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Heat equation with strongly inhomogeneous noise

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Abstract

We consider the heat equation in dimension one with singular drift and inhomogeneous space–time white noise. In particular, the quadratic variation measure of the white noise is not required to be absolutely continuous w.r.t. the Lebesgue measure, neither in space nor in time. Under some assumptions we give statements on strong and weak existence as well as strong and weak uniqueness of continuous solutions.

MSC: 60H15; 35R60; 60H20; 60J80

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1. Introduction

We consider the following stochastic partial differential equation (SPDE):

$$\begin{aligned}&\frac{\partial}{\partial t}u(t,x)=\frac{1}{2}\Delta u(t,x)+b(t,x,u(t,x))\,\frac{\sigma(\mathrm{d}t\,\mathrm{d}x)}{\mathrm{d}t\,\mathrm{d}x}\,(t,x)+a(t,x,u(t,x))\,\frac{\partial^2}{\partial t\partial x}\,w^{\varrho}(t,x),\\&u(0,x)=\eta(x),\quad t\geqslant 0,\quad x\in\mathbb{R},\end{aligned}\tag{1}$$

whose precise meaning will be given in Definition 2.1 below. Here $\Delta=\partial^2/\partial x^2$, a and $b:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ as well as $\eta:\mathbb{R}\to\mathbb{R}$ are continuous functions, $\sigma(\mathrm{d}t\,\mathrm{d}x)$ and $\varrho(\mathrm{d}t\,\mathrm{d}x)$ are positive Radon measures on $[0,\infty)\times\mathbb{R}$, $\sigma(\mathrm{d}t\,\mathrm{d}x)/\mathrm{d}t\,\mathrm{d}x$ is the Lebesgue density—possibly only existing as a generalized function (distribution)—of $\sigma(\mathrm{d}t\,\mathrm{d}x)$

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and $w^{\varrho} : [0,\infty) \times \mathbb{R} \times \Omega \to \mathbb{R}$ is an inhomogeneous two-parameter Brownian motion on $[0,\infty) \times \mathbb{R}$ based on $\varrho(\mathrm{d}t\,\mathrm{d}x)$. The latter object is characterized by the relation

$$W^{\varrho}((t,t'] \times (x,x']) = w^{\varrho}(t',x') - w^{\varrho}(t',x) - w^{\varrho}(t,x') + w^{\varrho}(t,x), \tag{2}$$

where W^{ϱ} is a white noise "measure" based on $\varrho(\mathrm{d}t\,\mathrm{d}x)$, that is a real-valued random function on the algebra $\mathscr{A} = \bigcup_{n \geqslant 1} \mathscr{B}([0,n] \times [-n,n]) \subset \mathscr{B}([0,\infty) \times \mathbb{R})$ satisfying the following two assertions for any disjoint sets $A, A' \in \mathscr{A}$:

- $W^{\varrho}(A) \sim N(0, \varrho(A))$,
- $W^{\varrho}(A)$, $W^{\varrho}(A')$ are independent and $W^{\varrho}(A \cup A') = W^{\varrho}(A) + W^{\varrho}(A')$.

Walsh (1986, p. 260) constructed W^{ϱ} as a Gaussian process on $\mathscr{A}$. In view of (2), w^{ϱ} can formally be associated with the "distribution function" and $\dot{w}^{\varrho}(t,x) = (\partial^2/\partial t \partial x) w^{\varrho}(t,x)$ with the "density" of W^{ϱ}. The latter is usually called white noise and coincides with W^{ϱ} in distribution sense. If $\varrho(\mathrm{d}t\,\mathrm{d}x) = \sigma(\mathrm{d}t\,\mathrm{d}x) = \mathrm{d}t\,\mathrm{d}x$, then $(\sigma(\mathrm{d}t\,\mathrm{d}x)/\mathrm{d}t\,\mathrm{d}x)(t,x) \equiv 1$ and w^{ϱ} is just the homogeneous two-parameter Brownian motion w on $[0,\infty) \times \mathbb{R}$. In this case, Eq. (1) turns into

$$\begin{aligned} &\frac{\partial}{\partial t} u(t,x) = \frac{1}{2} \Delta u(t,x) + b(t,x,u(t,x)) + a(t,x,u(t,x)) \frac{\partial^2}{\partial t \partial x} w(t,x) \\ &u(0,x) = \eta(x), \quad t \geqslant 0, \quad x \in \mathbb{R} \end{aligned} \tag{3}$$

and has been studied several times w.r.t. existence and uniqueness of solutions (see Walsh, 1986; Iwata, 1987; Mueller and Perkins, 1992; Shiga, 1994; Mytnik, 1998 among others).

One motivation to study Eq. (3) is the link to population systems. If $b \equiv 0$ and $a(t,x,u) = \sqrt{\varrho(t,x)u}$, Eq. (3) describes the evolution of an infinitesimal system (high-density/short-lifetime limit) of critical binary branching Brownian particles where the branching intensity of an infinitesimal particle at position x at time t is given by $\varrho(t,x)$; (see Konno and Shiga, 1988; Méléard and Roelly-Coppoletta, 1988 or Reimers, 1989). There the medium $\varrho(.,.)$ was assumed to be constant or a regular function. But media occurring in nature often have a more fractal shape. In particular, they should be modelled as a singular measure $\varrho(\mathrm{d}t\,\mathrm{d}x)$. In our example the branching intensity of a particle is then given by the formal expression $(\varrho(\mathrm{d}t\,\mathrm{d}x)/\mathrm{d}t\,\mathrm{d}x)(t,x)$. Accordingly, the whole system evolves according to equation (1) with $b \equiv 0$ and $a(t,x,u) = \sqrt{u}$, formally justified by the following fact. If $\varrho(\mathrm{d}t\,\mathrm{d}x)$ has a continuous $\mathrm{d}t\,\mathrm{d}x$-density $\varrho(t,x)$, then we have $(\partial^2/\partial t \partial x) w^{\varrho}(t,x) = \sqrt{\varrho(t,x)}(\partial^2/\partial t \partial x) w(t,x)$ in the sense of Definition 2.1 below. Dawson and Fleischmann (1991) characterized the infinitesimal particle system corresponding to a medium of the form $\varrho(\mathrm{d}t\,\mathrm{d}x) = \varrho_t(\mathrm{d}x)\,\mathrm{d}t$ as a measure-valued process $\bar{u}_t(\mathrm{d}x)$, the so-called catalytic super-Brownian motion. On the one hand, if $u(t,x)$ is a continuous solution to SPDE (1) with $b \equiv 0$ and $a(t,x,u) = \sqrt{u}$, it is also the $\mathrm{d}t\,\mathrm{d}x$-density of $\bar{u}_t(\mathrm{d}x)\,\mathrm{d}t$. This follows from the characterization of $\bar{u}_t(\mathrm{d}x)$ as unique solution to a certain martingale problem (cf. Zähle, 2004, Proposition 2.5). On the other hand, if $\varrho_t(\mathrm{d}x) \equiv \delta_c(\mathrm{d}x)$ with $c \in \mathbb{R}$ fixed, $\bar{u}_t(\mathrm{d}x)\,\mathrm{d}t$ possesses a $\mathrm{d}t\,\mathrm{d}x$-density with discontinuities on $[0,\infty) \times \{c\}$ (see Fleischmann and Le Gall, 1995, p. 82). So we cannot get a continuous solution to SPDE (1) with $b \equiv 0$ and $a(t,x,u) = \sqrt{u}$ for all measures $\varrho(\mathrm{d}t\,\mathrm{d}x) = \varrho_t(\mathrm{d}x)\,\mathrm{d}t$. However, there is a large class of non-atomic singular measures $\varrho_t(\mathrm{d}x)\,\mathrm{d}t$ for which $\bar{u}_t(\mathrm{d}x)\,\mathrm{d}t$ has a continuous $\mathrm{d}t\,\mathrm{d}x$-density which solves the SPDE. In fact, $\varrho_t(\mathrm{d}x)$ only has to satisfy one of the following two equivalent conditions

- $\exists \alpha \in (0,1]\ \forall T > 0\ \exists c_T > 0$: $\sup_{t\leqslant T} \sup_{x\in\mathbb{R}} \varrho_t(B(x,r)) \leqslant c_T r^{\alpha}\ \ \forall r \in (0,1]$;
- $\exists \alpha \in (0,1]\ \forall T > 0\ \exists c_T > 0$: $\sup_{t\leqslant T} \sup_{x\in\mathbb{R}} \int_{B(x,1)} |x-y|^{-\alpha} \varrho_t(\mathrm{d}y) \leqslant c_T$

on the concentration of mass (cf. Zähle, 2004), where $B(x,r) := (x-r, x+r)$.

In the present paper, we worry about continuous solutions to SPDE (1) with more general coefficients (than $b \equiv 0$, $a(t,x,u) = \sqrt{u}$) under similar conditions on $\varrho(\mathrm{d}t\,\mathrm{d}x)$ and $\sigma(\mathrm{d}t\,\mathrm{d}x)$. More precisely, $\varrho(\mathrm{d}t\,\mathrm{d}x)$ and $\sigma(\mathrm{d}t\,\mathrm{d}x)$ will always be assumed to satisfy condition (A), respectively (B), cf. Definition 1.1. For a metric space E let $\mathscr{M}(E)$ denote the spaces of positive Radon measures on E and define $\mathscr{M}_{\mathrm{uni}}(\mathbb{R}) = \{\mu \in \mathscr{M}(\mathbb{R}) : \sup_{x\in\mathbb{R}} \mu(B(x,1)) < \infty\}$.

Definition 1.1. A measure $\mu(\mathrm{d}t\,\mathrm{d}x) \in \mathscr{M}([0,\infty) \times \mathbb{R})$ is said to satisfy condition (A) (respectively (B)) if $\mu(\mathrm{d}t\,\mathrm{d}x) = \mu_1(t,\mathrm{d}x)\mu_2(\mathrm{d}t)$, where μ_1 is a kernel from $[0,\infty)$ to $\mathbb{R}$ with $\mu_1(t,\mathrm{d}x) \in \mathscr{M}_{\mathrm{uni}}(\mathbb{R})\ \forall t \geqslant 0$ and $\mu_2(\mathrm{d}t) \in \mathscr{M}([0,\infty))$, such that: $\exists \alpha_1, \alpha_2 \in [0,1]$ $\forall T > 0\ \exists c_T > 0$:

(a) $\sup_{t\leqslant T} \sup_{x\in\mathbb{R}} \mu_1(t, B(x,r)) \leqslant c_T r^{\alpha_1}\ \ \forall r \in (0,1]$;
(b) $\sup_{t\leqslant T} \mu_2([0,\infty) \cap B(t,r)) \leqslant c_T r^{\alpha_2}\ \ \forall r \in (0,1]$;
(c) $\alpha_1/2 + \alpha_2 > 1$ (respectively $\alpha_1/2 + \alpha_2 > \frac{1}{2}$).

It is clear that every $\varrho(\mathrm{d}t\,\mathrm{d}x) \in \mathscr{M}([0,\infty)\times\mathbb{R})$, that has a bounded space–time Lebesgue density, fulfills (A). But $\varrho(\mathrm{d}t\,\mathrm{d}x)$ does not need to be absolutely continuous w.r.t. $\mathrm{d}t\,\mathrm{d}x$. It just may not be too singular. Note that the Hausdorff dimension of the closed support $\mathrm{supp}(v)$ of a measure $v(d\xi) \in \mathscr{M}(\mathbb{R})$ must be at least γ if $\exists c > 0$ $\forall \xi \in \mathrm{supp}(v)$: $v(B(\xi,r)) \leqslant c r^{\gamma}\ \forall r \in (0,1]$. To give an example, let $\mathscr{C}_\lambda$ be the "λ-Cantor measure" on $[0,1]$, that is $\mathscr{C}_\lambda(.) := \mathscr{H}^{\gamma}(. \cap C(\lambda))$ where $C(\lambda)$ is the λ-Cantor set (cf. Mattila, 1995, 4.13), $\mathscr{H}^{\gamma}$ is the γ-dimensional Hausdorff measure on $\mathbb{R}$ and $\gamma = \log 2/|\log \lambda|$ is $C(\lambda)$'s Hausdorff dimension. $\mathscr{C}_\lambda$ is a finite singular measure with closed support $C(\lambda)$ and satisfies: $\exists c > 0\ \forall x \in \mathrm{supp}(\mathscr{C}_\lambda)$: $\mathscr{C}_\lambda(B(x,r)) \leqslant c r^{\gamma}\ \forall r \in (0,1]$. Therefore, $\varrho(\mathrm{d}t\,\mathrm{d}x) = \mathscr{C}_{\lambda_1}(\mathrm{d}x)\mathscr{C}_{\lambda_2}(\mathrm{d}t)$ satisfies condition (A) whenever $\lambda_1, \lambda_2 \in (0, \frac{1}{2})$ such that $\log 2/|2\log \lambda_1| + \log 2/|\log \lambda_2| > 1$. The mentioned measures also satisfy condition (B) since condition (A) is stronger than (B). Measures $\sigma(\mathrm{d}t\,\mathrm{d}x)$ satisfying condition (B) may even have spatial atoms ($\alpha_1 = 0$). For instance, $\sigma(\mathrm{d}t\,\mathrm{d}x) = \delta_0(\mathrm{d}x)\,\mathrm{d}t$ or $\sigma(\mathrm{d}t\,\mathrm{d}x) = \delta_0(\mathrm{d}x)\mathscr{C}_{\lambda_2}(\mathrm{d}t)$ with $\alpha_2 := \log 2/|\log \lambda_2| > \frac{1}{2}$ fulfill (B).

2. Preliminaries and main results

The formulation of Eq. (1) is rather vague. On the one hand, the Lebesgue density of $\sigma(\mathrm{d}t\,\mathrm{d}x)$ might fail to exists. On the other hand, w^{ϱ} is not differentiable in general.

The way out is to regard SPDE (1) as a stochastic integral equation. We shall adopt the notion of Walsh (1986) involving stochastic integrals against martingale measures. If M is an orthogonal martingale measure, we denote its quadratic variation measure by $\langle M\rangle(\mathrm{d}t\,\mathrm{d}x)$ and the stochastic integral of an admissible integrand f against M by $\int\int f(r,y)M(\mathrm{d}r\,\mathrm{d}y)$. Note that the white noise "measure" W^{ϱ} from Section 1 induces an orthogonal martingale measure, denoted by W^{ϱ} either, with $\langle W^{\varrho}\rangle(\mathrm{d}t\,\mathrm{d}x)=\varrho(\mathrm{d}t\,\mathrm{d}x)$. The probability domain of W^{ϱ} will be denoted by $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},\mathbf{P})$ where $(\mathscr{F}_t)_{t\geqslant 0}$ is a filtration in $\mathscr{F}$ satisfying the usual conditions. For an investigation of orthogonal martingale measures see also El Karoui and Méléard (1990).

Let $C(\mathbb{R})$ denote the space of real-valued continuous functions on $\mathbb{R}$. Subscripts b, c and superscripts $+$, ∞ refer to the subspaces of bounded, with compact support, non-negative and infinitely often differentiable (resp.) functions. Set $C_{\mathrm{tem}}(\mathbb{R})=\{\psi\in C(\mathbb{R})\colon\ |\psi|_{(-\lambda)}<\infty\ \forall\lambda>0\}$ where $|\psi|_{(-\lambda)}:=\|\mathrm{e}^{-\lambda|.|}\psi(.)\|_\infty$ and $\|.\|_\infty$ is the usual supremum norm. We equip $C_{\mathrm{tem}}(\mathbb{R})$ with the metric $d_{\mathrm{tem}}(\phi,\psi)=\sum_{k=1}^{\infty}2^{-k}(|\phi-\psi|_{(-1/k)}\wedge 1)$ and set $\langle\phi,\psi\rangle=\int_{\mathbb{R}}\phi(x)\psi(x)\,\mathrm{d}x$.

Definition 2.1. A $C_{\mathrm{tem}}(\mathbb{R})$-valued continuous process $(u(t,.)\colon\ t\geqslant 0)$ is said to be a strong solution to SPDE (1) with initial condition $\eta\in C_{\mathrm{tem}}(\mathbb{R})$ if, given the noise $[W^{\varrho},\Omega,\mathscr{F},(\mathscr{F}_t),\mathbf{P}]$, it is $(\mathscr{F}_t)$-adapted and

$$\begin{aligned}\langle u(t,.),\psi\rangle=\langle\eta,\psi\rangle&+\int_0^t\left\langle u(r,.),\frac{1}{2}\Delta\psi\right\rangle\mathrm{d}r\\&+\int_0^t\int_{\mathbb{R}}b(r,y,u(r,y))\psi(y)\sigma(\mathrm{d}r\,\mathrm{d}y)\\&+\int_0^t\int_{\mathbb{R}}a(r,y,u(r,y))\psi(y)W^{\varrho}(\mathrm{d}r\,\mathrm{d}y)\end{aligned}\tag{4}$$

holds for all $t\geqslant 0$ and $\psi\in C_c^{\infty}(\mathbb{R})$, $\mathbf{P}$-almost surely. We say u is a weak solution to SPDE (1) with initial condition $\eta\in C_{\mathrm{tem}}(\mathbb{R})$ if one can find any noise $[W^{\varrho},\Omega,\mathscr{F},(\mathscr{F}_t),\mathbf{P}]$ such that u is $(\mathscr{F}_t)$-adapted and (4) holds.

Definition 2.2. A solution to SPDE (1) is said to be strongly unique if any two solutions w.r.t. a given noise are indistinguishable. We say a solution is weakly unique if any two solutions coincide in law.

Let us now turn to our main results. We first assume the coefficients to be Lipschitz continuous. In this case we can find strongly unique strong solutions. Conditions (A) and (B) were introduced in Definition 1.1.

Theorem 2.3. *Let a and b be continuous. Assume for every $T>0$ there exist finite constants $c_T,L_T>0$ such that for all $t\leqslant T$ and $x,u,u'\in\mathbb{R}$,*

$$|a(t,x,u)|+|b(t,x,u)|\leqslant c_T(1+|u|),\tag{5}$$

$$|a(t,x,u)-a(t,x,u')|+|b(t,x,u)-b(t,x,u')|\leqslant L_T|u-u'|.\tag{6}$$

Let $\eta \in C_{\mathrm{tem}}(\mathbb{R})$, $\varrho(\mathrm{d}t\,\mathrm{d}x)$ *satisfy condition* (A) *with* α_1, α_2 *and* $\sigma(\mathrm{d}t\,\mathrm{d}x)$ *satisfy condition* (B) *with* β_1, β_2. *Then SPDE* (1) *with initial condition* η *has a strongly unique strong solution. Moreover, the solution is locally jointly Hölder-*γ*-continuous for all* $\gamma \in (0, (\alpha/2) \wedge \beta)$ *where* $\alpha := \alpha_1/2 + \alpha_2 - 1$, $\beta := \beta_1/2 + \beta_2 - \frac{1}{2}$.

Under slightly stronger assumptions we obtain non-negativity of the solution.

Theorem 2.4. *Let* a *and* b *be continuous and* $\kappa > 0$. *Suppose* $\rho(\mathrm{d}t\,\mathrm{d}x)$ *and* $\sigma(\mathrm{d}t\,\mathrm{d}x)$ *satisfy condition* (A) *and* (B), *respectively. Assume for every* $T > 0$ *there exist finite constants* $c_T, L_T > 0$ *such that for all* $t \leqslant T$ *and* $x, x', u, u' \in \mathbb{R}$ *inequalities* (5) *and*

$$|a(t,x,u) - a(t,x',u')| + |b(t,x,u) - b(t,x',u')| \leqslant L_T(|x - x'|^{\kappa} + |u - u'|)$$

hold. If moreover $a(t,x,0) = 0$, $b(t,x,0) \geqslant 0$ $(\forall t \geqslant 0, x \in \mathbb{R})$ *and* $\eta \in C^+_{\mathrm{tem}}(\mathbb{R})$, *then the unique solution from Theorem* 2.3 *is* **P***-almost surely non-negative.*

If one is only interested in weak solutions, condition (6) can be dropped.

Theorem 2.5. *Let* $a(t,x,u) = a(u)$ *and* $b(t,x,u) = b(u)$ *be continuous and assume there exists a finite constant* $c > 0$ *such that for all* $u \in \mathbb{R}$

$$|a(u)| + |b(u)| \leqslant c(1 + |u|). \tag{7}$$

Let $\eta \in C_{\mathrm{tem}}(\mathbb{R})$, $\varrho(\mathrm{d}t\,\mathrm{d}x)$ *satisfy condition* (A) *with* α_1, α_2 *and* $\sigma(\mathrm{d}t\,\mathrm{d}x)$ *satisfy condition* (B) *with* β_1, β_2. *Then SPDE* (1) *with initial condition* η *has a weak solution. The solution is locally jointly Hölder-*γ*-continuous for all* $\gamma \in (0, (\alpha/2) \wedge \beta)$ *where* $\alpha := \alpha_1/2 + \alpha_2 - 1$, $\beta := \beta_1/2 + \beta_2 - 1/2$. *If in addition* $\eta \in C^+_{\mathrm{tem}}(\mathbb{R})$, $a(0) = 0$ *and* $b(0) \geqslant 0$, *then the solution is* **P***-almost surely non-negative.*

In the Lipschitz case strong uniqueness of solutions can be obtained comparatively easily. In the non-Lipschitz case, however, the question of uniqueness becomes much more delicate. While statements on strong uniqueness do not exist so far, weak uniqueness could be established for Eq. (3) with $b \equiv 0$, $a(t,x,u) = u^{\gamma}$ and $\gamma \in [\frac{1}{2}, 1)$. For the case $\gamma = \frac{1}{2}$ see Roelly-Coppoletta (1986), the case $\gamma \in (\frac{1}{2}, 1)$ was studied by Mytnik (1998). We here give a generalization of the result on $\gamma = \frac{1}{2}$. As mentioned in Section 1, the interest in this case is due to the relation to the catalytic super-Brownian motion.

Theorem 2.6. *Let* $\eta \in C^+_{\mathrm{tem}}(\mathbb{R})$ *and* $\varrho(\mathrm{d}t\,\mathrm{d}x)$ *satisfy condition* (A). *Then the* (*non-negative*) *weak solution to SPDE* (1) *with* $b \equiv 0$, $a(t,x,u) = \sqrt{|u|}$ *and initial condition* η *is weakly unique.*

We will not prove Theorem 2.6 since a very similar result was proved in the appendix of Zähle (2004) with help of the method of duality. There $\varrho_2(\mathrm{d}t)$ was required to be the Lebesgue measure $\mathrm{d}t$ and the state space of $u = (u(t,.) : t \geqslant 0)$ was the space $C_{\mathrm{int}}(\mathbb{R})$ of Lebesgue-integrable continuous functions instead of $C_{\mathrm{tem}}(\mathbb{R})$. However, the adaption of the proof to our setting is not difficult. In particular, one has to take Lemma 3.6 (stated below) into account and the state space of u's dual process should be chosen to be the space $C_{\mathrm{rap}}(\mathbb{R})$ of rapidly decreasing continuous functions instead of $C_b(\mathbb{R})$.

The remainder of the paper is organized as follows. In the next section, we give a series of auxiliary lemmas. In Section 4, we shall establish the equivalence of SPDE (1) in the sense of Definition 2.1 to both a certain martingale problem and a certain stochastic integral equation. Sections 5–7 are devoted to the proofs of Theorems 2.3–2.5, respectively. Note that corresponding results on existence of solutions to SPDE (3) can be found in Iwata (1987), Mueller and Perkins (1992) and Shiga (1994). The analogue of the non-negativity result was proved by Shiga (1994).

3. Auxiliary lemmas

In this section, we give basic tools for the proofs of our main results. We start with two lemmas provided by Shiga (1994, Lemma 6.3) and Iwata (1987, Lemma 5.4). We equip $C([0,\infty),C_{\mathrm{tem}}(\mathbb{R}))$ with the usual metric inducing the topology of compact uniform convergence. The statements on the Hölder-continuity were not explicitly stated in Shiga (1994) and Iwata (1987) but they are, to some extent, by-products of the proofs.

Lemma 3.1. *Let $X=(X(t,x)\colon t\geqslant 0,\, x\in\mathbb{R})$ be a real-valued process such that $X(0,.)\in C_{\mathrm{tem}}(\mathbb{R})$ $\mathbf{P}$-almost surely. Assume there are constants $q,\varepsilon>0$ such that for every $\lambda,T>0$ there exists a finite constant $c_{\lambda,T}>0$ satisfying*

$$\mathbf{E}[|X(t,x)-X(t',x')|^q]\leqslant c_{\lambda,T}(|t-t'|^{2+\varepsilon}+|x-x'|^{2+\varepsilon})\mathrm{e}^{\lambda|x|}$$

for all $t,t'\leqslant T$ and $x,x'\in\mathbb{R}$ with $|x-x'|\leqslant 1$. Then X has a modification $\tilde{X}$ such that $(\tilde{X}(t,.)\colon t\geqslant 0)$ is $C_{\mathrm{tem}}(\mathbb{R})$-valued continuous. Moreover, $\tilde{X}$ is locally jointly Hölder-γ-continuous for each $\gamma\in(0,\varepsilon/q)$.

Lemma 3.2. *Let $X_1,X_2,\ldots$ be the coordinate processes of probability measures $\mathbf{P}_1$, $\mathbf{P}_2,\ldots$ on $C([0,\infty),C_{\mathrm{tem}}(\mathbb{R}))$. Assume there are constants $\varepsilon,q>0$ such that for every $\lambda,T>0$ there is a finite constant $c_{\lambda,T}>0$ satisfying*

$$\sup_{n\geqslant 1}\mathbf{E}_n[|X_n(t,x)-X_n(t',x')|^q]\leqslant c_{\lambda,T}(|t-t'|^{2+\varepsilon}+|x-x'|^{2+\varepsilon})\mathrm{e}^{\lambda|x|}$$

for all $t,t'\leqslant T$ and $x,x'\in\mathbb{R}$ with $|x-x'|\leqslant 1$. If $(\mathbf{P}_n\circ X_n(0,.)^{-1})$ is tight in $C_{\mathrm{tem}}(\mathbb{R})$, then $(\mathbf{P}_n)$ is tight in $C([0,\infty),C_{\mathrm{tem}}(\mathbb{R}))$. Also, the coordinate process of any limit point $\mathbf{P}$ is locally jointly Hölder-γ-continuous for each $\gamma\in(0,\varepsilon/q)$.

Lemma 3.3. *Let $\varrho(\mathrm{d}t\,\mathrm{d}x)\in\mathscr{M}([0,\infty)\times\mathbb{R})$ be as in Definition 1.1 with $\alpha_2>0$ instead of (c). Then the orthogonal martingale measure W^ϱ from Section 2 is a continuous one. In particular, the stochastic integral $\int\int f(r,y)W^\varrho(\mathrm{d}r\,\mathrm{d}y)$ is a continuous orthogonal martingale measure for every predictable $f:[0,\infty)\times\mathbb{R}\times\Omega\to\mathbb{R}$ with $\mathbf{E}[\int_0^t\int_{\mathbb{R}}f^2(r,y)\varrho(\mathrm{d}r\,\mathrm{d}y)]<\infty$ $\forall t\geqslant 0$.*

Proof. Let B be a bounded Borel set in $\mathbb{R}$ and recall $W^\varrho(A)\sim N(0,\varrho(A))$ for $A\in\mathscr{A}$. Then, if $W_t^\varrho(B):=W^\varrho((0,t]\times B)$, we have for $t\leqslant t'\leqslant T$ and $m\geqslant 1$;

$$\mathbf{E}[|W_t^\varrho(B)-W_{t'}^\varrho(B)|^{2m}]=\mathbf{E}[|W^\varrho(B\times(t,t'])|^{2m}]\leqslant c_{B,T}|t-t'|^{m\alpha_2}.$$

Hence, for m sufficiently large, Kolmogorov's theorem gives a continuous modification of $(W_t^{\varrho}(B){:}\ t\in[0,T])$ for every $T>0$. □

Lemma 3.4. *Consider* $\mu_1(\mathrm{d}x)\in\mathscr{M}_{\mathrm{uni}}(\mathbb{R})$, $\alpha_1\in(0,1]$, $\gamma>0$ *and*

(a) $\exists c>0$: $\sup_{x\in\mathbb{R}}\mu_1(B(x,r))\leqslant cr^{\alpha_1},\ r\in(0,1]$;
(b) $\exists c>0$: $\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\mathrm{e}^{-(x-y)^2/r}\mu_1(\mathrm{d}y)\leqslant cr^{\alpha_1/2},\ r\in(0,1]$;
(c) $\exists c_\lambda>0$: $\sup_{x\in\mathbb{R}}\mathrm{e}^{-\lambda|x|}\int_{\mathbb{R}}\mathrm{e}^{-(x-y)^2/r}\mathrm{e}^{\lambda|y|}\mu_1(\mathrm{d}y)\leqslant c_\lambda r^{\alpha_1/2},\ r\in(0,1]$;
(d) $\exists c_\lambda>0$: $\sup_{x,x'\in\mathbb{R}}\mathrm{e}^{-\lambda|x-x'|-\lambda|x|}\int_{\mathbb{R}}\mathrm{e}^{-(x'-y)^2/r}\mathrm{e}^{\lambda|y|}\mu_1(\mathrm{d}y)\leqslant c_\lambda r^{\alpha_1/2},\ r\in(0,1]$.

Then, (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c), (d) *for every* $\lambda>0$.

Proof. (a) $\Leftrightarrow$ (b) was proved in Zähle (2004, Lemma 3.1). The proof of (b) $\Rightarrow$ (c), (d) is not hard and will be omitted. □

Remark 3.5. If we assume (b) of Lemma 3.4, then we also have

(b)' $\exists c>0$: $\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\mathrm{e}^{-(x-y)^2/r}\mu_1(\mathrm{d}y)\leqslant c(r^{1/2}\vee r^{\alpha_1/2}),\ r>0$.

The remark is justified by the fact that elements of $\mathscr{M}_{\mathrm{uni}}(\mathbb{R})$ are globally (i.e. on balls with radius bigger than one) bounded by a multiple of the Lebesgue measure.

Lemma 3.6. *Let* $\mu_2(\mathrm{d}t)\in\mathscr{M}([0,\infty))$. *If there exists an* $\alpha_2\in(0,1]$ *such that*

$$\forall T>0\ \exists c_T>0:\quad \sup_{t\leqslant T}\mu_2([0,\infty)\cap B(t,r))\leqslant c_T r^{\alpha_2}\quad \forall r\in(0,1]$$

holds, then there is for every $T>0$ *a finite* $c_T>0$ *such that for all* $0\leqslant t\leqslant T$,

(a) $\int_s^t(t-r)^{-\gamma}\mu_2(\mathrm{d}r)\leqslant c_T(t-s)^{\alpha_2-\gamma}\ \forall\gamma\in[0,\alpha_2],\ 0\leqslant s\leqslant t$;
(b) $\int_s^v(t-r)^{-\gamma}\mu_2(\mathrm{d}r)\leqslant c_T(t-v)^{-(\gamma-\alpha_2)}\ \forall\gamma\in(\alpha_2,\infty),\ 0\leqslant s\leqslant v<t$;
(c) $\int_0^t r^{\delta}(t-r)^{-\gamma}\mu_2(\mathrm{d}r)\leqslant c_T t^{\delta+\alpha_2-\gamma}(\theta^{\delta}+(1-\theta)^{\alpha_2-\gamma})\ \forall\gamma\in[0,\alpha_2],\ \theta\in[0,1],\ \delta>0$;
(d) $\int_0^T\mathrm{e}^{-\gamma r}\mu_2(\mathrm{d}r)\leqslant c_T\gamma^{-\alpha_2}\ \forall\gamma>0$.

Proof. (a) By means of a substitution $v=u^{-1/\gamma}$ we obtain

$$\begin{aligned}\int_s^t\frac{1}{(t-r)^{\gamma}}\mu_2(\mathrm{d}r)&=\int_0^{\infty}\mu_2\left(r{:}\ \mathbf{1}_{[s,t]}(r)\frac{1}{(t-r)^{\gamma}}\geqslant u\right)\mathrm{d}u\\&\leqslant\int_0^{(t-s)^{-\gamma}}\mu_2([s,t])\,\mathrm{d}u+\int_{(t-s)^{-\gamma}}^{\infty}\mu_2\left(r{:}\,\mathbf{1}_{[s,t]}(r)\frac{1}{(t-r)^{\gamma}}\geqslant u\right)\mathrm{d}u\end{aligned}$$

$$\leqslant (t-s)^{-\gamma} c_T (t-s)^{\alpha_2} + \int_{(t-s)^{-\gamma}}^{\infty} \mu_2([t-u^{-1/\gamma}, t])\,\mathrm{d}u$$

$$= c_T (t-s)^{\alpha_2-\gamma} + \int_0^{t-s} \gamma v^{-\gamma-1} \mu_2([t-v,t])\,\mathrm{d}v$$

$$\leqslant c_T (t-s)^{\alpha_2-\gamma} + \int_0^{t-s} \gamma v^{-\gamma-1} c_T v^{\alpha_2}\,\mathrm{d}v \leqslant c'_T (t-s)^{\alpha_2-\gamma}.$$

(b) The proof goes along the lines of the proof of (a) with the obvious changes.
(c) Elementary estimates and part (a) yield

$$\int_0^t \frac{r^\delta}{(t-r)^\gamma} \mu_2(\mathrm{d}r) = \int_0^{\theta t} \frac{r^\delta}{(t-r)^\gamma} \mu_2(\mathrm{d}r) + \int_{\theta t}^t \frac{r^\delta}{(t-r)^\gamma} \mu_2(\mathrm{d}r)$$

$$\leqslant (\theta t)^\delta \int_0^{\theta t} \frac{1}{(t-r)^\gamma} \mu_2(\mathrm{d}r) + t^\delta \int_{\theta t}^t \frac{1}{(t-r)^\gamma} \mu_2(\mathrm{d}r)$$

$$\leqslant (\theta t)^\delta \int_0^{t} \frac{1}{(t-r)^\gamma} \mu_2(\mathrm{d}r) + t^\delta \int_{\theta t}^t \frac{1}{(t-r)^\gamma} \mu_2(\mathrm{d}r)$$

$$\leqslant (\theta t)^\delta c_T t^{\alpha_2-\gamma} + t^\delta c_T ((1-\theta)t)^{\alpha_2-\gamma}$$

$$\leqslant c_T t^{\delta+\alpha_2-\gamma} (\theta^{\,\delta} + (1-\theta)^{\alpha_2-\gamma}).$$

(d) With help of a substitution $v = \log(1/u)$ we obtain

$$\int_0^T \mathrm{e}^{-\gamma r} \mu_2(\mathrm{d}r) = \int_0^\infty \mu_2(r\colon \mathbf{1}_{[0,T]}(r)\mathrm{e}^{-\gamma r} \geqslant u)\,\mathrm{d}u \leqslant \int_0^1 \mu_2\left(\left[0, T \wedge \frac{1}{\gamma}\log\frac{1}{u}\right]\right)\mathrm{d}u$$

$$= \int_0^\infty \mu_2\left(\left[0, T\wedge \frac{1}{\gamma}v\right]\right)\mathrm{e}^{-v}\,\mathrm{d}v = \int_0^{T\gamma} \mu_2\left(\left[0, T\wedge \frac{1}{\gamma}v\right]\right)\mathrm{e}^{-v}\,\mathrm{d}v$$

$$+ \int_{T\gamma}^\infty \mu_2\left(\left[0, T\wedge \frac{1}{\gamma}v\right]\right)\mathrm{e}^{-v}\,\mathrm{d}v \leqslant \int_0^{T\gamma} c_T \left(\frac{1}{\gamma}v\right)^{\alpha_2} \mathrm{e}^{-v}\,\mathrm{d}v$$

$$+ c_T T^{\alpha_2} \int_{T\gamma}^\infty \mathrm{e}^{-v}\,\mathrm{d}v \leqslant c_T \gamma^{-\alpha_2} + c_T T^{\alpha_2} \mathrm{e}^{-T\gamma} \leqslant c_{T,\alpha_2} \gamma^{-\alpha_2}$$

where the last inequality follows from $\mathrm{e}^{-T\gamma} \leqslant c_{T,\alpha_2}\gamma^{-\alpha_2}$ $\forall \gamma > 0$. □

Define the heat kernel p by $p_t(x,y) = (2\pi t)^{-1/2}\mathrm{e}^{-(x-y)^2/(2t)}$ for $t > 0$ and $x, y \in \mathbb{R}$. Also, set $p_t \equiv 0$ for $t < 0$.

Lemma 3.7. *Let* $\varrho(\mathrm{d}t\,\mathrm{d}x)$ *satisfy condition* (A) *with* α_1, α_2 *and* $\sigma(\mathrm{d}t\,\mathrm{d}x)$ *satisfy condition* (B) *with* β_1, β_2. *Set* $\alpha := \alpha_1/2 + \alpha_2 - 1$ *and* $\beta := \beta_1/2 + \beta_2 - 1/2$. *Then for every* $\lambda \geqslant 0$ *and* $T > 0$ *there exists a finite constant* $c_{\lambda,T} > 0$ *such that for all*

$0 \leqslant t \leqslant t' \leqslant T$ *and* $x, x' \in \mathbb{R}$,

$$\int_0^{t'} \int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 \mathrm{e}^{\lambda|y|} \varrho(\mathrm{d}r\,\mathrm{d}y)$$
$$\leqslant c_{\lambda,T}(|t-t'|^{\alpha} + |x-x'|^{2\alpha})\mathrm{e}^{\lambda|x|}\mathrm{e}^{\lambda|x-x'|}, \tag{8}$$

$$\int_0^{t'} \int_{\mathbb{R}} |p_{t-r}(x,y) - p_{t'-r}(x',y)| \mathrm{e}^{\lambda|y|} \sigma(\mathrm{d}r\,\mathrm{d}y)$$
$$\leqslant c_{\lambda,T}(|t-t'|^{\beta} + |x-x'|^{2\beta})\mathrm{e}^{\lambda|x|}\mathrm{e}^{\lambda|x-x'|}. \tag{9}$$

Proof. Inequality (8) was proved in Zähle (2004, Lemma 3.4) for the case $\varrho_2(\mathrm{d}t) = \mathrm{d}t$ and $\lambda = 0$. By means of Lemma 3.4(c) and (d) and Lemma 3.6(a) and (b) the proof can easily be extended to our general setting. Eq. (9) can be obtained analogously. □

Lemma 3.8. *Let* $k \geqslant 1$ *and* $\mu_2^i(\mathrm{d}t) \in \mathscr{M}([0,\infty))$ $(1 \leqslant i \leqslant k)$. *Assume there exist* $\alpha_2^i \in (0,1]$ $(1 \leqslant i \leqslant k)$ *such that*

$$\forall T > 0\ \exists \hat{c}_T > 0: \quad \sup_{t \leqslant T} \mu_2^i([0,\infty) \cap B(t,r)) \leqslant \hat{c}_T r^{\alpha_2^i} \quad \forall r \in (0,1], \quad 1 \leqslant i \leqslant k.$$

Let $g_n : [0,\infty) \to [0,\infty)$ *be measurable functions* $(n \geqslant 1)$ *and assume* g_1 *is bounded. If there are constants* $\gamma^i \in [0, \alpha_2^i)$ $(1 \leqslant i \leqslant k)$ *and* $c_0 \geqslant 0$ *such that for every* $T > 0$ *there exists a constant* $c_T > 0$ *with*

$$g_{n+1}(t) \leqslant c_T \left(c_0 + \sum_{i=1}^{k} \sup_{s \leqslant t} \int_0^s \frac{1}{(s-r)^{\gamma^i}}\, g_n(r) \mu_2^i(\mathrm{d}r) \right) \quad \forall t \leqslant T, \ n \geqslant 1, \tag{10}$$

then for every $T > 0$ *there exist constants* $q_T \in (0,1)$ *and* $\tilde{c}_T > 0$ (*depending on* T, c_T, $\hat{c}_T$, α_i, γ_i, $\|g_1\|_\infty$, *and being independent of* c_0) *such that*

$$\sup_{t \leqslant T} g_n(t) \leqslant \tilde{c}_T(c_0 + q_T^n) \quad \forall n \geqslant 1.$$

Take note of the following special case. If $g_n = g\ \forall n \geqslant 1$, then (10) implies $\sup_{t \leqslant T} g(t) \leqslant \tilde{c}_T c_0\ \forall T > 0$. Lemma 3.8 is hence a sort of Gronwall lemma. Be aware that the following proof still works (with the obvious changes) if one replaces $\sup_{s \leqslant t} \int_0^s 1/(s-r)^{\gamma^i}$ by $\int_0^t 1/(t-r)^{\gamma^i}$ in the assumptions of the lemma.

Proof of Lemma 3.8. First of all note that in Lemma 3.6(c) the constant $c_T > 0$ is independent of $\delta > 0$. Set $\delta = \min\{\alpha_2^i - \gamma^i : 1 \leqslant i \leqslant k\} > 0$. By Lemma 3.6(a) and (c) we can choose finite constants $c'_T, c''_T > 0$ such that for all $t \leqslant T$, $1 \leqslant i \leqslant k$, $j \geqslant 1$ and $\theta \in (0,1)$,

$$\sup_{s \leqslant t} \int_0^s \frac{1}{(s-r)^{\gamma^i}} \mu_2^i(\mathrm{d}r) \leqslant c'_T t^{\alpha_2^i - \gamma^i} \leqslant c''_T t^{\delta}, \tag{11}$$

$$\sup_{s\leqslant t}\int_0^s \frac{r^{j\delta}}{(s-r)^{\gamma^i}}\mu_2^i(\mathrm{d}r)\leqslant c_T' t^{j\delta+\alpha_2^i-\gamma^i}(\theta^{j\delta}+(1-\theta)^{\alpha_2^i-\gamma^i})$$
$$\leqslant c_T'' t^{(j+1)\delta}(\theta^{j\delta}+(1-\theta)^{\delta}). \tag{12}$$

Set $\bar{c}_T := c_T'' \vee (c_T''\|g_1\|_\infty)$. By assumption and (11) we obtain for all $r_2\in[0,T]$

$$g_2(r_2)\leqslant c_T\left(c_0+\sum_{i=1}^k \sup_{s_1\leqslant r_2}\int_0^{s_1}\frac{1}{(s_1-r_1)^{\gamma^i}}g_1(r_1)\mu_2^i(\mathrm{d}r_1)\right)$$
$$\leqslant c_T(c_0+k\bar{c}_T r_2^\delta)=c_0c_T+c_Tk\bar{c}_Tr_2^\delta.$$

Using this inequality, the assumption and (12) we obtain for all $r_3\in[0,T]$

$$g_3(r_3)\leqslant c_T\left(c_0+\sum_{i=1}^k \sup_{s_2\leqslant r_3}\int_0^{s_2}\frac{1}{(s_2-r_2)^{\gamma^i}}g_2(r_2)\mu_2^i(\mathrm{d}r_2)\right)$$
$$\leqslant c_T\left(c_0+\sum_{i=1}^k \sup_{s_2\leqslant r_3}\int_0^{s_2}\frac{1}{(s_2-r_2)^{\gamma^i}}c_T(c_0+k\bar{c}_Tr_2^\delta)\mu_2^i(\mathrm{d}r_2)\right)$$
$$\leqslant c_T\left(c_0+c_Tc_0\sum_{i=1}^k \sup_{s_2\leqslant r_3}\int_0^{s_2}\frac{1}{(s_2-r_2)^{\gamma^i}}\mu_2^i(\mathrm{d}r_2)\right.$$
$$\left.+c_Tk\bar{c}_T\sum_{i=1}^k \sup_{s_2\leqslant r_3}\int_0^{s_2}\frac{r_2^\delta}{(s_2-r_2)^{\gamma^i}}\mu_2^i(\mathrm{d}r_2)\right)$$
$$\leqslant c_T(c_0+c_Tc_0k\bar{c}_Tr_3^\delta+c_Tk^2\bar{c}_T^2r_3^{2\delta}(\theta^\delta+(1-\theta)^\delta))$$
$$=c_0[c_T+c_T^2k\bar{c}_Tr_3^\delta]+c_T^2k^2\bar{c}_T^2r_3^{2\delta}(\theta^\delta+(1-\theta)^\delta).$$

Going ahead recursively we obtain for all $n\geqslant 1$ and $r_n\in[0,T]$

$$g_n(r_n)\leqslant c_T\left(c_0+\sum_{i=1}^k \sup_{s_{n-1}\leqslant r_n}\int_0^{s_{n-1}}\frac{1}{(s_{n-1}-r_{n-1})^{\gamma^i}}g_{n-1}(r_{n-1})\mu_2^i(\mathrm{d}r_{n-1})\right)$$
$$\leqslant c_0[c_T+c_T^2k\bar{c}_Tr_n^\delta+c_T^3k^2\bar{c}_T{}^2r_n^{2\delta}(\theta^\delta+(1-\theta)^\delta)$$
$$+c_T^4k^3\bar{c}_T{}^3r_n^{3\delta}(\theta^{2\delta}+(1-\theta)^\delta)(\theta^\delta+(1-\theta)^\delta)+\cdots$$
$$+c_T^{n-1}k^{n-2}\bar{c}_T{}^{n-2}r_n^{(n-2)\delta}$$
$$\times(\theta^{(n-2)\delta}+(1-\theta)^\delta)(\theta^{(n-3)\delta}+(1-\theta)^\delta)\cdots(\theta^\delta+(1-\theta)^\delta)]$$
$$+c_T^{n-1}k^{n-1}\bar{c}_T^{n-1}r_n^{(n-1)\delta}$$
$$\times(\theta^{(n-1)\delta}+(1-\theta)^\delta)(\theta^{(n-2)\delta}+(1-\theta)^\delta)\cdots(\theta^\delta+(1-\theta)^\delta).$$

Setting $K_T := c_T k \bar{c}_T T^\delta$ yields for every $n \geqslant 1$ and $r_n \in [0,T]$

$$\begin{aligned} g_n(r_n) \leqslant\; & c_0[c_T + c_T K_T + c_T K_T^2(\theta^{\delta} + (1-\theta)^\delta) + \cdots \\ & + c_T K_T^{n-2}(\theta^{(n-2)\delta} + (1-\theta)^\delta)(\theta^{(n-3)\delta} + (1-\theta)^\delta)\cdots(\theta^{\delta} + (1-\theta)^\delta)] \\ & + K_T^{n-1}(\theta^{(n-1)\delta} + (1-\theta)^\delta)(\theta^{(n-2)\delta} + (1-\theta)^\delta)\cdots(\theta^{\delta} + (1-\theta)^\delta). \end{aligned} \tag{13}$$

Pick $\varepsilon \in (0, K_T^{-1} \wedge 2)$, set $\theta = 1 - (\varepsilon/2)^{1/\delta}$ and choose $j_\varepsilon \geqslant 1$ in such a manner that $\theta^{j\delta} \leq \varepsilon/2$ holds for all $j \geqslant j_\varepsilon$. Thus $(\theta^{j\delta} + (1-\theta)^\delta) \leqslant \varepsilon$ holds for all $j \geqslant j_\varepsilon$. Set $M_\varepsilon = \sup_{j=1,\dots,j_\varepsilon - 1}(\theta^{j\delta} + (1-\theta)^\delta)(\theta^{(j-1)\delta} + (1-\theta)^\delta)\cdots(\theta^{\delta} + (1-\theta)^\delta)\varepsilon^{-(j-1)}$ and define $q_{\varepsilon,T} = \varepsilon K_T$ $(\in (0,1))$. Then we obtain by (13)

$$\begin{aligned} g_n(r_n) &\leqslant c_T c_0[1 + K_T + K_T^2 M_\varepsilon \varepsilon^2 + \cdots + K_T^{n-2} M_\varepsilon \varepsilon^{n-2}] + K_T^{n-1} M_\varepsilon \varepsilon^{n-1} \\ &\leqslant c_{\varepsilon,T}(c_0[1 + K_T + K_T^2 \varepsilon^2 + \cdots + K_T^{n-2}\varepsilon^{n-2}] + K_T^{n-1}\varepsilon^{n-1}) \\ &\leqslant c_{\varepsilon,T}(c_0[1 + K_T + q_{\varepsilon,T}^2 + \cdots + q_{\varepsilon,T}^{n-2}] + q_{\varepsilon,T}^{n-1}) \leqslant \tilde{c}_{\varepsilon,T}(c_0 + q_{\varepsilon,T}^n) \end{aligned}$$

for all $r_n \in [0,T]$ and $n \geqslant 1$. $\square$

4. SPDE (1) reformulated

This section is devoted to the equivalence of SPDE (1) to both a certain martingale problem and a certain stochastic integral equation (SIE).

Definition 4.1. Let $\eta \in C_{\text{tem}}(\mathbb{R})$. The law of an $(\mathscr{F}_t)$-adapted $C_{\text{tem}}(\mathbb{R})$-valued continuous process $(u(t,.)\colon t \geqslant 0)$ on any filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t))$ is said to be solution to the (a,b,η)-martingale problem if under this law

$$\begin{aligned} M_t(\psi) &= \langle u(t,.), \psi\rangle - \langle \eta, \psi\rangle \\ &\quad - \int_0^t \left\langle u(r,.), \frac{1}{2}\Delta\psi \right\rangle \mathrm{d}r - \int_0^t \int_{\mathbb{R}} b(r,y,u(r,y))\psi(y)\sigma(\mathrm{d}r\,\mathrm{d}y), \qquad t \geqslant 0 \end{aligned}$$

is a square-integrable continuous $(\mathscr{F}_t)$-martingale having

$$\langle M(\psi)\rangle_t = \int_0^t \int_{\mathbb{R}} a^2(r,y,u(r,y))\psi^2(y)\varrho(\mathrm{d}r\,\mathrm{d}y), \qquad t \geqslant 0$$

as its quadratic variation process for all $\psi \in C_c^\infty(\mathbb{R})$. We say the solution is unique if any two solutions coincide (in law).

Let $(P_t)_{t \geqslant 0}$ denote the (heat) semigroup corresponding to the heat kernel p, that is $P_t\psi(x) = \int_{\mathbb{R}} p_t(x,y)\psi(y)\,\mathrm{d}y$ for all $t > 0$, $x \in \mathbb{R}$ and $\psi \in C_{\text{tem}}(\mathbb{R})$. Note that $\frac{1}{2}\Delta$ is the generator of $(P_t)_{t \geqslant 0}$ on $C_0(\mathbb{R})$ where $C_0(\mathbb{R})$ is the space of real-valued continuous functions on $\mathbb{R}$ vanishing at infinity.

Definition 4.2. A $C_{\text{tem}}(\mathbb{R})$-valued continuous process $(u(t,.):t \geqslant 0)$ is said to be solution to SIE (14) with initial condition $\eta \in C_{\text{tem}}(\mathbb{R})$ if, given the noise $[W^{\varrho}, \Omega, \mathscr{F}, (\mathscr{F}_t), \mathbf{P}]$, it is $(\mathscr{F}_t)$-adapted and

$$u(t,x) = P_t\eta(x) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y)b(r,y,u(r,y))\sigma(\mathrm{d}r\,\mathrm{d}y) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y)a(r,y,u(r,y))W^{\varrho}(\mathrm{d}r\,\mathrm{d}y) \tag{14}$$

holds for all $t \geqslant 0$ and $x \in \mathbb{R}$, $\mathbf{P}$-almost surely. We say the solution is unique if any two solutions w.r.t. a given noise are indistinguishable.

Proposition 4.3. *Every weak solution to SPDE* (1) *with initial condition η in the sense of Definition* 2.1 *is a solution to the* (a,b,η)*-martingale problem and vice versa.*

Proof. A solution to the SPDE is easily seen to be a solution to the martingale problem. For the other direction one can generalize Section 4.2 of Zähle (2004). Note that the equivalence of an SPDE and a martingale problem has been considered much earlier (see El Karoui and Méléard, 1990). □

Proposition 4.4. *Assume* (5) *and* $\sup_{t\leqslant T}\sup_{x\in\mathbb{R}} \mu(B(x,1)\times B(t,1)) < \infty$ *for all* $T > 0$ *and* $\mu \in \{\varrho, \sigma\}$. *Then every strong solution to SPDE* (1) *with initial condition η in the sense of Definition* 2.1 *is a solution to SIE* (14) *with initial condition η and vice versa.*

Proof. The proof goes along the lines of the proof of Theorem 2.1 in Shiga (1994) with the obvious changes. □

5. Proof of Theorem 2.3

We shall prove that SIE (14) has a unique solution and so, by Proposition 4.4, the same is true for SPDE (1). Given the continuous orthogonal martingale measure $W^{\varrho} = [W^{\varrho}, \Omega, \mathscr{F}, (\mathscr{F}_t), \mathbf{P}]$, let $\mathscr{P}$ be the space of $(\mathscr{F}_t)$-predictable functions $u : [0,\infty) \times \mathbb{R} \times \Omega \to \mathbb{R}$ with $\|u\|_{\lambda,T,m} < \infty$ for all $\lambda, T > 0$ and $m \geqslant 1$ where

$$\|u\|_{\lambda,T,m} := \left(\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathrm{e}^{-\lambda|x|}\mathbf{E}[|u(t,x)|^{2m}] \right)^{1/(2m)}.$$

We identify $u, u' \in \mathscr{P}$ if $u(t,x) = u'(t,x)$ holds $\mathbf{P}$-almost surely for every fixed $(t,x) \in [0,\infty) \times \mathbb{R}$. Then $d_{\mathscr{P}}(f,f') = \sum_{k,l,m=1}^{\infty} 2^{-(k+l+m)}(1 \wedge \|f-f'\|_{1/k,l,m})$ provides a metric on $\mathscr{P}$ w.r.t. which $\mathscr{P}$ is complete. For the sake of a Picard–Lindelöf iteration we introduce the functional

$$\begin{aligned}\Phi(u)(t,x) &:= P_t\eta(x) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) b(r,y,u(r,y))\sigma(\mathrm{d}r\,\mathrm{d}y) \\ &\quad + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) a(r,y,u(r,y)) W^{\varrho}(\mathrm{d}r\,\mathrm{d}y) \\ &=: P_t\eta(x) + \Phi_2(u)(t,x) + \Phi_3(u)(t,x).\end{aligned}$$

For $u \in \mathscr{P}$ the stochastic integral is well defined since the integrand is admissible w.r.t. W^{ϱ}. This is guaranteed by the following estimate:

$$\begin{aligned}&\mathbf{E}\left[\int_0^t \int_{\mathbb{R}} p_{t-r}^2(x,y) a^2(r,y,u(r,y)) \langle W^{\varrho}\rangle(\mathrm{d}r\,\mathrm{d}y)\right] \\ &\leqslant \int_0^t \int_{\mathbb{R}} p_{t-r}^2(x,y) \mathrm{e}^{|y|}\mathrm{e}^{-|y|} c\mathbf{E}[(1+|u(r,y)|)^2]\varrho(\mathrm{d}r\,\mathrm{d}y) \\ &\leqslant c'(1+\|u\|_{1,t,1}^2)\frac{1}{2\pi}\int_0^t \frac{1}{t-r}\int_{\mathbb{R}} \mathrm{e}^{-(x-y)^2/(t-r)}\mathrm{e}^{|y|}\varrho_1(r,\mathrm{d}y)\varrho_2(\mathrm{d}r) \\ &\leqslant c_t' \int_0^t \frac{1}{t-r}\,\mathrm{e}^{|x|}(t-r)^{\alpha_1/2}\varrho_2(\mathrm{d}r) \leqslant c_t t^{\alpha_1/2+\alpha_2-1}\mathrm{e}^{|x|} < \infty \quad \forall t \geqslant 0\end{aligned}$$

for which we used (5), Lemma 3.4(a) $\Rightarrow$ (c) and Lemma 3.6(a). In Step 2 below we will also see that $\Phi(\mathscr{P}) \subset \mathscr{P}$.

Step 1: We first prove that $\Phi(u)$ may be assumed to be $C_{\mathrm{tem}}(\mathbb{R})$-valued continuous whenever $u \in \mathscr{P}$. Using Burckholder–Davis–Gundy's inequality, Hölder's inequality ($(m-1)/m + 1/m = 1$), (5) and (8), we get for all $\lambda > 0$, $0 \leqslant t \leqslant t' \leqslant T$ and $x, x' \in \mathbb{R}$:

$$\begin{aligned}&\mathbf{E}[|\Phi_3(u)(t,x) - \Phi_3(u)(t',x')|^{2m}] \\ &\leqslant c\mathbf{E}\left[\left|\int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 a^2(r,y,u(r,y))\varrho(\mathrm{d}r\,\mathrm{d}y)\right|^m\right] \\ &\leqslant c\left(\int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2\varrho(\mathrm{d}r\,\mathrm{d}y)\right)^{m-1} \int_0^{t'}\int_{\mathbb{R}} \mathrm{e}^{\lambda|y|}\mathrm{e}^{-\lambda|y|} \\ &\quad \times \int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 \mathbf{E}[c'(1+u(r,y))^{2m}]\varrho(\mathrm{d}r\,\mathrm{d}y) \\ &\leqslant c_T'(|t-t'|^{\alpha} + |x-x'|^{2\alpha})^{m-1} \\ &\quad \times \int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 \mathrm{e}^{\lambda|y|}(1+\|u\|_{\lambda,r,m}^{2m})\varrho(\mathrm{d}r\,\mathrm{d}y) \\ &\leqslant c_{\lambda,T}(|t-t'|^{\alpha} + |x-x'|^{2\alpha})^m \mathrm{e}^{\lambda|x-x'|}\mathrm{e}^{\lambda|x|}.\end{aligned}$$

For m sufficiently large, Lemma 3.1 thus provides a $C_{\text{tem}}(\mathbb{R})$-valued continuous modification of $\Phi_3(u)$. Proceeding analogously and using (9) instead of (8) we also get the following estimate for all $\lambda > 0$, $0 \leqslant t \leqslant t' \leqslant T$ and $x, x' \in \mathbb{R}$

$$\mathbf{E}[|\Phi_2(u)(t,x) - \Phi_2(u)(t',x')|^{2m}] \leqslant c_{\lambda,T}(|t-t'|^{\beta} + |x-x'|^{2\beta})^{2m} \mathrm{e}^{\lambda|x-x'|}\mathrm{e}^{\lambda|x|}$$

and so a $C_{\text{tem}}(\mathbb{R})$-valued continuous modification of $\Phi_2(u)$. By Iwata (1987, Lemma 3.1) $(P_t\eta(.){:}t \geqslant 0)$ is $C_{\text{tem}}(\mathbb{R})$-valued continuous, too. Hence $\Phi(u)$ has a $C_{\text{tem}}(\mathbb{R})$-valued continuous modification. Also, due to the obtained estimates, $\Phi(u)$ is locally Hölder-γ-continuous on $(0,\infty) \times \mathbb{R}$ for all $\gamma \in (0, (\alpha/2) \wedge \beta)$.

Step 2: As already mentioned, we intend a Picard–Lindelöf iteration w.r.t. the functional Φ. Set $u_0 = P_{\cdot}\eta(.)$ and $u_{n+1} = \Phi(u_n)$ for every $n \geqslant 0$. We shall show that $u_n \in \mathscr{P}$ for every $n \geqslant 0$ and, in particular, that

$$\sup_{n \geqslant 1} \|u_n\|_{\lambda,T,m} \leqslant c_{\lambda,T,m} < \infty \quad \forall \lambda, T > 0, \;\; m \geqslant 1 \tag{15}$$

holds whenever $\eta \in C_{\text{tem}}(\mathbb{R})$.

To show $u_n \in \mathscr{P}$ for every $n \geqslant 0$ it is enough to show $\Phi(u) \in \mathscr{P}$ whenever $u \in \mathscr{P}$ since $u_0 = P_{\cdot}\eta(.) \in \mathscr{P}$. By Step 1, $\Phi(u)$ is jointly continuous. Thus u is $(\mathscr{F}_t)$-predictable since it is also $(\mathscr{F}_t)$-adapted. If we also had

$$\|\Phi(u)\|_{\lambda,T,m}^{2m} \leqslant c_{\lambda,T,m} \left\{ 1 + \sup_{t \in [0,T]} \int_0^t \frac{1}{(t-r)^{1-\alpha_1/2}} \|u\|_{\lambda,r,m}^{2m} \varrho_2(\mathrm{d}r) \right.$$
$$\left. + \sup_{t \in [0,T]} \int_0^t \frac{1}{(t-r)^{1/2-\beta_1/2}} \|u\|_{\lambda,r,m}^{2m} \sigma_2(\mathrm{d}r) \right\} \tag{16}$$

for all $\lambda, T > 0$ and $m \geqslant 1$, then $\Phi(u) \in \mathscr{P}$ would follow from Lemma 3.6(a). We prove (16). It is not hard to show $\|P_{\cdot}\eta(.)\|_{\lambda,T,m}^{2m} \leqslant c_{\lambda,T,m} < \infty$. By Burckholder–Davis–Gundy's inequality, Hölder's inequality $((m-1)/m + 1/m = 1)$, Lemma 3.4(a) $\Rightarrow$ (c) and (5) we can estimate $\|\Phi_3(u)\|_{\lambda,T,m}^{2m}$ by

$$c \sup_{t \leqslant T} \sup_{x \in \mathbb{R}} \mathrm{e}^{-\lambda|x|} \mathbf{E}\left[\left| \int_0^t \int_{\mathbb{R}} p_{t-r}^2(x,y) a^2(r,y,u(r,y)) \varrho(\mathrm{d}r\,\mathrm{d}y) \right|^m \right]$$
$$\leqslant c \sup_{t \leqslant T} \sup_{x \in \mathbb{R}} \mathrm{e}^{-\lambda|x|} \left(\int_0^t \int_{\mathbb{R}} p_{t-r}^2(x,y) \varrho(\mathrm{d}r\,\mathrm{d}y) \right)^{m-1}$$
$$\times \int_0^t \int_{\mathbb{R}} \mathrm{e}^{\lambda|y|} p_{t-r}^2(x,y) \mathrm{e}^{-\lambda|y|} \mathbf{E}[(1+u(r,y))^{2m}] \varrho_1(r,\mathrm{d}y) \varrho_2(\mathrm{d}r)$$
$$\leqslant c_{T,m} \left\{ 1 + \sup_{t \leqslant T} \sup_{x \in \mathbb{R}} \int_0^t \left(\mathrm{e}^{-\lambda|x|} \int_{\mathbb{R}} \mathrm{e}^{\lambda|y|} p_{t-r}^2(x,y) \varrho_1(r,\mathrm{d}y) \right) \|u\|_{\lambda,r,m}^{2m} \varrho_2(\mathrm{d}r) \right\}$$
$$\leqslant c_{\lambda,T,m} \left\{ 1 + \sup_{t \leqslant T} \int_0^t \frac{1}{(t-r)^{1-\alpha_1/2}} \|u\|_{\lambda,r,m}^{2m} \varrho_2(\mathrm{d}r) \right\}.$$

In the same way we get the analogous estimate for $\|\Phi_2(u)\|^{2m}_{\lambda,T,m}$. On the whole, we reach (16) and so $\Phi(u)\in\mathscr{P}$. In particular, $u_n\in\mathscr{P}$ for every $n\geqslant 0$. The uniform estimate (15) is an immediate consequence of (16) and Lemma 3.8.

Step 3: We here intend to show $\lim_{n\to\infty}\|u_{n+1}-u_n\|_{\lambda,T,m}=0$ for all $\lambda,T>0$ and $m\geqslant 1$. By Lemma 3.8 it suffices to show that for every $\lambda,T>0$ and $m\geqslant 1$ there exists a constant $c_{\lambda,T,m}>0$ (being independent of n) such that

$$\|u_{n+1}-u_n\|^{2m}_{\lambda,T,m}\leqslant c_{\lambda,T,m}\left\{\sup_{t\leqslant T}\int_0^t\frac{1}{(t-r)^{1-\alpha_1/2}}\|u_n-u_{n-1}\|^{2m}_{\lambda,r,m}\varrho_2(\mathrm{d}r)\right.$$
$$\left.+\sup_{t\leqslant T}\int_0^t\frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_n-u_{n-1}\|^{2m}_{\lambda,r,m}\sigma_2(\mathrm{d}r)\right\}\tag{17}$$

holds for all $n\geqslant 0$. Proceeding as in Step 2 we obtain

$$\|\Phi_2(u_n)-\Phi_2(u_{n-1})\|^{2m}_{\lambda,T,m}\leqslant c_{\lambda,T,m}\sup_{t\leqslant T}\int_0^t\frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_n-u_{n-1}\|^{2m}_{\lambda,r,m}\sigma_2(\mathrm{d}r),$$

$$\|\Phi_3(u_n)-\Phi_3(u_{n-1})\|^{2m}_{\lambda,T,m}\leqslant c_{\lambda,T,m}\sup_{t\leqslant T}\int_0^t\frac{1}{(t-r)^{1-\alpha_1/2}}\|u_n-u_{n-1}\|^{2m}_{\lambda,r,m}\varrho_2(\mathrm{d}r),$$

where we used the Lipschitz condition (6) instead of (5). This proves (17).

Step 4: By Steps 2 and 3, (u_n) is a Cauchy sequence in $(\mathscr{P},d_{\mathscr{P}})$. So there exists $u_\infty\in\mathscr{P}$ such that $\lim_{n\to\infty}\|u_\infty-u_n\|_{\lambda,T,m}=0\ \forall\lambda,T>0,m\geqslant 1$. Then it is not hard to show that $u:=\Phi(u_\infty)$ satisfies $\|u-\Phi(u)\|_{\lambda,T,m}=0$, and so $u(t,x)=\Phi(u)(t,x)$ holds for all (t,x) from a fixed countable dense subset of $[0,\infty)\times\mathbb{R}$, $\mathbf{P}$-almost surely. By Step 1, u may be assumed to be $C_{\mathrm{tem}}(\mathbb{R})$-valued continuous and so $u(t,x)=\Phi(u)(t,x)$ even holds for all (t,x), $\mathbf{P}$-almost surely. Consequently, u is a solution of SIE (14). Step 1 also gives the desired local Hölder-continuity.

Step 5: It remains to show strong uniqueness of solutions. Let u,u' be two solutions to SPDE (1) and so to $u=\Phi(u)$. Fix some $\lambda>0$ and define for every $K>0$ the stopping time

$$\tau_K=\inf\left\{t>0\colon\sup_{x\in\mathbb{R}}|u(t,x)|\mathrm{e}^{-(\lambda/2)|x|}\geqslant K\ \text{ or }\ \sup_{x\in\mathbb{R}}|u'(t,x)|\mathrm{e}^{-(\lambda/2)|x|}\geqslant K\right\}$$

and $u_K(t,.)=\mathbf{1}_{t<\tau_K}u(t,.)$, $u'_K(t,.)=\mathbf{1}_{t<\tau_K}u'(t,.)$. As in Step 3 we get

$$\begin{aligned}\|u_K-u'_K\|^2_{\lambda,t,1}&=\|\Phi(u_K)-\Phi(u'_K)\|^2_{\lambda,t,1}\\&\leqslant c_{\lambda,T,1}\left\{\sup_{s\leqslant t}\int_0^{s\wedge\tau_K}\frac{1}{(s-r)^{1-\alpha_1/2}}\|u_K-u'_K\|^2_{\lambda,r,1}\varrho_2(\mathrm{d}r)\right.\\&\qquad\left.+\sup_{s\leqslant t}\int_0^{s\wedge\tau_K}\frac{1}{(s-r)^{1/2-\beta_1/2}}\|u_K-u'_K\|^2_{\lambda,r,1}\sigma_2(\mathrm{d}r)\right\}\end{aligned}$$

for all $t\leqslant T$ and $T>0$. Since $|u_K(t,x)-u'_K(t,x)|^2\leqslant 4K^2\mathrm{e}^{\lambda|x|}$ for all (t,x), Lemma 3.8 gives $\|u_K-u'_K\|_{\lambda,t,1}=0$ for all $t\geqslant 0$. In particular, $u_K(t,x)=u'_K(t,x)$ holds for all (t,x) from any fixed countable dense subset of $[0,\infty)\times\mathbb{R}$, $\mathbf{P}$-almost surely. But u and u' are $C_{\mathrm{tem}}(\mathbb{R})$-valued continuous, in particular $\tau_K\to\infty$ as $K\to\infty$, $\mathbf{P}$-almost surely.

Thus, $u(t,x)=u'(t,x)$ holds for all (t,x), **P**-almost surely. This completes the proof of Theorem 2.3.

6. Proof of Theorem 2.4

Shiga (1994, Theorem 2.3) established non-negativity of solutions to SPDE (3). In order to prove Theorem 2.4 we adapt his arguments. We first prepare the actual proof. Define

$$\Delta_\varepsilon := \varepsilon^{-1}(P_\varepsilon - I),$$

$$P_t^\varepsilon := \mathrm{e}^{t\Delta_\varepsilon} = \mathrm{e}^{-t/\varepsilon}\mathrm{e}^{t/\varepsilon P_\varepsilon} = \mathrm{e}^{-t/\varepsilon}\sum_{n=0}^{\infty}\frac{(t/\varepsilon)^n}{n!}P_{n\varepsilon} = \mathrm{e}^{-t/\varepsilon}I + Q_t^\varepsilon,$$

$$Q_t^\varepsilon f := \int_{\mathbb{R}} q_t^\varepsilon(.,y)f(y)\,\mathrm{d}y, \qquad q_t^\varepsilon(x,y) := \mathrm{e}^{-t/\varepsilon}\sum_{n=1}^{\infty}\frac{(t/\varepsilon)^n}{n!}\,p_{n\varepsilon}(x,y)$$

for all $t \geqslant 0$ and $\varepsilon > 0$. Since P_ε is a non-negative, conservative and contractive linear operator on $(C_0(\mathbb{R}),\|.\|_\infty)$, $(P_t^\varepsilon)_{t\geqslant 0}$ provides a Feller semigroup on $(C_0(\mathbb{R}),\|.\|_\infty)$ with generator Δ_ε.

Lemma 6.1. *There exists a finite constant $c>0$ such that $\mathrm{e}^{-h}\sum_{n=1}^{\infty}(h^n/n!)(h^\gamma/n^\gamma) \leqslant c$ holds for all $\gamma\in[0,1]$ and $h \geqslant 0$.*

Proof. For $h\in[0,1]$ the claim is trivial. Suppose $h>1$. Define $[h]$ to be the unique integer satisfying $[h] \leqslant h < [h]+1$. We plainly have $(h/n)^\gamma \leqslant h/n$ if $h/n \geqslant 1$ (i.e. $n \leqslant [h] \leqslant h$) and $(h/n)^\gamma \leqslant 1$ if $h/n<1$ (i.e. $n \geqslant [h]+1>h$). Therefore,

$$\begin{aligned}\mathrm{e}^{-h}\sum_{n=1}^{\infty}\frac{h^n}{n!}\frac{h^\gamma}{n^\gamma} &\leqslant \mathrm{e}^{-h}\left(\sum_{n=1}^{[h]}\frac{h^n}{n!}\frac{h}{n} + \sum_{[h]+1}^{\infty}\frac{h^n}{n!}\right)\\ &\leqslant \mathrm{e}^{-h}\left(\sum_{n=1}^{[h]}\frac{h^{n+1}}{(n+1)!}\frac{n+1}{n} + \sum_{[h]+1}^{\infty}\frac{h^n}{n!}\right)\\ &\leqslant \mathrm{e}^{-h}\left(\sum_{n=2}^{\infty}\frac{h^n}{n!}2 + \sum_{n=1}^{\infty}\frac{h^n}{n!}\right) \leqslant c. \qquad \square\end{aligned}$$

Lemma 6.2. *Let $i\in\{1,2\}$. For every $\lambda \geqslant 0$ and $R>0$ there exists a finite constant $c_{\lambda,R}>0$ such that for all $x\in\mathbb{R}$, $t>0$ and $\varepsilon\in(0,1]$,*

$$\sup_{r\in[0,R]}\int_{\mathbb{R}}\int_{\mathbb{R}} q_t^\varepsilon(x,z)^i\,p_\varepsilon(z,y)\mathrm{e}^{\lambda|\xi|}\,\mathrm{d}z\varrho_1(r,\mathrm{d}y) \leqslant c_{\lambda,R}\frac{1}{t^{i/2-\alpha_1/2}}\mathrm{e}^{\lambda|x|}, \qquad \xi\in\{y,z\}.$$

Proof. We only prove the case $i=2$ and $\xi=z$. The other cases can be shown analogously. Using Lemma 6.1, Hölder's inequality, Lemma 3.4 and Remark 3.5 we obtain the following estimate:

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} q_t^{\varepsilon}(x,z)^2\, p_\varepsilon(z,y)\mathrm{e}^{\lambda|z|}\,\mathrm{d}z\varrho_1(r,\mathrm{d}y)\\
&\quad=\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{-2t/\varepsilon}\left(\sum_{n=1}^{\infty}\frac{(t/\varepsilon)^n}{n!}p_{n\varepsilon}(x,z)\right)^2 p_\varepsilon(y,z)\mathrm{e}^{\lambda|z|}\,\mathrm{d}z\varrho_1(r,\mathrm{d}y)\\
&\quad\leqslant \mathrm{e}^{-t/\varepsilon}\sum_{n=1}^{\infty}\frac{(t/\varepsilon)^n}{n!}\int_{\mathbb{R}}\int_{\mathbb{R}} p_{n\varepsilon}(x,z)\,p_\varepsilon(z,y)\mathrm{e}^{\lambda|z|}\,\mathrm{d}z\varrho_1(r,\mathrm{d}y)\frac{\mathrm{e}^{-t/\varepsilon}}{t^{1/2}}\sum_{n=1}^{\infty}\frac{(t/\varepsilon)^n}{n!}\frac{(t/\varepsilon)^{1/2}}{n^{1/2}}\\
&\quad\leqslant \mathrm{e}^{-t/\varepsilon}\sum_{n=1}^{\infty}\frac{(t/\varepsilon)^n}{n!}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} p_{n\varepsilon}(x,z)\,p_\varepsilon(z,y)\,\mathrm{d}z\right)^{1/2}\\
&\qquad\times\left(\int_{\mathbb{R}} p_{n\varepsilon}(x,z)\,p_\varepsilon(z,y)\mathrm{e}^{2\lambda|z|}\,\mathrm{d}z\right)^{1/2}\varrho_1(r,\mathrm{d}y)\frac{c}{t^{1/2}}\\
&\quad\leqslant c\frac{1}{t^{1/2}}\,\mathrm{e}^{-t/\varepsilon}\sum_{n=1}^{\infty}\frac{(t/\varepsilon)^n}{n!}\int_{\mathbb{R}} p_{n\varepsilon+\varepsilon}(x,y)^{1/2}\frac{1}{(2\pi n\varepsilon)^{1/4}}\mathrm{e}^{\lambda|y|}\varrho_1(r,\mathrm{d}y)\\
&\quad\leqslant c'\frac{1}{t^{1/2}}\,\mathrm{e}^{-t/\varepsilon}\sum_{n=1}^{\infty}\frac{(t/\varepsilon)^n}{n!}c_{\lambda,R}\frac{1}{(n\varepsilon+\varepsilon)^{1/4-\alpha_1/2}}\mathrm{e}^{\lambda|x|}\frac{1}{(2\pi n\varepsilon)^{1/4}}\\
&\quad\leqslant c'_{\lambda,R}\frac{1}{t^{1-\alpha_1/2}}\left(\mathrm{e}^{-t/\varepsilon}\sum_{n=1}^{\infty}\frac{(t/\varepsilon)^n}{n!}\frac{(t/\varepsilon)^{1/2-\alpha_1/2}}{n^{1/2-\alpha_1/2}}\right)\left(\frac{n\varepsilon}{n\varepsilon+\varepsilon}\right)^{1/4-\alpha_1/2}\mathrm{e}^{\lambda|x|}\\
&\quad\leqslant c''_{\lambda,R}\frac{\mathrm{e}^{\lambda|x|}}{t^{1-\alpha_1/2}}
\end{aligned}
$$

for all $r\in[0,R]$. $\square$

Lemma 6.3. *For every $\delta>0$ and $T>0$ we have*

$$\lim_{\varepsilon\downarrow 0}\sup_{x,y\in\mathbb{R}}\sup_{t\leqslant T} t^{1/2+\delta}\left|\int_{\mathbb{R}} q_t^{\varepsilon}(x,z)\,p_\varepsilon(y,z)\,\mathrm{d}z - p_t(x,y)\right| = 0.$$

Lemma 6.4. *Let $\gamma>0$ and $\lambda\geqslant 0$. Then we have for all $\varepsilon\in(0,1]$*

$$\sup_{x\in\mathbb{R}}\int_{\mathbb{R}} p_\varepsilon(x,y)|x-y|^{\gamma}\mathrm{e}^{\lambda|x-y|}\,\mathrm{d}y\leqslant c_\lambda\varepsilon^{\gamma/4}.$$

The technical proofs of Lemmas 6.3 and 6.4 will be omitted.

Proof of Theorem 2.4. Define time measures $\sigma_x^\varepsilon(\mathrm{d}t) := \int_{\mathbb{R}} p_\varepsilon(x,y)\sigma(\mathrm{d}t\,\mathrm{d}y)$ and time white noises $W_x^\varepsilon(\mathrm{d}t) := \int_{\mathbb{R}} p_\varepsilon(x,y)W^\varrho(\mathrm{d}t\,\mathrm{d}y)$ (formally) for every $\varepsilon > 0$. Note that $\sigma_x^\varepsilon(t) := \int_0^t \sigma_x^\varepsilon(\mathrm{d}r)$ provides a (deterministic) non-decreasing continuous process and $W_x^\varepsilon(t) = \int_0^t W_x^\varepsilon(\mathrm{d}r) := \int_0^t \int_{\mathbb{R}} p_\varepsilon(x,y)W^\varrho(\mathrm{d}r\,\mathrm{d}y)$ provides a continuous square-integrable martingale with quadratic variation process $\langle W_x^\varepsilon\rangle(t) = \int_0^t \int_{\mathbb{R}} p_\varepsilon^2(x,y)\varrho(\mathrm{d}r\,\mathrm{d}y)$ for all $\varepsilon > 0$.

Then the strategy is as follows. First (Step 1) we shall prove that for every fixed $\varepsilon > 0$ the following family of SODEs with index $x \in \mathbb{R}$:

$$u_\varepsilon(t,x) = \eta(x) + \int_0^t \Delta_\varepsilon u_\varepsilon(r,x)\,\mathrm{d}r + \int_0^t b(r,x,u_\varepsilon(r,x))\sigma_x^\varepsilon(\mathrm{d}r) + \int_0^t a(r,x,u_\varepsilon(r,x))W_x^\varepsilon(\mathrm{d}r) \tag{18}$$

has a unique $C_{\mathrm{tem}}(\mathbb{R})$-valued continuous solution u_ε. Then we established that this solution is non-negative (Step 2). In Step 3 we approximate the unique solution u of SPDE (1) by u_ε ($\varepsilon \downarrow 0$) whereby the desired non-negativity of u will follow. The approximation of u by u_ε is not surprising since the equation family (18) is equivalent to the following mollified version of SPDE (1):

$$\begin{aligned}\langle u_\varepsilon(t,.),\psi\rangle = \langle \eta,\psi\rangle &+ \int_0^t \langle u_\varepsilon(r,.),\Delta_\varepsilon\psi\rangle\,\mathrm{d}r \\ &+ \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} b(r,z,u_\varepsilon(r,z))\psi(z)p_\varepsilon(y,z)\,\mathrm{d}z\sigma(\mathrm{d}r\,\mathrm{d}y) \\ &+ \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} a(r,z,u_\varepsilon(r,z))\psi(z)p_\varepsilon(y,z)\,\mathrm{d}zW^\varrho(\mathrm{d}r\,\mathrm{d}y)\end{aligned} \tag{19}$$

for every $t \geqslant 0$ and $\psi \in C_c^\infty(\mathbb{R})$. The key for the proof of the equivalence is that $\langle P_\varepsilon\phi,\psi\rangle = \langle\phi,P_\varepsilon\psi\rangle$ (and so $\langle\Delta_\varepsilon\phi,\psi\rangle = \langle\phi,\Delta_\varepsilon\psi\rangle$) holds for all $\phi \in C_{\mathrm{tem}}(\mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R})$. We omit the details.

Step 1: We here establish a unique solution to (18). The crucial point is that (19), and so (18), is equivalent to the following mollified version of SIE (14):

$$\begin{aligned}u_\varepsilon(t,x) = P_t^\varepsilon\eta(x) &+ \int_0^t\int_{\mathbb{R}} \mathrm{e}^{-(t-r)/\varepsilon} b(r,x,u_\varepsilon(r,x))p_\varepsilon(x,y)\sigma(\mathrm{d}r\,\mathrm{d}y) \\ &+ \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} q_{t-r}^\varepsilon(x,z)b(r,z,u_\varepsilon(r,z))p_\varepsilon(y,z)\,\mathrm{d}z\sigma(\mathrm{d}r\,\mathrm{d}y) \\ &+ \int_0^t\int_{\mathbb{R}} \mathrm{e}^{-(t-r)/\varepsilon} a(r,x,u_\varepsilon(r,x))p_\varepsilon(x,y)W^\varrho(\mathrm{d}r\,\mathrm{d}y) \\ &+ \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} q_{t-r}^\varepsilon(x,z)a(r,z,u_\varepsilon(r,z))p_\varepsilon(y,z)\,\mathrm{d}zW^\varrho(\mathrm{d}r\,\mathrm{d}y).\end{aligned} \tag{20}$$

The proof of the equivalence works analogously to the proof of Proposition 4.4 (recall that Δ_ε was the generator of (P_t^ε)). Then mimic the proof of Theorem 2.3 to obtain a unique $C_{\mathrm{tem}}(\mathbb{R})$-valued continuous solution to SIE (20). This time one has to choose

$\Phi^\varepsilon(u) := P^\varepsilon_{.}\eta(.) + \Phi^\varepsilon_{2,1}(u) + \Phi^\varepsilon_{2,2}(u) + \Phi^\varepsilon_{3,1}(u) + \Phi^\varepsilon_{3,2}(u) :=$ r.h.s. of (20). Note that the essential technical tools are Lemmas 3.4 and 3.6 as before, as well as Lemma 6.2 and an analogue of Lemma 3.7. In particular, one obtains $\sup_{\varepsilon\in(0,1]}\|u_\varepsilon\|_{\lambda,T,m} < \infty$ for all $\lambda, T > 0$ and $m \geqslant 1$.

Step 2: Let us turn to the non-negativity of solutions to (18). Choose a sequence $(x_n) \subset (-\infty, 0)$ in such a manner that $x_0 = -1$, $x_n \uparrow 0$ and $\int_{x_{n-1}}^{x_n} x^{-2}\,\mathrm{d}x = n$. For every $n \geqslant 1$ pick a real-valued continuous function g_n on $\mathbb{R}$ such that $\mathrm{supp}(g_n) \subset (x_{n-1}, x_n)$, $0 \leqslant g_n(x) \leqslant 2x^{-2}/n$ and $\int_{x_{n-1}}^{x_n} g_n(x)\,\mathrm{d}x = 1$. Set $f_n(x) = \int_x^0 \int_y^0 g_n(z)\,\mathrm{d}z\,\mathrm{d}y$ for all $x < 0$ and $f_n(x) = 0$ for all $x \geqslant 0$. The functions f_n are obviously in $C^2(\mathbb{R})$. We have $f_n'(x) = -\int_x^0 g_n(y)\,\mathrm{d}y$ and $f_n''(x) = g_n(x)$ for every $x < 0$ and $f_n'(x) = f_n''(x) = 0$ for every $x \geqslant 0$. One should interpret f_n, f_n' and f_n'' as approximations of $f(x) := -\min\{0, x\}$, "$f'(x)$"$= -\mathbf{1}_{(-\infty,0]}(x)$ and "$f''(x)$"$=\delta_0(x)$, respectively. In particular, we have $0 \leqslant f(x) - f_n(x) \leqslant -x_{n-1} \downarrow 0$ as $n \to \infty$ and $-f_n'(x) \in [0,1]$ for all $x \in \mathbb{R}$. Set $\tilde u_\varepsilon(t,x) = \mathrm{e}^{-|x|}u_\varepsilon(t,x)$. Then $\tilde u_\varepsilon(t,x) = \mathrm{e}^{-|x|}[\eta(x) + A_x^\varepsilon(t) + M_x^\varepsilon(t)] := \mathrm{e}^{-|x|}$[r.h.s. of (18)] is a semimartingale for every $x \in \mathbb{R}$ and Itô's formula yields

$$f_n(\tilde u_\varepsilon(t,x)) = f_n(\mathrm{e}^{-|x|}\eta(x)) + \int_0^t f_n'(\tilde u_\varepsilon(r,x))\mathrm{e}^{-|x|}\,\mathrm{d}A_x^\varepsilon(r)$$
$$+ \int_0^t f_n'(\tilde u_\varepsilon(r,x))\mathrm{e}^{-|x|}\,\mathrm{d}M_x^\varepsilon(r) + \frac{1}{2}\int_0^t f_n''(\tilde u_\varepsilon(r,x))\mathrm{e}^{-2|x|}\,\mathrm{d}\langle M_x^\varepsilon\rangle(r).$$

Taking expectation and using $f_n''=g_n$, $f_n'(u)=0$ for $u \geqslant 0$, $-b(r,y,u) \leqslant L_T|u|$ for $u \in \mathbb{R}$, $|a(r,y,u)| \leqslant L_T|u|$ for $u \in \mathbb{R}$, $-f_n'(u) \in [0,1]$ for $u \in \mathbb{R}$, Lemma 3.4 and $-u \leqslant f(u)$ for $u \in \mathbb{R}$ we obtain

$$\begin{aligned}
&\mathbf{E}[f_n(\tilde u_\varepsilon(t,x))]\\
&\quad= \mathbf{E}\left[\int_0^t f_n'(\tilde u_\varepsilon(r,x))\mathrm{e}^{-|x|}\,\mathrm{d}A_x^\varepsilon(r)\right] + \frac{1}{2}\mathbf{E}\left[\int_0^t f_n''(\tilde u_\varepsilon(r,x))\mathrm{e}^{-2|x|}\,\mathrm{d}\langle M_x^\varepsilon\rangle(r)\right]\\
&\quad= \mathbf{E}\left[\int_0^t f_n'(\tilde u_\varepsilon(r,x))\,(\Delta_\varepsilon u_\varepsilon(r,x))\,\mathrm{e}^{-|x|}\,\mathrm{d}r\right]\\
&\qquad+ \mathbf{E}\left[\int_0^t\int_{\mathbb{R}} f_n'(\tilde u_\varepsilon(r,x))b(r,x,u_\varepsilon(r,x))\,p_\varepsilon(x,y)\mathrm{e}^{-|x|}\sigma(\mathrm{d}r\,\mathrm{d}y)\right]\\
&\qquad+\frac{1}{2}\mathbf{E}\left[\int_0^t\int_{\mathbb{R}} f_n''(\tilde u_\varepsilon(r,x))a^2(r,x,u_\varepsilon(r,x))\,p_\varepsilon^2(x,y)\mathrm{e}^{-2|x|}\varrho(\mathrm{d}r\,\mathrm{d}y)\right]\\
&\quad=\frac{1}{\varepsilon}\mathbf{E}\left[\int_0^t f_n'(\tilde u_\varepsilon(r,x))\int_{\mathbb{R}} p_\varepsilon(x,y)\tilde u_\varepsilon(r,y)\mathrm{e}^{(|y|-|x|)}\,\mathrm{d}y\,\mathrm{d}r\right.\\
&\qquad\left.-\int_0^t f_n'(\tilde u_\varepsilon(r,x))\tilde u_\varepsilon(r,x)\,\mathrm{d}r\right]
\end{aligned}$$

$$+\mathbf{E}\left[\int_0^t\int_{\mathbb{R}}(-f_n'(\tilde u_\varepsilon(r,x)))(-b(r,x,u_\varepsilon(r,x)))\mathrm{e}^{-|x|}p_\varepsilon(x,y)\sigma(\mathrm{d}r\,\mathrm{d}y)\right]$$

$$+\frac{1}{2}\mathbf{E}\left[\int_0^t\int_{\mathbb{R}}g_n(\tilde u_\varepsilon(r,x))a^2(r,x,u_\varepsilon(r,x))p_\varepsilon^2(x,y)\mathrm{e}^{-2|x|}\varrho(\mathrm{d}r\,\mathrm{d}y)\right]$$

$$\leqslant\frac{1}{\varepsilon}\mathbf{E}\left[\int_0^t f_n'(\tilde u_\varepsilon(r,x))\int_{\mathbb{R}}p_\varepsilon(x,y)\tilde u_\varepsilon(r,y)\mathrm{e}^{|y|}\,\mathrm{d}y\,\mathrm{d}r\right]\mathrm{e}^{-|x|}$$

$$+\mathbf{E}\left[\int_0^t\int_{\mathbb{R}}(-f_n'(\tilde u_\varepsilon(r,x))L_T|\tilde u_\varepsilon(r,x)|)p_\varepsilon(x,y)\sigma(\mathrm{d}r\,\mathrm{d}y)\right]$$

$$+\frac{1}{2}\mathbf{E}\left[\int_0^t\int_{\mathbb{R}}\frac{2|\tilde u_\varepsilon(r,x)|^{-2}}{n}L_T^2|\tilde u_\varepsilon(r,x)|^2p_\varepsilon^2(x,y)\varrho(\mathrm{d}r\,\mathrm{d}y)\right]$$

$$\leqslant\frac{1}{\varepsilon}\mathbf{E}\left[\int_0^t(-f_n'(\tilde u_\varepsilon(r,x)))\int_{\mathbb{R}}p_\varepsilon(x,y)\mathrm{e}^{|y|}(-\tilde u_\varepsilon(r,y))\,\mathrm{d}y\,\mathrm{d}r\right]\mathrm{e}^{-|x|}$$

$$+c_{\varepsilon,T}\mathbf{E}\left[\int_0^t(-f_n'(\tilde u_\varepsilon(r,x)))(-\tilde u_\varepsilon(r,x))\sigma_2(\mathrm{d}r)\right]$$

$$+\frac{L_T^2}{n}\int_0^t\int_{\mathbb{R}}p_\varepsilon^2(x,y)\varrho(\mathrm{d}r\,\mathrm{d}y)$$

$$\leqslant c_\varepsilon\int_0^t\sup_{y\in\mathbb{R}}\mathbf{E}[f(\tilde u_\varepsilon(r,y))]\,\mathrm{d}r+c_{\varepsilon,T}\int_0^t\sup_{x\in\mathbb{R}}\mathbf{E}[f(\tilde u_\varepsilon(r,x))]\sigma_2(\mathrm{d}r)+\frac{\bar c_{\varepsilon,T}}{n}$$

for all $t\leqslant T$ and $x\in\mathbb{R}$, for each $T>0$. Letting $n\to\infty$ we infer by the dominated convergence theorem and the convergence of f_n to f that

$$\|\mathbf{E}[f(\tilde u_\varepsilon(t,.))]\|_\infty\leqslant c_\varepsilon\int_0^t\|\mathbf{E}[f(\tilde u_\varepsilon(r,.))]\|_\infty\,\mathrm{d}r+c_{\varepsilon,T}\int_0^t\|\mathbf{E}[f(\tilde u_\varepsilon(r,.))]\|_\infty\sigma_2(\mathrm{d}r)$$

holds for all $t\leqslant T$, for each $T>0$. Therefore, we deduce by Lemma 3.8 that $\sup_{t\leqslant T}\|\mathbf{E}[f(\tilde u_\varepsilon(t,.))]\|_\infty=0$ holds for each $T>0$. Since $f\geqslant0$, we conclude $f(\tilde u_\varepsilon(t,x))=0$ $\mathbf{P}$-almost surely, for all (t,x). Hence, $f(\tilde u_\varepsilon(t,x))=0$ holds for all rational couples (t,x), $\mathbf{P}$-almost surely. The joint continuity of $\tilde u_\varepsilon$ finally implies $\tilde u_\varepsilon(t,x)\geqslant0$ (and so $u_\varepsilon(t,x)\geqslant0$) for all (t,x), $\mathbf{P}$-almost surely.

Step 3: We now approximate u by the u_ε. Plainly,

$$\mathrm{e}^{-\lambda|x|}\mathbf{E}[|u_\varepsilon(t,x)-u(t,x)|^2]\leqslant\mathrm{e}^{-\lambda|x|}2\left\{|P_t^\varepsilon\eta(x)-P_t\eta(x)|^2\right.$$

$$+\mathbf{E}\left[\left|\int_0^t\int\mathrm{e}^{-(t-r)/\varepsilon}b(r,x,u_\varepsilon(r,x))p_\varepsilon(x,y)\sigma(\mathrm{d}r\,\mathrm{d}y)\right|^2\right]$$

$$+\mathbf{E}\left[\left|\int_0^t\iint q_{t-r}^\varepsilon(x,z)(b(r,z,u_\varepsilon(r,z))-b(r,z,u(r,z)))p_\varepsilon(y,z)\,\mathrm{d}z\sigma(\mathrm{d}r\,\mathrm{d}y)\right|^2\right]$$

$$+\mathbf{E}\left[\left|\int_0^t \iint q^{\varepsilon}_{t-r}(x,z)(b(r,z,u(r,z)) - b(r,y,u(r,y)))\, p_{\varepsilon}(y,z)\, \mathrm{d}z\sigma(\mathrm{d}r\,\mathrm{d}y)\right|^2\right]$$

$$+\mathbf{E}\left[\left|\int_0^t \int \left(\int_{\mathbb{R}} q^{\varepsilon}_{t-r}(x,z)\, p_{\varepsilon}(y,z)\, \mathrm{d}z - p_{t-r}(x,y)\right) b(r,y,u(r,y))\sigma(\mathrm{d}r\,\mathrm{d}y)\right|^2\right]$$

$$+\mathbf{E}\left[\left|\int_0^t \int \mathrm{e}^{-(t-r)/\varepsilon} a(r,x,u_{\varepsilon}(r,x))\, p_{\varepsilon}(x,y) W^{\varrho}(\mathrm{d}r\,\mathrm{d}y)\right|^2\right]$$

$$+\mathbf{E}\left[\left|\int_0^t \iint q^{\varepsilon}_{t-r}(x,z)(a(r,z,u_{\varepsilon}(r,z)) - a(r,z,u(r,z)))\, p_{\varepsilon}(y,z)\, \mathrm{d}z W^{\varrho}(\mathrm{d}r\,\mathrm{d}y)\right|^2\right]$$

$$+\mathbf{E}\left[\left|\int_0^t \iint q^{\varepsilon}_{t-r}(x,z)(a(r,z,u(r,z)) - a(r,y,u(r,y)))\, p_{\varepsilon}(y,z)\, \mathrm{d}z W^{\varrho}(\mathrm{d}r\,\mathrm{d}y)\right|^2\right]$$

$$+\mathbf{E}\left[\left|\int_0^t \int \left(\int_{\mathbb{R}} q^{\varepsilon}_{t-r}(x,z)\, p_{\varepsilon}(y,z)\, \mathrm{d}z - p_{t-r}(x,y)\right) a(r,y,u(r,y)) W^{\varrho}(\mathrm{d}r\,\mathrm{d}y)\right|^2\right]\Bigg\}$$

$$=: \mathrm{e}^{-\lambda|x|} 2\{I^{\varepsilon}_1(t,x) + \cdots + I^{\varepsilon}_9(t,x)\}.$$

By Lemmas 3.4, 3.6, 6.2 ($i = 2$, $\xi = z$), 6.4 and Hölder's inequality we obtain for all $t \leqslant T$;

$$\begin{aligned}
I^{\varepsilon}_6(t,x) &= \mathbf{E}\left[\int_0^t \int_{\mathbb{R}} \mathrm{e}^{-2(t-r)/\varepsilon} a^2(r,x,u_{\varepsilon}(r,x))\, p^2_{\varepsilon}(x,y) \varrho(\mathrm{d}r\,\mathrm{d}y)\right] \\
&\leqslant c \int_0^t \mathrm{e}^{-2(t-r)/\varepsilon} \int_{\mathbb{R}} p^2_{\varepsilon}(x,y) \mathrm{e}^{\lambda|x|} \varrho_1(r,\mathrm{d}y) \tilde{c}(1 + \|u_{\varepsilon}\|_{\lambda,r,1})^2 \varrho_2(\mathrm{d}r) \\
&\leqslant c \int_0^t \mathrm{e}^{-2(t-r)/\varepsilon} \frac{1}{2\pi\varepsilon} \int_{\mathbb{R}} \mathrm{e}^{-(x-y)^2/\varepsilon} \varrho_1(r,\mathrm{d}y) \tilde{c}_{\lambda,T} \varrho_2(\mathrm{d}r) \mathrm{e}^{\lambda|x|} \\
&\leqslant \tilde{c}_{\lambda,T} \frac{1}{\varepsilon^{1-\alpha_1/2}} \int_0^t \mathrm{e}^{-2(t-r)/\varepsilon} \varrho_2(\mathrm{d}r) \mathrm{e}^{\lambda|x|} \leqslant c_{\lambda,T}\, \varepsilon^{\alpha_1/2+\alpha_2-1} \mathrm{e}^{\lambda|x|},
\end{aligned}$$

$$\begin{aligned}
I^{\varepsilon}_7(t,x) &= \mathbf{E}\Bigg[\int_0^t \int_{\mathbb{R}} \Bigg(\int_{\mathbb{R}} q^{\varepsilon}_{t-r}(x,z) \\
&\quad \times (a(r,z,u_{\varepsilon}(r,z)) - a(r,z,u(r,z)))\, p_{\varepsilon}(y,z)\, \mathrm{d}z\Bigg)^2 \varrho(\mathrm{d}r\,\mathrm{d}y)\Bigg] \\
&\leqslant c\mathbf{E}\Bigg[\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} q^{\varepsilon}_{t-r}(x,z)^2 \mathrm{e}^{\lambda|z|} p_{\varepsilon}(y,z)\, \mathrm{d}z \\
&\quad \times \int_{\mathbb{R}} p_{\varepsilon}(y,z) \mathrm{e}^{-\lambda|z|} |u_{\varepsilon}(r,z) - u(r,z)|^2\, \mathrm{d}z \varrho(\mathrm{d}r\,\mathrm{d}y)\Bigg]
\end{aligned}$$

$$\leqslant \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} q_{t-r}^{\varepsilon}(x,z)^2 e^{\lambda|z|} p_{\varepsilon}(y,z)\,dz\varrho_1(r,dy)\|u_{\varepsilon}-u\|_{\lambda,r,1}^2 \varrho_2(dr)$$

$$\leqslant c_{\lambda,T} \int_0^t \frac{1}{(t-r)^{1-\alpha_1/2}} \|u_{\varepsilon}-u\|_{\lambda,r,1}^2 \varrho_2(dr) e^{\lambda|x|}$$

and

$$I_8^{\varepsilon}(t,x) = \mathbf{E}\left[\int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} q_{t-r}^{\varepsilon}(x,z) \right.\right.$$

$$\left.\left. \times\ (a(r,z,u(r,z)) - a(r,y,u(r,y)))\,p_{\varepsilon}(y,z)\,dz\right)^2 \varrho(dr\,dy)\right]$$

$$\leqslant c \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} q_{t-r}^{\varepsilon}(x,z)^2 e^{\lambda|z|} p_{\varepsilon}(y,z)\,dz$$

$$\times \int_{\mathbb{R}} p_{\varepsilon}(y,z) 2L_t^2(|z-y|^{2\kappa} + \mathbf{E}[|u(r,z)-u(r,y)|^2]) e^{-\lambda|z|}\,dz\varrho(dr\,dy)$$

$$\leqslant c_T' \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} q_{t-r}^{\varepsilon}(x,z)^2 e^{\lambda|z|} p_{\varepsilon}(y,z)\,dz$$

$$\times \int_{\mathbb{R}} p_{\varepsilon}(y,z)(|z-y|^{2\kappa} + c_T|z-y|^{2\alpha} e^{\lambda|z-y|} e^{\lambda|z|}) e^{-\lambda|z|}\,dz\varrho(dr\,dy)$$

$$\leqslant c_T'' \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} q_{t-r}^{\varepsilon}(x,z)^2 e^{\lambda|z|} p_{\varepsilon}(y,z)\,dz$$

$$\times \left(\int_{\mathbb{R}} p_{\varepsilon}(y,z)|z-y|^{2\kappa}\,dz + \int_{\mathbb{R}} p_{\varepsilon}(y,z)|z-y|^{2\alpha} e^{\lambda|z-y|}\,dz\right) \varrho(dr\,dy)$$

$$\leqslant c_{\lambda,T} e^{\lambda|x|}(\varepsilon^{\kappa/2} + \varepsilon^{\alpha/2}).$$

For the estimate of I_8 we used

$$\mathbf{E}[|u(r,z)-u(r,y)|^2] \leqslant c_T|z-y|^{2\alpha} e^{\lambda|z-y|} e^{\lambda|z|} \quad \forall r \leqslant T \quad \text{and} \quad z,y \in \mathbb{R}$$

which follows from Step 1 of the proof of Theorem 2.3. Further, using Lemmas 6.3 and 6.2 ($i=1,\ \xi=y$) we can estimate $I_9^{\varepsilon}(t,x)$ by

$$\mathbf{E}\left[\int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} q_{t-r}^{\varepsilon}(x,z) p_{\varepsilon}(y,z)\,dz - p_{t-r}(x,y)\right)^2 a^2(r,y,u(r,y))\varrho(dr\,dy)\right]$$

$$\leqslant c \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} q_{t-r}^{\varepsilon}(x,z) p_{\varepsilon}(y,z)\,dz - p_{t-r}(x,y)\right)^2$$

$$\times e^{\lambda|y|} e^{-\lambda|y|} \mathbf{E}[(1+u(r,y))^2]\varrho(dr\,dy)$$

$$\leqslant c' \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} q^{\varepsilon}_{t-r}(x,z) p_{\varepsilon}(y,z)\,\mathrm{d}z - p_{t-r}(x,y) \right)^2 \mathrm{e}^{\lambda|y|} \varrho_1(r,\mathrm{d}y) \|u\|^2_{\lambda,r,1} \varrho_2(\mathrm{d}r)$$

$$\leqslant c'_T \int_0^t \int_{\mathbb{R}} (t-r)^{1/2+\delta} \left| \int_{\mathbb{R}} q^{\varepsilon}_{t-r}(x,z) p_{\varepsilon}(y,z)\,\mathrm{d}z - p_{t-r}(x,y) \right|$$

$$\times \frac{1}{(t-r)^{1/2+\delta}} \left(\int_{\mathbb{R}} q^{\varepsilon}_{t-r}(x,z) p_{\varepsilon}(y,z)\,\mathrm{d}z + p_{t-r}(x,y) \right) \mathrm{e}^{\lambda|y|} \varrho_1(r,\mathrm{d}y) \varrho_2(\mathrm{d}r)$$

$$\leqslant c'_{\lambda,T} \int_0^t \int_{\mathbb{R}} h_{\delta,T}(\varepsilon) \times c''_T \frac{1}{(t-r)^{1/2+\delta+1/2-\alpha_1/2}} \mathrm{e}^{\lambda|x|} \varrho_2(\mathrm{d}r) \leqslant c_{\lambda,T} \mathrm{e}^{\lambda|x|} h_{\delta,T}(\varepsilon)$$

for any $\delta \in (0, \alpha_1/2 + \alpha_2 - 1)$, where $h_{\delta,T}(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

Proceeding in the same way we get analogous estimates for $I_2^{\varepsilon}(t,x)$, $I_3^{\varepsilon}(t,x)$, $I_4^{\varepsilon}(t,x)$ and $I_5^{\varepsilon}(t,x)$. On the whole, we obtain

$$\|u_\varepsilon - u\|^2_{\lambda,t,1} \leqslant c_{\lambda,T} \left\{ h_T(\varepsilon) + \sup_{s\in[0,t]} \int_0^s \frac{1}{(s-r)^{1-\alpha_1/2}} \|u_\varepsilon - u\|^2_{\lambda,r,1} \varrho_2(\mathrm{d}r) \right.$$

$$\left. + \sup_{s\in[0,t]} \int_0^s \frac{1}{(s-r)^{1/2-\beta_1/2}} \|u_\varepsilon - u\|^2_{\lambda,r,1} \sigma_2(\mathrm{d}r) \right\}$$

for all $t \leqslant T$, for any $\lambda > 0$ and some $h_T(.)$ satisfying $h_T(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. Lemma 3.8 then gives $\|u_\varepsilon - u\|_{\lambda,T,1} \leqslant \tilde{c}_{\lambda,T} h_T(\varepsilon)$ ($\downarrow 0$ as $\varepsilon \downarrow 0$) for every $T > 0$. Since u_ε and u are jointly continuous and u_ε is non-negative for every $\varepsilon > 0$, u is non-negative, too. We are done. □

7. Proof of Theorem 2.5

We may and do pick two sequences (a_n) and (b_n) of Lipschitz continuous functions approximating a and b, respectively, uniformly on compacts. Also, a_n and b_n can be chosen in such a manner that they fulfill (7) with a common constant c for all $n \geqslant 1$. By Theorem 2.3 there is for every $n \geqslant 1$ a unique strong $C_{\mathrm{tem}}(\mathbb{R})$-valued solutions u_n to SPDE (1) with a, b replaced by a_n, b_n. Let $(\mathbf{P}_n)$ denote the sequence of probability measures on $C([0,\infty), C_{\mathrm{tem}}(\mathbb{R}))$ induced by (u_n) and set $X_n(t,x)=$

$$\int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) b_n(u_n(r,y)) \sigma(\mathrm{d}r\,\mathrm{d}y) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) a_n(u_n(r,y)) W^{\varrho}(\mathrm{d}r\,\mathrm{d}y)$$

for every $n \geqslant 1$. As in Step 1 of the proof of Theorem 2.3 one can show that

$$\sup_{n\geqslant 1} \mathbf{E}_n[|X_n(t,x) - X_n(t',x')|^{2m}]$$

$$\leqslant c_{\lambda,T,m}(|t-t'|^{(\alpha m)\wedge(\beta 2m)} + |x-x'|^{2((\alpha m)\wedge(\beta 2m))}) \mathrm{e}^{\lambda|x|}$$

holds for all $t, t' \leqslant T$, $x, x' \in \mathbb{R}: |x - x'| \leqslant 1$, $\lambda > 0$ and $m \geqslant 1$. Thus, for m sufficiently large, Lemma 3.2 implies tightness of $(\mathbf{P}_n)_{n\geqslant 1}$. Any weak limit point is locally

Hölder-γ-continuous on $(0,\infty)\times\mathbb{R}$ for each $\gamma\in(0,(\alpha/2)\wedge\beta)$ which is also a consequence of Lemma 3.2. In order to complete the proof of Theorem 2.5 it remains to show that any weak limit point u is a weak solution to SPDE (1). By Proposition 4.3 it suffices to show that any weak limit point u solves the martingale problem in Definition 4.1. However, it is more or less standard to conclude the martingale characterization (Definition 4.1) of u from the one of u_n. We omit the details. Note, however, that an essential step is to show $\sup_{n\geqslant 1}\|u_n\|_{\lambda,T,m}<\infty$ (for some $\lambda>0$ and $m\geqslant 1$, and all $T>0$).

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References

Dawson, D.A., Fleischmann, K., 1991. Critical branching in a highly fluctuating random medium. Probab. Theory Related Fields 90 (2), 241–274.
El Karoui, N., Méléard, S., 1990. Martingale measures and stochastic calculus. Probab. Theory Related Fields 84, 83–101.
Fleischmann, K., Le Gall, J.-F., 1995. A new approach to the single point catalytic super-Brownian motion. Probab. Theory Related Fields 102 (1), 63–82.
Iwata, K., 1987. An infinite stochastic differential equation with state space $C(R)$. Probab. Theory Related Fields 74, 141–159.
Konno, N., Shiga, T., 1988. Stochastic partial differential equations for some measure-valued diffusions. Probab. Theory Related Fields 79, 201–225.
Mattila, P., 1995. Geometry of Sets and Measures in Euclidean Spaces. Cambridge University Press, Cambridge.
Méléard, S., Roelly-Coppoletta, S., 1988. A generalized equation for a continuous measure branching process. In: Lecture Notes in Math., Vol. 1390. Springer, Berlin, pp. 171–185.
Mueller, C., Perkins, E.A., 1992. The compact support property for solutions to the heat equation with noise. Probab. Theory Related Fields 93 (3), 325–358.
Mytnik, L., 1998. Weak uniqueness for the heat equation with noise. Ann. Probab. 26 (3), 968–984.
Reimers, M., 1989. One dimensional stochastic partial differential equations and the branching measure diffusion. Probab. Theory Related Fields 81, 319–340.
Roelly-Coppoletta, S., 1986. A criterion of convergence of measure-valued processes: application to measure branching processes. Stochastics 17, 43–65.
Shiga, T., 1994. Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. Canad. J. Math. 46 (2), 415–437.
Walsh, J.B., 1986. An Introduction to stochastic partial differential equations. In: Lecture Notes in Math., Vol. 1180. Springer, Berlin, pp. 266–439.
Zähle, H., 2004. Space-time regularity of catalytic super-Brownian motion. Math. Nachr., in press.