



High-frequency asymptotics for path-dependent functionals of Itô semimartingales

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Abstract

The estimation of local characteristics of Itô semimartingales has received a great deal of attention in both academia and industry over the past decades. In various papers limit theorems were derived for functionals of increments and ranges in the infill asymptotics setting. In this paper we establish the asymptotic theory for a wide class of statistics that are built from the incremental process of an Itô semimartingale. More specifically, we will show the law of large numbers and the associated stable central limit theorem for the path dependent functionals in the continuous setting, and discuss the asymptotic theory for range-based statistics in the discontinuous framework. Some examples from economics and physics demonstrate the potential applicability of our theoretical results in practice.

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1. Introduction

In the last decade limit theory for high frequency observations of Itô semimartingales has received a lot of attention in the scientific literature. Such observation schemes of semimartingales, also called *infill asymptotics*, naturally appear in financial, biological and physical applications among many others. For instance, a seminal work of Delbaen and Schachermayer [8] states that price processes must follow a semimartingale model under no arbitrage conditions.

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A general Itô semimartingale exhibits a representation of the form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + Z_t,$$

where μ represents the drift, σ is the volatility, W is a Brownian motion and Z denotes a pure jump process. Irrespective of the application field, researchers are interested in understanding the fine structure of the underlying Itô semimartingale model based on high frequency observations

$$X_0, X_{\Delta_n}, X_{2\Delta_n}, \dots, X_{\lfloor t/\Delta_n \rfloor \Delta_n},$$

where $\Delta_n \rightarrow 0$, which refers to infill asymptotics. For various testing and estimation problems, the class of generalised multipower variations turned out to be a very important probabilistic tool. In their most general form, generalised multipower variations are defined as

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor - d + 1} f\left(a_n(X_{i\Delta_n} - X_{(i-1)\Delta_n}), \dots, a_n(X_{(i+d-1)\Delta_n} - X_{(i+d-2)\Delta_n})\right),$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth function and the scaling a_n depends on whether the process X has jumps or not. In the continuous case the proper scaling is $a_n = \Delta_n^{-1/2}$. Probabilistic properties of generalised multipower variations in continuous and discontinuous settings have been studied in [4,14,17] among many others. We refer to a recent book [15] for a comprehensive study of high frequency asymptotics for Itô semimartingales. Such probabilistic results found manifold applications in the statistical analysis of semimartingale models. Estimation of the quadratic variation (see e.g. [14]), volatility forecasting (see e.g. [2,3]), and tests for the presence of the jump component (see e.g. [1,5]) are the most prominent applications among many others.

The aim of this paper is to study the asymptotic behaviour of *path dependent* high frequency functionals of Itô semimartingales. This framework is motivated by the fact that in some situations we cannot directly observe the semimartingale X , but only its path dependent functional over short time windows. Let us give two examples. In various applied sciences integrated diffusions (i.e. integrated Itô semimartingales) appear as a natural class of models for a given random phenomena. For example in physics, when a medium’s surface (such as the arctic sea ice) is modelled as a stochastic process, a sonar’s measurement of the reflection of this surface is given by the local time of the surface’s slope process (see e.g. [19,10]). Since this local time process is typically an Itô process again (see e.g. [21,18]), limit theorems for local averages are required in order to make inference on the structure of the original surface process (see e.g. [11] for a detailed discussion). Because only discrete (high frequency) observations of such integrated diffusions are available, one cannot recover the original path of the underlying Itô semimartingales from it. Another example of path dependent functionals are ranges whose statistical properties have been studied in [12,20] in the case of low frequency observations of a scaled Brownian motion. We also refer to an early result by William Feller [9], which characterises the distribution of the range of the Brownian motion.

In this paper we will consider functionals of the incremental process built from X , i.e.

$$V(X, g)_t^n = \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} g\left(\{a_n(X_{(i-1+s)\Delta_n} - X_{(i-1)\Delta_n}); s \in [0, 1]\}\right),$$

where g is now operating on $C([0, 1])$ and the scaling a_n equals $\Delta_n^{-1/2}$ when X is a continuous Itô semimartingale. Obviously, this class of statistics extends the classical concept of power

variations to path dependent functionals. The function $g(x) = \sup_{t \in [0,1]} x(t) - \inf_{t \in [0,1]} x(t)$ recovers the case of realised ranges as have been considered in [6,7] in the context of quadratic variation estimation. In this work we will prove the law of large numbers for the functional $V(X, g)_t^n$ and show the associated stable central limit theorem in the continuous framework. We remark that extending the analysis to general path dependent functionals increases the complexity of the proofs, which is due to the topological structure of the space $C([0, 1])$. Furthermore, a general asymptotic statement in the discontinuous case seems to be out of reach (in contrast to very general results for classical power variations studied in [14]). For this reason, we restrict our attention to range statistics of discontinuous Itô semimartingales where the jump process follows a compound Poisson process, as they seem to be useful in financial applications (see [7]). Finally, we present some applications of the probabilistic results, in particular in the context of integrated diffusions and realised ranges.

The paper is organised as follows. In Section 2 we state the two main theorems for general functionals of continuous Itô semimartingales, establishing the limits in probability as well as the associated stable central limit theorem. In Section 3 we apply the limit theory to three most prominent practical examples including general range statistics and integrated diffusions. Section 4 is devoted to the limit theorems for realised ranges of discontinuous Itô semimartingales. The proofs of the main results are collected in Section 5.

2. Limit theorems for continuous Itô semimartingales

Before we present the main results we start by introducing some notation. We denote by $C([0, 1])$ the space of continuous real valued functions on the interval $[0, 1]$, and by $\|\cdot\|_\infty$ the supremum norm on $C([0, 1])$. A function $f : C([0, 1]) \rightarrow \mathbb{R}$ is said to have polynomial growth if $|f(x)| \leq C(1 + \|x\|_\infty^p)$ for some $C, p > 0$. For any $x, y \in C([0, 1])$ and $f : C([0, 1]) \rightarrow \mathbb{R}$, the expression $f'_y(x)$ denotes the Gâteaux derivative of f at point x in the direction of y , i.e. $f'_y(x) := \lim_{h \rightarrow 0} (f(x + hy) - f(x))/h$ whenever this limit exists.

For any processes Y^n, Y we denote by $Y^n \xrightarrow{ucp} Y$ the uniform convergence in probability, i.e. $\sup_{t \in [0, T]} |Y_t^n - Y_t| \xrightarrow{\mathbb{P}} 0$ for all $T > 0$. Throughout this paper we frequently use the notion of stable convergence, which is due to Renyi [22]. A sequence of random variables $(Y_n)_{n \geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a Polish space (E, \mathcal{E}) is said to converge stably in law to Y ($Y_n \xrightarrow{dst} Y$), where Y is defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space, if and only if for any bounded, continuous function f and any bounded \mathcal{F} -measurable random variable Z it holds that

$$\mathbb{E}[f(Y_n)Z] \rightarrow \mathbb{E}'[f(Y)Z], \quad n \rightarrow \infty. \tag{2.1}$$

Typically, we will deal with spaces $E = \mathbb{D}([0, T], \mathbb{R})$ equipped with the uniform topology when the process Y is continuous. Notice that stable convergence is a stronger mode of convergence than weak convergence. In fact, the statement $Y_n \xrightarrow{dst} Y$ is equivalent to the joint weak convergence $(Y_n, Z) \xrightarrow{d} (Y, Z)$ for any \mathcal{F} -measurable random variable Z .

2.1. Law of large numbers

Throughout this section we are considering a stochastic process X defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions that follows the distribution

of a diffusion

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

for $t \geq 0$, where X_0 is a constant, W is a Brownian motion, μ is a predictable, locally bounded process and σ is an adapted, càdlàg process. Given a function $g : C([0, 1]) \rightarrow \mathbb{R}$ and a vanishing sequence $(\Delta_n)_{n \in \mathbb{N}}$ we define the sequence of processes

$$V(X, g)_t^n := \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} g(\Delta_n^{-\frac{1}{2}} d_i^n(X)), \tag{2.2}$$

$$d_i^n(X) := \{X_{(i-1+s)\Delta_n} - X_{(i-1)\Delta_n}\}_{s \in [0, 1]}. \tag{2.3}$$

For any $z \in \mathbb{R}$ and $g \in C([0, 1])$ we introduce the quantity

$$\rho_z(g) := \mathbb{E}[g(\{z W_s; s \in [0, 1]\})], \tag{2.4}$$

whenever the latter expectation is finite. Our first result is the law of large numbers for the functional $V(X, g)_t^n$.

Theorem 2.1 (Law of Large Numbers). *Let g be a locally uniformly continuous functional, i.e. for $x, y \in C([0, 1])$,*

- (i) *given $K, \epsilon > 0$ there exists $\delta > 0$ such that for $\|x\|_\infty, \|y\|_\infty \leq K, \|x - y\|_\infty \leq \delta$ it follows that $|g(x) - g(y)| \leq \epsilon$,*

and have polynomial growth. Then it holds that

$$V(X, g)_t^n \xrightarrow{ucp} V(X, g)_t := \int_0^t \rho_{\sigma_s}(g) ds, \tag{2.5}$$

where the quantity $\rho_z(g)$ is defined at (2.4).

Remark 2.1. Our notion of locally uniform continuity is slightly unusual. Instead of requiring uniform continuity on neighbourhoods or compact sets we demand it on balls $B_{\leq K}(0) = \{x \in C([0, 1]); \|x\|_\infty \leq K\}$ for $K > 0$, which are not compact with respect to the uniform topology. This type of locally uniform continuity is not required in the classical limit theory for functionals of increments of X (see e.g. [4]) since on finite dimensional spaces continuity on closed balls implies uniform continuity. We remark that our locally uniform continuity assumption is satisfied whenever

$$|g(x) - g(y)| \leq C \|x - y\|_\infty^\delta$$

for all $x, y \in C[0, 1]$ and some $C, \delta > 0$. This condition is satisfied for all practical examples.

Remark 2.2. The law of large numbers in **Theorem 2.1** extends to a multivariate setting in a straightforward manner. Assume that X is a d -dimensional continuous Itô semimartingale, where μ is \mathbb{R}^d -valued predictable, locally bounded process, σ is $\mathbb{R}^{d \times d}$ -valued adapted, càdlàg process and W is a d -dimensional Brownian motion. For a function $g : C([0, 1])^d \rightarrow \mathbb{R}^k$, define the statistic $V(X, g)_t^n$ as at (2.2). Furthermore, for any matrix $\Sigma \in \mathbb{R}^{d \times d}$, we set

$$\rho_\Sigma(g) := \mathbb{E}[g(\{\Sigma W_s; s \in [0, 1]\})].$$

Then, under assumptions of Theorem 2.1, it holds that

$$V(X, g)_t^n \xrightarrow{ucp} V(X, g)_t := \int_0^t \rho_{\sigma_s}(g) ds.$$

This convergence is proved exactly in the same manner as in the univariate case.

2.2. Central limit theorem

Having determined the limit in probability we now turn to the associated stable central limit theorem.

Theorem 2.2 (Central Limit Theorem). *Let g satisfy the conditions of Theorem 2.1. Moreover, we assume that*

- (ii) *given $K, \epsilon > 0$ there exists $\delta > 0$ such that for $\|x\|_\infty, \|y\|_\infty \leq K, \|x - y\|_\infty \leq \delta, \|v\|_\infty \leq 1$ it follows that $|g'_v(x) - g'_v(y)| \leq \epsilon,$*
- (iii) *there exist $C, p > 0$ such that $|g'_v(x)| \leq C(1 + \|x\|_\infty^p)$ for $\|v\|_\infty \leq 1.$*

Let σ be a continuous Itô semimartingale of the form

$$\sigma_t = \sigma_0 + \int_0^t \tilde{\mu}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \tilde{v}_s dV_s,$$

where $\tilde{\mu}, \tilde{\sigma}$ and \tilde{v} are adapted, càdlàg processes and V is another Brownian motion independent of W . Then it follows that

$$\Delta_n^{-\frac{1}{2}} (V(X, g)^n - V(X, g)) \xrightarrow{d_{st}} U(X, g) \tag{2.6}$$

where $U(X, g)_t := \int_0^t u_s^1 ds + \int_0^t u_s^2 dW_s + \int_0^t u_s^3 dW'_s$ with

$$\begin{aligned} u_s^1 &:= \mu_s \rho_{\sigma_s}^{(2)}(g') + \frac{1}{2} \tilde{\sigma}_s \rho_{\sigma_s}^{(3)}(g') - \frac{1}{2} \tilde{\sigma}_s \rho_{\sigma_s}^{(2)}(g') \\ u_s^2 &:= \rho_{\sigma_s}^{(1)}(g), \\ u_s^3 &:= \sqrt{\rho_{\sigma_s}(g^2) - \rho_{\sigma_s}^2(g) - (\rho_{\sigma_s}^{(1)}(g))^2}, \end{aligned}$$

and, for $z \in \mathbb{R}$ and $f(x, y) := g'_y(x),$

$$\begin{aligned} \rho_z^{(1)}(g) &:= \mathbb{E} \left[g(\{z W_s; s \in [0, 1]\}) W_1 \right], \\ \rho_z^{(2)}(f) &:= \mathbb{E} \left[f(\{z W_s; s \in [0, 1]\}, \{s; s \in [0, 1]\}) \right], \\ \rho_z^{(3)}(f) &:= \mathbb{E} \left[f(\{z W_s; s \in [0, 1]\}, \{W_s^2; s \in [0, 1]\}) \right]. \end{aligned}$$

Furthermore, W' is a Brownian motion defined on an extension of $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}),$ which is independent of $\mathcal{F}.$

Some remarks on the application of this probabilistic result are in order.

Remark 2.3. When $g(x) \equiv f(x(1))$ for some function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that f, f' have polynomial growth, we recover the stable central limit theorem for functionals of increments of $X.$

More precisely, it holds that

$$\rho_z^{(1)}(g) = \mathbb{E}[f(zW_1)W_1], \quad \rho_z^{(2)}(g') = \mathbb{E}[f'(zW_1)], \quad \rho_z^{(3)}(g') = \mathbb{E}[f'(zW_1)W_1^2],$$

and we obtain the one-dimensional analogue of the asymptotic theory presented in [17].

Remark 2.4. The assumption on the structural form of σ in Theorem 2.2 is necessary for the proof of the central limit theorem. It can be extended to include general discontinuous Itô semimartingales, without affecting the limit, at the costs of more complicated proof. Such more general assumption has been used in e.g. [4].

Remark 2.5. In general, Theorem 2.2 cannot be applied for statistical inference, since the distribution of the limit $U(X, g)_t$ is unknown. However, when g is an even functional, i.e. $g(x) = g(-x)$ for all $x \in C([0, 1])$, things become different. In this case it holds that

$$\rho_z^{(1)}(g) = \rho_z^{(2)}(g') = \rho_z^{(3)}(g') = 0$$

for all $z \in \mathbb{R}$, since $W \stackrel{d}{=} -W$ and expectations of odd functionals of W are 0. Hence, the limiting process $U(X, g)$ has the form

$$U(X, g)_t = \int_0^t \sqrt{\rho_{\sigma_s}(g^2) - \rho_{\sigma_s}^2(g)} \, dW'_s,$$

which is, conditionally on \mathcal{F} , a Gaussian martingale with mean 0. For a fixed $t > 0$, the result of Theorem 2.2 can be transformed into a standard central limit theorem when g is even. A slight modification of Theorem 2.1 shows that

$$V_t^n := \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - 1} \left\{ g^2(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - g(\Delta_n^{-\frac{1}{2}} d_i^n(X))g(\Delta_n^{-\frac{1}{2}} d_{i+1}^n(X)) \right\} \\ \xrightarrow{ucp} \int_0^t \rho_{\sigma_s}(g^2) - \rho_{\sigma_s}^2(g) \, ds.$$

(This should be compared with the asymptotic theory for bipower variation established in [4].) For any fixed $t > 0$, we then deduce a standard central limit theorem

$$\frac{\Delta_n^{-\frac{1}{2}} (V(X, g)_t^n - V(X, g)_t)}{\sqrt{V_t^n}} \xrightarrow{d} \mathcal{N}(0, 1)$$

by properties of stable convergence. The latter can be used to obtain confidence regions for the quantity $V(X, g)_t$.

Remark 2.6. We continue with the multivariate extension as in Remark 2.2. We do not present the proofs as they follow along the lines of the univariate case. Recalling the notation of the multivariate setting in Remark 2.2, we will again require a structural assumption on σ from Theorem 2.2, except the process $\tilde{\sigma}$ is now $\mathbb{R}^{d \times d \times d}$ -valued (and also the processes $\tilde{\mu}, \tilde{\nu}$ attain the corresponding dimension). For a function $h : C([0, 1])^d \rightarrow \mathbb{R}$, we set $f_h(x, y) = h'_y(x)$ and define

$$\rho_{\Sigma}^{(1)}(h) := \mathbb{E} \left[h(\{z W_s; s \in [0, 1]\}) W_1 \right], \\ \rho_{\Sigma, \nu}^{(2)}(h) := \mathbb{E} \left[f_h(\{\Sigma W_s; s \in [0, 1]\}, \{s\nu; s \in [0, 1]\}) \right],$$

$$\rho_{\Sigma, \Upsilon}^{(3)}(h) := \mathbb{E} \left[f_h \left(\left\{ \Sigma W_s; s \in [0, 1] \right\}, \left\{ \int_0^s \Upsilon W_u dW_u; s \in [0, 1] \right\} \right) \right],$$

with $v \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d \times d}$, $\Upsilon \in \mathbb{R}^{d \times d \times d}$. Then, under assumptions of **Theorem 2.2**, we obtain the stable convergence

$$\Delta_n^{-\frac{1}{2}} (V(X, g)^n - V(X, g)) \xrightarrow{d_{st}} \int_0^t u_s^1 ds + \int_0^t u_s^2 dW_s + \int_0^t u_s^3 dW'_s,$$

where W' is a k -dimensional Brownian motion defined on an extension of $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, which is independent of \mathcal{F} , and the processes u^1, u^2, u^3 are $\mathbb{R}^k, \mathbb{R}^{k \times d}, \mathbb{R}^{k \times k}$ -valued respectively, which are defined as follows:

$$\begin{aligned} u_s^{1,j} &= \rho_{\sigma_s, \mu_s}^{(2)}(g_j) + \rho_{\sigma_s, \tilde{\sigma}_s}^{(3)}(g_j), \\ u_s^{2,(j,l)} &= \rho_{\sigma_s}^{(1),l}(g_j), \\ u_s^3(u_s^3)^* &= w_s - u_s^2(u_s^2)^*, \\ w_s^{(j,j')} &= \rho_{\sigma_s}(g_j g_{j'}) - \rho_{\sigma_s}(g_j) \rho_{\sigma_s}(g_{j'}), \end{aligned}$$

where $g = (g_1, \dots, g_k)$ and $1 \leq j, j' \leq k, 1 \leq l \leq d$. When all involved processes are univariate we readily deduce the result of **Theorem 2.2**, since the derivative $h'_y(x)$ is linear in y and it holds that $2 \int_0^s W_u dW_u = W_s^2 - s$ in the one-dimensional case.

3. Examples and applications

In this section we present some examples that demonstrate the applicability of the limit theory for path dependent functionals of continuous Itô semimartingales. For comparison reasons we start with the classical results on power variations.

Example 1. Here we consider the power variation case which corresponds to $g(x) \equiv f(x(1))$ with $f(x) = |x|^p, p > 0$. Recalling the asymptotic theory from [4] we conclude that

$$\Delta_n^{1-\frac{p}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^p \xrightarrow{ucp} \lambda^{1,p} \int_0^t |\sigma_s|^p ds$$

where $\lambda^{1,p} = \mathbb{E}[|W_1|^p]$. Moreover, the following stable central limit theorem holds

$$\Delta_n^{-\frac{1}{2}} \left(\frac{\Delta_n^{1-\frac{p}{2}}}{\lambda^{1,p}} \sum_{i=0}^{\lfloor t/\Delta_n \rfloor} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^p - \int_0^t |\sigma_s|^p ds \right) \xrightarrow{d_{st}} \sqrt{\Lambda^{1,p}} \int_0^t |\sigma_s|^p dW'_s,$$

where $\Lambda^{1,p} := \frac{\lambda^{1,2p} - (\lambda^{1,p})^2}{(\lambda^{1,p})^2}$. Later on we will compare the efficiency of power variation with other estimators presented in the following examples.

Example 2. Let $g : C([0, 1]) \rightarrow \mathbb{R}$ be defined as $g(x) := f(\int_0^1 x(s) ds)$ for a continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that f, f' have polynomial growth. Then condition (i) of **Theorem 2.1** is obviously satisfied. Furthermore, it holds that

$$g'_y(x) = f' \left(\int_0^1 x(s) ds \right) \int_0^1 y(s) ds, \quad \forall x, y \in C([0, 1]),$$

and conditions (ii) and (iii) of **Theorem 2.2** are fulfilled since f' is continuous and has polynomial growth. In particular, for $f(x) = |x|^p$ with $p > 0$ we obtain that

$$\Delta_n^{1-\frac{p}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left| \Delta_n^{-1} \int_{(i-1)\Delta_n}^{i\Delta_n} X_s \, ds - X_{(i-1)\Delta_n} \right|^p \xrightarrow{ucp} \lambda^{2,p} \int_0^t |\sigma_s|^p \, ds$$

where $\lambda^{2,p} = \mathbb{E}[\left| \int_0^1 W_s \, ds \right|^p]$. Furthermore, for $p > 1$ we deduce the corresponding stable central limit theorem (cf. **Remark 2.5**)

$$\begin{aligned} \Delta_n^{-\frac{1}{2}} \left(\frac{\Delta_n^{1-\frac{p}{2}}}{\lambda^{2,p}} \sum_{i=0}^{\lfloor t/\Delta_n \rfloor} \left| \Delta_n^{-1} \int_{(i-1)\Delta_n}^{i\Delta_n} X_s \, ds - X_{(i-1)\Delta_n} \right|^p - \int_0^t |\sigma_s|^p \, ds \right) \\ \xrightarrow{d_{st}} \sqrt{\Lambda^{2,p}} \int_0^t |\sigma_s|^p \, dW'_s, \end{aligned}$$

with $\Lambda^{2,p} := \frac{\lambda^{2,2p} - (\lambda^{2,p})^2}{(\lambda^{2,p})^2}$.

Example 3. Range-based statistics have been recently successfully applied in financial econometrics. Nowadays, not only asset prices are recorded at moderate frequencies (5–10 min), but also the highs and the lows over the same periods. It turns out that using the observed ranges instead of the observed returns results in more efficient estimation procedures. The infill asymptotics of range-based functionals have been originally studied in [6]. Here the functional $g : C([0, 1]) \rightarrow \mathbb{R}$ is a function of the range, i.e. $g(x) = f(\sup_{t \in [0, 1]} x(t) - \inf_{t \in [0, 1]} x(t))$ for a continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$, such that f, f' have polynomial growth. Then the law of large numbers in **Theorem 2.1** readily applies, but the central limit theorem cannot be directly deduced from **Theorem 2.2**, because the range is not Gâteaux differentiable in general.

However, we may apply the following result: Let $x, y \in C([0, 1])$ be functions such that the set $M := \{t \in [0, 1] : t = \operatorname{argmax}_{s \in [0, 1]} x(s)\}$ is finite, then it holds that (cf. [6])

$$\frac{1}{h} \left(\sup_{0 \leq s \leq 1} (x(s) + hy(s)) - \sup_{0 \leq s \leq 1} x(s) \right) = \max_{t \in M} y(t).$$

In the proofs (see again [6]) the function x plays the role of the Brownian motion, which attains its maximum (resp. minimum) at a unique point almost surely. Let $t_{max} := \operatorname{argmax}_{s \in [0, 1]} W_s$ and $t_{min} := \operatorname{argmin}_{s \in [0, 1]} W_s$. Then the assertion of **Theorem 2.2** remains valid in the range case when σ is everywhere invertible (cf. [7]) with

$$\begin{aligned} \rho_x^{(1)}(g) &= \mathbb{E} \left[f \left(x \left(\sup_{0 \leq t \leq 1} W_s - \inf_{0 \leq s \leq 1} W_s \right) \right) W_1 \right], \\ \rho_x^{(2)}(g') &= \mathbb{E} \left[f' \left(x \left(\sup_{0 \leq t \leq 1} W_s - \inf_{0 \leq s \leq 1} W_s \right) \right) (t_{max} - t_{min}) \right], \\ \rho_x^{(3)}(g') &= \mathbb{E} \left[f' \left(x \left(\sup_{0 \leq t \leq 1} W_s - \inf_{0 \leq s \leq 1} W_s \right) \right) (W_{t_{max}}^2 - W_{t_{min}}^2) \right], \end{aligned}$$

which extends the asymptotic theory presented in [7] to general functions of the range. In particular, for $f(x) = |x|^p$ with $p > 0$ we obtain that

$$\Delta_n^{1-\frac{p}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \sup_{s, u \in [(i-1)\Delta_n, i\Delta_n]} (X_s - X_u)^p \xrightarrow{ucp} \lambda^{3,p} \int_0^t |\sigma_s|^p \, ds \tag{3.1}$$

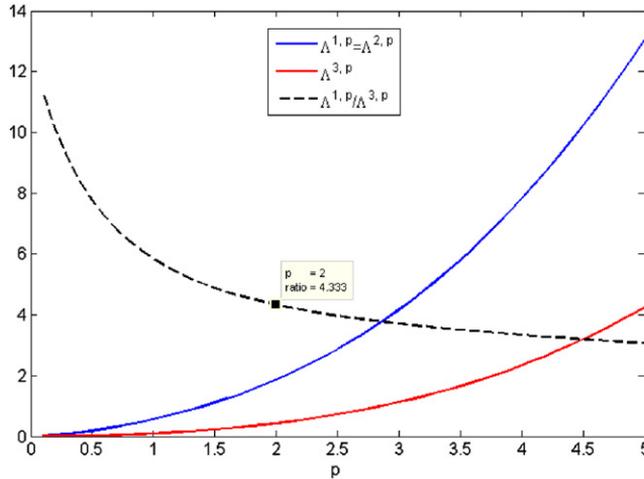


Fig. 1. The parameters $\Delta^{1,p} = \Delta^{2,p}$, $\Delta^{3,p}$ and their ratio.

where $\lambda^{3,p} = \mathbb{E}[\sup_{s,u \in [0,1]} (W_s - W_u)^p]$. Furthermore, since the function f is even, we deduce the following central limit theorem

$$\Delta_n^{-\frac{1}{2}} \left(\frac{\Delta_n^{1-\frac{p}{2}}}{\lambda^{3,p}} \sum_{i=0}^{\lfloor t/\Delta_n \rfloor} \sup_{s,u \in [(i-1)\Delta_n, i\Delta_n]} (X_s - X_u)^p - \int_0^t |\sigma_s|^p ds \right) \xrightarrow{d_{st}} \sqrt{\Delta^{3,p}} \int_0^t |\sigma_s|^p dW'_s$$

where $\Delta^{3,p} := \frac{\lambda^{3,2p} - (\lambda^{3,p})^2}{(\lambda^{3,p})^2}$. This recovers the analysis presented in [7].

Comparison of Examples 1–3

When comparing different estimators of integrated powers of volatility presented in the previous examples, we see that $\Delta^{i,p}$, $i = 1, 2, 3$ serve as a convenient measure of their efficiency. We remark however that this comparison is not fair as the sampling schemes of Example 1 and Examples 2–3 are not comparable.

Since $\int_0^1 W_s ds \sim \mathcal{N}(0, 1/3)$, it follows that

$$\lambda^{1,p} = 3^{p/2} \lambda^{2,p}$$

and $\Delta^{1,p}$ coincides with $\Delta^{2,p}$. However, $\Delta^{3,p}$ is considerably smaller so as expected, range based estimation is asymptotically superior. For example in the case $p = 2$ we find $\Delta^{1,p}, \Delta^{2,p} = 2$, whereas $\Delta^{3,p} \approx 0.4$. The smaller p is, the more pronounced this relative difference becomes. Fig. 1 illustrates these relationships.

Example 4. In various applied sciences integrated diffusions appear as a natural model of a random phenomena. For example in physics, when a medium’s surface (such as the arctic sea ice) is modelled as a stochastic process, a sonar’s measurement of the reflection of this surface is given by the local time of the surface’s slope process (see e.g. [19,10]). Since this local time process is typically an Itô process again (see e.g. [21,18]) and since the observations are given as local averages, limit theorems for local averages are required in order to make inference on the

structure of the original surface process (see e.g. [11]). So let us define the local averages of an Itô process X as

$$\bar{X}_i^n := \frac{1}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} X_s ds.$$

A natural candidate estimator for the quadratic variation of X is given by

$$\sum_{i=2}^{\lfloor t/\Delta_n \rfloor} (\bar{X}_i^n - \bar{X}_{i-1}^n)^2.$$

We note that this estimator does not directly exhibit a representation as in Example 2. However, when we use the decomposition

$$\begin{aligned} \bar{X}_i^n - \bar{X}_{i-1}^n &= \frac{1}{\Delta_n} \left(\int_{(i-1)\Delta_n}^{i\Delta_n} X_s - X_{(i-1)\Delta_n} ds - \int_{(i-2)\Delta_n}^{(i-1)\Delta_n} X_s - X_{(i-2)\Delta_n} ds \right. \\ &\quad \left. + (X_{(i-1)\Delta_n} - X_{(i-2)\Delta_n}) \right), \end{aligned}$$

Theorem 2.1, and the bipower concept of Remark 2.5, we deduce the ucp convergence

$$\sum_{i=2}^{\lfloor t/\Delta_n \rfloor} (\bar{X}_i^n - \bar{X}_{i-1}^n)^2 \xrightarrow{ucp} \frac{2}{3} \int_0^t \sigma_s^2 ds.$$

This clearly provides a way of estimating the quadratic variation of X from observations of an integrated diffusion.

4. Limit theorems for Itô semimartingales with jumps

In this section we study the behaviour of certain path-dependent functionals of discontinuous Itô semimartingales. As the general theory is much more difficult to establish compared to the work of [14], we restrict our attention to ranges of Itô semimartingales with jumps. For simplicity of exposition, we will further restrict ourselves to finite activity jump processes. For more general jump specifications we refer to a much more complex routine of [14], which might be also applicable to range statistics after modifications.

4.1. Law of large numbers

Consider now a stochastic process X defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions that follows the distribution of a diffusion with a jump component in the form of a compound Poisson process $Z_t = \sum_{i=1}^{N_t} J_i$ where N is a Poisson process with intensity λ and i.i.d. jump sizes J_i , i.e.

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + Z_t,$$

where W is a Brownian motion independent of N , μ is a predictable, locally bounded process and σ is an adapted, càdlàg process. For a positive exponent $p > 0$ we define

$$R(X, p)_t^n := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \sup_{s, u \in [(i-1)\Delta_n, i\Delta_n]} |X_s - X_u|^p$$

for $t \geq 0, n \in \mathbb{N}$. Our first result is the following law of large numbers.

Theorem 4.1. *We have that*

$$R(X, p)_t^n \xrightarrow{\mathbb{P}} R(X, p)_t := \begin{cases} \lambda^{3,2} \int_0^t \sigma_s^2 ds + \sum_{i=1}^{N_t} J_i^2 & p = 2 \\ \sum_{i=1}^{N_t} |J_i|^p & p > 2 \end{cases} \tag{4.1}$$

where $\lambda^{3,2} = \mathbb{E}[\sup_{s,u \in [0, 1]} |W_s - W_u|^2]$.

For $p < 2$ we obtain infinity in the limit whenever $\int_0^t \sigma_s^2 ds > 0$. We remark that the first convergence of Theorem 4.1 has been already proved in [7] in the context of range based estimation of quadratic variation. Very similar results have been established for the classical power variations in [14].

4.2. Central limit theorem

Having determined the limit in probability we now turn our attention to the associated stable central limit theorems. In order to introduce the weak limit theory we require some further notation. We denote by $(T_i)_{i \geq 1}$ the successive jump times of the Poisson process N . Furthermore, we introduce two Brownian motions $(W'_t)_{t \geq 0}$, $(\tilde{W}_t)_{t \geq 0}$ and a sequence $(\kappa_i)_{i \geq 1}$ of i.i.d. $\mathcal{U}([0, 1])$ -distributed random variables, which are mutually independent, and independent of \mathcal{F} . Finally, we introduce the process

$$U(X, p)_t = p \sum_{i=1}^{N_t} |J_i|^{p-1} \left\{ \sup_{\substack{0 \leq s \leq \kappa_i \\ \kappa_i \leq u \leq 1}} ((\tilde{W}_{i+\kappa_i} - \tilde{W}_{i+s})\sigma_{T_i-} + (\tilde{W}_{i+u} - \tilde{W}_{i+\kappa_i})\sigma_{T_i}) \mathbf{1}_{\{J_i > 0\}} \right. \\ \left. + \sup_{\substack{0 \leq s \leq \kappa_i \\ \kappa_i \leq u \leq 1}} (-(\tilde{W}_{i+\kappa_i} - \tilde{W}_{i+s})\sigma_{T_i-} - (\tilde{W}_{i+u} - \tilde{W}_{i+\kappa_i})\sigma_{T_i}) \mathbf{1}_{\{J_i < 0\}} \right\} \tag{4.2}$$

that is defined on the extension of the original space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The central limit theorem is as follows.

Theorem 4.2 (Central Limit Theorem). (i) *For $p > 3$ and fixed $t > 0$ we obtain the stable convergence*

$$\Delta_n^{-\frac{1}{2}} (R(X, p)_t^n - R(X, p)_t) \xrightarrow{d_{st}} U(X, p)_t. \tag{4.3}$$

(ii) *Let $p = 2$. Assume that the invertible volatility process σ follows the distribution of a discontinuous Itô semimartingale*

$$\sigma_t = \sigma_0 + \int_0^t \tilde{\mu}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \tilde{v}_s dV_s + \tilde{Z}_t,$$

where $\tilde{\mu}$, $\tilde{\sigma}$ and \tilde{v} are adapted, càdlàg processes, V is another Brownian motion independent of W and $\tilde{Z}_t = \sum_{i=0}^{\tilde{N}_t} \tilde{J}_i$ is a compound Poisson processes with \tilde{N} being independent of W (\tilde{N} and N are possibly correlated). Then, for any fixed $t > 0$, we obtain the stable convergence

$$\Delta_n^{-\frac{1}{2}} (R(X, 2)_t^n - R(X, 2)_t) \xrightarrow{d_{st}} U(X, 2)_t + \sqrt{\lambda^{3,4} - (\lambda^{3,2})^2} \int_0^t \sigma_s^2 dW'_s. \tag{4.4}$$

We remark that [Theorem 4.2](#) is similar in fashion to central limit theorems for classical power variations; see [14]. We do believe that [Theorem 4.2](#) remains valid for a rather general Itô semimartingale model (i.e. not only in the finite activity case), but the proofs become considerably longer.

After local estimation of σ and jump sizes J_i , the conditional law of $U(X, p)_t$ given \mathcal{F} can be simulated. However, unlike for the mixed normal case in the classical power variation framework, the knowledge of the conditional law of $U(X, p)_t$ is not sufficient for statistical inference (e.g. construction of confidence regions).

5. Proofs

First of all, note that without loss of generality we may assume that the processes $\mu, \sigma, \tilde{\mu}, \tilde{\sigma}, \tilde{\nu}$ are bounded. This follows from a standard localisation procedure (see e.g. [4]). Below, all positive constants are denoted by C or C_p if they depend on an external parameter p , although they may change from line to line.

Proof of Theorem 2.1. We begin with some preliminary observations. Denoting

$$A_t := \int_0^t \mu_s \, ds, \quad M_t := \int_0^t \sigma_s \, dW_s,$$

we find that for $p > 0$,

$$\begin{aligned} \mathbb{E} \left[\left(\Delta_n^{-\frac{1}{2}} \|d_i^n(X)\|_\infty \right)^p \right] &\leq C_p \Delta_n^{-\frac{p}{2}} \left(\mathbb{E} \left[\|d_i^n(A)\|_\infty^p \right] + \mathbb{E} \left[\|d_i^n(M)\|_\infty^p \right] \right) \\ &\leq C_p \left(\Delta_n^{\frac{p}{2}} \|\mu\|_\infty^p + \Delta_n^{-\frac{p}{2}} \mathbb{E} \left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 \, ds \right)^{\frac{p}{2}} \right] \right) \\ &\leq C_p \left(\Delta_n^{\frac{p}{2}} \|\mu\|_\infty^p + \|\sigma\|_\infty^p \right) < \infty \end{aligned} \tag{5.1}$$

where we used the Burkholder–Davis–Gundy inequality and the boundedness of μ and σ . Now, by the assumption of polynomial growth, $|g(x)| \leq C(1 + \|x\|_\infty^p)$ for $p > 0$ so

$$\mathbb{E} \left[g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) \right] \leq C \left(1 + \Delta_n^{-\frac{p}{2}} \mathbb{E} \left[\|d_i^n(X)\|_\infty^p \right] \right) < \infty. \tag{5.2}$$

Define $\beta_i^n := \Delta_n^{-\frac{1}{2}} \sigma_{(i-1)\Delta_n} d_i^n(W)$, an approximation of $\Delta_n^{-\frac{1}{2}} d_i^n(X)$. As in (5.1), (5.2) we find that

$$\mathbb{E} \left[\|\beta_i^n\|_\infty^p \right] \leq C_p, \quad p > 0, \tag{5.3}$$

$$\mathbb{E} \left[|g(\beta_i^n)| \right] \leq C. \tag{5.4}$$

β_i^n will serve as a convenient approximation because of its simple form and

$$\begin{aligned} &\mathbb{E} \left[\|\beta_i^n - \Delta_n^{-\frac{1}{2}} d_i^n(X)\|_\infty^p \right] \\ &= \Delta_n^{-\frac{p}{2}} \mathbb{E} \left[\sup_{[(i-1)\Delta_n, i\Delta_n]} \left| \int_{(i-1)\Delta_n}^t \mu_s \, ds + \int_{(i-1)\Delta_n}^t (\sigma_s - \sigma_{(i-1)\Delta_n}) \, dW_s \right|^p \right] \end{aligned}$$

$$\begin{aligned} &\leq C \left(\|\mu\|_\infty^p \Delta_n^{\frac{p}{2}} + \Delta_n^{-\frac{p}{2}} \mathbb{E} \left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_s - \sigma_{(i-1)\Delta_n})^2 ds \right)^{\frac{p}{2}} \right] \right) \\ &\rightarrow 0, \end{aligned} \tag{5.5}$$

where we used again the Burkholder–Davis–Gundy inequality and for the last step that σ is càdlàg. Returning to the claimed convergence in ucp, let

$$\begin{aligned} U_t^n &:= \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} [g(\beta_i^n) \mid \mathcal{F}_{(i-1)\Delta_n}], \\ R_t^{1,n} &:= \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (g(\beta_i^n) - \mathbb{E} [g(\beta_i^n) \mid \mathcal{F}_{(i-1)\Delta_n}]), \\ R_t^{2,n} &:= \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - g(\beta_i^n)), \end{aligned}$$

for all $t \geq 0, n \in \mathbb{N}$. Clearly, $V(X, g)_t^n = U_t^n + R_t^{1,n} + R_t^{2,n}$. In order to prove (2.5), we will first show that the approximation U^n converges to $V(X, g)$ and afterwards that the error terms R^1, R^2 vanish. By definition, $\mathbb{E} [g(\beta_i^n) \mid \mathcal{F}_{(i-1)\Delta_n}] = \rho_{\sigma_{(i-1)\Delta_n}}(g)$, and therefore

$$U_t^n = \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \rho_{\sigma_{(i-1)\Delta_n}}(g) \xrightarrow{ucp} V(X, g)_t = \int_0^t \rho_{\sigma_s}(g) ds$$

due to continuity of the function $\rho(g)$. Turning to the claimed disappearance of $R^{1,n}$ we exploit its martingale property and apply Doob’s maximal inequality to get that

$$\mathbb{P} \left[\sup_{0 \leq t \leq T} |R_t^{1,n}| > \epsilon \right] \leq C \frac{\Delta_n^2}{\epsilon^2} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E} [g(\beta_i^n)^2] \leq C_T \Delta_n \epsilon^{-2} \rightarrow 0$$

for each $\epsilon > 0$. Regarding $R^{2,n}$, Chebyshev’s inequality gives that for $\epsilon > 0$,

$$\mathbb{P} \left[\sup_{0 \leq t \leq T} |R_t^{2,n}| > \epsilon \right] \leq \frac{\Delta_n}{\epsilon} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E} \left[|g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - g(\beta_i^n)| \right].$$

Now we make use of the locally uniform continuity of g . For $K, \hat{\epsilon} > 0$ choose $\delta > 0$ as in (i). Defining $A^{i,n,K} := \{\|\beta_i^n\|_\infty + \|\Delta_n^{-\frac{1}{2}} d_i^n(X)\|_\infty \leq K\}$ as well as $A^{i,n,K,\delta} := A^{i,n,K} \cap \{\|\beta_i^n - \Delta_n^{-\frac{1}{2}} d_i^n(X)\|_\infty \leq \delta\}$ and denoting $\Delta_i^n g := g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - g(\beta_i^n)$, we find that

$$\begin{aligned} \mathbb{E} [|\Delta_i^n g|] &= \mathbb{E} [|\Delta_i^n g| (\mathbf{1}_{A^{i,n,K,\delta}} + \mathbf{1}_{A^{i,n,K} \setminus A^{i,n,K,\delta}} + \mathbf{1}_{\Omega \setminus A^{i,n,K}})] \\ &\leq \hat{\epsilon} + C \left(\mathbb{E} \left[\|\beta_i^n - \Delta_n^{-\frac{1}{2}} d_i^n(X)\|_\infty \right] / \delta + 1/K \right). \end{aligned} \tag{5.6}$$

Hence, choosing K and n large, and then $\hat{\epsilon}$ small, we see that $\mathbb{P} \left[\sup_{0 \leq t \leq T} |R_t^{2,n}| > \epsilon \right]$ vanishes as $n \rightarrow \infty$ and we are done. \square

Proof of Theorem 2.2. Thanks to σ following a diffusion process the approximation β_i^n is now sharper than in (5.5):

$$\begin{aligned}
 & \Delta_n^{-\frac{p}{2}} \mathbb{E} \left[\|\beta_i^n - \Delta_n^{-\frac{1}{2}} d_i^n(X)\|_\infty^p \right] \\
 &= \Delta_n^{-p} \mathbb{E} \left[\sup_{t \in [(i-1)\Delta_n, i\Delta_n]} \left| \int_{(i-1)\Delta_n}^t \mu_s \, ds + \int_{(i-1)\Delta_n}^t (\sigma_s - \sigma_{(i-1)\Delta_n}) \, dW_s \right|^p \right] \\
 &\leq \Delta_n^{-p} C \left(\|\mu\|_\infty^p \Delta_n^p + C \mathbb{E} \left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_s - \sigma_{(i-1)\Delta_n})^2 \, ds \right)^{\frac{p}{2}} \right] \right) \\
 &\leq C \left(\|\mu\|_\infty^p + \Delta_n^{-\frac{p}{2}} \mathbb{E} \left[\sup_{s \in [(i-1)\Delta_n, i\Delta_n]} |\sigma_s - \sigma_{(i-1)\Delta_n}|^p \right] \right) \\
 &\leq C \left(\|\mu\|_\infty^p + \Delta_n^{\frac{p}{2}} \|\tilde{\mu}\|_\infty + \Delta_n^{-\frac{p}{2}} \mathbb{E} \left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} (\tilde{\sigma}_s - \tilde{\sigma}_{(i-1)\Delta_n})^2 \, ds \right)^{\frac{p}{2}} \right] \right) \\
 &\leq C.
 \end{aligned} \tag{5.7}$$

Again, we used the Burkholder–Davis–Gundy inequality and the càdlàg property of $\tilde{\sigma}$. In order to prove (2.6) we split up the original term $\Delta_n^{-\frac{1}{2}} (V(X, g)_t^n - V(X, g)_t)$ into an approximation and several error terms:

$$\begin{aligned}
 \Delta_n^{-\frac{1}{2}} (V(X, g)_t^n - V(X, g)_t) &= \Delta_n^{-\frac{1}{2}} \left(\Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - \int_0^t \rho_{\sigma_s}(g) \, ds \right) \\
 &=: U_t^n + R_t^{1,n} + R_t^{2,n} + R_t^{3,n} + R_t^{4,n},
 \end{aligned}$$

where

$$\begin{aligned}
 U_t^n &= \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (g(\beta_i^n) - \mathbb{E}[g(\beta_i^n) | \mathcal{F}_{(i-1)\Delta_n}]), \\
 R_t^{1,n} &= \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - g(\beta_i^n) - \mathbb{E} \left[g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - g(\beta_i^n) | \mathcal{F}_{(i-1)\Delta_n} \right] \right), \\
 R_t^{2,n} &= \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - g(\beta_i^n) | \mathcal{F}_{(i-1)\Delta_n} \right], \\
 R_t^{3,n} &= \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\Delta_n \mathbb{E}[g(\beta_i^n) | \mathcal{F}_{(i-1)\Delta_n}] - \int_{(i-1)\Delta_n}^{i\Delta_n} \rho_{\sigma_s}(g) \, ds \right), \\
 R_t^{4,n} &= \Delta_n^{-\frac{1}{2}} \int_{\lfloor t/\Delta_n \rfloor \Delta_n}^t \rho_{\sigma_s}(g) \, ds.
 \end{aligned}$$

Obviously, $R_t^{4,n} \xrightarrow{ucp} 0$ due to the boundedness of σ and the continuity of ρ . Furthermore, we also have

Lemma 5.1. Under conditions of Theorem 2.2 we obtain

- (i) $U_t^n \xrightarrow{d_{st}} \int_0^t \rho_{\sigma_s}^{(1)}(g) dW_s + \int_0^t \sqrt{\rho_{\sigma_s}(g^2) - \rho_{\sigma_s}(g)^2 - (\rho_{\sigma_s}^{(1)}(g))^2} dW'_s,$
- (ii) $R_t^{1,n} \xrightarrow{ucp} 0,$
- (iii) $R_t^{2,n} \xrightarrow{ucp} \int_0^t \mu_s \rho_{\sigma_s}^{(2)}(g') ds + \frac{1}{2} \int_0^t \tilde{\sigma}_s \rho_{\sigma_s}^{(3)}(g') ds - \frac{1}{2} \int_0^t \tilde{\sigma}_s \rho_{\sigma_s}^{(2)}(g') ds,$
- (iv) $R_t^{3,n} \xrightarrow{ucp} 0.$

Proof. (i) Defining $\xi_i^n := \Delta_n^{\frac{1}{2}}(g(\beta_i^n) - \mathbb{E}[g(\beta_i^n) | \mathcal{F}_{(i-1)\Delta_n}])$ we have $U_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \xi_i^n$. Now we will verify the conditions of Jacod’s theorem of stable convergence for semimartingales (see [13]). Introducing the notation $\Delta_i^n W := W_{i\Delta_n} - W_{(i-1)\Delta_n}$ we find that

$$\begin{aligned} &\mathbb{E}[\xi_i^n | \mathcal{F}_{(i-1)\Delta_n}] = 0, \\ &\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[(\xi_i^n)^2 | \mathcal{F}_{(i-1)\Delta_n}] \\ &= \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\mathbb{E}[g(\beta_i^n)^2 | \mathcal{F}_{(i-1)\Delta_n}] - \mathbb{E}[g(\beta_i^n) | \mathcal{F}_{(i-1)\Delta_n}]^2 \right) \\ &= \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\rho_{\sigma_{(i-1)\Delta_n}}(g^2) - \rho_{\sigma_{(i-1)\Delta_n}}(g)^2 \right) \\ &\xrightarrow{ucp} \int_0^t \left(\rho_{\sigma_s}(g^2) - \rho_{\sigma_s}(g)^2 \right) ds, \\ &\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[\xi_i^n \Delta_i^n W | \mathcal{F}_{(i-1)\Delta_n}] = \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[g(\beta_i^n) \Delta_i^n W | \mathcal{F}_{(i-1)\Delta_n}] \\ &= \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[g(\{x W_s; s \in [0, 1]\}) W_1] |_{x=\sigma_{(i-1)\Delta_n}} \\ &\xrightarrow{ucp} \int_0^t \rho_{\sigma_s}^{(1)}(g) ds, \\ &\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[(\xi_i^n)^2 \mathbf{1}_{\{|\xi_i^n| > \epsilon\}} | \mathcal{F}_{(i-1)\Delta_n}] \\ &\leq \frac{\Delta_n^2}{\epsilon^2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[(g(\beta_i^n) - \mathbb{E}[g(\beta_i^n) | \mathcal{F}_{(i-1)\Delta_n}])^4 | \mathcal{F}_{(i-1)\Delta_n}] \leq \Delta_n C / \epsilon^2 \rightarrow 0. \end{aligned}$$

Finally, let $N \in \mathcal{M}_b(W)^\perp$, the space of all bounded (\mathbb{P}, \mathbb{F}) -martingales that have zero quadratic covariation with W . Define $M_u := \mathbb{E}[g(\beta_i^n) | \mathcal{F}_u]$ for $u \geq (j - 1)\Delta_n$. By the martingale representation theorem we deduce the identity

$$M_u = M_{(i-1)\Delta_n} + \int_{(i-1)\Delta_n}^u \eta_s dW_s$$

for a suitable predictable process η . By the Itô isometry we conclude that

$$\begin{aligned} \mathbb{E}[g(\beta_i^n) \Delta_i^n N | \mathcal{F}_{(i-1)\Delta_n}] &= \mathbb{E}[M_{i\Delta_n} \Delta_i^n N | \mathcal{F}_{(i-1)\Delta_n}] \\ &= \mathbb{E}[\Delta_i^n M \Delta_i^n N | \mathcal{F}_{(i-1)\Delta_n}] = 0. \end{aligned}$$

Hence, Jacod’s convergence theorem (see [16, Theorem IX.7.28]) gives

$$U_t^n \xrightarrow{d_{st}} \int_0^t \rho_{\sigma_s}^{(1)}(g) \, dW_s + \int_0^t \sqrt{\rho_{\sigma_s}(g^2) - \rho_{\sigma_s}(g)^2 - (\rho_{\sigma_s}^{(1)}(g))^2} \, dW'_s. \quad \square$$

(ii) Let $\eta_i^n := \Delta_n^{-\frac{1}{2}}(g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - g(\beta_i^n))$ so $R_t^{1,n} = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\eta_i^n - \mathbb{E}[\eta_i^n | \mathcal{F}_{(i-1)\Delta_n}])$. Since $R^{1,n}$ is a martingale we may apply Doob’s inequality to obtain

$$\mathbb{P} \left[\sup_{t \leq T} |R_t^{1,n}| > \epsilon \right] \leq C \frac{\Delta_n}{\epsilon^2} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E} \left[(g(\Delta_n^{-\frac{1}{2}} d_i^n(X)) - g(\beta_i^n))^2 \right].$$

By the same argument as in (5.6), making use of the locally uniform continuity of g , we find that the last term converges to 0. \square

(iii) By the assumed Gâteaux differentiability of g the mean-value theorem gives

$$g(y) - g(x) = g'_{y-x}(x + \hat{t}(y - x))$$

for some $\hat{t} \in [0, 1]$. Let us again use the notation $f(x; y) := g'_y(x)$. We expand $R^{2,n} = R^{2.1,n} + R^{2.2,n}$ where

$$\begin{aligned} R_t^{2.1,n} &:= \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[f'(\beta_i^n; \Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n) | \mathcal{F}_{(i-1)\Delta_n} \right], \\ R_t^{2.2,n} &:= \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \\ &\quad \times \left[(f'(\chi_i^n; \Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n) - f'(\beta_i^n; \Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n)) | \mathcal{F}_{(i-1)\Delta_n} \right], \end{aligned}$$

with $\chi_i^n = \beta_i^n + \hat{t}_i^n (\Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n)$ and $\hat{t}_i^n \in [0, 1]$. Decompose also $\Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n = V_i^n(1) + V_i^n(2)$ where

$$\begin{aligned} V_i^n(1)_t &:= \Delta_n^{-\frac{1}{2}} \left(t \Delta_n \mu_{(i-1)\Delta_n} \right. \\ &\quad \left. + \int_{(i-1)\Delta_n}^{(i-1+t)\Delta_n} (\tilde{\sigma}_{(i-1)\Delta_n}(W_s - W_{(i-1)\Delta_n}) + \tilde{v}_{(i-1)\Delta_n}(V_s - V_{(i-1)\Delta_n})) \, dW_s \right) \\ &= \Delta_n^{-\frac{1}{2}} \left(t \Delta_n \mu_{(i-1)\Delta_n} + \frac{1}{2} \tilde{\sigma}_{(i-1)\Delta_n} ((W_{(i-1+t)\Delta_n} - W_{(i-1)\Delta_n})^2 - t \Delta_n) \right. \\ &\quad \left. + \tilde{v}_{(i-1)\Delta_n} \int_{(i-1)\Delta_n}^{(i-1+t)\Delta_n} (V_s - V_{(i-1)\Delta_n}) \, dW_s \right), \\ V_i^n(2)_t &:= \Delta_n^{-\frac{1}{2}} \left(\int_{(i-1)\Delta_n}^{(i-1+t)\Delta_n} (\mu_s - \mu_{(i-1)\Delta_n}) \, ds \right. \\ &\quad \left. + \left(\int_{(i-1)\Delta_n}^s (\tilde{v}_u - \tilde{v}_{(i-1)\Delta_n}) \, dV_u \right) dW_s \right. \\ &\quad \left. + \left(\int_{(i-1)\Delta_n}^s \tilde{\mu}_u \, du + \int_{(i-1)\Delta_n}^s (\tilde{\sigma}_u - \tilde{\sigma}_{(i-1)\Delta_n}) \, dW_u \right) dW_s \right) \end{aligned}$$

for $t \in [0, 1]$. Now, by the linearity of f in the second argument

$$\begin{aligned} & \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[f(\beta_i^n; V_i^n(1)) \mid \mathcal{F}_{(i-1)\Delta_n} \right] \\ &= \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\mu_{(i-1)\Delta_n} \mathbb{E} \left[f(\{x W_s; s \in [0, 1]\}; \{s; s \in [0, 1]\}) \Big|_{x=\sigma_{(i-1)\Delta_n}} \right] \right. \\ & \quad \left. + \frac{1}{2} \tilde{\sigma}_{(i-1)\Delta_n} \mathbb{E} \left[f(\{x W_s; s \in [0, 1]\}; \{W_s^2 - s; s \in [0, 1]\}) \Big|_{x=\sigma_{(i-1)\Delta_n}} \right] \right) \\ &= \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\mu_{(i-1)\Delta_n} \rho_{\sigma_{(i-1)\Delta_n}}^{(2)}(f) + \frac{1}{2} \tilde{\sigma}_{(i-1)\Delta_n} \left(\rho_{\sigma_{(i-1)\Delta_n}}^{(3)}(f) - \rho_{\sigma_{(i-1)\Delta_n}}^{(2)}(f) \right) \right) \\ & \xrightarrow{ucp} \int_0^t \mu_s \rho_{\sigma_s}^{(2)}(f) ds + \frac{1}{2} \int_0^t \tilde{\sigma}_s \rho_{\sigma_s}^{(3)}(f) ds - \frac{1}{2} \int_0^t \tilde{\sigma}_s \rho_{\sigma_s}^{(2)}(f) ds, \end{aligned}$$

where we used the independence of W and V . Due to linearity we observe the identity

$$f(\beta_i^n; V_i^n(2)) = f(\beta_i^n; V_i^n(2)/\|V_i^n(2)\|_\infty) \|V_i^n(2)\|_\infty,$$

whenever $\|V_i^n(2)\|_\infty > 0$ and 0 otherwise. Hence, we deduce that

$$\begin{aligned} & \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[|f(\beta_i^n; V_i^n(2))| \right] \\ &= \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[|f(\beta_i^n; V_i^n(2)/\|V_i^n(2)\|_\infty)| \|V_i^n(2)\|_\infty \right] \\ &\leq \Delta_n^{\frac{1}{2}} C \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[\|V_i^n(2)\|_\infty^2 \right]^{\frac{1}{2}} \\ &\leq C \left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[\|V_i^n(2)\|_\infty^2 \right] \right)^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

by the Cauchy–Schwarz inequality and the polynomial growth of f . So we are only left to prove that $R^{2.2,n} \xrightarrow{ucp} 0$. Defining $\xi_i^n := \frac{\Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n}{\|\Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n\|_\infty}$ for $\|\Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n\|_\infty > 0$ and 0 otherwise, we get

$$\begin{aligned} & \left| R_i^{2.2,n} \right| \\ &\leq \Delta_n^{\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[|f(\chi_i^n; \xi_i^n) - f(\beta_i^n; \xi_i^n)| \|\Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n\|_\infty \mid \mathcal{F}_{(i-1)\Delta_n} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathbb{P}\left[\sup_{t \leq T} |R_t^{2.2,n}| > \epsilon\right] \\ & \leq \frac{\Delta_n^{\frac{1}{2}}}{\epsilon} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}\left[\|f(\chi_i^n; \xi_i^n) - f(\beta_i^n; \xi_i^n)\| \Delta_n^{-\frac{1}{2}} d_i^n(X) - \beta_i^n\|_{\infty}\right] \\ & \leq C \frac{\Delta_n}{\epsilon} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \sqrt{\mathbb{E}\left[\left(f(\chi_i^n; \xi_i^n) - f(\beta_i^n; \xi_i^n)\right)^2\right]} \rightarrow 0, \end{aligned}$$

where again we used Jensen’s and Cauchy–Schwarz inequality and the local Hölder continuity of f (uniformly on the unit circle in the second argument). Putting everything together, we have thus proven (iii). \square

(iv) We want to show that

$$R_t^{3,n} = \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\Delta_n \mathbb{E}\left[g(\beta_i^n) \mid \mathcal{F}_{(i-1)\Delta_n}\right] - \int_{(i-1)\Delta_n}^{i\Delta_n} \rho_{\sigma_s}(g) ds \right) \xrightarrow{ucp} 0.$$

Define $\mu_i^n := \Delta_n^{-\frac{1}{2}} \int_{(i-1)\Delta_n}^{i\Delta_n} (\rho_{\sigma_{(i-1)\Delta_n}}(g) - \rho_{\sigma_s}(g)) ds$ so $R_t^{3,n} = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mu_i^n$. Thanks to the differentiability of g and the polynomial growth of g and g' , we find that

$$\lim_{h \rightarrow 0} \frac{\rho_{x+h}(g) - \rho_x(g)}{h} = \mathbb{E}[g'_W(xW)],$$

so the derivative of $\rho_x(g) =: \psi(x)$ exists. Similarly, using that g' is continuous, linear in the second argument and of polynomial growth in the first argument, we see that ψ' is continuous. This allows us to expand $\mu_i^n =: \mu_i^n(1) + \mu_i^n(2)$ where

$$\begin{aligned} \mu_i^n(1) &= \Delta_n^{-\frac{1}{2}} \psi'(\sigma_{(i-1)\Delta_n}) \int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_{(i-1)\Delta_n} - \sigma_s) ds, \\ \mu_i^n(2) &= \Delta_n^{-\frac{1}{2}} \int_{(i-1)\Delta_n}^{i\Delta_n} (\psi'(\chi_s^n) - \psi'(\sigma_{(i-1)\Delta_n})) (\sigma_{(i-1)\Delta_n} - \sigma_s) ds \end{aligned}$$

with $|\chi_s^n - \sigma_{(i-1)\Delta_n}| \leq |\sigma_{i\Delta_n} - \sigma_{(i-1)\Delta_n}|$. Now, decompose $-\mu_i^n(1)$ into a martingale increment $\mu_i^n(1.2)$ and a remainder term $\mu_i^n(1.1)$, i.e. $-\mu_i^n(1) = \mu_i^n(1.1) + \mu_i^n(1.2)$ with

$$\begin{aligned} \mu_i^n(1.1) &= \Delta_n^{-\frac{1}{2}} \psi'(\sigma_{(i-1)\Delta_n}) \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\int_{(i-1)\Delta_n}^s \tilde{\mu}_u du \right) ds, \\ \mu_i^n(1.2) &= \Delta_n^{-\frac{1}{2}} \psi'(\sigma_{(i-1)\Delta_n}) \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\int_{(i-1)\Delta_n}^s \tilde{\sigma}_u dW_u + \int_{(i-1)\Delta_n}^s \tilde{v}_u dV_u \right) ds. \end{aligned}$$

Observing that $\mu_i^n(1.1) \leq \Delta_n^{\frac{3}{2}} \sup_{|x| \leq |\sigma|} |\psi'(x)| \|\tilde{\mu}\|_{\infty}$, its convergence to 0 follows immediately. With the help of Doob’s inequality,

$$\begin{aligned} \mathbb{P}\left[\sup_{t \leq T} \left| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mu_i^n(1.2) \right| > \epsilon\right] &\leq C/\epsilon^2 \mathbb{E}\left[\left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mu_i^n(1.2)\right)^2\right] \\ &= C/\epsilon^2 \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[(\mu_i^n(1.2))^2\right] \\ &\leq C \frac{\Delta_n^{\frac{1}{2}}}{\epsilon^2} \sup_{|x| \leq \|\sigma\|_\infty} \psi'(x) (\|\tilde{\sigma}\|^2 + \|\tilde{v}\|^2) \rightarrow 0 \end{aligned}$$

so $\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mu_i^n(1) \xrightarrow{ucp} 0$. Regarding $\mu_i^n(2)$, since ψ' is uniformly continuous on $[-\|\sigma\|_\infty, \|\sigma\|_\infty]$ choose $\delta > 0$ for a given $\epsilon > 0$ such that for all $s, t \leq T$ we have $|\sigma_s - \sigma_t| \leq \delta \Rightarrow |\psi'(\sigma_s) - \psi'(\sigma_t)| \leq \epsilon$. Now,

$$\begin{aligned} |\mu_i^n(2)| &\leq \Delta_n^{-\frac{1}{2}} \epsilon \int_{(i-1)\Delta_n}^{i\Delta_n} |\sigma_{(i-1)\Delta_n} - \sigma_s| ds \\ &\quad + 2 \Delta_n^{-\frac{1}{2}} / \delta \sup_{|x| \leq \|\sigma\|_\infty} |\psi'(x)| \int_{(i-1)\Delta_n}^{i\Delta_n} |\sigma_{(i-1)\Delta_n} - \sigma_s|^2 ds, \end{aligned}$$

leading to

$$\begin{aligned} \mathbb{P}\left[\sup_{t \leq T} \left| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mu_i^n(2) \right| > \hat{\epsilon}\right] &\leq \Delta_n^{-\frac{1}{2}} \epsilon / \hat{\epsilon} \mathbb{E}\left[\int_0^T |\sigma_{(i-1)\Delta_n} - \sigma_s| ds\right] \\ &\quad + 2 \Delta_n^{-\frac{1}{2}} \sup_{|x| \leq \|\sigma\|_\infty} |\psi'(x)| \mathbb{E}\left[\int_0^T |\sigma_{(i-1)\Delta_n} - \sigma_s|^2 ds\right] / (\hat{\epsilon} \delta) \\ &\leq C_T \left(\epsilon / \hat{\epsilon} + \Delta_n^{\frac{1}{2}} \sup_{|x| \leq \|\sigma\|_\infty} |\psi'(x)| / (\hat{\epsilon} \delta)\right), \end{aligned}$$

where we used Fubini’s theorem. So choosing first ϵ small and then n large finishes the proof of (iv), the last step in the proof of (2.6). \square

Proof of Theorems 4.1 and 4.2. As in the previous proof we may assume without loss of generality that the processes $\mu, \sigma, \tilde{\mu}, \tilde{\sigma}, \tilde{v}$ as well as the jump sizes J, \tilde{J} are uniformly bounded in (ω, t) . This is again justified by a standard localisation procedure (see e.g. [4]). Moreover, by the same localisation procedure, we may assume without loss of generality that the jump sizes J are bounded from below, i.e.

$$|J_i| > \epsilon, \quad 1 \leq i \leq N_t,$$

for some $\epsilon > 0$. Now, let $I_i^n = [(i - 1)\Delta_n, i\Delta_n]$ and $\Omega_n := \{\omega \in \Omega : \#\{j \in \mathbb{N} : T_j \in J_i^n\} \leq 1\}$, where T_j denotes the arrival time of the j ’th jump of the Poisson process N . We clearly have

$$\lim_{n \rightarrow \infty} \mathbb{P}[\Omega_n] = \mathbb{P}[\Omega] = 1.$$

Note that each interval I_i^n contains at most one jump of X on Ω_n and each jump is at least of size ϵ .

Proof of Theorem 4.1. The assertion $R(X, p)_t^n \xrightarrow{\mathbb{P}} R(X, p)_t$ for $p = 2$ has been already proved in [7], so we show the result for $p > 2$. We write $X_t = X^c + Z_t$, where X^c denotes the continuous

part X and Z stands for the jump part. We define

$$K_n = \{i \leq \lfloor t/\Delta_n \rfloor : \exists T_j \in I_i^n\}.$$

Note that the cardinality of K_n is finite almost surely as it is bounded by N_t . We decompose the statistic $R(X, p)_t^n$ as

$$R(X, p)_t^n = \sum_{i \in K_n^c} \sup_{s, u \in I_i^n} |X_s^c - X_u^c|^p + \sum_{i \in K_n} \sup_{s, u \in I_i^n} |X_s - X_u|^p. \tag{5.8}$$

By Burkholder–Davis–Gundy inequality we conclude that $\mathbb{E}[\sup_{s, u \in I_i^n} |X_s^c - X_u^c|^p] \leq C_p \Delta_n^{p/2}$, and since $p > 2$, we obtain the convergence

$$\sum_{i \in K_n^c} \sup_{s, u \in I_i^n} |X_s^c - X_u^c|^p \xrightarrow{\mathbb{P}} 0.$$

Moreover, on Ω_n , we have that

$$\sum_{i \in K_n} \sup_{s, u \in I_i^n} |Z_s - Z_u|^p - R(X, p)_t = \sum_{i=N \Delta_n \lfloor t/\Delta_n \rfloor}^{N_t} |J_i|^p \xrightarrow{\mathbb{P}} 0.$$

Finally, we obtain by mean value theorem that

$$\begin{aligned} & \mathbb{E} \left| \sum_{i \in K_n} \sup_{s, u \in I_i^n} |Z_s - Z_u|^p - \sup_{s, u \in I_i^n} |X_s - X_u|^p \right| \\ & \leq p \mathbb{E} \sum_{i \in K_n} \sup_{s, u \in I_i^n} \max(|Z_s - Z_u|, |X_s - X_u|)^{p-1} \sup_{s, u \in I_i^n} |X_s^c - X_u^c| \\ & \xrightarrow{\mathbb{P}} 0, \end{aligned}$$

since the set K_n is finite. Due to $\Omega_n \rightarrow \Omega$, we thus conclude the assertion of [Theorem 4.1](#). □

Proof of Theorem 4.2. (i) We again use the decomposition (5.8) of $R(X, p)_t^n$. It holds that

$$\Delta_n^{-1/2} \mathbb{E} \left[\sum_{i \in K_n^c} \sup_{s, u \in I_i^n} |X_s^c - X_u^c|^p \right] \leq \Delta_n^{(p-1)/2-1} \rightarrow 0,$$

since $p > 3$. On the other hand we also have

$$\Delta_n^{-1/2} \left(\sum_{i \in K_n} \sup_{s, u \in I_i^n} |Z_s - Z_u|^p - R(X, p)_t \right) \xrightarrow{\mathbb{P}} 0.$$

Now, since all jump sizes $|J_i|$ are bounded by ϵ from below and X^c is continuous, we obtain by mean value theorem for all $i \in K_n$ on Ω_n :

$$\begin{aligned} & \sup_{s, u \in I_i^n} (X_s - X_u)^p - \sup_{s, u \in I_i^n} (Z_s - Z_u)^p \\ & = p |\Delta Z_{T_i}|^{p-1} \left(\sup_{\substack{s, u \in I_i^n \\ s < T_i^n \leq u}} (X_u^c - X_s^c) \mathbf{1}_{\{\Delta Z_{T_i} > 0\}} + \sup_{\substack{s, u \in I_i^n \\ u < T_i^n \leq s}} (X_u^c - X_s^c) \mathbf{1}_{\{\Delta Z_{T_i} < 0\}} \right) \\ & \quad + o_{\mathbb{P}}(\Delta_n^{1/2}), \end{aligned}$$

where T_i^n denotes the jump time of N_t in the interval I_i^n . Hence, we deduce the decomposition

$$\Delta_n^{-1/2} \sum_{i \in K_n} \left(\sup_{s,u \in I_i^n} (X_s - X_u)^p - \sup_{s,u \in I_i^n} (Z_s - Z_u)^p \right) = \sum_{i \in K_n} (\zeta_i^n + \bar{\zeta}_i^n) + o_{\mathbb{P}}(1),$$

where

$$\zeta_i^n = p \Delta_n^{-1/2} |\Delta Z_{T_i^n}|^{p-1} \left(\sup_{\substack{s,u \in I_i^n \\ s < T_i^n \leq u}} (\sigma_{T_i^n-}(W_{T_i^n} - W_s) + \sigma_{T_i^n}(W_u - W_{T_i^n})) \mathbf{1}_{\{\Delta Z_{T_i^n} > 0\}} \right. \\ \left. + \sup_{\substack{s,u \in I_i^n \\ u < T_i^n \leq s}} (-\sigma_{T_i^n-}(W_{T_i^n} - W_u) - \sigma_{T_i^n}(W_s - W_{T_i^n})) \mathbf{1}_{\{\Delta Z_{T_i^n} < 0\}} \right)$$

and the quantity $\bar{\zeta}_i^n$ is defined via the identity

$$\zeta_i^n + \bar{\zeta}_i^n = p \Delta_n^{-1/2} |\Delta Z_{T_i^n}|^{p-1} \times \left(\sup_{\substack{s,u \in I_i^n \\ s < T_i^n \leq u}} (X_u^c - X_s^c) \mathbf{1}_{\{\Delta Z_{T_i^n} > 0\}} + \sup_{\substack{s,u \in I_i^n \\ u < T_i^n \leq s}} (X_u^c - X_s^c) \mathbf{1}_{\{\Delta Z_{T_i^n} < 0\}} \right).$$

Obviously the term ζ_i^n serves as the first order approximation while $\bar{\zeta}_i^n$ is the error term. Since N and W are independent, we obtain the stable convergence

$$(\kappa_i^n, \tilde{W}_i^n)_{i \geq 1} := \left(\Delta_n^{-1} \{T_i - \Delta_n \lfloor T_i / \Delta_n \rfloor\}, \Delta_n^{-1/2} \{W_{(i-1+s)\Delta_n} - W_{(i-1)\Delta_n}\}_{s \in [0,1]} \right)_{i \geq 1} \\ \xrightarrow{d_{st}} (\kappa_i, \{\tilde{W}_{i-1+s} - \tilde{W}_{i-1}\}_{s \in [0,1]})_{i \geq 1}, \tag{5.9}$$

where κ_i and \tilde{W} were defined in Section 4.2. This result is an immediate consequence of [15, Lemma 6.2], but it can be easily shown in a straightforward manner. Now, by properties of stable convergence and continuous mapping theorem, we conclude that

$$\sum_{i \in K_n} \zeta_i^n \xrightarrow{d_{st}} U(X, p)_t$$

for any fixed $t > 0$. Indeed this can be deduced from the stable convergence in (5.9), by defining the function $f_i, : \mathbb{R}^3 \times [0, 1] \times C([0, 1])$ via

$$f_i(x, y, z) = px_1 \sup_{0 \leq s < y \leq u \leq 1} \left(x_2(z(y) - z(s)) + x_3(z(u) - z(y)) \right)$$

and observing that

$$\zeta_i^n = f_i \left((|\Delta Z_{T_i^n}|^{p-1} \mathbf{1}_{\{\Delta Z_{T_i^n} > 0\}}, \sigma_{T_i^n-}, \sigma_{T_i^n}), \kappa_i^n, \tilde{W}_i^n \right) \\ + f_i \left((|\Delta Z_{T_i^n}|^{p-1} \mathbf{1}_{\{\Delta Z_{T_i^n} < 0\}}, \sigma_{T_i^n-}, \sigma_{T_i^n}), \kappa_i^n, -\tilde{W}_i^n \right).$$

Hence, to complete the proof we need to show that $\sum_{i \in K_n} \bar{\zeta}_i^n \xrightarrow{\mathbb{P}} 0$. Since the processes μ , σ and the jump sizes J_i are uniformly bounded, we deduce by Burkholder–Davis–Gundy inequality that

$$\mathbb{E}[|\bar{\zeta}_i^n|^2] \leq C \Delta_n^{-1} \left(\Delta_n^2 + \int_{T_i^n}^{i \Delta_n} (\sigma_u - \sigma_{T_i^n})^2 du + \int_{(i-1)\Delta_n}^{T_i^n} (\sigma_u - \sigma_{T_i^n-})^2 du \right),$$

where the right side converges to 0, because σ is càdlàg. This completes the proof since K_n is finite. \square

(ii) Now let us consider the case $p = 2$. According to the previous proof and the limiting results of [6] for the continuous case, we obtain the following asymptotic decomposition

$$\Delta_n^{-1/2} (R(X, 2)_t^n - R(X, 2)_t) = \sum_{i \in K_n} \zeta_i^n + \sum_{i \in K_n^c} \tilde{\zeta}_i^n + o_{\mathbb{P}}(1),$$

where ζ_i^n has been defined in the previous step (now with $p = 2$) and $\tilde{\zeta}_i^n$ serves as the first order approximation in the continuous case, i.e.

$$\tilde{\zeta}_i^n = \Delta_n^{-1/2} \sigma_{(i-1)\Delta_n}^2 \left(\sup_{s, u \in I_i^n} (W_u - W_s)^2 - \Delta_n^{-1} \lambda_{3,2} \right).$$

Now, we need to prove joint stable convergence of the vector $(\sum_{i \in K_n} \zeta_i^n, \sum_{i \in K_n^c} \tilde{\zeta}_i^n)$. This problem is closely related to [14, Lemma 5.8]. Indeed, following exactly the same proof steps, which are based on certain conditioning arguments, it is sufficient to prove the stable central limit theorem for each component of the vector (indeed, the two stable limits are independent conditionally on \mathcal{F}). But the stable convergence for the first component follows from the previous step and the stable convergence for the second component has been shown in [6] under the conditions of Theorem 4.2. This completes the proof. \square

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