Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 136 (2021) 92–124

stochastic processes and their applications

www.elsevier.com/locate/spa

Maximal moments and uniform modulus of continuity for stable random fields

Snigdha Panigrahi[a,*], Parthanil Roy[b], Yimin Xiao[c]

[a] Department of Statistics, University of Michigan, Ann Arbor, MI 48109, USA
[b] Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore 560059, India
[c] Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824-1027, USA

Received 2 October 2018; received in revised form 26 November 2020; accepted 13 February 2021
Available online 4 March 2021

Dedicated to the memory of Professor Wenbo Li.

Abstract

In this work, we provide sharp bounds on the rate of growth of maximal moments for stationary symmetric stable random fields when the underlying nonsingular group action (or its restriction to a suitable lower rank subgroup) has a nontrivial dissipative component. We also investigate the relationship between this rate of growth and the path regularity properties of self-similar stable random fields with stationary increments, and establish uniform modulus of continuity of such fields. In the process, a new notion of weak effective dimension is introduced for stable random fields and is connected to maximal moments and path properties.

Keywords: Random field; Stable process; Uniform modulus of continuity; Extreme value theory; Nonsingular group actions

1. Introduction

A real-valued stochastic process $\{X(t) : t \in \mathbb{T}^d\}$ ($\mathbb{T} = \mathbb{Z}$ or $[0, 1]$ or $\mathbb{R}$) is called a *symmetric α-stable* ($S\alpha S$) random field if each of its finite linear combination follows an $S\alpha S$ distribution. In general, the parameter α satisfies $0 < \alpha \leq 2$, although in this paper, we assume our random fields to be non-Gaussian and therefore $0 < \alpha < 2$. See, for example, [38] for detailed discussions on non-Gaussian stable distributions and processes.

Sample path continuity and Hölder regularity of stochastic processes and random fields have been studied for many years. The main tool behind such investigation has been a powerful chaining argument that is mainly applicable to Gaussian and other light-tailed processes;

* Corresponding author.
E-mail address: psnigdha@umich.edu (S. Panigrahi).

https://doi.org/10.1016/j.spa.2021.02.002
0304-4149/

see [2,14,21,42]. Recently, there has been a significant interest in establishing uniform modulus of continuity of sample paths for stable and other non-Gaussian infinitely divisible processes; see, for instance, [4,6,7,45].

Motivated by [15,45] modified the existing chaining argument and made it amenable to heavy-tailed random fields. This technique uses estimates of the lower order moments of the maximum increments over the two consecutive steps of the chain to obtain a uniform modulus of continuity for stable and other heavy-tailed random fields.

In this context, it was stated in [45] (see Page 173 therein) that for a stationary α-stable sequence $\{\xi_k : k \geq 1\}$, it is an open problem to give sharp upper and lower bounds for the maximal moment sequence $\mathbb{E}\big(\max_{1\leq k\leq n} |\xi_k|^\gamma\big)$ for $\gamma \in (0, \alpha)$. [45] also presented two approaches of partial solution to this open problem: one using results of [42] in this setup and another one based on his own improvement of Talagrand's results (more specifically, Lemma 3.5 of [45]). However both of these methods lead to weaker path continuity results and we have been able to improve them significantly in this paper as described below.

The dependence structure of a stationary $S\alpha S$ random field is determined, to a large extent, by the underlying *nonsingular* (also known as *quasi-invariant*) group action discovered by [28]. As a result, various probabilistic facets (e.g., mixing features (see [30,31,37,44]), large deviations issues (see [11,23]), growth of maxima (see [3,24,34,35]), extremal point processes (see [27,32,39]), functional central limit theorem (see [13,24]), statistical aspects (see [5]), etc.) of a stationary $S\alpha S$ random field have been connected to ergodic theoretic properties of the underlying nonsingular action.

We have been able to extend the connection mentioned in the above paragraph to the investigation of maximal moments and path properties for stable random fields when the group action (or its restriction to a lower rank subgroup F; see Section 2.2) has a nontrivial dissipative component. In particular, we have partially solved the open problem mentioned above and derived sharp bounds on the moments of the maximal process for stationary $S\alpha S$ discrete-parameter random fields generated by dissipative actions; see Theorem 3.1. Our machineries include structure theorem for finitely generated abelian groups, ergodic theory of quasi-invariant actions on σ-finite standard measure spaces, and a new notion of weak effective dimension introduced in this paper. This work easily extends to the continuous parameter case (see Theorem A.1 and Remark A.2) provided the random field is measurable and stationary.

Partial solution to the open problem in the discrete-parameter case allows us to prove results on uniform modulus of continuity for a class of self-similar $S\alpha S$ random fields with stationary increments; see Section 4. To this end, we have introduced a novel notion, namely that of *weak effective dimension*, for stable random fields; see Definition 3.3. This notion encompasses the concept of effective dimension (defined by [34]) as a special case and connects naturally to maximal moments (see Theorem 3.4) and path properties (see Corollary 4.4) of stable random fields. In some sense, our new notion is better than the effective dimension, which is always an integer (and hence more restrictive) whereas weak effective dimension need not be so.

We would like to mention that our bounds (on both growth-rate of maximal moments as well as uniform modulus of continuity) are significantly better than the existing ones for stable random fields that are not full-dimensional (see Definition 3.3) to the extent that we improve the leading (polynomial) term of these bounds. For full-dimensional fields generated by dissipative actions (i.e., for mixed moving average fields), however, the improvement is in the logarithmic term. On the other hand, stationary $S\alpha S$ random fields generated by conservative actions have much more complex structures and hence cannot be put into a unified framework. Instead, it is better go with a case-by-case approach.

This paper is organized as follows. In Section 2, we recall a result of [45], explain how it naturally leads to a problem on rate of growth of the maximal moment sequences and describe the ergodic theoretic and group theoretic connections to this extreme value theoretic problem. Section 2.3 contains a brief summary of the contributions of our work. In Section 3, we state the results on the asymptotic behavior of the maximal moments of stationary $S\alpha S$ random fields as the index parameter runs over d-dimensional hypercubes of increasing edge-length even though the proofs are deferred to Section 6 to increase the readability of this paper. In Section 4, we establish results on uniform modulus of continuity for self-similar $S\alpha S$ random fields whose first order increments are stationary. A few important examples are discussed in Section 5. Finally in the Appendix, we present a result on the growth-rate of maximal moments in the continuous parameter case.

Throughout this paper, we will use K to denote a positive and finite constant which may differ in each occurrence, even in two consecutive ones. Some specific constants will be denoted by $c, c_1, c_2, \dots, K_1, K_2, \dots$, etc. For two sequences of nonzero real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ the notation $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$. For $u, v \in \mathbb{R}^d$, $u = (u_1, u_2, \dots, u_d) \le v = (v_1, v_2, \dots, v_d)$ means $u_i \le v_i$ for all $i = 1, 2, \dots, d$. The vectors $\mathbf{0} = (0, 0, \dots, 0)$, $\mathbf{1} = (1, 1, \dots, 1)$ are elements of $\mathbb{Z}^d$. We shall abuse the notation and use $[u, v]$ to denote the set $\{t \in \mathbb{Z}^d : u \le t \le v\}$ or the set $\{t \in \mathbb{R}^d : u \le t \le v\}$ depending on the context (the former notation is used throughout the main body of the paper while latter one is used only in the Appendix). For $\alpha \in (0, 2)$ and a σ-finite standard measure space $(S, \mathcal{S}, \mu)$, we define the space $L^\alpha(S, \mu) := \{f : S \to \mathbb{R} \text{ measurable} : \|f\|_\alpha < \infty\}$, where $\|f\|_\alpha := \left(\int_S |f(s)|^\alpha \, \mu(ds)\right)^{1/\alpha}$. Note that $\|\cdot\|_\alpha$ is a norm if and only if $\alpha \in [1, 2)$ making the corresponding $L^\alpha(S, \mu)$ a Banach space but not a Hilbert space. For two random variables Y, Z, we write $Y \overset{\mathcal{L}}{=} Z$ if Y and Z are identically distributed. For two stochastic processes $\{Y(t)\}_{t\in T}$ and $\{Z(t)\}_{t\in T}$, the notation $\{Y(t)\}_{t\in T} \overset{\mathcal{L}}{=} \{Z(t)\}_{t\in T}$ (or simply $Y(t) \overset{\mathcal{L}}{=} Z(t)$, $t \in T$) means that they have the same finite-dimensional distributions.

2. Preliminaries

2.1. A chaining argument for path properties

We start with an important special case of the main result of [45]. This result was proved using a modification of the chaining arguments used in the proofs of Kolmogorov's continuity theorem, Dudley's entropy theorem and other results on path regularity properties in the light-tailed situations; see [2,14,21,42]. To this end, Let $\{X(t)\}_{t\in T}$ be a random field indexed by T.

We start by introducing some notation. Let $T = [0, 1]^d$ be endowed with the metric $\rho(s, t) = \max_{1\le i\le d} |s_i - t_i|$. Define, for all $n \ge 1$, $D_n = \left\{2^{-n}u : u \in [\mathbf{0}, (2^n - 1)\mathbf{1}] \cap \mathbb{Z}^d\right\}$. For every $u \in D_n$,

$$O_{n-1}(u) := \{u' \in D_{n-1} : \rho(u, u') \le 2^{-n}\}.$$

The following particular case of Theorem 2.1 of [45] will play a significant role in this paper.

Proposition 2.1. *Let $\mathbf{X} = \{X(t)\}_{t\in T}$ be a real-valued random field indexed by $T = [0, 1]^d$. Suppose $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function which is regularly varying at the origin with index $\delta > 0$ (i.e., $\lim_{h\to 0+} \sigma(ch)/\sigma(h) = c^\delta$ for all $c > 0$). If there are constants $\gamma > 0$, and $K > 0$ such that*

$$\mathbb{E}\left(\max_{\tau_n\in D_n} \max_{\tau'_{n-1}\in O_{n-1}(\tau_n)} |X(\tau_n) - X(\tau'_{n-1})|^\gamma\right) \le K \left(\sigma(2^{-n})\right)^\gamma \tag{2.1}$$

for all integers $n \geq 1$, then for all $\epsilon > 0$,

$$\lim_{h\to 0+} \frac{\sup_{t\in T}\sup_{\rho(s,t)\leq h}|X(t)-X(s)|}{\sigma(h)(\log 1/h)^{(1+\epsilon)/\gamma}} = 0 \tag{2.2}$$

almost surely.

In this paper, we will focus on studying the maximal moments of $S\alpha S$ random fields indexed by $\mathbb{Z}^d$ or $\mathbb{R}^d$ so that we can apply Proposition 2.1 to self-similar $S\alpha S$ random fields with stationary increments. Recall that a random field $\{X(t)\}_{t\in\mathbb{R}^d}$ is called H-self-similar ($H > 0$) if $\{X(ct)\}_{t\in\mathbb{R}^d} \stackrel{\mathcal{L}}{=} \{c^H X(t)\}_{t\in\mathbb{R}^d}$ for all $c > 0$. $\{X(t)\}_{t\in\mathbb{R}^d}$ is said to have stationary increments if $\{X(t+u)-X(u)\}_{t\in\mathbb{R}^d} \stackrel{\mathcal{L}}{=} \{X(t)-X(0)\}_{t\in\mathbb{R}^d}$, for each $u \in \mathbb{R}^d$.

We shall now apply Proposition 2.1 to a self-similar $S\alpha S$ random field $\{X(t)\}_{t\in\mathbb{R}^d}$ with stationary increments. Using the self-similarity of $\{X(t)\}_{t\in\mathbb{R}^d}$, it follows (see the proof of Theorem 4.1) that for all $\gamma \in (0, \alpha\wedge 1)$ and for all $n \geq 1$,

$$\begin{aligned} &\mathbb{E}\left(\max_{\tau_n\in D_n}\max_{\tau'_{n-1}\in O_{n-1}(\tau_n)}|X(\tau_n)-X(\tau'_{n-1})|^\gamma\right) \\ &\leq 2^{-nH\gamma}\sum_{v\in V}\mathbb{E}\left(\max_{t\in[\mathbf{1},2^n\mathbf{1}]\cap\mathbb{Z}^d}|Y^{(v)}(t)|^\gamma\right), \end{aligned} \tag{2.3}$$

where $\mathbf{Y}^{(v)} = \{Y^{(v)}(t)\}_{t\in\mathbb{Z}^d}$ is the discrete-parameter increment field defined by

$$Y^{(v)}(t) = X(t+v) - X(t), \quad t \in \mathbb{Z}^d$$

in the direction $v \in V := \{-1,0,1\}^d \setminus \{\mathbf{0}\}$.

The crucial observation is that due to the stationarity of the increments, each discrete-parameter field $\mathbf{Y}^{(v)}$ is stationary. Therefore, in order to estimate the quantity in (2.3), it suffices to establish sharp upper bounds on

$$\mathbb{E}\left(\max_{t\in[\mathbf{0},(2^n-1)\mathbf{1}]\cap\mathbb{Z}^d}|Y(t)|^\gamma\right), \tag{2.4}$$

where $\mathbf{Y} = \{Y(t)\}_{t\in\mathbb{Z}^d}$ is a stationary $S\alpha S$ random field, $n \geq 1$ and $\gamma \in (0, \alpha\wedge 1)$. This translates an investigation of sample path regularity properties into an extreme value theoretic question. Along this direction, some partial results were obtained in [45] which are applicable to stable random fields with certain specific dependence structures.

2.2. *Related work on partial maxima of stable fields*

In this work, we have improved upon the results in [45] and computed the exact rate of growth of the maximal moment sequence (2.4) for a large class of stationary $S\alpha S$ random fields. We have thus partially solved the problem of characterizing path properties of such random fields as posed in [45] (see pages 173–174 therein). The main tools used in our solution are ergodic theoretic and algebraic in nature as described below. We provide an overview of these techniques and related work below.

It was established by [28,29] that every stationary SαS random field $\mathbf{Y} = \{Y(t)\}_{t\in\mathbb{Z}^d}$ has an integral representation of the form

$$Y(t) \stackrel{d}{=} \int_S c_t(s)\left(\frac{d\mu\circ\phi_t}{d\mu}(s)\right)^{1/\alpha} f\circ\phi_t(s)M(ds), \quad t\in\mathbb{Z}^d, \tag{2.5}$$

where M is a $S\alpha S$ random measure on some standard Borel space $(S, \mathcal{S})$ with σ-finite control measure μ, $f \in L^\alpha(S, \mu)$, $\{\phi_t\}_{t \in \mathbb{Z}^d}$ is a nonsingular $\mathbb{Z}^d$-action on $(S, \mathcal{S}, \mu)$ (i.e., each $\phi_t : S \to S$ is a measurable map, ϕ_0 is the identity map on S, $\phi_{u+v} = \phi_u \circ \phi_v$ for all $u, v \in \mathbb{Z}^d$ and each $\mu \circ \phi_t$ is equivalent to μ), and $\{c_t\}_{t \in \mathbb{Z}^d}$ is a measurable cocycle for $\{\phi_t\}$ (i.e., each c_t is a ± 1-valued measurable map defined on S satisfying $c_{u+v}(s) = c_u(\phi_v(s))c_v(s)$ for all $u, v \in \mathbb{Z}^d$ and for all $s \in S$). See, for example, [1,16,43] and [46] for discussions on nonsingular (also known as *quasi-invariant*) group actions.

We say that a stationary $S\alpha S$ random field $\{Y(t)\}_{t \in \mathbb{Z}^d}$ is generated by a nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}$ on (S, μ) if it has an integral representation of the form (2.5) satisfying the full support condition $\bigcup_{t \in \mathbb{Z}^d} \text{Support}(f \circ \phi_t) = S$, which will be assumed without loss of generality. As mentioned in Section 1, the Rosiński Representation (2.5) is very useful in determining various probabilistic properties of $\mathbf{Y}$. In this work, we shall use this representation to estimate the maximal moment (2.4) and uniform modulus of continuity of $S\alpha S$ random fields.

A measurable set $W \subseteq S$ is called a *wandering set* for the nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}_{t \in \mathbb{Z}^d}$ if $\{\phi_t(W) : \ t \in \mathbb{Z}^d\}$ is a pairwise disjoint collection. The set S can be decomposed into two disjoint and invariant parts as follows: $S = \mathcal{C} \cup \mathcal{D}$, where $\mathcal{D} = \bigcup_{t \in \mathbb{Z}^d} \phi_t(W^*)$ for some wandering set $W^* \subseteq S$, and $\mathcal{C}$ has no wandering subset of positive μ-measure; see [1] and [16]. This decomposition is called the *Hopf decomposition*, and the sets $\mathcal{C}$ and $\mathcal{D}$ are called *c*onservative and *d*issipative parts (of $\{\phi_t\}_{t \in G}$), respectively. The action is called conservative if $S = \mathcal{C}$ and dissipative if $S = \mathcal{D}$.

Denote by f_t $(t \in \mathbb{Z}^d)$ the functions on S in the representation (2.5):

$$f_t(s) = c_t(s) \left(\frac{d\mu \circ \phi_t}{d\mu}(s) \right)^{1/\alpha} f \circ \phi_t(s), \quad t \in \mathbb{Z}^d.$$

The Hopf decomposition of $\{\phi_t\}_{t \in \mathbb{Z}^d}$ induces the following unique (in law) decomposition of the random field $\mathbf{Y}$

$$Y(t) \stackrel{d}{=} \int_{\mathcal{C}} f_t(s) M(ds) + \int_{\mathcal{D}} f_t(s) M(ds) := Y^{\mathcal{C}}(t) + Y^{\mathcal{D}}(t), \quad t \in \mathbb{Z}^d, \tag{2.6}$$

where the two random fields $\mathbf{Y}^{\mathcal{C}}$ and $\mathbf{Y}^{\mathcal{D}}$ are independent and are generated by conservative and dissipative $\mathbb{Z}^d$-actions, respectively; see [28,29], and [34]. This decomposition reduces the study of stationary $S\alpha S$ random fields to that of the ones generated by conservative and dissipative actions.

It was argued by [35] (see also [34]) that stationary $S\alpha S$ random fields generated by conservative actions have longer memory than those generated by dissipative actions and therefore, the following dichotomy were observed:

$$n^{-d/\alpha} \max_{\|t\|_\infty \le n} |Y(t)| \Rightarrow \begin{cases} c_{\mathbf{Y}} Z_\alpha, & \text{if } \mathbf{Y} \text{ is generated by a dissipative action,} \\ 0, & \text{if } \mathbf{Y} \text{ is generated by a conservative action} \end{cases} \tag{2.7}$$

as $n \to \infty$. In the limit above, Z_α is a standard Fréchet type extreme value random variable with distribution

$$\mathbb{P}(Z_\alpha \le x) = e^{-x^{-\alpha}}, \quad x > 0, \tag{2.8}$$

and $c_{\mathbf{Y}}$ is a positive constant depending on the random field $\mathbf{Y}$. In fact, this limiting behavior of the maximal process is closely tied with the limit of the deterministic sequence

$$\{b_n\}_{n \ge 1} = \left\{ \left(\int_S \max_{\mathbf{0} \le t \le (n-1)\mathbf{1}} |f_t(s)|^\alpha \mu(ds) \right)^{1/\alpha} \right\}_{n \ge 1}, \tag{2.9}$$

which has been proved by [34,35] to satisfy

$$n^{-d/\alpha} b_n \to \begin{cases} \tilde{c}_{\mathbf{Y}} & \text{if action is dissipative,} \\ 0 & \text{if action is conservative,} \end{cases} \tag{2.10}$$

where $\tilde{c}_{\mathbf{Y}}$ is a positive constant.

For conservative actions, the actual rate of growth of the partial maxima sequence M_n depends on further properties of the action as investigated in [34].

The work mentioned above hinges on some group theoretic preliminaries, as discussed briefly below. Let

$$A = \left\{\phi_t : t \in \mathbb{Z}^d\right\}$$

be a subgroup of the group of invertible nonsingular transformations on (S, μ) and define a group homomorphism, $\Phi : \mathbb{Z}^d \to A$ by $\Phi(t) = \phi_t$ for all $t \in \mathbb{Z}^d$. Let

$$\mathcal{K} = \mathrm{Ker}(\Phi) = \left\{t \in \mathbb{Z}^d : \phi_t = 1_S\right\},$$

where 1_S denotes the identity map on S. Then $\mathcal{K}$ is a free abelian group and by the first isomorphism theorem of groups, we have

$$A \cong \mathbb{Z}^d/\mathcal{K}.$$

Now, by the structure theorem of finitely generated abelian groups (see, for example, Theorem 8.5 in Chapter I of [18]), we get,

$$A = \bar{F} \oplus \bar{N},$$

where $\bar{F}$ is a free abelian group and $\bar{N}$ is a finite group. Assume $rank(\bar{F}) = p \geq 1$ and $|\bar{N}| = l$. Since, $\bar{F}$ is free abelian, there exists an injective group homomorphism,

$$\Psi : \bar{F} \to \mathbb{Z}^d,$$

such that $\Phi \circ \Psi = 1_{\bar{F}}$. Then $F = \Psi(\bar{F})$ is a free subgroup of $\mathbb{Z}^d$ of rank p. The subgroup F can be regarded as an effective index set and its rank p is an upper bound on effective dimension of the random field giving more precise information on the rate of growth of the partial maximum than the actual dimension d. Depending on the nature of the action restricted to F, the deterministic sequence b_n controlling the rate of partial maxima shows the following asymptotic behavior:

$$n^{-p/\alpha} b_n \to \begin{cases} c & \text{if action restricted to } F \text{ is not conservative,} \\ 0 & \text{if action restricted to } F \text{ is conservative,} \end{cases} \tag{2.11}$$

where c is a positive and finite constant.

We will call p the effective dimension of the field as long as the restricted action $\{\phi_t\}_{t \in F}$ is not conservative. Otherwise, p should be regarded as an upper bound on the effective dimension. By Corollary 4.4.6 of [38], the sequence b_n is completely determined by the field and it does not depend on the choice of the integral representation. Therefore, thanks to (2.11), the same comment applies to the effective dimension p. However, explicit computation of p will need the integral representation to be of the Rosiński form (2.5).

More specifically, Theorem 5.4 in [34] (which is summarized below) sharpens the description of the asymptotic behavior of the partial maxima of a random field when the action is conservative by observing the behavior of the action when restricted to the free subgroup F of

$\mathbb{Z}^d$, leading to the conclusion that $\max_{\|t\|_\infty \le n} |Y(t)| = O(n^{p/\alpha})$ when the effective F-action is not conservative, and is $o(n^{p/\alpha})$ in the conservative case. That is,

$$n^{-p/\alpha} \max_{\|t\|_\infty \le n} |Y(t)| \Rightarrow \begin{cases} c_{\mathbf{Y}} Z_\alpha & \text{if } \{\phi_t\}_{t\in F} \text{ is not conservative,} \\ 0 & \text{if } \{\phi_t\}_{t\in F} \text{ is conservative.} \end{cases}$$

Similar rates of growth are computed for continuous-parameter random fields in [8] improving upon the works of [36] and [33].

2.3. Our contributions

This work provides the rates of growth of the βth moment of the partial maxima sequence denoted as

$$M_n = \max_{\mathbf{0}\le t\le (n-1)\mathbf{1}} |Y(t)| \tag{2.12}$$

whenever $0 < \beta < \alpha$ for a stationary $S\alpha S$ process $\mathbf{Y} = \{Y(t)\}_{t\in\mathbb{Z}^d}$ when the underlying group action (or its restriction to the subgroup F) has a nontrivial dissipative component. This partially solves an open problem mentioned in [45]. Theorem 3.1 in Section 3 shows that the βth moment of maxima of such discrete random fields are $O(n^{d\beta/\alpha})$ for a nonconservative action and $o(n^{d\beta/\alpha})$ for a conservative one. In the case of a conservative action we look at properties of the underlying action restricted to the free subgroup F with effective dimension p and obtain a better estimate for the maximal moments; see Theorem 3.6. We also introduce the concept of *weak effective dimension* generalizing the notion of *effective dimension* (see Section 5 of [34] and also Section 2.2) and relate it to maximal moments (see Theorem 3.4) of stable random fields. We also provide easy extensions of our results to the continuous parameter case in the Appendix.

Finally, we use the rates of growth of the partial maxima sequence for stationary random fields $\mathbf{Y}^{(v)}$ to derive path properties of a real valued H-self-similar $S\alpha S$ random field $\mathbf{X}$ with stationary increments. Our main result is Theorem 4.1 which establishes uniform modulus of continuity for a large class of such random fields. As a consequence (see Corollary 4.3), we prove that the paths of $\mathbf{X}$ are uniformly Hölder continuous of all orders $< H - \frac{p}{\alpha}$ when the corresponding increment processes $\mathbf{Y}^{(v)}$ are generated by actions with effective dimension p. Corollary 4.4 connects path properties with weak effective dimension in a natural fashion. The short memory case (i.e., when the effective dimension $p = d$), on the other hand, is considered in Corollary 4.2. These results show that in presence of stronger dependence, the sample paths of $\mathbf{X}$ become smoother because stronger dependence prevents erratic jumps.

3. Maximal moments of stationary $S\alpha S$ random fields

In this section, we give sharp upper and lower bounds on maximal moments of stationary $S\alpha S$ random fields when the maximum is taken over hypercubes of increasing size. Our results significantly improve the existing bounds given in Lemma 3.5 of [45] and hence the ones in [42]. This is achieved through exploitation of underlying nonsingular actions, and their ergodic theory and group theory. We also introduce the notion of weak effective dimension of stationary $S\alpha S$ random fields in this section and apply it to estimate maximal moments.

The following is our main result on the asymptotic behavior of the maximal moments of stationary $S\alpha S$ random fields indexed by $\mathbb{Z}^d$. The proof is deferred to Section 6 in order to increase the readability of our paper.

Theorem 3.1. *Let* $\mathbf{Y} = \{Y(t)\}_{t\in\mathbb{Z}^d}$ *be a stationary* $S\alpha S$ *random field with* $0 < \alpha < 2$ *and having integral representation* (2.5).

1. *If* $\mathbf{Y}$ *is generated by a dissipative action then, for all* $\beta \in (0, \alpha)$*, there exists* $C \in (0, \infty)$ *such that*

$$n^{-d\beta/\alpha}\mathbb{E}\big[M_n^\beta\big] \to C \quad \text{as } n \to \infty. \tag{3.1}$$

2. *If* $\mathbf{Y}$ *is generated by a conservative action, then for all* $\beta \in (0, \alpha)$*,*

$$n^{-d\beta/\alpha}\mathbb{E}\big[M_n^\beta\big] \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Remark 3.2. When $\mathbf{Y}$ is generated by a dissipative action, then by Theorem 3.3 of [34], $\mathbf{Y}$ has a mixed moving average representation (see also [40] and [29]) given by

$$\mathbf{Y} \stackrel{d}{=} \left\{\int_{W\times\mathbb{Z}^d} f(v, t+s)M(dv, ds)\right\}_{t\in\mathbb{Z}^d},$$

where the function $f \in L^\alpha(W \times \mathbb{Z}^d, \nu \otimes l)$, l is the counting measure on $\mathbb{Z}^d$, ν is a σ-finite measure on a standard Borel space W, and M has control measure $\nu \otimes l$. The limiting constant in (3.1) is

$$C = \tilde{c}_{\mathbf{Y}}^\beta C_\alpha^{\beta/\alpha}\mathbb{E}\left[Z_\alpha^\beta\right] = \tilde{c}_{\mathbf{Y}}^\beta C_\alpha^{\beta/\alpha}\mathbb{E}\left[Z_{\alpha/\beta}\right], \tag{3.3}$$

where Z_α is an α-Fréchet random variable defined in (2.8),

$$C_\alpha = \begin{cases} \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1, \\ \dfrac{2}{\pi} & \text{if } \alpha = 1, \end{cases} \tag{3.4}$$

and $\tilde{c}_{\mathbf{Y}}$ is the constant in (2.10) (see Proposition 4.1 of [34]) given by

$$\tilde{c}_{\mathbf{Y}} = \left(\int_W \sup_{s\in\mathbb{Z}^d} |f(v, s)|^\alpha \nu(dv)\right)^{1/\alpha} \in (0, \infty).$$

In order to compare the convergence of moments in (3.1) with the weak convergence in (2.7), we would like to mention that $c_{\mathbf{Y}} = \tilde{c}_{\mathbf{Y}}\, C_\alpha^{1/\alpha}$.

Theorem 3.1 partially solves an open problem mentioned in [45] (see pages 173–174 therein) when the underlying group action (or its restriction to a lower rank subgroup) is dissipative. Note that as long as the action is not conservative, the same asymptotics will hold for the maximal moment sequence. In the next result, we present a solution to the problem in a more general situation. Before we describe the next theorem, we introduce the notion of weak effective dimension of a stationary $S\alpha S$ random field. This notion should be considered significantly better than the effective dimension (defined by [34]), which is always an integer whereas weak effective dimension need not be an integer. As we will see in Corollary 4.4 below, the following rougher estimate on b_n suffices for our purpose.

Definition 3.3. We say that a stationary $S\alpha S$ random field has weak effective dimension bounded by $\theta_2 \in (0, d]$ if there exist constants $c_1 > 0$, $c_2 > 0$ and $\theta_1 \in (0, \theta_2]$ such that the sequence b_n defined by (2.9) satisfies

$$c_1 n^{\theta_1} \le b_n^\alpha \le c_2 n^{\theta_2} \tag{3.5}$$

for all sufficiently large n. If (3.5) is satisfied with $\theta_2 = \theta_1$, then we call θ_2 the weak effective dimension of the random field. If further weak effective dimension $\theta_2 = d$, then we say that the stationary $S\alpha S$ random field is full-dimensional.

Clearly, Proposition 4.1 in [34] ensures that any stationary $S\alpha S$ random field with a non-trivial dissipative (equivalently, mixed moving average) part is full-dimensional. The rationale behind this nomenclature (and also behind restricting the value of θ_2 in the interval $(0, d]$) can be explained by the following calculation:

$$b_n = \left(\int_S \max_{\mathbf{0} \le t \le (n-1)\mathbf{1}} |f_t(x)|^\alpha \mu(dx) \right)^{1/\alpha}$$
$$= \left(\int_S \max_{\mathbf{0} \le t \le (n-1)\mathbf{1}} \left[|f \circ \phi_t(x)|^\alpha \frac{d\mu \circ \phi_t}{d\mu}(x) \right] \mu(dx) \right)^{1/\alpha}.$$

Bounding the maximum by the sum and using Fubini's Theorem, we get

$$b_n^\alpha \le \sum_{\mathbf{0} \le t \le (n-1)\mathbf{1}} \int_S \left[|f \circ \phi_t(x)|^\alpha \frac{d\mu \circ \phi_t}{d\mu}(x) \right] \mu(dx)$$
$$= \sum_{\mathbf{0} \le t \le (n-1)\mathbf{1}} \int_S |f \circ \phi_t(x)|^\alpha d\mu \circ \phi_t(x)$$
$$= \sum_{\mathbf{0} \le t \le (n-1)\mathbf{1}} \int_S |f(x)|^\alpha d\mu(x) = n^d \|f\|_\alpha^\alpha.$$

If a stable random field has effective dimension (as described in Section 2) p, then thanks to Proposition 5.1 in [34], we can take $\theta_1 = \theta_2 = p$ in Definition 3.3 making this notion coincide with its weaker version introduced in Definition 3.3. The connection of weak effective dimension to asymptotics of maximal moments is given in the following result, which is also applicable to $\mathbf{Y}$ generated by conservative actions although the computation of θ_2 has to be carried out on a case-by-case basis.

Theorem 3.4. *Consider a stationary $S\alpha S$ random field with $0 < \alpha < 2$, $\mathbf{Y} = \{Y(t)\}_{t \in \mathbb{Z}^d}$ with integral representation as (2.5). If the field has weak effective dimension bounded by θ_2, then for all $n \ge 1$,*

$$n^{-\theta_2 \beta/\alpha} \mathbb{E}\left[M_n^\beta\right] \le K', \tag{3.6}$$

where K' is a finite constant.

Remark 3.5. By Theorem 2.1 of [20] (see also Equation (3.4) in [35]), as long as $\alpha \in (0, 1)$,

$$\mathbb{E}(M_n^\beta) \le K_2 b_n^\beta$$

always holds for some $K_2 \in (0, \infty)$, for all $\beta \in (0, \alpha)$ and for all $n \ge 1$. Therefore, the lower bound in (3.5) is not required when $0 < \alpha < 1$.

Now we consider the case when the underlying group action is conservative and establish refined results on maximal moments in terms of the effective dimension p of $\mathbf{Y}$. This is the place where algebra (more specifically, *structure theorem for finitely generated abelian groups*) plays a significant role in the asymptotic properties of maximal moments.

Theorem 3.6. *Let $\mathbf{Y} = \{Y(t)\}_{t\in\mathbb{Z}^d}$ be a stationary $S\alpha S$ random field with $0 < \alpha < 2$, with integral representation written in terms of functions $\{f_t\}$ as in (2.5).*

1. *If the underlying action $\{\phi_t\}_{t\in F}$ is dissipative when restricted to free subgroup F with rank p, then*
$$n^{-p\beta/\alpha}\mathbb{E}\big[M_n^\beta\big] \to C \quad \text{as } n \to \infty, \tag{3.7}$$
where $C = c^\beta C_\alpha^{\beta/\alpha}\mathbb{E}\left[Z_\alpha^\beta\right] \in (0,\infty)$, with Z_α denoting an α-Fréchet random variable defined in (2.8), $c = \lim_{n\to\infty} n^{-p/\alpha} b_n$, and C_α as defined in (3.4).
2. *If the underlying action $\{\phi_t\}_{t\in F}$ is conservative when restricted to free subgroup F with rank p, then*
$$n^{-p\beta/\alpha}\mathbb{E}\big[M_n^\beta\big] \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

Remark 3.7. The asymptotic properties of maximal moments can easily be extended to stationary measurable symmetric α-stable random fields indexed by $\mathbb{R}^d$. This can be done based on the works of [33,36] and [8]. Since the results (and the proofs) are similar to those presented in this section, we have included them (only $d = 1$ case for simplicity of presentation) in the Theorem A.1 in Appendix.

4. Uniform modulus of continuity

This section combines the maximal moment estimates in Section 3 with Proposition 2.1 to establish uniform modulus of continuity of self-similar $S\alpha S$ random fields with stationary increments. We would like to mention once again that this is mainly carried out through the following theorem, which has three corollaries (see Corollaries 4.2, 4.3 and 4.4).

Theorem 4.1. *Let $\mathbf{X} = \left\{X(t)\right\}_{t\in\mathbb{R}^d}$ be a real-valued H-self-similar $S\alpha S$ random field with stationary increments and with the following integral representation*
$$X(t) \overset{d}{=} \int_E f_t(s)M(ds), \quad t \in \mathbb{R}^d, \tag{4.1}$$
where M is a $S\alpha S$ random measure on a measurable space $(E,\mathcal{E})$ with a σ-finite control measure m, while $f_t \in L^\alpha(m,\mathcal{E})$ for all $t \in \mathbb{R}^d$.

Let $V = \{v = (v_1,\dots,v_d) : v_i \in \{-1,0,1\}\}\setminus\{(0,\dots,0)\}$ be the set of vertices of unit cubes in $[-1,1]^d$, excluding the origin $\mathbf{0}$. Define for each $v \in V$, the random field $\mathbf{Y}^{(v)} = \{Y^{(v)}(t), t \in \mathbb{R}^d\}$ by $Y^{(v)}(t) = X(t+v) - X(t)$, with the integral representation given by
$$Y^{(v)}(t) = \int_E f_t^{(v)}(x)M(dx),$$
where $f_t^{(v)} = f_{v+t} - f_t$ for all $t \in \mathbb{R}^d$. If either

1. *$0 < \alpha < 1$ and there exist constants $0 < \theta_2 < \alpha H$ and $K > 0$ such that for all $v \in V$,*
$$b_n^{(v)} := \left(\int_E \max_{\mathbf{0}\le t\le(n-1)\mathbf{1}} \left|f_t^{(v)}(x)\right|^\alpha m(dx)\right)^{1/\alpha} \le K\, n^{\theta_2/\alpha}$$
for all sufficiently large n, or
2. *$1 \le \alpha < 2$ and there exists $\theta_2 \in (0,\alpha H)$ such that for all $v \in V$ the increment field $\{Y^{(v)}(t)\}$ has weak effective dimension bounded by θ_2,*

then for any $0 < \gamma < \alpha$,

$$\limsup_{h\to 0+} \frac{\sup_{t\in T}\sup_{|s-t|_\infty\le h}|X(t)-X(s)|}{h^{(H-\theta_2/\alpha)}(\log 1/h)^{1/\gamma}} = 0 \quad \text{a.s.}, \tag{4.2}$$

where $T=[0,1]^d$ *and* $|s-t|_\infty = \max_{1\le j\le d}|s_j-t_j|$ *is the* ℓ^∞ *metric on* $\mathbb{R}^d$.

Proof. We first give the proof under condition (2) (i.e., when $1 \le \alpha < 2$). Recall that $\{D_n, n\ge 0\}$ as,

$$D_n = \left\{\left(\frac{k_1}{2^n}, \frac{k_2}{2^n}, \ldots, \frac{k_d}{2^n}\right) : 0 \le k_j \le 2^n-1, 1\le j\le d\right\}.$$

Observe that for any $0<\gamma<\alpha$,

$$\begin{aligned}
&\mathbb{E}\left(\max_{\tau_n\in D_n}\max_{\tau'_{n-1}\in O_{n-1}(\tau_n)}|X(\tau_n)-X(\tau'_{n-1})|^\gamma\right)\\
&\le \sum_{v\in V}\mathbb{E}\left(\max_{0\le k_j\le 2^n-1,\forall j=1,\ldots,d}\left|X\left(\left(\frac{k_1}{2^n},\ldots,\frac{k_d}{2^n}\right)+\frac{v}{2^n}\right)\right.\right.\\
&\qquad\qquad\left.\left.-X\left(\frac{k_1}{2^n},\ldots,\frac{k_d}{2^n}\right)\right|^\gamma\right) \qquad (4.3)\\
&= 2^{-n\gamma H}\sum_{v\in V}\mathbb{E}\left(\max_{0\le k_j\le 2^n-1,\forall j=1,\ldots,d}\left|Y^{(v)}\left((k_1,\ldots,k_d)\right)\right|^\gamma\right)\\
&= 2^{-n\gamma H}\sum_{v\in V}\mathbb{E}\left[\left(M^{(v)}_{2^n}\right)^\gamma\right],
\end{aligned}$$

where $M^{(v)}$ is the partial maxima sequence of the stationary $S\alpha S$ random field $\mathbf{Y}^{(v)}$, and where the first equality follows from the self-similarity of $\mathbf{X}$.

Under the assumption of Theorem 4.1 we have that for some positive constants θ_1, θ_2, c_1 and c_2,

$$c_1 n^{\theta_1/\alpha} \le b_n^{(v)} \le c_2 n^{\theta_2/\alpha}.$$

It follows from Theorem 3.4 that the sequence $\mathbb{E}\big[\big(b_n^{(v)^{-1}} M_n^{(v)}\big)^\gamma\big]$ is bounded above by a constant $K'>0$. Hence

$$\mathbb{E}\left[\left(M_n^{(v)}\right)^\gamma\right] \le K'\left(b_n^{(v)}\right)^\gamma \le K\, n^{\theta_2\gamma/\alpha}$$

for a finite constant $K>0$. Noting that the cardinality of V is $|V| = 3^d-1$, we have

$$\mathbb{E}\left(\max_{\tau_n\in D_n}\max_{\tau'_{n-1}\in O_{n-1}(\tau_n)}|X(\tau_n)-X(\tau'_{n-1})|^\gamma\right) \le (3^d-1)K\,2^{-n\gamma(H-\theta_2/\alpha)}.$$

It follows immediately from Proposition 2.1 that for any $\epsilon>0$ and $\gamma\in(0,\alpha)$,

$$\limsup_{h\to 0+}\frac{\sup_{t\in T}\sup_{|s-t|_\infty\le h}|X(t)-X(s)|}{h^{(H-\theta_2/\alpha)}(\log 1/h)^{(1+\epsilon)/\gamma}} = 0 \ \ a.s.$$

Since $\epsilon>0$ and γ are arbitrary, (4.2) follows.

Under condition (1), the same proof will go through because when $0<\alpha<1$, the lower bound on $b_n^{(v)}$ is not needed for establishing $\mathbb{E}\big[\big(M_n^{(v)}\big)^\gamma\big] \le K\,n^{\theta_2\gamma/\alpha}$ for some $K>0$ (see Remark 3.5). This completes the proof of Theorem 4.1. □

The above theorem has three important consequences (see below) that describe how the uniform modulus of continuity changes for self-similar stable random fields with stationary increments as we pass from a dissipative action to a conservative one in the integral representation of the increment fields. The more the strength of conservativity of the action, the lower the value of the (weak) effective dimension of the increment fields and the smoother the paths of the original field due to longer memory will be.

Corollary 4.2. *Let $\mathbf{X} = \{X(t)\}_{t\in\mathbb{R}^d}$ be a real-valued H-self-similar $S\alpha S$ random field with stationary increments and with the integral representation (4.1). If, for every vertex $v \in V$, the increment process $\mathbf{Y}^{(v)}$ defined as in Theorem 4.1 is generated by a dissipative action and $\alpha > \dfrac{d}{H}$, then for any $0 < \gamma < \alpha$,*

$$\limsup_{h\to 0+} \frac{\sup_{t\in T} \sup_{|s-t|_\infty \le h} |X(t) - X(s)|}{h^{(H-d/\alpha)}(\log 1/h)^{1/\gamma}} = 0 \ \ a.s. \tag{4.4}$$

Proof. Considering the same chaining sequence as in the proof of Theorem 4.1 with the ℓ^∞ metric, we may proceed similarly as in (4.3) to derive that for any $0 < \gamma < \alpha$,

$$\mathbb{E}\left(\max_{\tau_n \in D_n} \max_{\tau'_{n-1}\in O_{n-1}(\tau_n)} |X(\tau_n) - X(\tau'_{n-1})|^\gamma \right) \le 2^{-n\gamma H} \sum_{v\in V} \mathbb{E}\left[\left(M^{(v)}_{2^n-1}\right)^\gamma \right],$$

where $M^{(v)}$ is the partial maxima sequence of the stationary $S\alpha S$ random field $\mathbf{Y}^{(v)}$. From Theorem 3.1 in Section 3, when $\mathbf{Y}^{(v)}$ is generated by a dissipative action, we have

$$\lim_{n\to\infty} \mathbb{E}\left[(2^n - 1)^{-\gamma d/\alpha} \left(M^{(v)}_{2^n-1}\right)^\gamma \right] = c, \tag{4.5}$$

where $c > 0$ is a finite constant. Hence, there exists a finite constant K such that

$$\mathbb{E}\left(\max_{\tau_n \in D_n} \max_{\tau'_{n-1}\in O_{n-1}(\tau_n)} |X(\tau_n) - X(\tau'_{n-1})|^\gamma \right) \le K\, 2^{-n\gamma(H-d/\alpha)},$$

for all sufficiently large n. It is now clear that (4.4) follows from Proposition 2.1. □

Corollary 4.3. *Let $\mathbf{X} = \{X(t)\}_{t\in\mathbb{R}^d}$ be a real-valued H-self-similar random field with stationary increments as in Theorem 4.1. If, for every vertex $v \in V$, the increment process $\mathbf{Y}^{(v)}$ has effective dimension $p \le d$ and $\alpha > \dfrac{p}{H}$, then for any $0 < \gamma < \alpha$,*

$$\limsup_{h\to 0+} \frac{\sup_{t\in T} \sup_{|s-t|_\infty \le h} |X(t) - X(s)|}{h^{(H-p/\alpha)}(\log 1/h)^{1/\gamma}} = 0 \ \ a.s.$$

Proof. The proof follows similarly along the lines of Corollary 4.2 by using the bound on moments in terms of the effective dimension in Theorem 3.6. □

Corollary 4.4. *Let $\mathbf{X} = \{X(t)\}_{t\in\mathbb{R}^d}$ be a real-valued H-self-similar random field with stationary increments as in Theorem 4.1. If for every vertex $v \in V$, the increment field $\mathbf{Y}^{(v)}$ has weak effective dimension bounded by $\theta_2 \in (0, \alpha H)$, then for any $0 < \gamma < \alpha$,*

$$\limsup_{h\to 0+} \frac{\sup_{t\in T} \sup_{|s-t|_\infty \le h} |X(t) - X(s)|}{h^{(H-\theta_2/\alpha)}(\log 1/h)^{1/\gamma}} = 0 \ \ a.s. \tag{4.6}$$

Proof. The proof follows immediately by using the same arguments as the second part of Theorem 4.1. □

Remark 4.5. (i) If the (weak) effective dimension of the increment fields in Corollaries 4.3 and 4.4 is strictly less than d (i.e., when we are not in the full-dimensional case), our uniform modulus of continuity results improves the leading (polynomial) term of the existing ones (see, for example, [45] and the references therein). On the other hand, in the full-dimensional case (i.e., in Corollary 4.2), we better the logarithmic term in the modulus of continuity.

(ii) From the proof of Corollary 4.4, it transpires that even when the weak effective dimension of $\mathbf{Y}^{(v)}$ is bounded by $\theta_2(v)$ (that may depend on $v \in V$), (4.6) holds with θ_2 replaced by $\max_{v\in V}\theta_2(v)$ as long as this maximum is strictly less than αH. A similar comment applies to Corollary 4.3.

5. Examples

The theorems in Section 4 can be applied to various classes of self-similar random fields with stationary increments. In the following, we give three examples of them: linear fractional stable motion, linear fractional stable field indexed $\mathbb{R}^2$, and harmonizable fractional stable fields indexed by $\mathbb{R}^d$ for any $d \geq 1$. For further examples of self-similar processes with stationary increments, see [25,26,38].

5.1. Linear fractional stable motion

For any given constants $0 < \alpha < 2$ and $H \in (0,1)$, we define a $S\alpha S$ process $Z^H = \{Z^H(t)\}_{t\in\mathbb{R}_+}$ with values in $\mathbb{R}$ by

$$Z^H(t) = \kappa \int_{\mathbb{R}} \left\{(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\right\} M_\alpha(ds), \tag{5.1}$$

where $\kappa > 0$ is a normalizing constant, $t_+ = \max\{t,0\}$ and M_α is a $S\alpha S$ random measure with Lebesgue control measure.

Using (5.1) one can verify that the stable process Z^H is H-self-similar and has stationary increments. It is a stable analogue of fractional Brownian motion, and is called a linear fractional stable motion (LFSM). Many sample path properties of Z^H are different from those of fractional Brownian motion. For example, [19] showed that, if $H\alpha < 1$, then Z^H has a.s. unbounded sample functions on all intervals. [41] showed that, if $H\alpha > 1$, then the index of uniform Hölder continuity of Z^H is $H - \frac{1}{\alpha}$.

In order to apply the results in Section 5, we consider for every $v \in \{-1,1\}$ the increment process

$$Y^{(v)}(t) = \int_{\mathbb{R}} \left\{(t+v-s)_+^{H-1/\alpha} - (t-s)_+^{H-1/\alpha}\right\} M_\alpha(ds). \tag{5.2}$$

Then for any $n \geq 1$,

$$b_n^{(v)} = \left(\int_{\mathbb{R}} \max_{0\le k\le n-1} \left|(k+v-s)_+^{H-1/\alpha} - (k-s)_+^{H-1/\alpha}\right|^\alpha ds\right)^{1/\alpha}.$$

For simplicity, we only consider the case of $v = 1$ and write $b_n^{(v)}$ as b_n. The case of $v = -1$ can be treated the same way. For integers $k \in \{0,1,\ldots,n-1\}$, let $g_k(s) = (k+1-s)_+^{H-1/\alpha}$

$-(k-s)_+^{H-1/\alpha}$. It is easy to see that for each fixed $s \le k$, the sequence $g_k(s)$ is non-negative and non-increasing in k. We write b_n^α as

$$\begin{aligned} b_n^\alpha &= \int_{-\infty}^0 \max_{0\le k\le n-1} g_k(s)^\alpha ds + \sum_{\ell=0}^{n-1} \int_\ell^{\ell+1} \max_{0\le k\le n-1} g_k(s)^\alpha ds \\ &= \int_{-\infty}^0 g_0(s)^\alpha ds + \sum_{\ell=0}^{n-1} \int_\ell^{\ell+1} \max\Big\{g_\ell(s), g_{\ell+1}(s)\Big\}^\alpha ds, \end{aligned} \tag{5.3}$$

where $g_n \equiv 0$. Now it is elementary to verify that each of the $(n+1)$ integrals on the right hand side of (5.3) is a positive and finite constant depending only on α and H. Except the first and the last integral, all the other integrals are equal. Consequently, there is a positive and finite constant K such that

$$\lim_{n\to\infty} n^{-1/\alpha} b_n = K.$$

Hence, for $v \in \{1,-1\}$, the weak effective dimension of the stationary $S\alpha S$ process $\big\{Y^{(v)}(n)\big\}_{n\in\mathbb{Z}}$ is 1. It can be verified that

$$\sum_{k\in\mathbb{Z}} \big|g_k(s)\big|^\alpha < \infty \qquad \text{for a.e. } s \in \mathbb{R}.$$

It follows from Corollary 4.2 of Rosiński (1995) that $\big\{Y^{(v)}(n)\big\}_{n\in\mathbb{Z}}$ is generated by a dissipative flow. Moreover, Condition (2) of Theorem 4.1 is satisfied with $\theta_1 = \theta_2 = 1$. It follows from (4.6) that if $\alpha \in (1,2)$ and $H > 1/\alpha$, then for any $0 < \gamma < \alpha$,

$$\limsup_{h\to 0+} \frac{\sup_{t\in[0,1]} \sup_{|s-t|\le h} |Z^H(t) - Z^H(s)|}{h^{(H-1/\alpha)}(\log 1/h)^{1/\gamma}} = 0 \ \ a.s. \tag{5.4}$$

This result improves Theorem 2 in [15] for linear fractional stable motion Z^H with $H > 1/\alpha$. In this case, [41] established a stronger result on uniform modulus of continuity by using more delicate analysis.

5.2. A linear fractional stable field indexed by $\mathbb{R}^2$

In this subsection, we give a new example of a linear fractional stable field for which our results perform better than the existing ones. This happens because the increment fields have effective dimension 1. We would like to mention that this example can easily be extended to an $\mathbb{R}^d$-indexed field but for simplicity of presentation, we only consider the $d = 2$ case. Throughout this subsection v and t will denote the vectors $(v_1, v_2) \in \mathbb{R}^2$ and $(t_1, t_2) \in \mathbb{R}^2$, respectively.

For $0 < \alpha < 2$ and $H \in (0,1)$, we define an $S\alpha S$ random field $\{W^H(t_1,t_2)\}_{t_1,t_2\in\mathbb{R}_+}$ by

$$W^H(t_1,t_2) = \int_{\mathbb{R}} \Big\{(t_1 - t_2 - s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\Big\} M_\alpha(ds),$$

where M_α is a $S\alpha S$ random measure with Lebesgue control measure. It is easy to check that this random field is H-self-similar with stationary increments. For each direction $v = (v_1, v_2)$, the stationary $S\alpha S$ field

$$Y^{(v_1,v_2)}(t_1,t_2) := W^H(t_1+v_1, t_2+v_2) - W^H(t_1,t_2), \ \ (t_1,t_2) \in \mathbb{R}_+^2$$

has integral representation (2.5) with $c_{(t_1,t_2)} \equiv 1$,

$$f(s) = (v_1 - v_2 - s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}, \quad s \in \mathbb{R},$$

and

$$\phi_{(t_1,t_2)}(s) = s + t_2 - t_1, \quad s \in \mathbb{R}.$$

Applying an argument used in Example 3.5 of [8], it follows that each increment field $\{Y^{(v_1,v_2)}(t_1,t_2)\}$ has effective dimension 1. Therefore, Corollary 4.3 of our paper yields that for $\alpha \in (1,2)$, $H \in (1/\alpha, 1)$ and for all $\gamma \in (0,\alpha)$,

$$\limsup_{h\to 0+} \frac{\sup_{t\in[0,1]^2} \sup_{|s-t|_\infty \le h} |W^H(t) - W^H(s)|}{h^{(H-1/\alpha)}(\log 1/h)^{1/\gamma}} = 0 \ \ a.s. \tag{5.5}$$

Note that (5.5) cannot be obtained using the existing results (e.g., Corollary 3.7 of [45]) on uniform modulus of continuity for self-similar stable random fields with stationary increments.

5.3. Harmonizable fractional stable fields

For any given $\alpha \in (0,2)$ and $H \in (0,1)$, let $\widetilde{Z}^H = \{\widetilde{Z}^H(t)\}_{t\in\mathbb{R}^d}$ be the real-valued harmonizable fractional $S\alpha S$ field (HFαSF or HFSF, for brevity) with Hurst index H, defined by:

$$\widetilde{Z}^H(t) := \widetilde{\kappa}\, \mathrm{Re} \int_{\mathbb{R}^d} \frac{e^{i\langle t,x\rangle} - 1}{|x|^{H+d/\alpha}}\, \widetilde{M}_\alpha(dx), \tag{5.6}$$

where Re denotes the real-part, $\langle t,x\rangle$ the usual inner product of t and x, $\widetilde{M}_\alpha$ a complex-valued rotationally invariant α-stable random measure with Lebesgue control measure, and $\widetilde{\kappa}$ is the positive normalizing constant given by

$$\widetilde{\kappa} = 2^{-1/2}\left(\int_{\mathbb{R}^d} \frac{\big(1-\cos\langle \xi, x\rangle\big)^{\alpha/2}}{|x|^{\alpha H+d}}\, dx\right)^{-1/\alpha}, \tag{5.7}$$

where ξ is an arbitrary element of the unit sphere $\mathbb{S}^{d-1}$. When $d = 1$, $\widetilde{Z}^H$ is called a harmonizable fractional stable motion (cf. e.g., [38]).

For every $v \in \{-1,0,1\}^d \setminus \{\mathbf{0}\}$ consider the increment process

$$\widetilde{Y}^{(v)}(t) = \widetilde{\kappa}\, \mathrm{Re} \int_{\mathbb{R}^d} \frac{e^{i\langle t+v,x\rangle} - e^{i\langle t,x\rangle}}{|x|^{H+d/\alpha}}\, \widetilde{M}_\alpha(dx). \tag{5.8}$$

Then for any integer $n \ge 1$,

$$\begin{aligned} b_n^{(v)} &= \widetilde{\kappa}\left(\int_{\mathbb{R}^d} \max_{\mathbf{0}\le k\le (n-1)\mathbf{1}} \left|e^{i\langle k+v,x\rangle} - e^{i\langle k,x\rangle}\right|^\alpha \frac{dx}{|x|^{\alpha H+d}}\right)^{1/\alpha} \\ &= \widetilde{\kappa}\left(\int_{\mathbb{R}^d} \left|e^{i\langle v,x\rangle} - 1\right|^\alpha \frac{dx}{|x|^{\alpha H+d}}\right)^{1/\alpha}, \end{aligned}$$

which is independent of n. This implies that the weak effective dimension of the stationary $S\alpha S$ process $\{\widetilde{Y}^{(v)}(t)\}_{t\in\mathbb{Z}^d}$ is 0. Applying Corollary 3.2 of [34], one can verify that $\{\widetilde{Y}^{(v)}(t)\}_{t\in\mathbb{Z}^d}$ is generated by a conservative flow.

We remark that the results in Section 3 are not applicable for determining the precise magnitude of the maximal moments $\mathbb{E}\big[\max_{\mathbf{0}\le k\le (n-1)\mathbf{1}} |\widetilde{Y}^{(v)}(k)|^\gamma\big]$ for $\gamma \in (0,\alpha)$. Some partial results are known. It follows from Theorem 2.1 of [20] that if $0 < \alpha < 1$ then for any

$0 < \gamma < \alpha$, one has $K \le \mathbb{E}\Big[\max_{0\le k\le n-1} |Y^{(v)}(k)|^{\gamma}\Big] \le K'$, where K and K' are positive and finite constants. When $1 \le \alpha \le 2$, by appealing to the fact that $\widetilde{Y}^{(v)}(t)$ is conditionally Gaussian, see [6,7], or [15], we can modify the proof of Proposition 4.3 in [45] to derive the following upper and lower bounds

$$K \le \mathbb{E}\Big[\max_{\mathbf{0}\le k\le (n-1)\mathbf{1}} |\widetilde{Y}^{(v)}(k)|^{\gamma}\Big] \le K'\big(\log n\big)^{\gamma/2} \tag{5.9}$$

for some positive and finite constants K and K'. We omit a detailed verification of (5.9) here because it is lengthy and does not produce the optimal bounds. In the case of $\alpha = 2$, it can be proved by applying the Sudakov minoration (see Lemma 2.1.2 in [42]) that the upper bound in (5.9) is optimal. For $1 < \alpha < 2$, a lower bound for $\mathbb{E}\big[\max_{\mathbf{0}\le k\le (n-1)\mathbf{1}} |\widetilde{Y}^{(v)}(k)|\big]$ in terms of $\log n$ and $q = \alpha/(\alpha-1)$ can be derived from Theorem 2.2 of [20].

It follows from (5.9), (4.3) and Proposition 2.1 with $\sigma(h) = h^H\big|\log 1/h\big|^{1/2}$ that for any $\epsilon > 0$,

$$\limsup_{h\to 0+} \frac{\sup_{t\in[0,1]^d} \sup_{|s-t|\le h} |\widetilde{Z}^H(t) - \widetilde{Z}^H(s)|}{h^H(\log 1/h)^{\frac{1}{2}+\frac{1}{\alpha}+\epsilon}} = 0 \ \ a.s. \tag{5.10}$$

This improves Theorem 4.5 in [45] and extends Theorem 1 in [15] to the random field setting. However, it is an open problem to determine the exact uniform modulus of continuity for HFSM $\widetilde{Z}^H$.

We remark that, even though LFSM Z^H and HFSM $\widetilde{Z}^H$ are H-self-similar with stationary increments, their properties are very different. By the exact modulus of continuity in [41] and (5.10), it is clear that the laws of Z^H and $\widetilde{Z}^H$ are singular with respect to each other.

5.4. The Mittag-Leffler fractional SαS motions

We provide another application of Theorem 4.1 to the Mittag-Leffler fractional SαS motions introduced by [24] (see also [9,10] for related results). These processes form an important class of self-similar stable processes with stationary increments whose regularity properties are different from linear fractional stable motions and harmonizable fractional stable motions.

For a constant $\beta \in (0,1)$, let $S_\beta = \{S_\beta(t)\}_{t\ge 0}$ be a β-stable subordinator defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Define its inverse process $M_\beta = \{M_\beta(x)\}_{x\ge 0}$ by

$$M_\beta(x) = \inf\{t \ge 0 : S_\beta(t) \ge x\}, \qquad x \ge 0. \tag{5.11}$$

For each fixed $x \ge 0$, $M_\beta(x)$ is a stopping time and its distribution is the Mittag-Leffler distribution with the Laplace transform

$$\mathbb{E}'\Big(e^{\theta M_\beta(x)}\Big) = \sum_{n=0}^{\infty} \frac{(\theta t^\beta)^n}{\Gamma(1+n\beta)}, \qquad \theta \in \mathbb{R}.$$

It is known from [22] that the process M_β is β-self-similar and has a continuous and nondecreasing version. In the terminology of [24], M_β is called the Mittag-Leffler process.

The β-Mittag-Leffler (or β-ML) fractional $S\alpha S$ motion $Y_{\alpha,\beta} = \{Y_{\alpha,\beta}(t)\}_{t\ge 0}$ is defined by

$$Y_{\alpha,\beta}(t) = \int_{\Omega'\times[0,\infty)} M_\beta\big((t-x)_+, \omega'\big)\, dZ_{\alpha,\beta}\big(\omega', x\big), \quad t \ge 0, \tag{5.12}$$

where $Z_{\alpha,\beta}$ is a SαS random measure on $\Omega' \times [0,\infty)$ with control measure $\mathbb{P}' \times \nu$, with ν a measure on $[0,\infty)$ given by $\nu(dx) = (1-\beta)x^{-\beta}dx$, $x > 0$.

[24] proved that the β-ML fractional stable motion $Y_{\alpha,\beta}$ is H-self-similar with stationary increments, where

$$H = \beta + \frac{1-\beta}{\alpha}, \tag{5.13}$$

and is the scaling limit of partial sums of certain symmetric stationary infinitely divisible processes with regularly varying Lévy measures. Among many interesting results, they also established the following results on uniform modulus of continuity for $Y_{\alpha,\beta}$ in their Theorem 3.3: If $0 < \alpha < 1$, then

$$\sup_{0\le s<t\le 1/2} \frac{|Y_{\alpha,\beta}(t)Y_{\alpha,\beta}(s)|}{(t-s)^{\beta}|\log(t-s)^{1-\beta}} < \infty, \qquad \text{a.s.} \tag{5.14}$$

and if $\alpha < 2$, then

$$\sup_{0\le s<t\le 1/2} \frac{|Y_{\alpha,\beta}(t)Y_{\alpha,\beta}(s)|}{(t-s)^{\beta}|\log(t-s)^{\frac{3}{2}-\beta}} < \infty, \qquad \text{a.s.} \tag{5.15}$$

Since $\beta < H$, (5.14) and (5.15) show that the sample path of $Y_{\alpha,\beta}$ is rougher than that of HFSM $\widetilde{Z}^H$. This indicates that, even though both $Y_{\alpha,\beta}$ and $\widetilde{Z}^H$ are conservative, the dependence (or memory) in $\widetilde{Z}^H$ is stronger than in $Y_{\alpha,\beta}$.

The method for proving (5.14) and (5.15) in [24] is based on a random series representation for $Y_{\alpha,\beta}$ and a uniform modulus of continuity for the Mittag-Leffler process M_β. We provide an alternative approach for studying the uniform modulus of continuity of $Y_{\alpha,\beta}$.

For every $v \in \{-1, 1\}$, we consider the increment process

$$Y^{(v)}_{\alpha,\beta}(t) = \int_{\Omega'\times[0,\infty)} \Big[M_\beta\big((t+v-x)_+, \omega'\big) - M_\beta\big((t-x)_+, \omega'\big)\Big]\, dZ_{\alpha,\beta}(\omega', x). \tag{5.16}$$

By Theorem 3.5 of [24], the stationary sequence $\{Y^{(v)}(n)\}_{n\ge 0}$ is generated by a conservative null flow and is mixing.

For simplicity, we only consider $v = 1$ and write the corresponding $b_n^{(v)}$ as b_n. Then for any integer $n \ge 1$,

$$\begin{aligned} b_n^\alpha &= \int_{\Omega'\times[0,\infty)} \max_{0\le k\le n-1} \Big[M_\beta\big((k+1-x)_+, \omega'\big) - M_\beta\big((k-x)_+, \omega'\big)\Big]^\alpha \mathbb{P}'(d\omega')\nu(dx) \\ &= (1-\beta)\int_0^\infty \mathbb{E}'\bigg(\max_{0\le k\le n-1}\Big[M_\beta\big((k+1-x)_+\big) - M_\beta\big((k-x)_+\big)\Big]^\alpha\bigg)\frac{dx}{x^\beta} \\ &= (1-\beta)\sum_{\ell=0}^{n-1}\int_\ell^{\ell+1} \mathbb{E}'\bigg(\max_{\ell\le k\le n-1}\Big[M_\beta\big(k+1-x\big) - M_\beta\big((k-x)_+\big)\Big]^\alpha\bigg)\frac{dx}{x^\beta}. \end{aligned} \tag{5.17}$$

In the following, we obtain a lower bound of b_n and a slightly worse upper bound. To this end, we will make use of the following facts, where (i) follows from Lemma 1 of [12] and (ii) is from p. 246 in [24] or Exercise 5.6 in [17].

(i). There exist positive constants $c_3 < c_4$, depending on β such that the small tail probability of $S_\beta(1)$ satisfies

$$\exp\Big(-c_4\theta^{-\beta/(1-\beta)}\Big) \le \mathbb{P}'\big(S_\beta(1)\le\theta\big) \le \exp\Big(-c_3\theta^{-\beta/(1-\beta)}\Big), \qquad \theta\in(0,1). \tag{5.18}$$

(ii). For a constant $r > 0$, let $\delta_r = S_\beta(M_\beta(r)) - r$ be the overshoot of the level r by the β-stable subordinator $\{S_\beta(t)\}_{t\geq 0}$. The law of δ_r is given by

$$\mathbb{P}'(\delta_r \in d\eta) = \frac{\sin \beta\pi}{\pi} r^\beta (r+\eta)^{-1}\eta^{-\beta}\, d\eta, \quad \eta > 0. \tag{5.19}$$

(iii). By the strong Markov property of S_β, we have that for any $r > 0$ the process $\big\{S_\beta\big(M_\beta(r)+u\big) - S_\beta\big(M_\beta(r)\big)\big\}_{u\geq 0}$ is independent of the σ-algebra $\mathcal{F}_{M_\beta(r)}$ (i.e., the history up to the stopping time $M_\beta(r)$) and has the same law as $\{S_\beta(u)\}_{u\geq 0}$. In particular, $\big\{S_\beta\big(M_\beta(r)+u\big) - S_\beta\big(M_\beta(r)\big)\big\}_{u\geq 0}$ is independent of δ_r.

To get a lower bound for b_n, we fix $\ell \in \{0, 1, \ldots, n-1\}$ and $x \in [\ell, \ell+1]$ first and write the expectation in the last line of (5.17) as

$$\begin{aligned}
&\mathbb{E}'\Big(\max_{\ell\leq k\leq n-1}\Big[M_\beta\big(k+1-x\big) - M_\beta\big((k-x)_+\big)\Big]^\alpha\Big)\\
&= \alpha\int_0^\infty u^{\alpha-1}\mathbb{P}'\Big(\max_{\ell\leq k\leq n-1}\Big[M_\beta\big(k+1-x\big) - M_\beta\big((k-x)_+\big)\Big] > u\Big)\, du\\
&\geq \alpha\int_0^\infty u^{\alpha-1}\mathbb{P}'\Big(M_\beta\big(\ell+3-x\big) - M_\beta\big(\ell+2-x\big) > u\Big)\, du.
\end{aligned} \tag{5.20}$$

By the definition of M_β, the aforementioned facts (ii) and (iii), we have

$$\begin{aligned}
&\mathbb{P}'\Big(M_\beta\big(\ell+3-x\big) - M_\beta\big(\ell+2-x\big) > u\Big)\\
&= \mathbb{P}'\Big(S_\beta\big(M_\beta(\ell+2-x)+u\big) - S_\beta\big(M_\beta(\ell+2-x)\big) \leq 1-\delta_{\ell+2-x}\Big)\\
&= \frac{\sin \pi\beta}{\beta}\int_0^1 \frac{(\ell+2-x)^\beta}{(\ell+2-x+\eta)\,\eta^\beta}\mathbb{P}'\big(S_\beta(u) < 1-\eta\big)\, d\eta\\
&= \frac{\sin \pi\beta}{\beta}\int_0^1 \frac{(\ell+2-x)^\beta}{(\ell+2-x+\eta)\,\eta^\beta}\mathbb{P}'\big(M_\beta(1-\eta) > u\big)\, d\eta.
\end{aligned} \tag{5.21}$$

Plugging (5.21) into (5.20), we derive

$$\begin{aligned}
&\mathbb{E}'\Big(\max_{\ell\leq k\leq n-1}\Big[M_\beta\big((k+1-x)_+\big) - M_\beta\big((k-x)_+\big)\Big]^\alpha\Big)\\
&\geq \frac{\alpha \sin \pi\beta}{\beta}\int_0^\infty\int_0^1 u^{\alpha-1}\frac{(\ell+2-x)^\beta}{(\ell+2-x+\eta)\eta^\beta}\mathbb{P}'\big(M_\beta(1-\eta) > u\big)\, d\eta\, du\\
&\geq \frac{\sin \pi\beta}{\beta}\int_0^1 \frac{1}{(\ell+2-x+\eta)\,\eta^\beta}\mathbb{E}'\big(M_\beta(1-\eta)^\alpha\big)\, d\eta\\
&= \frac{\sin \pi\beta}{\beta}\,\mathbb{E}'\big(M_\beta(1)^\alpha\big)\int_0^1 \frac{(1-\eta)^{\alpha\beta}}{(\ell+2-x+\eta)\,\eta^\beta}\, d\eta\\
&\geq K > 0,
\end{aligned} \tag{5.22}$$

where $K > 0$ is a constant. In the above we have used the fact that $1 \leq \ell+2-x \leq 2$ and $\mathbb{E}'\big(M_\beta(1-\eta)^\alpha\big) = (1-\eta)^{\alpha\beta}\mathbb{E}'\big(M_\beta(1)^\alpha\big)$.

It follows from (5.17) and (5.22) that

$$b_n^\alpha \geq K\sum_{\ell=0}^{n-1}\int_\ell^{\ell+1}\frac{dx}{x^\beta} = K' n^{1-\beta}. \tag{5.23}$$

This gives a lower bound for b_n.

In order to derive an upper bound for b_n, we consider the increasing function

$$\Phi(x) = \exp\Big(\frac{c_3}{2}x^\gamma\Big) - 1,$$

where c_3 is the constant in (5.18) and $\gamma = \frac{1}{\alpha(1-\beta)}$. Then $\Phi(x)$ is strictly increasing, $\Phi(0) = 0$, and convex over the interval (a, ∞) for some constant $a \geq 0$ ($a = 0$ if $\gamma \geq 1$ and Φ is an Orlicz function). The inverse function of Φ is given by $\Phi^{-1}(y) = \big(\ln(1+y)\big)^{1/\gamma}$. It is a decreasing function and is concave on $(\Phi(a), \infty)$.

To consider the maximum moment in the last line of (5.17), we denote $\xi_k = M_\beta\big(k+1-x\big) - M_\beta\big((k-x)_+\big)$ for $k \geq \ell$. Then similar to (5.21), we apply the aforementioned facts (ii) and (iii) to obtain that for any $k \geq \ell+1$ and $u > 1$,

$$\begin{aligned}
\mathbb{P}'\big(\xi_k > u\big) &= \mathbb{P}'\Big(S_\beta\big(M_\beta(k-x+u)\big) - S_\beta\big(M_\beta(k-x)\big) \leq 1 - \delta_{k-x}\Big)\\
&= \frac{\sin\pi\beta}{\beta}\int_0^1 \frac{(k-x)^\beta}{(k-x+\eta)\,\eta^\beta}\mathbb{P}'\big(S_\beta(u) < 1-\eta\big)\,d\eta\\
&\leq \frac{\sin\pi\beta}{\beta}\int_0^1 \frac{(k-x)^\beta}{(k-x+\eta)\,\eta^\beta}\mathbb{P}'\big(S_\beta(1) < (1-\eta)/u^{1/\beta}\big)\,d\eta\\
&\leq \frac{\sin\pi\beta}{\beta}\int_0^1 \frac{(k-x)^\beta}{(k-x+\eta)\,\eta^\beta}\exp\Big(-c_3\Big(\frac{u}{(1-\eta)^\beta}\Big)^{1/(1-\beta)}\Big)\,d\eta\\
&\leq K(k-\ell)^{-1+\beta}\exp\Big(-c_3u^{1/(1-\beta)}\Big).
\end{aligned} \tag{5.24}$$

For the case of $k = \ell$, we have $\xi_\ell = M_\beta\big(\ell+1-x\big)$, hence

$$\mathbb{P}'\big(\xi_\ell > u\big) = \mathbb{P}'\big(S_\beta(u) \leq \ell+1-x\big) \leq \exp\Big(-c_3u^{1/(1-\beta)}\Big).$$

Let $a' > a$ be a constant. Then $\Phi(\xi_k^\alpha + a') > \Phi(a)$. It follows from (5.24) that for $k \geq \ell+1$,

$$\begin{aligned}
\mathbb{E}'\Big(\Phi(\xi_k^\alpha + a')\Big) &= \int_0^\infty \Phi'(u)\mathbb{P}'\Big(\xi_k^\alpha + a' > u\Big)\,du\\
&\leq K(k-\ell)^{-1+\beta}\Big(1 + \int_{a'+1}^\infty \Phi'(u)\exp\Big(-c_3(u-a')^{1/(1-\beta)}\Big)du\Big)\\
&\leq K'(k-\ell)^{-1+\beta},
\end{aligned} \tag{5.25}$$

where K, K' are finite constants. For $k = \ell$, the expectation is finite. Hence, by Jensen's inequality and (5.25), the maximum moment in the last line of (5.17) can be bounded by

$$\begin{aligned}
\mathbb{E}'\Big(\max_{\ell\leq k\leq n-1}\xi_k^\alpha\Big) &\leq \Phi^{-1}\bigg(\mathbb{E}'\Big(\Phi\Big(\max_{\ell\leq k\leq n-1}\xi_k^\alpha + a'\Big)\Big)\bigg)\\
&\leq \Phi^{-1}\bigg(\sum_{k=\ell}^{n-1}\mathbb{E}'\Big(\Phi(\xi_k^\alpha + a')\Big)\bigg)\\
&\leq \Phi^{-1}\bigg(K'\Big(1 + \sum_{k=\ell+1}^{n-1}(k-\ell)^{-1+\beta}\Big)\bigg)\\
&\leq K\big(\ln(1+n)\big)^{1/\gamma}.
\end{aligned} \tag{5.26}$$

Combining (5.17) and (5.26) yields

$$b_n^\alpha \le K \sum_{\ell=0}^{n-1} \frac{1}{\ell^\beta} \big(\ln n\big)^{1/\gamma} \le K\, n^{1-\beta} \big(\ln(1+n)\big)^{1/\gamma}. \tag{5.27}$$

Therefore we have proven that for all $n \ge 1$

$$K'\, n^{1-\beta} \le b_n^\alpha \le K\, n^{1-\beta} \big(\ln(1+n)\big)^{1/\gamma}, \tag{5.28}$$

where $\gamma = \frac{1}{\alpha(1-\beta)}$. We remark that the upper bound (5.28) may not be sharp, but we are not able to improve.

Following the proof of Theorem 3.4 (see Section 6.2), we get that (5.28) yields for all $\zeta \in (0, \alpha)$,

$$\mathbb{E}(M_{v,n}^\zeta) \le K_1 b_n^\zeta \le K_2 n^{(1-\beta)\zeta/\alpha} \big(\ln(1+n)\big)^{\zeta/\alpha\gamma}, \tag{5.29}$$

where $M_{v,n}$ is the partial maxima sequence (2.12) for $\{Y_{\alpha,\beta}^{(v)}(n)\}_{n\ge 0}$. (When $0 < \alpha < 1$, (5.29) follows directly from Remark 3.5.) Then it follows from the proof of Theorem 4.1 that for any $\varepsilon > 0$,

$$\limsup_{h\to 0+} \frac{\sup_{s,t\in[0,1], s-t|\le h} |Y_{\alpha,\beta}(t) - Y_{\alpha,\beta}(s)|}{h^\beta (\log 1/h)^{\gamma + \frac{1}{\alpha} + \varepsilon}} = 0 \quad \text{a.s.}, \tag{5.30}$$

where $\gamma = \frac{1}{\alpha(1-\beta)}$.

To compare (5.30) with the results (5.14) and (5.15) of [24], we notice that the power of the logarithmic factor in (5.30) is $\frac{2-\beta}{\alpha(1-\beta)} + \varepsilon$ which is bigger than $1-\beta$ if $0 < \alpha < 1$. Thus, for the case $0 < \alpha < 1$, (5.14) is stronger than (5.30). When $1 \le \alpha < 2$, we have $\frac{2-\beta}{\alpha(1-\beta)} < \frac{3}{2} - \beta$ if $\beta > 0$ is small and α is close to 2. In such a case, (5.30) is stronger than (5.14). However, as in the case of the harmonizable fractional stable motion $\widetilde{Z}^H$, it is an open problem to determine the exact uniform modulus of continuity of $Y_{\alpha,\beta}$.

6. Proofs of theorems in Section 3

In this section, we prove Theorems 3.1, 3.4 and 3.6. The key idea is to encash the series representation given in [35] (and follow the proof of Theorem 4.1 therein) to obtain sharp tail bounds for the lower powers of maxima of stationary $S\alpha S$ random fields and then invoke the dominated convergence theorem.

6.1. *Proof of Theorem 3.1*

In the following, $\{\Gamma_j\}_{j\ge 1}$ denotes the arrival times of a unit rate Poisson process on $(0, \infty)$, $\{\xi_j\}_{j\ge 1}$ is a sequence of i.i.d. Rademacher random variables, and for each fixed $n \in \mathbb{N}$, $\{U_j^{(n)}\}_{j\ge 1}$ is a sequence of i.i.d. S-valued random variables with common law η_n whose density is given by

$$\frac{d\eta_n}{d\mu} = b_n^{-\alpha} \max_{\mathbf{0}\le t\le (n-1)\mathbf{1}} |f_t(s)|^\alpha, \quad s \in S.$$

All three of the above sequences are independent. For each fixed $n \in \mathbb{N}$, we will make use of the following series representation for the random vector $\big(Y_k, \mathbf{0} \le k \le (n-1)\mathbf{1}\big)$:

$$Y_k \stackrel{d}{=} b_n C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{m\in[\mathbf{0},(n-1)\mathbf{1}]} |f_m(U_j^{(n)})|}, \qquad \mathbf{0} \le k \le (n-1)\mathbf{1}. \tag{6.1}$$

See Section 3.10 in [38].

We first consider the case (1) when $\mathbf{Y}$ is generated by a dissipative action. It follows from (2.10) that the deterministic sequence $\{b_n\}_{n\ge1}$ satisfies

$$\lim_{n\to\infty} n^{-d/\alpha} b_n = \tilde{c}_{\mathbf{Y}}, \tag{6.2}$$

where $\tilde{c}_{\mathbf{Y}} > 0$ is a constant. This implies that b_n satisfies condition (4.6) in [35], namely

$$(\mathbf{LB}): \quad b_n \ge c n^{\theta} \text{ for some constant } c > 0$$

with $\theta = d/\alpha$. Additionally, condition (4.8) in [35] also holds, i.e., for all $\epsilon > 0$,

$$(\mathbf{LL}): \ \mathbb{P}\left[\text{for some } k \in [\mathbf{0}, (n-1)\mathbf{1}],\ \frac{f_k(U_j^{(n)})}{\max_{m\in[\mathbf{0},(n-1)\mathbf{1}]} |f_m(U_j^{(n)})|} > \epsilon,\ j = 1, 2\right] \to 0$$

as $n \to \infty$; see Remark 4.2 in [35] (or Remark 4.4 in [34]). Further, (6.2) implies that for any $p > \alpha$, there is a finite constant A such that

$$(\mathbf{UB}): \qquad n^d b_n^{-p} < n^d b_n^{-\alpha} \le A.$$

Note that **(LB)** is a lower bound on b_n that is essential in carrying out our arguments and **(UB)** yields **(LB)** when the underlying group action is dissipative. The condition **(LL)** ensures that only the first term of the series representation (6.1) contributes to the asymptotics of the maximal moments.

Let $K = d$, ϵ and δ be chosen such that

$$0 < \epsilon < \frac{\delta}{K}.$$

Then we obtain from (4.21) in [35] the following upper bound on the tail distribution of $b_n^{-1} M_n$ under **(LB)** and **(LL)**:

$$\mathbb{P}\big(b_n^{-1} M_n > \lambda\big) \le \mathbb{P}\big(C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} > \lambda(1-\delta)\big) + \phi_n(\epsilon, \lambda) + \psi_n(\epsilon, \delta, \lambda), \tag{6.3}$$

where

$$\phi_n(\epsilon, \lambda) = \mathbb{P}\Bigg(\exists\, k \in [\mathbf{0}, (n-1)\mathbf{1}],\quad \frac{\Gamma_j^{-1/\alpha} |f_k(U_j^{(n)})|}{\max_{m\in[\mathbf{0},(n-1)\mathbf{1}]} |f_m(U_j^{(n)})|} > \frac{\epsilon\lambda}{C_\alpha^{1/\alpha}} \quad \text{for at least 2 different } j\text{'s}\Bigg)$$

and

$$\psi_n(\epsilon, \delta, \lambda) = \mathbb{P}\Bigg(\max_{k\in[\mathbf{0},(n-1)\mathbf{1}]} \left| \sum_{j=1}^{\infty} \frac{\xi_j \Gamma_j^{-1/\alpha} |f_k(U_j^{(n)})|}{\max_{m\in[\mathbf{0},(n-1)\mathbf{1}]} |f_m(U_j^{(n)})|} \right| > \frac{\lambda}{C_\alpha^{1/\alpha} \|f\|_\alpha}, \quad \Gamma_1^{-1/\alpha} \le \frac{b_n \lambda(1-\delta)}{C_\alpha^{1/\alpha} \|f\|_\alpha} \text{ and } \Gamma_2^{-1/\alpha} \le \frac{b_n \lambda \epsilon}{C_\alpha^{1/\alpha} \|f\|_\alpha}\Bigg).$$

Note that

$$\phi_n(\epsilon, \lambda) \leq n^d \mathbb{P}\bigg(\Gamma_j^{-1/\alpha} > \frac{b_n \epsilon \lambda}{C_\alpha^{1/\alpha} \| f \|_\alpha} \quad \text{for at least 2 different } j\text{'s}\bigg). \tag{6.4}$$

In deriving the last inequality, we have applied the fact that for every $k \in [\mathbf{0}, (n-1)\mathbf{1}]$, the points

$$b_n \xi_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{\mathbf{0} \leq s \leq (n-1)\mathbf{1}} |f_s(U_j^{(n)})|}, \quad j = 1, 2, \ldots$$

have the same joint distribution as the points

$$\xi_j \| f \|_\alpha \Gamma_j^{-1/\alpha}, \quad j = 1, 2, \ldots$$

which represent a symmetric Poisson random measure on $\mathbb{R}$ with mean measure

$$\Lambda((x, \infty)) = x^{-\alpha} \| f \|_\alpha^\alpha / 2, \quad \text{for } x > 0. \tag{6.5}$$

In the above, the function f is given in (2.5) and $\| f \|_\alpha = \left(\int_S |f(s)|^\alpha \mu(ds)\right)^{1/\alpha}$. Similarly, we have

$$\psi_n(\epsilon, \delta, \lambda) \leq n^d \mathbb{P}\bigg(\Big|\sum_{j=1}^{\infty} \xi_j \Gamma_j^{-1/\alpha}\Big| > \frac{b_n \lambda}{C_\alpha^{1/\alpha} \| f \|_\alpha}, \ \Gamma_1^{-1/\alpha} \leq \frac{b_n \lambda (1-\delta)}{C_\alpha^{1/\alpha} \| f \|_\alpha}, \ \text{and } \Gamma_2^{-1/\alpha} \leq \frac{b_n \lambda \epsilon}{C_\alpha^{1/\alpha} \| f \|_\alpha}\bigg). \tag{6.6}$$

For any $0 < \beta < \alpha$, by using the tail bound in (6.3) we have

$$\begin{aligned} \mathbb{E}\big[b_n^{-\beta} M_n^\beta\big] &= \int_0^\infty \mathbb{P}\big(b_n^{-1} M_n > \tau^{1/\beta}\big) d\tau \\ &\leq \int_0^\infty \mathbb{P}\big(C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} > \tau^{1/\beta}(1-\delta)\big) d\tau \\ &\quad + \int_0^\infty \phi_n(\epsilon, \tau^{1/\beta}) d\tau + \int_0^\infty \psi_n(\epsilon, \delta, \tau^{1/\beta}) d\tau \\ &:= T_1(\delta) + T_2^{(n)}(\epsilon) + T_3^{(n)}(\epsilon, \delta). \end{aligned} \tag{6.7}$$

It is shown in [35] that for every $\tau > 0$,

$$\phi_n(\epsilon, \tau^{1/\beta}), \text{ and } \psi_n(\epsilon, \delta, \tau^{1/\beta}) \text{ converge to } 0,$$

as $n \to \infty$ for choices of ϵ adequately smaller in comparison to δ.

Next we present non-trivial integrable bounds on $(1, \infty)$ for integrands $\phi_n(\epsilon, \tau^{1/\beta})$ and $\psi_n(\epsilon, \delta, \tau^{1/\beta})$ in $T_2^{(n)}(\epsilon)$ and $T_3^{(n)}(\epsilon, \delta)$ in (6.7) respectively, and use the trivial bound of 1 on $(0, 1)$. Finally, we apply the dominated convergence theorem to show that the terms $T_2^{(n)}(\epsilon)$ and $T_3^{(n)}(\epsilon, \delta)$ converge to 0 as $n \to \infty$.

We begin by providing an integrable upper bound for $\phi_n(\epsilon, \tau^{1/\beta})$ on $(1, \infty)$. It follows from (6.4) that

$$\begin{aligned} \phi_n(\epsilon, \tau^{1/\beta}) &\leq n^d \mathbb{P}\bigg(\sum_{j=1}^{\infty} 1_{\xi_j \| f \|_\alpha \Gamma_j^{-1/\alpha}} \Big\{\big(-\infty, -C_\alpha^{-1/\alpha} b_n \epsilon \tau^{1/\beta}\big) \\ &\qquad \cup \big(C_\alpha^{-1/\alpha} b_n \epsilon \tau^{1/\beta}, \infty\big)\Big\} \geq 2\bigg) \\ &= n^d \mathbb{P}(\mathrm{Poi}(\Lambda(B_n)) \geq 2), \end{aligned} \tag{6.8}$$

where

$$B_n = \left(-\infty, -C_\alpha^{-1/\alpha} b_n \epsilon \tau^{1/\beta}\right) \cup \left(C_\alpha^{-1/\alpha} b_n \epsilon \tau^{1/\beta}, \infty\right)$$

and we have used the fact that

$$\sum_{j=1}^{\infty} 1_{\xi_j \|f\|_\alpha \Gamma_j^{-1/\alpha}}\{B_n\} \sim \mathrm{Poi}(\Lambda(B_n)).$$

Thus, the Markov inequality and definition (6.5) of the mean measure Λ imply

$$\begin{aligned} \phi_n(\epsilon, \tau^{1/\beta}) &\le n^d \frac{\mathbb{E}(\mathrm{Poi}(\Lambda(B_n)))}{2} = n^d \Lambda(B_n)/2 \\ &= n^d b_n^{-\alpha} \frac{C_\alpha \epsilon^{-\alpha}}{\tau^{\alpha/\beta}} \frac{\|f\|_\alpha^\alpha}{2} \le A \frac{C_\alpha^{-1} \epsilon^{-\alpha} \|f\|_\alpha^\alpha}{2\tau^{\alpha/\beta}}. \end{aligned} \tag{6.9}$$

The last inequality in (6.9) follows using **(UB)** and yields an integrable upper bound in τ on $(1, \infty)$. We apply the dominated convergence theorem to $T_2^{(n)}(\epsilon)$ as

$$T_2^{(n)}(\epsilon) = \int_0^1 \phi_n(\epsilon, \tau^{1/\beta}) d\tau + \int_1^\infty \phi_n(\epsilon, \tau^{1/\beta}) d\tau$$

by using the trivial bound of 1 on $(0, 1)$ and the bound derived in (6.9) on $(1, \infty)$ to conclude

$$T_2^{(n)}(\epsilon) \to 0 \ \text{ as } \ n \to \infty.$$

We next derive an upper bound for $\psi_n(\epsilon, \delta, \tau^{1/\beta})$. It follows from (6.6) that $\psi_n(\epsilon, \delta, \tau^{1/\beta})$ is bounded from above by

$$\begin{aligned} & n^d \mathbb{P}\Bigg(\left| C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \right| > \frac{b_n \tau^{1/\beta}}{\|f\|_\alpha},\ C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} \le \frac{b_n \tau^{1/\beta}(1-\delta)}{\|f\|_\alpha}, \\ &\qquad\qquad \text{and } C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} \le \frac{b_n \tau^{1/\beta} \epsilon}{\|f\|_\alpha} \text{ for all } j \ge 2 \Bigg) \\ & \le n^d \mathbb{P}\left(C_\alpha^{1/\alpha} \left| \sum_{j=K+1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \right| > \frac{b_n \tau^{1/\beta}(\delta - \epsilon(K-1))}{\|f\|_\alpha} \right) \\ & \le n^d \mathbb{P}\left(C_\alpha^{1/\alpha} \left| \sum_{j=K+1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \right| > \frac{b_n \tau^{1/\beta} \epsilon}{\|f\|_\alpha} \right) \\ & \le n^d b_n^{-p} \frac{\|f\|_\alpha^p \mathbb{E}\left| C_\alpha^{1/\alpha} \sum_{j=K+1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \right|^p}{\tau^{p/\beta} \epsilon^p} \\ & \le A \|f\|_\alpha^p \frac{\mathbb{E}\left| C_\alpha^{1/\alpha} \sum_{j=K+1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \right|^p}{\tau^{p/\beta} \epsilon^p} \quad \text{using } \textbf{(UB)}. \end{aligned} \tag{6.10}$$

For any p such that $\alpha < p < \alpha(K+1)$,

$$\mathbb{E}\left| C_\alpha^{1/\alpha} \sum_{j=K+1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \right|^p < \infty$$

(see p.1451 of [35]). Therefore (6.10) gives an integrable upper bound for $\psi_n(\epsilon, \delta, \tau^{1/\beta})$ on $(1, \infty)$. By a similar argument using the dominated convergence theorem, we have

$$T_3^{(n)}(\epsilon) \to 0 \text{ as } n \to \infty.$$

Using (6.7), we complete the proof by noting

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{E}\big[b_n^{-\beta}M_n^{\beta}\big] &\le \int_0^\infty \mathbb{P}\big(\Gamma_1^{-1/\alpha} > C_\alpha^{-1/\alpha}\tau^{1/\beta}(1-\delta)\big)d\tau \\ &= \int_0^\infty \big(1-\exp\big(-C_\alpha\tau^{-\alpha/\beta}(1-\delta)^{-\alpha}\big)\big)\,d\tau.\end{aligned}$$

By letting $\delta \to 0^+$, and applying the dominated convergence theorem again and using (6.2), we have

$$\limsup_{n\to\infty}\mathbb{E}\big[n^{-d\beta/\alpha}M_n^{\beta}\big] \le \tilde{c}_{\mathbf{Y}}^{\beta}C_\alpha^{\beta/\alpha}\mathbb{E}\big[Z_{\alpha/\beta}\big].$$

The argument for establishing a corresponding lower bound is similar. We start with the following lower bound for the tail distribution of $b_n^{-1}M_n$ from [35],

$$\mathbb{P}\big(b_n^{-1}M_n > \lambda\big) \ge \mathbb{P}\big(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \lambda(1+\delta)\big) - \phi_n(\epsilon,\lambda) - \widetilde{\psi}_n(\epsilon,\delta,\lambda), \tag{6.11}$$

where $\phi_n(\epsilon,\lambda)$ is the same as in (6.4) and $\widetilde{\psi}_n(\epsilon,\delta,\lambda)$ is defined by

$$\begin{aligned}\widetilde{\psi}_n(\epsilon,\delta,\lambda) &= \mathbb{P}\Bigg(\max_{k\in[\mathbf{0},(n-1)\mathbf{1}]}\Bigg|\sum_{j=1}^{\infty}\frac{\xi_j\Gamma_j^{-1/\alpha}|f_k(U_j^{(n)})|}{\max_{m\in[\mathbf{0},(n-1)\mathbf{1}]}|f_m(U_j^{(n)})|}\Bigg| > \frac{\lambda}{C_\alpha^{1/\alpha}\|f\|_\alpha}, \\ &\qquad \Gamma_1^{-1/\alpha} \le \frac{b_n\lambda(1+\delta)}{C_\alpha^{1/\alpha}\|f\|_\alpha},\ \text{and}\ \Gamma_2^{-1/\alpha} \le \frac{b_n\lambda\epsilon}{C_\alpha^{1/\alpha}\|f\|_\alpha}\Bigg) \\ &\le n^d\,\mathbb{P}\Bigg(\Bigg|\sum_{j=1}^{\infty}\xi_j\Gamma_j^{-1/\alpha}\Bigg| > \frac{b_n\lambda}{C_\alpha^{1/\alpha}\|f\|_\alpha},\ \Gamma_1^{-1/\alpha} \le \frac{b_n\lambda(1+\delta)}{C_\alpha^{1/\alpha}\|f\|_\alpha}, \\ &\qquad \text{and}\ \Gamma_2^{-1/\alpha} \le \frac{b_n\lambda\epsilon}{C_\alpha^{1/\alpha}\|f\|_\alpha}\Bigg).\end{aligned}$$

By a similar argument leading to (6.7), we obtain

$$\begin{aligned}\mathbb{E}\big[b_n^{-\beta}M_n^{\beta}\big] &\ge \int_0^\infty \mathbb{P}\big(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \tau^{1/\beta}(1+\delta)\big)d\tau \\ &\quad - \int_0^\infty \phi_n(\epsilon,\tau^{1/\beta})d\tau - \int_0^\infty \widetilde{\psi}_n(\epsilon,\delta,\tau^{1/\beta})d\tau \\ &:= \widetilde{T}_1(\delta) - T_2^{(n)}(\epsilon) - \widetilde{T}_3^{(n)}(\epsilon,\delta).\end{aligned} \tag{6.12}$$

By applying the dominated convergence theorem with the integrable bounds derived in (6.8) and (6.10), we derive

$$\liminf_{n\to\infty}\mathbb{E}\big[n^{-d\beta/\alpha}M_n^{\beta}\big] \ge \tilde{c}_{\mathbf{Y}}^{\beta}C_\alpha^{\beta/\alpha}\mathbb{E}\left[Z_{\alpha/\beta}\right].$$

Combining the above inequalities, we prove (3.1), that is

$$n^{-d\beta/\alpha}\mathbb{E}\big[M_n^{\beta}\big] \to C \text{ as } n \to \infty.$$

In the case of a conservative action, let $\mathbf{W}$ be a stationary $S\alpha S$ random field independent of $\mathbf{Y}$, having a similar integral representation with $S\alpha S$ measure M' on space S' with control measure μ', independent of M in the integral representation of $\mathbf{Y}$. That is,

$$W_t = \int_{S'} c_t'(s)\left(\frac{d\mu'\circ\phi_t'}{d\mu'}(s)\right)^{1/\alpha} g\circ\phi_t'(s)M'(ds),\quad t\in\mathbb{Z}^d.$$

Denoting the above integrand by $g_t(s)$, further let $\mathbf{W}$ be such that the sequence

$$b_n^W = \left(\int_{S'} \max_{\mathbf{0} \le t \le (n-1)\mathbf{1}} |g_t(s)|^\alpha \mu'(ds) \right)^{1/\alpha}, \quad n \ge 1,$$

satisfies **(LB)** for some $\theta > 0$.

Define $\mathbf{Z} = \mathbf{W} + \mathbf{Y}$. Then $\mathbf{Z}$ inherits its natural integral representation on $S \cup S'$ and the naturally defined action on that space is a stationary $S\alpha S$ random field generated by a conservative $\mathbb{Z}^d$-action. The deterministic maximal sequence b_n^Z corresponding to conservative $\mathbf{Z}$ satisfies **(LB)** as

$$b_n^Z \ge b_n^W \text{ for all } n.$$

Using symmetry, we have

$$\mathbb{P}\left(M_n^Z > x\right) \ge \frac{1}{2} \mathbb{P}\left(M_n > x\right) \tag{6.13}$$

and

$$\begin{aligned} \mathbb{E}\big[n^{-d\beta/\alpha} M_n^\beta\big] &= \int_0^\infty \mathbb{P}\left(n^{-d/\alpha} M_n > \tau^{1/\beta}\right) d\tau \\ &\le 2 \int_0^1 \mathbb{P}\left((b_n^Z)^{-1} M_n^Z > C\tau^{1/\beta}\right) d\tau \\ &\qquad + 2 \int_1^\infty \mathbb{P}\left((b_n^Z)^{-1} M_n^Z > C\tau^{1/\beta}\right) d\tau \\ &= S_n^{(1)} + S_n^{(2)} \end{aligned}$$

with the second step following from (6.13) and that $n^{-d/\alpha} b_n^Z$ converges to 0 and hence is bounded by a constant $1/C$ say. We use the fact from [35] that

$$n^{-d/\alpha} M_n \Rightarrow 0 \text{ as } n \to \infty,$$

and conclude (3.2) via a dominated convergence argument by using the trivial bound

$$\mathbb{P}\left((b_n^Z)^{-1} M_n^Z > C\tau^{1/\beta}\right) \le 1$$

for $\tau \in (0, 1)$ and obtaining a non-trivial integrable bound for the same on $(1, \infty)$. Again with a similar choice of ϵ as in the dissipative case, we have

$$\begin{aligned} \mathbb{P}\big(M_n^Z > C b_n^Z \tau^{1/\beta}\big) \le\ & \mathbb{P}\big(\Gamma_1^{-1/\alpha} > C\tau^{1/\beta}\epsilon\big) \\ & + \mathbb{P}\big(M_n^Z > C b_n^Z \tau^{1/\beta},\ \Gamma_1^{-1/\alpha} \le C\tau^{1/\beta}\epsilon\big), \end{aligned}$$

where M_n^Z is the maxima, and b_n^Z is the corresponding deterministic maximal sequence for $\mathbf{Z}$. Let $\mathbf{Z}$ have a series representation in terms of arrival times of a unit Poisson process, Γ_j and Rademacher variables ξ_j. Now choose K large enough so that $\alpha(K+1) > d/\theta$. For p satisfying

$$\frac{d}{\theta} < p < \alpha(K+1),$$

using a technique similar to (6.10) by an application of Markov's inequality, we derive an integrable upper bound for $\tau \in (1, \infty)$ as

$$
\begin{aligned}
&\mathbb{P}\Big(M_n^Z > Cb_n^Z \tau^{1/\beta},\ \ \Gamma_1^{-1/\alpha} \le C\tau^{1/\beta}\epsilon\Big)\\
&\le n^d \mathbb{P}\bigg(\Big|C_\alpha^{1/\alpha}\sum_{j=1}^{\infty}\xi_j \Gamma_j^{-1/\alpha}\Big| > \frac{Cb_n^Z\tau^{1/\beta}}{\|f^Z\|_\alpha},\ C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} \le \frac{Cb_n^Z\tau^{1/\beta}\epsilon}{\|f^Z\|_\alpha}\bigg)\\
&\le n^d \mathbb{P}\bigg(\Big|C_\alpha^{1/\alpha}\sum_{j=1}^{\infty}\xi_j \Gamma_j^{-1/\alpha}\Big| > \frac{Cb_n^Z\tau^{1/\beta}}{\|f^Z\|_\alpha},\\
&\qquad\qquad C_\alpha^{1/\alpha}\Gamma_j^{-1/\alpha} \le \frac{Cb_n^Z\tau^{1/\beta}\epsilon}{\|f^Z\|_\alpha} \text{ for all } j \in \mathbb{N}\bigg) \qquad (6.14)\\
&\le n^d \mathbb{P}\bigg(C_\alpha^{1/\alpha}\Big|\sum_{j=K+1}^{\infty}\xi_j \Gamma_j^{-1/\alpha}\Big| > C\|f^Z\|_\alpha^{-1} b_n^Z \tau^{1/\beta}(1-K\epsilon)\bigg)\\
&\le n^d (b_n^Z)^{-p} C^p \frac{\|f^Z\|_\alpha^p \mathbb{E}\Big|C_\alpha^{1/\alpha}\sum_{j=K+1}^{\infty}\epsilon_j \Gamma_j^{-1/\alpha}\Big|^p}{\tau^{p/\beta}\epsilon^p}\\
&\le AC^p \|f^Z\|_\alpha^p \frac{\mathbb{E}\Big|C_\alpha^{1/\alpha}\sum_{j=K+1}^{\infty}\epsilon_j \Gamma_j^{-1/\alpha}\Big|^p}{\tau^{p/\beta}\epsilon^p}.
\end{aligned}
$$

Observing that

$$\int_0^\infty \mathbb{P}\big(\Gamma_1^{-1/\alpha} > \epsilon\tau^{1/\beta}\big)d\tau = \epsilon^{-\beta}\mathbb{E}\big[Z_{\alpha/\beta}\big] = \epsilon^{-\beta}\Gamma(1-\beta/\alpha) < \infty,$$

and using integrable bound for

$$\mathbb{P}\big(M_n^Z > \tau^{1/\beta} b_n^Z,\ \ \Gamma_1^{-1/\alpha} \le \epsilon\tau^{1/\beta}\big)$$

as derived in (6.14), we obtain a nontrivial bound for $S_n^{(2)}$. Applying the dominated convergence theorem with the trivial bound 1 for $S_n^{(1)}$ and an integrable bound for $S_n^{(2)}$, we conclude (3.2). This completes the proof.

6.2. Proof of Theorem 3.4

The proof again follows by noting that

$$
\begin{aligned}
\mathbb{E}\big[b_n^{-\beta} M_n^\beta\big] &= \int_0^\infty \mathbb{P}\big(b_n^{-1} M_n > \tau^{1/\beta}\big)d\tau\\
&\le \int_0^\infty \Big\{ \mathbb{P}\big(\Gamma_1^{-1/\alpha} > \tau^{1/\beta}\epsilon\big)\\
&\qquad + \mathbb{P}\big(M_n > \tau^{1/\beta} b_n,\ \ \Gamma_1^{-1/\alpha} \le \tau^{1/\beta}\epsilon\big)\Big\}\, d\tau\\
&= \epsilon^{-\beta}\Gamma(1-\beta/\alpha) + \int_0^1 \mathbb{P}\big(M_n > \tau^{1/\beta} b_n,\ \ \Gamma_1^{-1/\alpha} \le \tau^{1/\beta}\epsilon\big)d\tau\\
&\qquad + \int_1^\infty \mathbb{P}\big(M_n > \tau^{1/\beta} b_n,\ \ \Gamma_1^{-1/\alpha} \le \tau^{1/\beta}\epsilon\big)d\tau.
\end{aligned}
$$

The integral over [0, 1] is bounded by 1. To bound the integral over $(1,\infty)$, we choose K large enough so that $\alpha(K+1) > \frac{\alpha d}{\theta_1}$. Fix ϵ satisfying $0<\epsilon<\frac{1}{K}$ and p satisfying

$$\frac{\alpha d}{\theta_1} < p < \alpha(K+1).$$

The same argument as in (6.14), together with the lower bound in (3.5), gives

$$\mathbb{P}\big(M_n > \tau^{1/\beta} b_n,\ \ \Gamma_1^{-1/\alpha} \le \tau^{1/\beta}\epsilon\big) \le \frac{B}{\tau^{p/\beta}\epsilon^p},$$

where

$$B = A\|f\|_\alpha^p \mathbb{E}\left|C_\alpha^{1/\alpha}\sum_{j=K+1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\right|^p.$$

It follows from above that

$$\begin{aligned}\mathbb{E}\big[b_n^{-\beta}M_n^\beta\big] &\le \epsilon^{-\beta}\Gamma(1-\beta/\alpha) + 1 + \int_1^\infty \frac{B}{\tau^{p/\beta}\epsilon^p}d\tau \\ &= K_1 < \infty.\end{aligned}$$

Hence

$$\mathbb{E}\big[M_n^\beta\big] \le K_1\cdot b_n^\beta \le K_1c_2\cdot n^{\beta\theta_2/\alpha}$$

for all sufficiently large n, say $n \ge n_0$. Taking $K' = \max\{c_2K_1; \mathbb{E}\big[M_k^\beta\big], k\le n_0\}$ yields (3.6).

6.3. *Proof of Theorem 3.6*

(1) When the action $\{\phi_t\}_{t\in F}$ restricted to the free group F is dissipative, then by Proposition 5.1 of [34], the sequence $\{b_n\}_{n\ge 0}$ satisfies

$$\lim_{n\to\infty} n^{-p/\alpha}b_n = c,\ \text{ a constant},$$

which implies that b_n satisfies **(LB)** with $\theta = p/\alpha$. Also, (4.17) of [34] holds; see the proof of Theorem 5.4 in [34].

Now we choose K such that $\alpha(K+1) > d\alpha/p$, use the same tail bound as in (6.3) and apply the dominated convergence theorem using integrable bounds on

$$\phi_n(\epsilon,\tau^{1/\beta}) \le n^d b_n^{-\alpha}\frac{C_\alpha^{-1}\epsilon^{-\alpha}}{\tau^{\alpha/\beta}} \le K_2C_\alpha^{-1}\epsilon^{-\alpha}\tau^{-\alpha/\beta},$$

$$\begin{aligned}\psi_n(\epsilon,\delta,\tau) &\le n^d b_n^{-p'}\frac{\|f\|_\alpha^{p'}\mathbb{E}\left|C_\alpha^{1/\alpha}\sum_{j=K+1}^\infty\epsilon_j\Gamma_j^{-1/\alpha}\right|^{p'}}{\tau^{p'/\beta}\epsilon^{p'}} \\ &\le K_3\epsilon^{-p'}\tau^{-p'/\beta},\end{aligned}$$

for p' satisfying

$$\frac{d\alpha}{p} \le p' \le \alpha(K+1).$$

Then as in the proof of (3.1), (3.7) follows.

(2) When the action $\{\phi_t\}_{t\in F}$ is conservative, we can obtain a stationary $S\alpha S$ random field $\mathbf{Z}$ generated by a conservative $\mathbb{Z}^d$-action such that b_n^Z satisfies **(LB)** for some $\theta>0$ and

$$n^{-p/\alpha}b_n^Z \to 0 \text{ as } n\to\infty.$$

Again by the exact argument used to prove (3.2), we obtain (3.8).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We are extremely grateful to the anonymous reviewers for their detailed comments, which significantly improved the paper. In particular, we would like to thank one reviewer for encouraging us to work on Section 5.4. The research of P. Roy was partially supported by the project RARE-318984 (a Marie Curie FP7 IRSES Fellowship), a SERB, India grant MTR/2017/000513 and a SwarnaJayanti Fellowship from DST, India. The research of Y. Xiao was partially supported by NSF (the National Science Foundation) grants DMS-1607089 and DMS-1855185.

Appendix. Maximal moments for continuous parameter case

Here we present the theorem on the rate of growth of moments of maximum of $S\alpha S$ process indexed by continuous time in $\mathbb{R}$, which can be easily extended to the class of fields indexed by $\mathbb{R}^d$ (see Remark A.2).

Theorem A.1. *Let $\mathbf{Y} = \{Y(t)\}_{t\in\mathbb{R}}$ be a stationary measurable $S\alpha S$ process with $0 < \alpha < 2$ and having integral representation as*

$$Y(t) \overset{d}{=} \int_S f_t(s)M(ds) = \int_S c_t(s)\left(\frac{d\mu\circ\phi_t}{d\mu}(s)\right)^{1/\alpha} f\circ\phi_t(s)M(ds), \quad t\in\mathbb{R},$$

where $f \in L^\alpha(S,\mu)$, $\{\phi_t\}_{t\in\mathbb{R}}$ is a nonsingular flow, $\{c_t\}_{t\in\mathbb{R}}$ is a ± 1-valued cocycle with respect to $\{\phi_t\}_{t\in\mathbb{R}}$ and M is an $S\alpha S$ measure with control measure μ; see [28].

1. *If $\mathbf{Y}$ is generated by a dissipative flow, then for $0 < \beta < \alpha$,*

$$\mathbb{E}\big[T^{-\beta/\alpha}M_T^\beta\big] \to C \text{ as } T\to\infty, \tag{A.1}$$

   where C is a positive constant with an expression analogous to (3.3).

2. *If $\mathbf{Y}$ is generated by a conservative flow, then*

$$\mathbb{E}\big[T^{-\beta/\alpha}M_T^\beta\big] \to 0 \quad \text{as } T\to\infty. \tag{A.2}$$

Proof. Stationarity and measurability together imply continuity in probability for stable processes (see Proposition 3.1 of [33]). Therefore following [36], we shall approximate the stable process (and all of its functionals) by its dyadic skeletons even without writing it explicitly at times. This will ensure, in particular, that every quantity considered in this proof is measurable.

As in the proof of Theorem 3.1, we consider cases (1) and (2) separately. When $\mathbf{Y}$ is generated by a dissipative flow,

$$\{b_T\}_{T\geq 0} = \left\{\left(\int_S \sup_{0\leq t\leq T} |f_t(s)|^\alpha \mu(ds)\right)^{1/\alpha}\right\}_{T\geq 0}$$

satisfies $\lim_{T\to\infty} T^{-1/\alpha}b_T = c$, a constant. The above implies that b_T satisfies conditions (2.9) with $\theta = 1/\alpha$ and (2.12) in [36], analogous to **(LB)** and **(LL)** in Theorem 3.1 for fields indexed by $\mathbb{Z}^d$. For a choices of $\epsilon > 0$ and $0 < \delta < 1$ such that ϵ is chosen small enough as compared to δ and for $K = 0, 1, 2, \ldots$ satisfying

$$K < \frac{1}{\epsilon C_\alpha^{1/\alpha}},$$

we bound the tail distribution of $b_T^{-1} M_T$ as

$$\mathbb{P}\big(b_T^{-1} M_T > \lambda\big) \le \mathbb{P}\big(C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} > \lambda(1-\delta)\big) + \phi_T(\epsilon, \lambda) + \psi_T(\epsilon, \delta, \lambda), \tag{A.3}$$

taken from [36]. The quantities ϕ_T and ψ_T in (A.3) are defined and bounded as follows:

$$\begin{aligned} \phi_T(\epsilon, \lambda) &= \mathbb{P}\left(\text{for some } t \in [0, T],\ \frac{\Gamma_j^{-1/\alpha} |f_t(U_j^{(T)})|}{\sup_{s\in[0,T]} |f_s(U_j^{(T)})|} > \epsilon\lambda \right.\\ &\qquad\qquad \left. \text{for at least 2 different } j\text{'s} \right) \\ &\le \lfloor T \rfloor \mathbb{P}\left(\Gamma_j^{-1/\alpha} \sup_{0\le t\le 1} \frac{|f_t(U_j^{(T)})|}{\sup_{s\in[0,T]} |f_s(U_j^{(T)})|} > \epsilon\lambda \right.\\ &\qquad\qquad \left. \text{for at least 2 different } j\text{'s} \right), \end{aligned} \tag{A.4}$$

where $\lfloor T \rfloor$ denotes the smallest integer $\ge T$ and the inequality follows from the same argument as in (2.26) of [36]. Furthermore, the random points

$$b_T \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(T)})|}{\sup_{s\in[0,T]} |f_s(U_j^{(T)})|}, \quad j = 1, 2, \ldots$$

have the same distribution as

$$Z_j(t) = b_1 \Gamma_j^{-1/\alpha} \frac{|f_t(V_j)|}{\sup_{s\in[0,1]} |f_s(V_j)|}, \quad j = 1, 2, \ldots,$$

where $\{V_j\}$ is identically distributed as $\{U_j^{(1)}\}$ and independent of $\{\Gamma_j\}$. This and (A.4) imply that

$$\begin{aligned} \phi_T(\epsilon, \lambda) &\le \lfloor T \rfloor \mathbb{P}\left(b_1 \Gamma_j^{-1/\alpha} \sup_{0\le t\le 1} \frac{|f_t(V_j)|}{\sup_{s\in[0,1]} |f_s(V_j)|} > b_T \epsilon\lambda \right.\\ &\qquad\qquad \left. \text{for at least 2 different } j\text{'s} \right) \\ &= \lfloor T \rfloor \mathbb{P}\left(\sum_{j=1}^{\infty} 1_{\{\sup_{t\in[0,1]} |Z_j(t)|\}}(b_T\epsilon\lambda, \infty) \ge 2 \right). \end{aligned}$$

For set of interest

$$B(T) = \left\{ (z(t); t \in [0, 1]) : \sup_{t\in[0,1]} |z(t)| > b_T \epsilon\lambda \right\},$$

$\{Z_j(t), j \ge 1\}$ are points of a Poisson random measure with mean measure

$$\Lambda(B(T)) = \left(\frac{b_T \epsilon\lambda}{b_1} \right)^{-\alpha}.$$

Using the fact that $[T]b_T^{-\alpha} \leq K_4$ a constant, we have

$$\phi_T(\epsilon, \lambda) \leq K_4 b_1^{\alpha} \epsilon^{-\alpha} \lambda^{-\alpha}. \tag{A.5}$$

Similarly as above, we have for $0 < \epsilon < \delta/K$,

$$\begin{aligned}\psi_T(\epsilon, \delta, \lambda) &= \mathbb{P}\left(b_T \sup_{t\in[0,T]} \left| \sum_{j=1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \cdot \frac{|f_t(U_j^{(t)})|}{\sup_{s\in[0,T]} |f_s(U_j^{(s)})|} \right| > C_\alpha^{-1/\alpha} b_T \lambda; \right. \\ &\qquad \left. b_1 \Gamma_1^{-1/\alpha} \leq b_T \lambda(1-\delta) \text{ and } b_1 \Gamma_2^{-1/\alpha} \leq b_T \lambda \epsilon \right) \\ &\leq \lfloor T \rfloor \mathbb{P}\left(b_1 \sup_{t\in[0,1]} \left| \sum_{j=1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \cdot \frac{|f_t(V_j)|}{\sup_{s\in[0,1]} |f_s(V_j)|} \right| > C_\alpha^{-1/\alpha} b_T \lambda; \right. \\ &\qquad \left. b_1 \Gamma_1^{-1/\alpha} \leq b_T \lambda(1-\delta) \text{ and } b_1 \Gamma_2^{-1/\alpha} \leq b_T \lambda \epsilon \right).\end{aligned} \tag{A.6}$$

Using the same argument as in (2.29)–(2.33) of [36], leveraging on the observation that

$$b_T \Gamma_j^{-1/\alpha} \frac{|f_t(U_j^{(T)})|}{\sup_{s\in[0,T]} |f_s(U_j^{(T)})|}$$

are identically distributed as $Z_j(t)$ and applying an exponential Markov inequality in the penultimate step, we derive

$$\begin{aligned}\psi_T(\epsilon, \delta, \lambda) &\leq \lfloor T \rfloor \mathbb{P}\left(\sup_{t\in[0,1]} \left| \sum_{j=K+1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \cdot \frac{|f_t(V_j)|}{\sup_{s\in[0,1]} |f_s(V_j)|} \right| \right. \\ &\qquad \left. > b_T \big(1 - \epsilon C_\alpha^{1/\alpha}\big) b_1^{-1} C_\alpha^{-1/\alpha} \lambda \right) \\ &\leq 4\lfloor T \rfloor \int_0^{\infty} \exp(-x) \frac{x^K}{K!} \exp\left\{ - \frac{\big(1 - \epsilon C_\alpha^{1/\alpha}\big) \lambda \log 2}{(\gamma + 2x^{-1/\alpha} b_T) b_1 C_\alpha^{1/\alpha}} \right\} dx \\ &\leq 4\lfloor T \rfloor \left(C_1 \exp(-\zeta(\lambda) T^{\theta}) + \int_0^1 \frac{x^K}{K!} \exp\big(-x - C_2 \lambda x^{1/\alpha} T^{\theta}\big) dx \right),\end{aligned}$$

where $\zeta(\lambda)$ is an increasing function of λ.

For any $0 < \beta < \alpha$, using the tail bound in (A.3) we have

$$\begin{aligned}\mathbb{E}\big[b_T^{-\beta} M_T^{\beta}\big] &= \int_0^{\infty} \mathbb{P}\big(b_T^{-1} M_T > \tau^{1/\beta}\big) d\tau \\ &\leq \int_0^{\infty} \mathbb{P}\big(C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} > \tau^{1/\beta}(1-\delta)\big) d\tau \\ &\qquad + \int_0^{\infty} \phi_T(\epsilon, \tau^{1/\beta}) d\tau + \int_0^{\infty} \psi_T(\epsilon, \delta, \tau^{1/\beta}) d\tau \\ &= T_1(\delta) + T_2^{(T)}(\epsilon) + T_3^{(T)}(\epsilon, \delta).\end{aligned}$$

Again from [36], we know that, as $T \to \infty$, $\phi_T(\epsilon, \tau)$ and $\psi_T(\epsilon, \delta, \tau^{1/\beta})$ converge to 0 pointwise for all $\tau \in [0, \infty)$. Hence using the integrable bounds derived in (A.4) and (A.6) on $(1, \infty)$ and the trivial bound 1 on $(0, 1)$, we apply the dominated convergence theorem to conclude that

$$T_2^{(T)}(\epsilon),\ T_3^{(T)}(\epsilon, \delta) \to 0 \text{ as } T \to \infty,$$

which gives

$$\limsup_{T\to\infty} \mathbb{E}\big[b_T^{-\beta} M_T^{\beta}\big] \le \int_0^\infty \mathbb{P}(\Gamma_1^{-1/\alpha} > C_\alpha^{-1/\alpha}\tau^{1/\beta}(1-\delta))d\tau$$
$$= \int_0^\infty \big(1-\exp\big(-C_\alpha \tau^{-\alpha/\beta}(1-\delta)^{-\alpha}\big)\big)\, d\tau.$$

By letting $\delta \to 0^+$, and applying the dominated convergence theorem again gives

$$\limsup_{T\to\infty} \mathbb{E}\big[T^{-\beta/\alpha} M_T^{\beta}\big] \le c^\beta C_\alpha^{\beta/\alpha} \mathbb{E}\left[Z_{\alpha/\beta}\right].$$

On the other hand, we can use a similar lower tail bound

$$\mathbb{E}\big[b_T^{-\beta} M_T^{\beta}\big] \ge T_1(\delta) - T_2^{(T)}(\epsilon) - T_3^{(T)}(\epsilon,\delta).$$

and applying the dominated convergence theorem with the integrable bounds derived in (A.4) and (A.6), we have

$$\liminf_{T\to\infty} \mathbb{E}\big[T^{-\beta/\alpha} M_T^{\beta}\big] \ge c^\beta C_\alpha^{\beta/\alpha} \mathbb{E}\left[Z_{\alpha/\beta}\right].$$

This concludes the proof of (A.1).

(2) Consider a stationary $S\alpha S$ random field $\mathbf{W}$ independent of $\mathbf{Y}$, also given by the integral representation of the form

$$\mathbf{W} = \int_{S'} g_t(s) M'(ds), \quad t \in \mathbb{R},$$

where M' is a $S\alpha S$ random measure with control measure μ', independent of M in the integral representation of $\mathbf{Y}$ and generated by a conservative flow and also satisfying

$$b_T^W \ge cT^\theta \text{ for sufficiently large } T \tag{A.7}$$

for some $\theta > 0$. Define $\mathbf{Z} = \mathbf{Y} + \mathbf{W}$, a stationary $S\alpha S$ random process generated by a conservative $\mathbb{R}$-action with the natural integral representation on $S \cup S'$ corresponding to the naturally defined action on that space. Let b_T^Z be the corresponding deterministic maximal quantity defined for the process Z. As $b_T^Z \ge b_T^Y$ for all $T > 0$, the conservative process Z satisfies (A.7).

$$\mathbb{E}\big[T^{-\beta/\alpha} M_T^{\beta}\big] = \int_0^\infty \mathbb{P}\big(T^{-1/\alpha} M_T > \tau^{1/\beta}\big)d\tau$$
$$\le 2\int_0^1 \mathbb{P}\big((b_T^Z)^{-1} M_T^Z > C\tau^{1/\beta}\big)d\tau$$
$$+ 2\int_1^\infty \mathbb{P}\big((b_T^Z)^{-1} M_T^Z > C\tau^{1/\beta}\big)d\tau$$
$$= S_T^{(1)} + S_T^{(2)}$$

with the second step following from symmetry and the fact $T^{-1/\alpha} b_T^Z$ is bounded by C^{-1}, a constant. Using the bounding technique in (A.6), we have a similar integrable bound for

$$\mathbb{P}\big(M_T^Z > \tau^{1/\beta} b_T^Z,\ \ \Gamma_1^{-1/\alpha} \le \tau^{1/\beta}\epsilon\big).$$

This leads to (A.2) by a similar dominated convergence argument using the fact that

$$\mathbb{P}\big(T^{-1/\alpha} M_T > \tau^{1/\beta}\big) \to 0$$

as $T \to \infty$; see Theorem 2.2 of [36]. □

Remark A.2. The results presented in this section can easily be extended to stationary measurable symmetric α-stable random fields indexed by $\mathbb{R}^d$. For simplicity of presentation, we only dealt with the $d = 1$ case here. This extension to higher dimension can be done using the techniques of [33] and [8]. More specifically, the idea is to approximate the continuous parameter random field $\{X_t\}_{t\in\mathbb{R}^d}$ by its discrete parameter skeletons $\{X_t\}_{t\in 2^{-i}\mathbb{Z}^d}$, $i = 0, 1, 2, \ldots$.

In [8], the notion of effective dimension was extended to the continuous parameter case based on the following observation: the effective dimensions of $\{X_t\}_{t\in 2^{-i}\mathbb{Z}^d}$, $i = 0, 1, 2, \ldots$ are equal and hence can be defined as the *group theoretic dimension* of $\{X_t\}_{t\in\mathbb{R}^d}$. With this definition, Theorem A.1 can be extended to the higher-dimensional case connecting the rate of growth of maximal moments to the group theoretic dimension p. We can also define a continuous parameter analogue of weak effective dimension and relate it to the asymptotic properties of the maximal moments. In summary, all the results presented in Section 3 can be rewritten for stationary measurable $S\alpha S$ random fields indexed by $\mathbb{R}^d$.

References

[1] J. Aaronson, An Introduction to Infinite Ergodic Theory, in: Mathematical Surveys and Monographs, vol. 50, American Mathematical Society, Providence, 1997.
[2] R. Adler, J. Taylor, Random Fields and Geometry, Springer, New York, 2007.
[3] J. Athreya, M. Mj, P. Roy, Stable Random Fields, Patterson-Sullivan Measures and Extremal Cocycle Growth, 2019, arXiv:1809.08295.
[4] A. Ayache, F. Roueff, Y. Xiao, Linear fractional stable sheets: wavelet expansion and sample path properties, Stochastic Process. Appl. 119 (2009) 1168–1197.
[5] A. Bhattacharya, P. Roy, A large sample test for the length of memory of stationary symmetric stable random fields via nonsingular $\mathbb{Z}^d$-actions, J. Appl. Probab. 55 (2018) 179–195.
[6] H. Biermé, C. Lacaux, Hölder regularity for operator scaling stable random fields, Stochastic Process. Appl. 119 (2009) 2222–2248.
[7] H. Biermé, C. Lacaux, Modulus of continuity of some conditionally sub-Gaussian fields, application to stable random fields, Bernoulli 21 (2015) 1719–1759.
[8] A. Chakrabarty, P. Roy, Group-theoretic dimension of stationary symmetric α-stable random fields, J. Theoret. Probab. 26 (2013) 1–19.
[9] S. Cohen, G. Samorodnitsky, Random rewards, fractional Brownian local times and stable self-similar processes, Ann. Appl. Probab. 16 (2006) 1432–1461.
[10] C. Dombry, N. Guillotin-Plantard, Discrete approximation of a stable self-similar stationary increments process, Bernoulli 15 (2009) 195–222.
[11] V. Fasen, P. Roy, Stable random fields, point processes and large deviations, Stochastic Process. Appl. 126 (2016) 832–856.
[12] J. Hawkes, A lower Lipschitz condition for the stable subordinator, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 17 (1971) 23–32.
[13] P. Jung, T. Owada, G. Samorodnitsky, Functional central limit theorem for a class of negatively dependent heavy-tailed stationary infinitely divisible processes generated by conservative flows, Ann. Probab. 45 (2017) 2087–2130.
[14] D. Khoshnevisan, Multiparameter Processes: An Introduction to Random Fields, Springer Science & Business Media, 2002.
[15] N. Kôno, M. Maejima, Hölder continuity of sample paths of some self-similar stable processes, Tokyo J. Math. 14 (1991) 93–100.
[16] U. Krengel, Ergodic Theorems, De Gruyter, Berlin, New York, 1985.
[17] A.E. Kyprianou, Introductory Lectures on Fluctuations of Lévy Processes with Applications, Springer Science & Business Media, 2006.
[18] S. Lang, Algebra, revised 3rd ed., in: Graduate Texts in Mathematics, Springer-Verlag, New York, 2002.
[19] M. Maejima, A self-similar process with nowhere bounded sample paths, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 65 (1983) 115–119.

[20] M. Marcus, Extreme values for sequences of stable random variables, in: Statistical Extremes and Applications (Vimeiro, 1983), in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 131, Reidel, Dordrecht, 1984, pp. 311–324.
[21] M. Marcus, J. Rosen, Markov Processes, Gaussian Processes, and Local Times, Vol. 100, Cambridge University Press, 2006.
[22] M.M. Meerschaert, H.-P. Scheffler, Limit theorems for continuous-time random walks with infinite mean waiting times, J. Appl. Probab. (2004) 623–638.
[23] T. Mikosch, G. Samorodnitsky, Ruin probability with claims modeled by a stationary ergodic stable process, Ann. Probab. 28 (2000) 1814–1851.
[24] T. Owada, G. Samorodnitsky, Functional central limit theorem for heavy tailed stationary infinitely divisible processes generated by conservative flows, Ann. Probab. 43 (2015) 240–285.
[25] V. Pipiras, M.S. Taqqu, Decomposition of self-similar stable mixing moving averages, Probab. Theory Related Fields 123 (2002) 412–452.
[26] V. Pipiras, M.S. Taqqu, Stable Non-Gaussian Self-Similar Processes with Stationary Increments, in: SpringerBriefs in Probability and Mathematical Statistics, Springer, Cham, 2017.
[27] S. Resnick, G. Samorodnitsky, Point processes associated with stationary stable processes, Stochastic Process. Appl. 114 (2004) 191–210.
[28] J. Rosiński, On the structure of stationary stable processes, Ann. Probab. 23 (1995) 1163–1187.
[29] J. Rosiński, Decomposition of stationary α-stable random fields, Ann. Probab. 28 (2000) 1797–1813.
[30] J. Rosiński, G. Samorodnitsky, Classes of mixing stable processes, Bernoulli 2 (1996) 365–377.
[31] E. Roy, Ergodic properties of Poissonian ID processes, Ann. Probab. 35 (2007) 551–576.
[32] P. Roy, Ergodic theory, abelian groups and point processes induced by stable random fields, Ann. Probab. 38 (2010a) 770–793.
[33] P. Roy, Nonsingular group actions and stationary $S\alpha S$ random fields, Proc. Amer. Math. Soc. 138 (2010b) 2195–2202.
[34] P. Roy, G. Samorodnitsky, Stationary symmetric α-stable discrete parameter random fields, J. Theoret. Probab. 21 (2008) 212–233.
[35] G. Samorodnitsky, Extreme value theory, ergodic theory, and the boundary between short memory and long memory for stationary stable processes, Ann. Probab. 32 (2004a) 1438–1468.
[36] G. Samorodnitsky, Maxima of continuous time stationary stable processes, Adv. Appl. Probab. 36 (2004b) 805–823.
[37] G. Samorodnitsky, Null flows, positive flows and the structure of stationary symmetric stable processes, Ann. Probab. 33 (2005) 1782–1803.
[38] G. Samorodnitsky, M. Taqqu, Stable Non-Gaussian Random Processes, Chapman and Hall, New York, 1994.
[39] S. Sarkar, P. Roy, Stable random fields indexed by finitely generated free groups, Ann. Probab. 46 (2018) 2680–2714.
[40] D. Surgailis, J. Rosiński, V. Mandrekar, S. Cambanis, Stable mixed moving averages, Probab. Theory Related Fields 97 (1993) 543–558.
[41] K. Takashima, Sample path properties of ergodic self-similar processes, Osaka J. Math. 26 (1989) 159–189.
[42] M. Talagrand, The Generic Chaining: Upper and Lower Bounds of Stochastic Processes, Springer Science & Business Media, 2006.
[43] V. Varadarajan, Geometry of Quantum Theory, Vol. 2, Van Nostrand Reinhold, New York, 1970.
[44] Y. Wang, P. Roy, S.A. Stoev, Ergodic properties of sum- and max-stable stationary random fields via null and positive group actions, Ann. Probab. 41 (2013) 206–228.
[45] Y. Xiao, Uniform modulus of continuity of random fields, Monatsh. Math. 159 (2010) 163–184.
[46] R. Zimmer, Ergodic Theory and Semisimple Groups, Birkhäuser, Boston, 1984.