_x(\tau > y) dy \\ &\leq \sum_{n=0}^{\infty} P_x(\tau > n) \leq \frac{1}{\delta} < \infty. \quad \square \end{aligned}$$

**Lemma 3.5.** *We have  $\sup_{x \in \bar{D}} E_x[\ell_\tau^D] < \infty$ .*

**Proof.** Since  $\tau < +\infty$  a.s. we have, using the same notations as in the previous proof, for every  $x \in \bar{D}$ ,

$$\begin{aligned} E_x[\ell_\tau^D] &= \sum_{n=0}^{\infty} E_x[\mathbb{1}_{\{n \leq \tau < n+1\}} \ell_\tau^D] \\ &\leq \sum_{n=0}^{\infty} E_x[\mathbb{1}_{\{n \leq \tau\}} \ell_{n+1}^D] \\ &\leq \sum_{n=0}^{\infty} P_x(n \leq \tau)^{1/2} E_x[(\ell_{n+1}^D)^2]^{1/2} \\ &\leq \sum_{n=0}^{\infty} (1 - \delta)^{n/2} K_2(n+1)^{1/2} < +\infty \end{aligned}$$

by Proposition 2.6.  $\square$

**Remark.** These proofs can be adapted to show that  $\tau$  and  $\ell_\tau^D$  have moments of any order greater than 1.

An additive functional is uniquely defined by its characteristic measure if this measure is of finite energy (see Azéma, 1973 or Dynkin, 1981. See also Dthersin and Le Gall, 1997 for additive functionals of the Brownian snake.). If  $\mu$  is a measure on  $\mathcal{W}_x$ , the computations of Le Gall (1994a) give that the energy of  $\mu$  (with respect to the symmetric Markov process  $(W_s)$ ) is given by

$$\mathcal{E}(\mu) = 2E_x \left[ \int_0^\infty \left( \frac{d\mu_{(t)}}{dP_{x|\mathcal{G}_t}} \right)^2 dt \right].$$

Here,  $(\mathcal{G}_t)$  is the  $\sigma$ -field on  $\mathcal{W}_x$  generated by the coordinate mappings  $f \mapsto (f(r), 0 \leq r \leq t)$ . We denote by  $\mu_{(t)}$  the restriction of  $\mu$  to paths whose lifetime is greater than  $t$ , viewed as a measure on  $\mathcal{G}_t$ . Finally,  $P_{x|\mathcal{G}_t}$  represents the restriction of  $P_x$  to the  $\sigma$ -field  $\mathcal{G}_t$ .

**Remark.** The references above (Le Gall, 1994a; Dthersin and Le Gall, 1997) only deal with the case of a Brownian motion as spatial motion but it is not hard to check that the results also apply for other continuous Markov processes including reflecting Brownian motion.

Let  $\mu_1$  and  $\mu_2$  be the measures on  $\mathcal{W}_x$  defined for every bounded nonnegative measurable function  $\varphi$  by

$$\begin{aligned} \langle \mu_1, \varphi \rangle &= E_x[\varphi(X^{(\tau)})], \\ \langle \mu_2, \varphi \rangle &= E_x \left[ \int_0^\tau d\ell_s^D \varphi(X^{(s)}) \right], \end{aligned}$$

where we have written  $w^{(t)}$  for the path  $w_{\cdot \wedge t}$ .

**Proposition 3.6.**

$$\begin{aligned} \mathcal{E}(\mu_1) &= 2E_x(\tau), \\ \mathcal{E}(\mu_2) &= 2E_x \left[ \int_0^\tau dt E_{X_t}[\ell_\tau^D]^2 \right]. \end{aligned}$$

**Proof.** The first computation can be found in Le Gall (1994a). For the second equality, let  $\varphi$  be a  $\mathcal{G}_t$ -measurable nonnegative function. Then, we have

$$\begin{aligned} \langle \mu_{2(t)}, \varphi \rangle &= E_x \left[ \int_0^\tau \mathbb{1}_{\{s > t\}} \varphi(X^{(s)}) d\ell_s^D \right] \\ &= E_x \left[ \mathbb{1}_{\{t < \tau\}} \int_t^\tau \varphi(X^{(t)}) d\ell_s^D \right] \\ &= E_x[\mathbb{1}_{\{t < \tau\}} \varphi(X^{(t)})(\ell_\tau^D - \ell_t^D)] \\ &= E_x[\mathbb{1}_{\{t < \tau\}} \varphi(X^{(t)})E_{X_t}[\ell_\tau^D]]. \end{aligned}$$



So,

$$\frac{d\mu_{2(t)}}{dP_{x|g_t}} = \mathbb{1}_{\{t < \tau\}} E_{X_t}[\ell_\tau^D]$$

which gives the desired result.  $\square$

Lemmas 3.4 and 3.5 and Proposition 3.6 prove that  $\mu_1$  and  $\mu_2$  are of finite energy. Consequently, we can define two additive functionals of  $W$ , denoted, respectively, by  $L_s^F$  and  $\tilde{L}_s^F$ , by the following property.

**Definition 3.7.** For every bounded measurable function  $\varphi: \mathcal{W}_x \rightarrow \mathbb{R}_+$ ,

$$\mathbb{N}_x \left[ \int_0^\infty dL_s^F \varphi(W_s) \right] = E_x[\varphi(X^{(\tau)})],$$

$$\mathbb{N}_x \left[ \int_0^\infty d\tilde{L}_s^F \varphi(W_s) \right] = E_x \left[ \int_0^\tau d\ell_s^D \varphi(X^{(s)}) \right].$$

The additive functional  $L^F$  increases a.s. only when  $\zeta_s = \tau(W_s)$  whereas  $\tilde{L}^F$  increases a.s. only when  $\zeta_s < \tau$  and  $\hat{W}_s \in F^c$ .

We then define (under  $\mathbb{N}_x$ ) two measures on  $\partial D$ , for every bounded measurable function  $\phi$  on  $\partial D$ , by

$$\langle X_F^D, \phi \rangle = \int_0^\infty dL_s^F \phi(\hat{W}_s), \quad (2)$$

$$\langle \tilde{X}_F^D, \phi \rangle = \int_0^\infty d\tilde{L}_s^F \phi(\hat{W}_s). \quad (3)$$

We also set  $A_s^F = L_s^F + \tilde{L}_s^F$  and  $Z_F^D = X_F^D + \tilde{X}_F^D$ .

**Remark.** If we did not stop the trajectories at time  $\tau$ , the measure  $\mu_2$  which defines the local time  $\tilde{L}_t^F$  is not of finite energy and therefore formula (3) does not define correctly the additive functional.

#### 4. The Neumann–Dirichlet problem

In this section, we make the following hypothesis on  $F: \mathring{F} \neq \emptyset$ ,  $\partial F$  is  $\partial$ -polar for the Brownian motion and  $\mathring{F}$  is connected and has a  $\mathcal{C}^2$  boundary.

##### 4.1. Strong solutions

Let  $f$  and  $g$  be two bounded nonnegative functions on  $\partial D$ . We assume that  $f$  and  $g$  are continuous, respectively, on  $F$  and  $F^c$ . We say that  $u$  is a strong solution of the Neumann–Dirichlet problem  $ND(f, g)$  if

- (i)  $u \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$ .
- (ii)  $\Delta u = 4u^2$  on  $D$ .

- (iii)  $u = f$  on  $\tilde{F}$ .
- (iv)  $\partial u / \partial n = 2g$  on  $F^c$ .

We set  $\phi_{f,g} = f\mathbb{1}_F + g\mathbb{1}_{F^c}$  and, for every  $x \in \tilde{D}$ , we set

$$v(x) = \mathbb{N}_x[1 - \exp - \langle Z_F^D, \phi_{f,g} \rangle].$$

**Proposition 4.1.** *The function  $v$  is bounded on  $\tilde{D}$ .*

**Proof.** We have, for every  $x \in \tilde{D}$ ,

$$\begin{aligned} v(x) &\leq \mathbb{N}_x[\langle Z_F^D, \phi_{f,g} \rangle] \\ &= \mathbb{N}_x[\langle X_F^D, f \rangle] + \mathbb{N}_x[\langle \tilde{X}_F^D, g \rangle] \\ &= E_x[f(X_\tau)] + E_x \left[ \int_0^\tau d\ell_s^D g(X_s) \right], \end{aligned}$$

by Definition 3.7.

To conclude, notice that  $f$  and  $g$  are bounded and  $E_x[\ell_\tau^D]$  is finite.  $\square$

**Theorem 4.2.** *The function  $v$  is a solution of the integral equation*

$$\forall x \in \tilde{D}, \quad u(x) + 2E_x \left[ \int_0^\tau u(X_s)^2 ds \right] = E_x[f(X_\tau)] + E_x \left[ \int_0^\tau g(X_s) d\ell_s^D \right]. \quad (4)$$

**Proof.** We follow the ideas of Le Gall (1994b, Theorem 4.2). Under  $\mathbb{N}_x$ ,  $\zeta$  is distributed as a positive Brownian excursion. Thus we can define, under  $\mathbb{N}_x$  the length of that excursion, denoted by  $\sigma$ . By definition of the measure  $Z_F^D$  we have, for every  $x \in \tilde{D}$ ,

$$\begin{aligned} v(x) &= \mathbb{N}_x[1 - \exp - \langle Z_F^D, \phi_{f,g} \rangle] \\ &= \mathbb{N}_x \left[ 1 - \exp - \int_0^\sigma dA_s^F \phi_{f,g}(\hat{W}_s) \right] \\ &= \mathbb{N}_x \left[ \int_0^\sigma dA_s^F \phi_{f,g}(\hat{W}_s) \exp - \int_s^\sigma dA_u^F \phi_{f,g}(\hat{W}_u) \right]. \end{aligned}$$

We replace  $\exp - \int_s^\sigma dA_u^F \phi_{f,g}(\hat{W}_u)$  by its predictable projection

$$v(x) = \mathbb{N}_x \left[ \int_0^\sigma dA_s^F \phi_{f,g}(\hat{W}_s) \mathbb{E}_{\mathcal{W}_s}^* \left[ \exp - \int_0^\sigma dA_u^F \phi_{f,g}(\hat{W}_u) \right] \right].$$

Then, using Proposition 3.1 and the exponential formula for Poisson measures, we have

$$\begin{aligned} v(x) &= \mathbb{N}_x \left[ \int_0^\sigma dA_s^F \phi_{f,g}(\hat{W}_s) \exp - 2 \int_0^{\zeta_s} du \mathbb{N}_{\mathcal{W}_s(u)}(1 - \exp - \langle Z_F^D, \phi_{f,g} \rangle) \right] \\ &= \mathbb{N}_x \left[ \int_0^\sigma dA_s^F \phi_{f,g}(\hat{W}_s) \exp - 2 \int_0^{\zeta_s} du v(W_s(u)) \right]. \end{aligned}$$

Definition 3.7 of the additive functionals finally leads to the formula

$$\begin{aligned} v(x) = E_x \left[ \phi_{f,g}(X_\tau) \exp - 2 \int_0^\tau du v(X_u) \right] \\ + E_x \left[ \int_0^\tau d\ell_s^D \phi_{f,g}(X_s) \exp - 2 \int_0^s du v(X_u) \right] \end{aligned}$$

which can be written, by noticing that  $P_x$ -a.s.  $X_\tau \in F$  and  $P_x$ -a.s.  $d\ell_s^D$ -a.e on  $\{s < \tau\}$ ,  $X_s \in F^c$ ,

$$\begin{aligned} v(x) = E_x \left[ f(X_\tau) \exp - 2 \int_0^\tau du v(X_u) \right] \\ + E_x \left[ \int_0^\tau d\ell_s^D g(X_s) \exp - 2 \int_0^s du v(X_u) \right]. \end{aligned} \quad (5)$$

We rewrite formula (5) into

$$\begin{aligned} v(x) = E_x[f(X_\tau)] - E_x \left[ f(X_\tau) \left( 1 - \exp - 2 \int_0^\tau du v(X_u) \right) \right] \\ + E_x \left[ \int_0^\tau d\ell_s^D g(X_s) \right] - E_x \left[ \int_0^\tau d\ell_s^D g(X_s) \left( 1 - \exp - 2 \int_0^s du v(X_u) \right) \right]. \end{aligned}$$

Now, let us compute

$$\begin{aligned} E_x \left[ f(X_\tau) \left( 1 - \exp - 2 \int_0^\tau du v(X_u) \right) \right] \\ = 2E_x \left[ f(X_\tau) \int_0^\tau dr v(X_r) \exp - 2 \int_r^\tau du v(X_u) \right] \\ = 2 \int_0^\infty dr E_x \left[ \mathbb{1}_{\{r < \tau\}} v(X_r) f(X_\tau) \exp - 2 \int_r^\tau du v(X_u) \right] \\ = 2 \int_0^\infty dr E_x \left[ \mathbb{1}_{\{r < \tau\}} v(X_r) E_{X_r} \left[ f(X_\tau) \exp - 2 \int_0^\tau du v(X_u) \right] \right]. \end{aligned}$$

An analogous computation gives

$$\begin{aligned} E_x \left[ \int_0^\tau d\ell_s^D g(X_s) \left( 1 - \exp - 2 \int_0^s du v(X_u) \right) \right] \\ = 2 \int_0^\infty dr E_x \left[ \mathbb{1}_{\{r < \tau\}} v(X_r) E_{X_r} \left[ \int_0^\tau d\ell_s^D g(X_s) \exp - 2 \int_0^s du v(X_u) \right] \right]. \end{aligned}$$

Thus,

$$\begin{aligned} E_x \left[ f(X_\tau) \left( 1 - \exp - 2 \int_0^\tau du v(X_u) \right) \right] \\ + E_x \left[ \int_0^\tau d\ell_s^D g(X_s) \left( 1 - \exp - 2 \int_0^s du v(X_u) \right) \right] \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^\infty dr E_x \left[ \mathbf{1}_{\{r < \tau\}} v(X_r) \left( E_{X_r} \left[ f(X_\tau) \exp - 2 \int_0^\tau du v(X_u) \right] \right. \right. \\
&\quad \left. \left. + E_{X_r} \left[ \int_0^\tau d\ell_s^D g(X_s) \exp - 2 \int_0^s du v(X_u) \right] \right) \right] \\
&= 2 \int_0^\infty dr E_x [\mathbf{1}_{\{r < \tau\}} v(X_r)^2] = 2E_x \left[ \int_0^\tau v(X_r)^2 \right]
\end{aligned}$$

by formula (5).  $\square$

**Proposition 4.3.** *For every  $y \in \mathring{F}$ ,  $\lim_{x \rightarrow y} v(x) = f(y)$ .*

**Proof.** Let  $y \in \mathring{F}$ . Then, as  $\partial D$  is regular and  $D$  bounded, we have

$$\begin{aligned}
\lim_{x \rightarrow y} E_x \left[ \int_0^\tau ds v(X_s)^2 \right] &= 0, \\
\lim_{x \rightarrow y} E_x[f(X_\tau)] &= f(y), \\
\lim_{x \rightarrow y} E_x \left[ \int_0^\tau g(X_s) d\ell_s^D \right] &= 0
\end{aligned}$$

using the boundedness of  $v$  and Lemma 3.4, the continuity of  $f$ , and the boundedness of  $g$  and Lemma 3.5. Then taking the limit into formula (4) completes the proof.  $\square$

**Proposition 4.4.** *Let  $T$  be a stopping time. Then, for every  $x \in \bar{D}$ ,*

$$\begin{aligned}
v(x) + 2E_x \left[ \int_0^{T \wedge \tau} v(X_s)^2 ds \right] \\
= E_x[v(X_T) \mathbf{1}_{\{T < \tau\}}] + E_x[f(X_\tau) \mathbf{1}_{\{T \geq \tau\}}] + E_x \left[ \int_0^{T \wedge \tau} d\ell_s^D g(X_s) \right].
\end{aligned}$$

**Proof.** We apply the strong Markov property at time  $T \wedge \tau$  and then formula (4) to get

$$\begin{aligned}
E_x \left[ \int_0^\tau v(X_s)^2 ds \right] &= E_x \left[ \int_0^{T \wedge \tau} v(X_s)^2 ds \right] + E_x \left[ \int_{T \wedge \tau}^\tau v(X_s)^2 ds \right] \\
&= E_x \left[ \int_0^{T \wedge \tau} v(X_s)^2 ds \right] + E_x \left[ E_{X_{T \wedge \tau}} \left[ \int_0^\tau v(X_s)^2 ds \right] \right] \\
&= E_x \left[ \int_0^{T \wedge \tau} v(X_s)^2 ds \right] + \frac{1}{2} E_x \left[ E_{X_{T \wedge \tau}} \left[ \int_0^\tau d\ell_s^D g(X_s) \right] \right] \\
&\quad + \frac{1}{2} E_x[E_{X_{T \wedge \tau}}[f(X_\tau)]] - \frac{1}{2} E_x[v(X_{T \wedge \tau})] \\
&= E_x \left[ \int_0^{T \wedge \tau} v(X_s)^2 ds \right] + \frac{1}{2} E_x \left[ \int_{T \wedge \tau}^\tau d\ell_s^D g(X_s) \right] + \frac{1}{2} E_x[f(X_\tau)] - \frac{1}{2} E_x[v(X_{T \wedge \tau})].
\end{aligned}$$

Moreover, formula (4) directly gives that

$$E_x \left[ \int_0^\tau v(X_s)^2 ds \right] = \frac{1}{2} E_x \left[ \int_0^\tau d\ell_s^D g(X_s) \right] + \frac{1}{2} E_x[f(X_\tau)] - \frac{1}{2} v(x).$$

So, we have

$$v(x) + 2E_x \left[ \int_0^{T \wedge \tau} v(X_s)^2 ds \right] = E_x[v(X_{T \wedge \tau})] + E_x \left[ \int_0^{T \wedge \tau} g(X_s) d\ell_s^D \right].$$

Now, as  $X_\tau \in F$   $P_x$ -a.s., the desired result is a consequence of Proposition 4.3.  $\square$

**Proposition 4.5.** *The function  $v$  is  $\mathcal{C}^2$  on  $D$  and is a solution of  $\Delta u = u^2$  in  $D$ .*

**Proof.** Let  $x \in D$ . There exists  $r > 0$  such that  $\overline{B(x, r)} \subset D$ . We set  $T = \inf\{t \geq 0, X_t \notin B(x, r)\}$  and apply Proposition 4.4 at that stopping time. As  $T < \tau$  and  $\ell^D = 0$  on  $[0, T]$   $P_x$ -a.s., we have

$$v(x) + 2E_x \left[ \int_0^T v(X_s)^2 ds \right] = E_x[v(X_T)]$$

which is the equation of Le Gall (1994b, Theorem 4.2) as  $X$  is a standard Brownian motion on  $[0, T]$ . So, Corollary 4.3 of Le Gall (1994b) applies.  $\square$

**Theorem 4.6.** *If  $v \in \mathcal{C}^1(\bar{D})$ , then  $v$  is a strong solution of problem ND( $f, g$ ).*

**Proof.** Propositions 4.5 and 4.3 show that the first three conditions of ND( $f, g$ ) are satisfied. It remains to prove that  $\partial v / \partial n = 2g$  on  $F^c$ .

Let  $x \in \bar{D}$  and let  $T$  be a bounded stopping time such that  $T \leq \tau$   $P_x$ -a.s. We apply the optional stopping theorem to the martingale of Proposition 2.8 with  $\varphi = v$ :

$$E_x[v(X_T)] - v(x) - \frac{1}{2} E_x \left[ \int_0^T \Delta v(X_s) ds \right] + \frac{1}{2} E_x \left[ \int_0^T \frac{\partial v}{\partial n}(X_s) d\ell_s^D \right] = 0.$$

This gives, as  $\Delta v = 4v^2$  on  $D$ ,

$$E_x[v(X_T)] - v(x) - 2E_x \left[ \int_0^T v(X_s)^2 ds \right] + \frac{1}{2} E_x \left[ \int_0^T \frac{\partial v}{\partial n}(X_s) d\ell_s^D \right] = 0.$$

Now, by Proposition 4.4,

$$E_x[v(X_T)] - v(x) - 2E_x \left[ \int_0^T v(X_s)^2 ds \right] + E_x \left[ \int_0^T g(X_s) d\ell_s^D \right] = 0.$$

So, we have, for every  $x \in \bar{D}$  and every bounded stopping time  $T \leq \tau$ ,

$$E_x \left[ \int_0^T \left( \frac{1}{2} \frac{\partial v}{\partial n}(X_s) - g(X_s) \right) d\ell_s^D \right] = 0. \quad (6)$$

Now, let  $x \in F^c$  and suppose that  $\partial v / \partial n(x) - 2g(x) > 0$ . We consider the stopping time

$$T = \inf \left\{ t > 0 \mid X_t \in F^c \text{ and } \frac{\partial v}{\partial n}(X_t) - 2g(X_t) \leq 0 \right\} \wedge \tau \wedge 1.$$

Owing to the continuity of  $\partial v/\partial n$  and  $g$  and the regularity of  $\partial D$ , we have that  $P_x$ -a.s.  $T > 0$  and  $\ell_T^D > 0$ , and so

$$E_x \left[ \int_0^T \left( \frac{1}{2} \frac{\partial v}{\partial n}(X_s) - g(X_s) \right) d\ell_s^D \right] > 0$$

which contradicts formula (6).  $\square$

#### 4.2. Weak solutions

In general, we cannot prove that  $v \in \mathcal{C}^1(\bar{D})$  and so cannot apply Theorem 4.6 to obtain a strong solution of problem  $\text{ND}(f, g)$ . That is why we will consider now weak solutions.

Let  $u$  be a strong solution of problem  $\text{ND}(f, g)$  and  $\varphi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$  such that  $\Delta \varphi$  is bounded on  $D$ . Then, by the Green formula, we have

$$\begin{aligned} \int_D u(x) \Delta \varphi(x) \, dx - 4 \int_D u(x)^2 \varphi(x) \, dx \\ = \int_{\partial D} \frac{\partial \varphi}{\partial n}(y) u(y) \sigma(dy) - \int_{\partial D} \varphi(y) \frac{\partial u}{\partial n}(y) \sigma(dy) \end{aligned}$$

as  $\Delta u = 4u^2$  on  $D$ .

Now, suppose that  $\varphi = 0$  on  $\mathring{F}$  and  $\partial \varphi/\partial n = 0$  on  $F^c$ . Then, using the boundary conditions for  $u$ , we have

$$\begin{aligned} \int_D u(x) \Delta \varphi(x) \, dx - 4 \int_D u(x)^2 \varphi(x) \, dx \\ = \int_{\mathring{F}} \frac{\partial \varphi}{\partial n}(y) f(y) \sigma(dy) - 2 \int_{F^c} \varphi(y) g(y) \sigma(dy). \end{aligned}$$

This leads to the following definition.

**Definition 4.7.** We set

$$\mathcal{S} = \left\{ \varphi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D}) \text{ s.t. } \Delta \varphi \text{ is bounded on } D, \varphi = 0 \text{ on } \mathring{F}, \frac{\partial \varphi}{\partial n} = 0 \text{ on } F^c \right\}$$

the set of test functions.

We say that  $u$  is a weak solution of problem  $\text{ND}(f, g)$  if  $u$  is bounded and continuous on  $\bar{D} \setminus \partial F$  and if, for every test function  $\varphi \in \mathcal{S}$ , we have

$$\begin{aligned} \int_D u(x) \Delta \varphi(x) \, dx - 4 \int_D u(x)^2 \varphi(x) \, dx \\ = \int_{\mathring{F}} \frac{\partial \varphi}{\partial n}(y) f(y) \sigma(dy) - 2 \int_{F^c} \varphi(y) g(y) \sigma(dy). \end{aligned}$$

As  $\partial F$  is  $\partial$ -polar for the reflecting Brownian motion,  $X_\tau \in \mathring{F}$  a.s.

**Proposition 4.8.** *The function  $v$  is continuous on  $\bar{D} \setminus \partial F$ .*

**Proof.** Proposition 4.5 gives the continuity of  $v$  on  $D$ . Moreover, Proposition 4.3 and the continuity of  $f$  give the continuity of  $v$  on  $\mathring{F}$ . Now, let  $x_0 \in F^c$ . As  $F^c$  is an

open set of  $\partial D$ , there exists a ball  $B$  of  $\mathbb{R}^d$ , centered at  $x_0$ , such that  $B \cap F = \emptyset$ . For  $x \in B \cap \bar{D}$ , we have, by Proposition 4.4, for every  $t > 0$ ,

$$\begin{aligned} v(x) &= E_x[f(X_\tau)\mathbb{1}_{\{t \geq \tau\}}] + E_x \left[ \int_0^{t \wedge \tau} d\ell_s^D g(X_s) \right] \\ &\quad - 2E_x \left[ \int_0^{t \wedge \tau} v(X_s)^2 ds \right] + E_x[v(X_t)\mathbb{1}_{\{t < \tau\}}] \\ &= E_x[f(X_\tau)\mathbb{1}_{\{t \geq \tau\}}] + E_x \left[ \int_0^{t \wedge \tau} d\ell_s^D g(X_s) \right] \\ &\quad - 2E_x \left[ \int_0^{t \wedge \tau} v(X_s)^2 ds \right] - E_x[v(X_t)\mathbb{1}_{\{t \geq \tau\}}] + E_x[v(X_t)]. \end{aligned}$$

Then, as  $f, g$  and  $v$  are bounded, all the terms but the last one converge to 0 as  $t$  goes to 0, uniformly in  $x \in B \cap \bar{D}$ , and, by Proposition 2.3, for fixed  $t > 0$ , the last term is continuous on  $B \cap \bar{D}$ .  $\square$

We can be even more precise near  $F$  if  $f$  is Hölder.

**Proposition 4.9.** *Suppose  $f$  is Hölder of index  $\alpha > 0$ . Then, there exists two positive constant  $K$  and  $\gamma$  such that, for every  $x \in \mathring{F}$  and every  $y \in D$  such that  $|x - y| < 1$ ,*

$$|v(y) - f(x)| \leq K|y - x|^\gamma.$$

**Proof.** Let  $x \in \mathring{F}$  and  $y \in D$ . Using Eq. (4), we get

$$\begin{aligned} |v(y) - f(x)| &\leq E_y[|f(X_\tau) - f(x)|] + E_y \left[ \int_0^\tau |g(X_s)| d\ell_s^D \right] + 2E_y \left[ \int_0^\tau |v(X_s)|^2 ds \right] \\ &\leq C(E_y[|(X_\tau - x)|^\alpha] + E_y[\ell_\tau^D] + E_y[\tau]) \\ &\leq C(E_y[|(X_\tau - y)|^\alpha] + |y - x|^\alpha + E_y[\ell_\tau^D] + E_y[\tau]). \end{aligned}$$

Let us first consider the case  $y \in \partial D$ . Let  $T$  be the hitting time of the semi straight line  $\mathbb{R}_- = \{(t, 0) \in \mathbb{R}^2, t \leq 0\}$  by a planar Brownian motion starting from 1. A well-known result gives that there exists some constant  $C_3$  such that

$$h(M) = P(T > M) \leq \frac{C_3}{M^{1/4}}.$$

As  $\mathring{F}$  is supposed to be sufficiently smooth and connected, there exists a ball  $B_r$  on the boundary of radius  $r$  independent of  $y$  included in  $F$  and at a distance less than  $2d(y, \mathring{F})$  of  $y$ . If  $T_r$  represents the hitting time of  $B_r$  by the reflecting Brownian motion, we have, by a scaling argument, that, for every  $k \in (0, 2)$ ,  $\mathbb{P}_y(T_r > d(y, \mathring{F})^k)$  behaves, as  $y$  tends to a point in  $\mathring{F}$ , like  $h(d(y, \mathring{F})^{k-2})$  (to be more precise, we should separate the reflecting Brownian motion into a normal part and a tangent one to perform the right computation). So, there exists a constant  $C_4$  (depending on  $D$ ) such that

$$\mathbb{P}_y(\tau > d(y, \mathring{F})^{2/9}) \leq \mathbb{P}_y(T_r > d(y, \mathring{F})^{2/9}) \leq C_4 d(y, \mathring{F})^{4/9}.$$

Now,

$$\begin{aligned}\mathbb{E}_y[\tau] &= \mathbb{E}_y[\tau \mathbf{1}_{\tau \leq d(y, \mathring{F})^{2/9}}] + \mathbb{E}_y[\tau \mathbf{1}_{\tau > d(y, \mathring{F})^{2/9}}] \\ &\leq d(y, \mathring{F})^{2/9} + \mathbb{E}_y[\tau^2]^{1/2} \mathbb{P}_y(\tau > d(y, \mathring{F})^{4/9})^{1/2} \\ &\leq Cd(y, \mathring{F})^{2/9}\end{aligned}$$

using the previous upper bound and Lemma 3.4.

If  $y \notin \partial D$ , let

$$T_D = \inf\{t \geq 0 \mid X_t \in \partial D\}.$$

Using the martingale problem for the standard Brownian motion in  $\mathbb{R}^d$  and the optional stopping theorem, we have that

$$d\mathbb{E}_y[T_D] = \mathbb{E}_y[|X_{T_D} - y|^2].$$

Now, if  $P(y, z)$  is the Poisson kernel of  $D$ , we have

$$\mathbb{E}_y[|X_{T_D} - y|^2] = \int_{\partial D} \sigma(dz) P(y, z) |y - z|^2,$$

where  $\sigma$  is the surface measure on  $\partial D$ . Now, let us recall the well-known upper bound for the Poisson kernel: there exists a constant  $C_5$  such that

$$P(y, z) \leq C_5 \rho(y) |z - y|^{-d},$$

where  $\rho(y) = d(y, \partial D)$ .

Finally, we obtain that there exists a constant  $C_6$  such that

$$\mathbb{E}_y[T_D] \leq \mathbb{E}_y[|X_{T_D} - y|^2] \leq C_6 d(y, \mathring{F}).$$

Now, using the Markov property, we have, for  $0 < k < \frac{1}{2}$ ,

$$\begin{aligned}\mathbb{E}_y[\tau] &= \mathbb{E}_y[T_D] + \mathbb{E}_y[\mathbb{E}_{X_{T_D}}[\tau]] \\ &= \mathbb{E}_y[T_D] + \mathbb{E}_y \left[ \mathbf{1}_{|X_{T_D} - y| \leq d(y, \mathring{F})^k} \mathbb{E}_{X_{T_D}}[\tau] \right] + \mathbb{E}_y \left[ \mathbf{1}_{|X_{T_D} - y| > d(y, \mathring{F})^k} \mathbb{E}_{X_{T_D}}[\tau] \right] \\ &\leq Cd(y, \mathring{F}) + \sup_{\substack{z \in \partial D \\ d(z, \mathring{F}) \leq 2d(y, \mathring{F})^k}} \mathbb{E}_z[\tau] + C \mathbb{P}_y(|X_{T_D} - y| > d(y, \mathring{F})^k)\end{aligned}$$

using the previous inequalities and Lemma 3.4. Now, the upper bound for  $y \in \partial D$  and Markov inequality give

$$\begin{aligned}\mathbb{E}_y[\tau] &\leq Cd(y, \mathring{F}) + Cd(y, \mathring{F})^{2k/9} + C \frac{\mathbb{E}_y[|X_{T_D} - y|^2]}{d(y, \mathring{F})^{2k}} \\ &\leq Cd(y, \mathring{F}) + Cd(y, \mathring{F})^{2k/9} + Cd(y, \mathring{F})^{1-2k} \\ &\leq Cd(y, \mathring{F})^\beta\end{aligned}$$

for some  $\beta > 0$ .



For  $\mathbb{E}_y[\ell_\tau^d]$ , we use the computations of Lemma 3.5:

$$\mathbb{E}_y[\ell_\tau^D] \leq \sum_{n=0}^{+\infty} \mathbb{E}_y[(\ell_{n+1}^D)^2]^{1/2} \mathbb{P}_y(\tau \geq n)^{1/2} \leq K_2 \sum_{n=0}^{+\infty} (n+1)(1-\delta)^{(n-1)/2} \mathbb{P}_y(\tau \geq 1)^{1/2},$$

where  $\delta = \inf_{z \in \bar{D}} \mathbb{P}_z(\tau \geq 1) > 0$ . Using Markov inequality, we eventually get that there exists a constant  $C_7$  such that

$$\mathbb{E}_y[\ell_\tau^D] \leq C_7 d(y, \mathring{F})^{\beta/2}.$$

Moreover, the martingale problem of Proposition 2.8 applied to function  $\varphi(z) = |z - y|^2$  gives that

$$|X_t - y|^2 - dt - \int_0^t f(X_s) d\ell_s^D$$

is a martingale under  $P_y$ ,  $f$  representing the normal derivative of  $\varphi$  on  $\partial D$ . We only use that it is bounded by a constant independent of  $y$ . Applying the optional stopping theorem at time  $t \wedge \tau$  and letting  $t$  tend to infinity, we obtain that

$$\mathbb{E}_y[|X_\tau - y|^2] \leq C(\mathbb{E}_y[\tau] + \mathbb{E}_y[\ell_\tau^D]).$$

Now, Hölder inequality and the previous upper bounds give the desired result.  $\square$

**Theorem 4.10.** *If  $f$  is Hölder, then  $v$  is a weak solution of problem  $\text{ND}(f, g)$ .*

**Proof.** We already know that  $v$  is continuous. It remains to prove the integral formula.

Let  $\varphi \in \mathcal{S}$ . Then, by Proposition 2.8, we have, for every  $t \geq 0$ ,

$$E_x[\varphi(X_t)] - \varphi(x) = \frac{1}{2} E_x \left[ \int_0^t \Delta \varphi(X_s) ds \right] - \frac{1}{2} E_x \left[ \int_0^t \frac{\partial \varphi}{\partial n}(X_s) d\ell_s^D \right].$$

Multiplying by  $v$  and integrating on  $D$ , we get

$$\begin{aligned} & \int_D v(x) [E_x[\varphi(X_t)] - \varphi(x)] dx \\ &= \frac{1}{2} \int_0^t ds \int_D v(x) E_x[\Delta \varphi(X_s)] dx - \frac{1}{2} \int_D v(x) E_x \left[ \int_0^t \frac{\partial \varphi}{\partial n}(X_s) d\ell_s^D \right] dx. \end{aligned}$$

Now, by Proposition 2.2, we obtain

$$\begin{aligned} & \int_D \varphi(x) [E_x[v(X_t)] - v(x)] dx \\ &= \frac{1}{2} \int_0^t ds \int_D \Delta \varphi(x) E_x[v(X_s)] dx - \frac{1}{2} \int_D v(x) E_x \left[ \int_0^t \frac{\partial \varphi}{\partial n}(X_s) d\ell_s^D \right] dx \\ &= \frac{1}{2} \int_D dx \Delta \varphi(x) \int_0^t ds E_x[v(X_s)] - \frac{1}{2} \int_D v(x) E_x \left[ \int_0^t \frac{\partial \varphi}{\partial n}(X_s) d\ell_s^D \right] dx. \end{aligned}$$

But, Proposition 4.4 gives

$$\begin{aligned} E_x[v(X_t)] - v(x) &= E_x[v(X_t) \mathbb{1}_{\{t \geq \tau\}}] - E_x[v(X_\tau) \mathbb{1}_{\{t \geq \tau\}}] \\ &\quad - E_x \left[ \int_0^{t \wedge \tau} g(X_s) d\ell_s^D \right] + 2E_x \left[ \int_0^{t \wedge \tau} v(X_s)^2 ds \right]. \end{aligned}$$

Then, plugging that equality in the last formula and dividing by  $t$  leads to

$$\begin{aligned} & \int_D \varphi(x) \frac{1}{t} E_x[(v(X_t) - f(X_\tau)) \mathbf{1}_{\{t \geq \tau\}}] dx \\ & - \int_D dx \varphi(x) \frac{1}{t} E_x \left[ \int_0^{t \wedge \tau} g(X_s) d\ell_s^D \right] + 2 \int_D \varphi(x) \frac{1}{t} E_x \left[ \int_0^{t \wedge \tau} v(X_s)^2 ds \right] dx \\ & = \frac{1}{2} \int_D dx \Delta \varphi(x) \frac{1}{t} \int_0^t ds E_x[v(X_s)] - \frac{1}{2} \int_D dx v(x) \frac{1}{t} E_x \left[ \int_0^t \frac{\partial \varphi}{\partial n}(X_s) d\ell_s^D \right]. \end{aligned}$$

Before taking limits as  $t$  tends to 0, we must prove the following lemmas.

**Lemma 4.11.** *For every bounded function  $\phi$  on  $D$  and every bounded continuous function  $\psi$  on  $D$ ,*

$$\lim_{t \rightarrow 0} \int_D dx \phi(x) \frac{1}{t} E_x \left[ \int_0^t ds \psi(X_s) \right] = \int_D ds \phi(x) \psi(x).$$

**Lemma 4.12.** *For every bounded function  $\phi$  on  $D$  and every bounded continuous function  $\psi$  on  $D$ ,*

$$\lim_{t \rightarrow 0} \int_D dx \phi(x) \frac{1}{t} E_x \left[ \int_0^{t \wedge \tau} ds \psi(X_s) \right] = \int_D ds \phi(x) \psi(x).$$

**Proof.** Let us compute

$$\begin{aligned} & \left| \int_D dx \phi(x) \frac{1}{t} E_x \left[ \int_0^t ds \psi(X_s) \right] - \int_D dx \phi(x) \frac{1}{t} E_x \left[ \int_0^{t \wedge \tau} ds \psi(X_s) \right] \right| \\ & \leq \int_D dx |\phi(x)| \frac{1}{t} E_x \left[ \int_{t \wedge \tau}^t |\psi(X_s)| ds \right] \\ & = \int_D dx |\phi(x)| \frac{1}{t} E_x \left[ \int_{t \wedge \tau}^t |\psi(X_s)| \mathbf{1}_{\{t \geq \tau\}} ds \right] \\ & \leq C \int_D dx P_x(t \geq \tau). \end{aligned}$$

This last term tends to 0 as  $t$  tends to 0 by dominated convergence.  $\square$

**Lemma 4.13.** *For every continuous function  $\phi$  on  $\bar{D}$  and every continuous function  $\psi$  on  $\partial D$ ,*

$$\lim_{t \rightarrow 0} \int_D dx \phi(x) \frac{1}{t} E_x \left[ \int_0^t \psi(X_s) d\ell_s^D \right] = \int_{\partial D} \sigma(dy) \phi(y) \psi(y).$$

**Proof.** By Proposition 2.5 and Fubini's theorem, we have

$$\begin{aligned} & \int_D dx \phi(x) \frac{1}{t} E_x \left[ \int_0^t \psi(X_s) d\ell_s^D \right] \\ & = \int_D dx \phi(x) \frac{1}{t} \int_0^t ds \int_{\partial D} \psi(y) p(s, x, y) \sigma(dy) \end{aligned}$$

$$\begin{aligned}
&= \int_{\partial D} \sigma(dy) \psi(y) \frac{1}{t} \int_0^t ds \int_D dx \phi(x) p(s, x, y) \\
&= \int_{\partial D} \sigma(dy) \psi(y) \frac{1}{t} E_y \left[ \int_0^t ds \phi(X_s) \right].
\end{aligned}$$

This term tends to  $\int_{\partial D} \sigma(dy) \phi(y) \psi(y)$  as  $t$  goes to 0.  $\square$

**Lemma 4.14.** *For every function  $\phi \in \mathcal{S}$  and every continuous function  $\psi$  on  $\partial D$ ,*

$$\lim_{t \rightarrow 0} \int_D dx \phi(x) \frac{1}{t} E_x \left[ \int_0^{t \wedge \tau} \psi(X_s) d\ell_s^D \right] = \int_{\partial D} \sigma(dy) \phi(y) \psi(y).$$

**Proof.** Let us compute

$$\begin{aligned}
&\left| \int_D dx \phi(x) \frac{1}{t} E_x \left[ \int_0^t \psi(X_s) d\ell_s^D \right] - \int_D dx \phi(x) \frac{1}{t} E_x \left[ \int_0^{t \wedge \tau} \psi(X_s) d\ell_s^D \right] \right| \\
&\leq \int_D dx |\phi(x)| \frac{1}{t} E_x \left[ \mathbb{1}_{\{t \geq \tau\}} \int_{t \wedge \tau}^t |\psi(X_s)| d\ell_s^D \right] \\
&\leq C \int_D dx |\phi(x)| \frac{1}{t} E_x \left[ \mathbb{1}_{\{t \geq \tau\}} (\ell_t^D - \ell_\tau^D) \right] \\
&\leq C \int_D dx |\phi(x)| \frac{1}{\sqrt{t}} P_x(t \geq \tau)^{1/2}
\end{aligned}$$

by Proposition 2.6 and Cauchy–Schwarz inequality.

Let  $\rho(x) = d(x, \partial D)$ . We have

$$P_x(t \geq \tau) \leq \mathbb{P}_0 \left( \sup_{0 \leq s \leq t} |B_s| \geq \rho(x) \right) \leq C \left( \frac{\rho(x)}{\sqrt{t}} \right)^{d-2} e^{-\rho(x)^2/2t}$$

which proves that

$$\lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} P_x(t \geq \tau)^{1/2} = 0.$$

As  $D$  is a domain of class  $\mathcal{C}^2$ , there exists  $r > 0$  such that, for every  $z \in \partial D$ , there exists a ball of radius  $r$  containing  $z$  and included in  $\bar{D}$ . Let  $\mathcal{D}_r$  be the set of such domains. Let us also denote  $f_x^\lambda: z \mapsto \lambda(z - x)$ .

First, let us suppose that  $d(x, F) < 2/r$ . We will denote here by  $P_x^D$  the law of the reflecting Brownian motion in  $D$  starting at  $x$ . We have

$$\begin{aligned}
P_x^D \left( \sup_{0 \leq s \leq t} |X_s - x| \geq d(x, F) \right) &= P_x^{f_x^{1/d(x, F)}(D)} \left( \sup_{0 \leq s \leq t/d(x, F)^2} |X_s - x| \geq 1 \right) \\
&\leq \sup_{\substack{D' \in \mathcal{D}_2 \\ x \in D'}} P_x^{D'} \left( \sup_{0 \leq s \leq t/d(x, F)^2} |X_s - x| \geq 1 \right).
\end{aligned}$$

And,

$$\begin{aligned} \frac{1}{\sqrt{t}} \mathbb{P}_x(t \geq \tau)^{1/2} &\leq \frac{1}{\sqrt{t}} P_x^D \left( \sup_{0 \leq s \leq t} |X_s - x| \geq d(x, F) \right)^{1/2} \\ &\leq \frac{1}{d(x, F)} \frac{d(x, F)}{\sqrt{t}} \sup_{\substack{D' \in \mathcal{D}_2 \\ x \in D'}} P_x^{D'} \left( \sup_{0 \leq s \leq t/d(x, F)^2} |X_s - x| \geq 1 \right)^{1/2}. \end{aligned}$$

It is then easy to see that the function

$$u \mapsto \frac{1}{u} \sup_{\substack{D' \in \mathcal{D}_2 \\ x \in D'}} P_x^{D'} \left( \sup_{0 \leq s \leq u} |X_s - x| \geq 1 \right)$$

is continuous, tends to 0 as  $u$  tends to 0 and to  $+\infty$  and hence is bounded on  $(0, +\infty)$ . So, if  $d(x, F) < 2/r$ , there exists a constant  $C_8$  such that

$$\frac{1}{\sqrt{t}} \mathbb{P}_x(t \geq \tau)^{1/2} \leq \frac{C_8}{d(x, F)}.$$

For  $d(x, F) \geq 2/r$ , the Markov property shows that

$$\frac{1}{\sqrt{t}} \mathbb{P}_x(t \geq \tau)^{1/2}$$

is bounded by a constant independent of  $x$  and  $t$ .

As  $\phi$  is  $\mathcal{C}^1$  on  $\tilde{D}$  and vanishes on  $F$ ,  $\phi(x) \leq Kd(x, F)$ . The result then follows from Lebesgue's theorem.  $\square$

**Lemma 4.15.** *For every function  $\phi \in \mathcal{S}$ , we have*

$$\lim_{t \rightarrow 0} \int_D dx |\phi(x)| \frac{1}{t} E_x[|v(X_t) - f(X_t)| \mathbb{1}_{\{t \geq \tau\}}] = 0.$$

**Proof.** First, let us compute for every  $x \in \mathring{F}$  using Proposition 4.9

$$\begin{aligned} E_x[|v(X_t) - f(x)|] &= E_x[|v(X_t) - f(x)| \mathbb{1}_{\{|X_t - x| \leq 1\}}] + E_x[|v(X_t) - f(x)| \mathbb{1}_{\{|X_t - x| > 1\}}] \\ &\leq C(E_x[|X_t - x|^\gamma] + P_x(|X_t - x| > 1)). \end{aligned}$$

Let  $g(y) = |y - x|^\gamma$ . Let us compute

$$\begin{aligned} E_x[g(X_t)] &= \int_D dy p(t, x, y) g(y) \\ &= \int_D dy p(t, y, x) g(y), \end{aligned}$$

by symmetry of the function  $p(t, \cdot, \cdot)$ . Then, using the decomposition of Proposition 2.1, we have, on the one hand,

$$\begin{aligned} \int_D dy g(y) p_0(t, y, x) &\leq C \int_D dy g(y) \tilde{p}(at, y, x) \\ &= CE_x[g(B_{at})] \\ &\leq Ct^{\gamma/2} \end{aligned}$$

using a scaling argument.

On the other hand, we have

$$\begin{aligned} \int_D dy g(y) p_1(t, y, x) &= \int_D dy g(y) \int_0^t du \int_D dz p_0(t-u, y, z) q(u, z, x) \\ &= \int_0^t du \int_D dz q(u, z, x) \int_D dy g(y) p_0(t-u, y, z) \\ &\leq \int_0^t \frac{C du}{\sqrt{u}} = 2C\sqrt{t} \end{aligned}$$

if  $t \leq 1$ .

Furthermore, observe that  $P_x(|X_t - x| > 1)$  is exponentially small as  $t \rightarrow 0$ .

So, this computation and the Markov property show that it suffices to prove that

$$\lim_{t \rightarrow 0} \int_D dx |\phi(x)| t^{\gamma/2-1} P_x(t \geq \tau) = 0$$

which is done using the same arguments as in the previous proof.  $\square$

These limits finally give

$$\begin{aligned} & - \int_{\partial D} \sigma(dy) \varphi(y) g(y) + 2 \int_D \varphi(x) v(x)^2 dx \\ &= \frac{1}{2} \int_D dx \Delta \varphi(x) v(x) - \frac{1}{2} \int_{\partial D} \sigma(dy) v(y) \frac{\partial \varphi}{\partial n}(dy). \end{aligned}$$

The boundary conditions for  $\varphi$  end the proof.  $\square$

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