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Existence and uniqueness of stationary Lévy-driven CARMA processes

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Abstract

Necessary and sufficient conditions for the existence of a strictly stationary solution of the equations defining a general Lévy-driven continuous-parameter ARMA process with index set $\mathbb{R}$ are determined. Under these conditions the solution is shown to be unique and an explicit expression is given for the process as an integral with respect to the background driving Lévy process. The results generalize results obtained earlier for second-order processes and for processes defined by the Ornstein–Uhlenbeck equation.

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1. Introduction

Let $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process, i.e. a process with homogeneous independent increments, continuous in probability, with càdlàg sample paths and $L_0 = 0$. For integers p and q such that $p > q$, we define a (complex valued) CARMA(p, q) process $Y = (Y_t)_{t\in\mathbb{R}}$, driven by L, by the equation

$$Y_t = \mathbf{b}'\mathbf{X}_t, \quad t \in \mathbb{R}, \tag{1.1}$$

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where $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$ is a $\mathbb{C}^p$-valued process satisfying the stochastic differential equation,

$$\mathrm{d}\mathbf{X}_t = A\mathbf{X}_t\,\mathrm{d}t + \mathbf{e}\,\mathrm{d}L_t, \tag{1.2}$$

or equivalently

$$\mathbf{X}_t = \mathrm{e}^{A(t-s)}\mathbf{X}_s + \int_s^t \mathrm{e}^{A(t-u)}\mathbf{e}\,\mathrm{d}L_u, \quad \forall s \le t \in \mathbb{R}, \tag{1.3}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \qquad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix},$$

where $a_1, \ldots, a_p, b_0, \ldots, b_{p-1}$ are complex-valued coefficients such that $b_q \neq 0$ and $b_j = 0$ for $j > q$. For $p = 1$ the matrix A is to be understood as $A = (-a_1)$.

Eqs. (1.1) and (1.2) constitute the state-space representation of the formal pth-order stochastic differential equation,

$$a(D)Y_t = b(D)DL_t, \tag{1.4}$$

where D denotes differentiation with respect to t and $a(\cdot)$ and $b(\cdot)$ are the polynomials,

$$a(z) = z^p + a_1 z^{p-1} + \cdots + a_p, \tag{1.5}$$

and

$$b(z) = b_0 + b_1 z + \cdots + b_{p-1} z^{p-1}. \tag{1.6}$$

Eq. (1.4) is the natural continuous-time analogue of the pth-order linear difference equations used to define a discrete-time ARMA process (see e.g. [1]). However, since the derivatives on the right-hand side of (1.4) do not exist as random functions, we base the definition on the state-space formulation (1.1) and (1.2). The aim of the present paper is to establish necessary and sufficient conditions for the existence of a strictly stationary solution of Eqs. (1.1) and (1.2) for $(Y_t)_{t\in\mathbb{R}}$.

Under the assumptions that $EL_1^2 < \infty$ and $\mathbf{X}_0$ is independent of $(L_t)_{t>0}$, it is well known (see [2,3]) that necessary and sufficient conditions for existence of a covariance stationary solution $(\mathbf{X}_t)_{t\geq 0}$ of (1.2) are that the zeroes of the polynomial a (which are also the eigenvalues of the matrix A) have strictly negative real parts and that $\mathbf{X}(0)$ has the same mean and covariance as $\int_0^\infty \mathrm{e}^{Au}\mathbf{e}\,\mathrm{d}L_u$. Under these conditions $(Y_t)_{t\geq 0}$ defined by (1.1) is a weakly stationary CARMA process, said to be *causal* since for each $t > 0$, Y_t is a measurable function of $\mathbf{X}_0$ and $(L_s)_{s\leq t}$. Under the weaker assumption that $E|L_1|^r < \infty$ for some $r > 0$, Brockwell [4] showed that if $\mathbf{X}_0$ has the same distribution as $\int_0^\infty \mathrm{e}^{Au}\mathbf{e}\,\mathrm{d}L_u$ and is independent of $(L_t)_{t>0}$ and if the real parts of the zeroes of a are strictly negative, then the solution of (1.2) is strictly stationary and the corresponding process $(Y_t)_{t\geq 0}$ is a causal strictly stationary CARMA process driven by L.

The aim of the present paper is to dispense with the assumptions of the previous paragraph and to derive necessary and sufficient conditions for Eqs. (1.1) and (1.2) to have a strictly stationary, not necessarily causal, solution $Y = (Y_t)_{t\in\mathbb{R}}$. Observe that *a priori* we do not require the state vector $(\mathbf{X}_t)_{t\in\mathbb{R}}$ to be strictly stationary, and we will indeed encounter cases when $a(\cdot)$ and $b(\cdot)$ have common zeroes on the imaginary axis and in which strictly stationary solutions Y exist without a corresponding strictly stationary state vector $\mathbf{X}$. We shall also establish uniqueness of the solution Y and give an explicit representation for the solution as an integral with respect to L. The results generalize those of Wolfe [5] and Sato and Yamazato [6], who derived a necessary and sufficient condition for the existence of a stationary solution of the Lévy-driven Ornstein–Uhlenbeck equation.

The paper is organised as follows: under the condition that $a(\cdot)$ and $b(\cdot)$ have no common zeroes we derive necessary conditions for a strictly stationary solution Y to exist in Section 2, and give a necessary and sufficient criterion in Section 3, where also uniqueness of this solution is established. The *a priori* assumption of no common zeroes of $a(\cdot)$ and $b(\cdot)$ is then eliminated in Section 4. The special case when L is deterministic is treated separately in Section 5, in which case the characterisation is slightly different from that for random L.

2. Necessary conditions for a stationary solution

In this section we derive conditions on the polynomials $a(\cdot)$ and $b(\cdot)$ and the Lévy process L necessary for the existence of a strictly stationary solution $(Y_t)_{t\in\mathbb{R}}$ of Eqs. (1.1) and (1.2).

In the derivation of the results we make extensive use of the process obtained by sampling the process Y at integer times. The first lemma provides a set of difference equations satisfied by the sequence $(Y_n)_{n\in\mathbb{Z}}$ when $(Y_t)_{t\in\mathbb{R}}$ satisfies (1.1) and $(\mathbf{X}_t)_{t\in\mathbb{R}}$ satisfies (1.2). From (1.3) we have, for the sampled state vector,

$$\mathbf{X}_n = \mathrm{e}^{A}\mathbf{X}_{n-1} + \mathbf{R}_n, \quad n \in \mathbb{Z}, \tag{2.1}$$

where

$$\mathbf{R}_n := \int_{n-1}^{n} \mathrm{e}^{A(n-u)}\mathbf{e}\,\mathrm{d}L_u, \quad n \in \mathbb{Z}, \tag{2.2}$$

and $(\mathbf{R}_n)_{n\in\mathbb{Z}}$ is clearly an i.i.d. sequence.

Writing the polynomial $a(z)$ as $\prod_{i=1}^{p}(z-\lambda_i)$, where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of A, we introduce the polynomial,

$$\Phi(z) := \prod_{j=1}^{p}(1-\mathrm{e}^{\lambda_j}z) =: 1 - d_1 z - \cdots - d_p z^p, \quad z \in \mathbb{C},$$

which plays a key role in the difference equations for the sampled process (Y_n), given in the following lemma. As usual, we denote by B the backward shift operator, defined by $B(\mathbf{X}_n) = \mathbf{X}_{n-1}$.

Lemma 2.1. *Let Φ be defined as above. Then*

$$\Phi(B)(\mathbf{X}_n) = \mathbf{X}_n - d_1\mathbf{X}_{n-1} - \cdots - d_p\mathbf{X}_{n-p} = \sum_{r=0}^{p-1}\left(\mathrm{e}^{rA} - \sum_{j=1}^{r} d_j \mathrm{e}^{(r-j)A}\right)\mathbf{R}_{n-r}, \tag{2.3}$$

and, from (1.1),

$$\Phi(B)Y_n = Y_n - d_1 Y_{n-1} - \cdots - d_p Y_{n-p} = \mathbf{b}'\sum_{r=0}^{p-1}\left(\mathrm{e}^{rA} - \sum_{j=1}^{r} d_j \mathrm{e}^{(r-j)A}\right)\mathbf{R}_{n-r}. \tag{2.4}$$

The latter can be written as

$$\Phi(B)Y_n = Y_n - d_1 Y_{n-1} - \cdots - d_p Y_{n-p} = Z_n^1 + Z_{n-1}^2 + \cdots + Z_{n-p+1}^p, \tag{2.5}$$

where

$$Z_n^r := \int_{n-1}^n \mathbf{b}' \left(\mathrm{e}^{(r-1)A} - \sum_{j=1}^{r-1} d_j \mathrm{e}^{(r-1-j)A} \right) \mathrm{e}^{A(n-u)} \mathbf{e} \, \mathrm{d}L_u, \quad r = 1, \ldots, p. \tag{2.6}$$

Proof. It suffices to prove (2.3), from which the remaining assertions follow. For that, we shall first show that for any $m \in \mathbb{N}_0$ and for any complex numbers $c_1, \ldots, c_m$, we can write

$$\mathbf{X}_n = \sum_{r=1}^m c_r \mathbf{X}_{n-r} + \left(\mathrm{e}^{mA} - \sum_{r=1}^m c_r \mathrm{e}^{(m-r)A} \right) \mathbf{X}_{n-m} + \sum_{r=0}^{m-1} \left(\mathrm{e}^{rA} - \sum_{j=1}^r c_j \mathrm{e}^{(r-j)A} \right) \mathbf{R}_{n-r}. \tag{2.7}$$

For $m = 0$ this is clear. To show that validity of the statement for any particular m implies validity for $m + 1$, let c_{m+1} be an arbitrary complex number. We can then write, by (2.1),

$$\begin{aligned}
\mathbf{X}_n &= \sum_{r=1}^m c_r \mathbf{X}_{n-r} + \left(\mathrm{e}^{mA} - \sum_{r=1}^m c_r \mathrm{e}^{(m-r)A} \right) \left(\mathrm{e}^A \mathbf{X}_{n-(m+1)} + \mathbf{R}_{n-m} \right) \\
&\quad + \sum_{r=0}^{m-1} \left(\mathrm{e}^{rA} - \sum_{j=1}^r c_j \mathrm{e}^{(r-j)A} \right) \mathbf{R}_{n-r} \\
&= \sum_{r=1}^{m+1} c_r \mathbf{X}_{n-r} + \left(\mathrm{e}^{(m+1)A} - \sum_{r=1}^m c_r \mathrm{e}^{(m+1-r)A} - c_{m+1} \right) \mathbf{X}_{n-(m+1)} \\
&\quad + \sum_{r=0}^{m} \left(\mathrm{e}^{rA} - \sum_{j=1}^r c_j \mathrm{e}^{(r-j)A} \right) \mathbf{R}_{n-r},
\end{aligned}$$

completing the induction step.

Next, observe that the eigenvalues of e^{-A}, including repeated values, are $\mathrm{e}^{-\lambda_1}, \ldots, \mathrm{e}^{-\lambda_p}$, see e.g. [7], Propositions 4.4.4 and 11.2.3. Hence we see that

$$\Phi(z) = \prod_{j=1}^p (1 - \mathrm{e}^{\lambda_j} z) = \prod_{j=1}^p (-\mathrm{e}^{\lambda_j}) \prod_{j=1}^p (z - \mathrm{e}^{-\lambda_j}) = \left(\prod_{j=1}^p (-\mathrm{e}^{\lambda_j}) \right) \chi_{\exp(-A)}(z),$$

where $\chi_{\exp(-A)}$ denotes the characteristic polynomial of e^{-A}. From the Cayley–Hamilton theorem it then follows that $\Phi(\mathrm{e}^{-A}) = 0$, so that

$$\mathrm{e}^{-0A} - d_1 \mathrm{e}^{-A} - \cdots - d_p \mathrm{e}^{-pA} = 0.$$

Multiplying this by e^{pA} gives

$$\mathrm{e}^{pA} - d_1 \mathrm{e}^{(p-1)A} - \cdots - d_p \mathrm{e}^{0A} = 0,$$

and inserting this in (2.7) with $m = p$ and $c_r = d_r$ gives (2.3). $\square$

The following two lemmas provide analytical tools which are used in the subsequent derivations.

Lemma 2.2. *Let $l \in \mathbb{N}_0$. Then for every $c_1, \dots, c_{l+1} \in \mathbb{R}$ there exist $\delta_0, \dots, \delta_l \in \mathbb{R}$ such that*

$$c_{l+1}n^{l+1} + c_l n^l + \cdots + c_1 n = \sum_{v=0}^{l} \delta_v \sum_{u=0}^{n-1} u^v \quad \forall\, n \in \mathbb{N}.$$

If $c_{l+1} \neq 0$, then one can choose $\delta_l \neq 0$.

Proof. The assertion will be proved by induction on l. For $l = 0$ it suffices to choose $\delta_0 = c_1$. Now, assuming the claim is true for a particular value of l, choose any $c_{l+2} \in \mathbb{R}$. Then

$$\sum_{u=0}^{n-1} u^{l+1} = \frac{1}{l+2} \sum_{m=0}^{l+1} \binom{l+2}{m} n^{l+2-m} B_m = \frac{1}{l+2} n^{l+2} + \frac{1}{l+2} \sum_{m=1}^{l+1} \binom{l+2}{m} n^{l+2-m} B_m,$$

where $(B_m)_{m \in \mathbb{N}_0}$ denotes the sequence of Bernoulli numbers, defined by $\frac{x}{\mathrm{e}^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m$. Choosing $\delta_{l+1} := (l+2) c_{l+2}$, we conclude that

$$c_{l+2}n^{l+2} + c_{l+1}n^{l+1} + \cdots + c_1 n = \delta_{l+1} \sum_{u=0}^{n-1} u^{l+1} + c'_{l+1} n^{l+1} + \cdots + c'_1 n$$

for some $c'_1, \dots, c'_{l+1}$, and so by the induction hypothesis we obtain

$$c_{l+2}n^{l+2} + c_{l+1}n^{l+1} + \cdots + c_1 n = \sum_{v=0}^{l+1} \delta_v \sum_{u=0}^{n-1} u^v$$

for suitable $\delta_0, \dots, \delta_l$. □

Lemma 2.3. *Define A, $\mathbf{b}$ and $\mathbf{e}$ as in Section 1 and define the polynomials*

$$h_{k,p}(z) := \sum_{u=0}^{p-k} a_u z^{p-k-u} = \sum_{u=0}^{p-k} a_{p-k-u} z^u, \quad k = 1, \dots, p, \tag{2.8}$$

where we let $a_0 := 1$. Then for every vector $\mathbf{V} = [V_1, \dots, V_p]' \in \mathbb{C}^p$ we have

$$\mathbf{b}' \mathrm{e}^{At} \mathbf{V} = \frac{1}{2\pi \mathrm{i}} \int_\rho \frac{b(z)}{a(z)} \mathrm{e}^{tz} \sum_{k=1}^{p} V_k h_{k,p}(z)\, \mathrm{d}z, \tag{2.9}$$

where ρ is a simple closed curve that encircles all eigenvalues of the matrix A. In particular,

$$\mathbf{b}' \mathrm{e}^{At} \mathbf{e} = \frac{1}{2\pi \mathrm{i}} \int_\rho \frac{b(z)}{a(z)} \mathrm{e}^{tz}\, \mathrm{d}z, \tag{2.10}$$

which can be expressed as the sum of residues,

$$\mathbf{b}' \mathrm{e}^{At} \mathbf{e} = \sum_{\lambda} \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k \mathrm{e}^{\lambda t},$$

where $\sum_\lambda$ denotes the sum over distinct zeroes of $a(\cdot)$, $\mu(\lambda)$ is the multiplicity of the zero λ and $\sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k \mathrm{e}^{\lambda t}$ is the residue of $z \mapsto \mathrm{e}^{zt} b(z)/a(z)$ at λ, i.e.

$$\sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k \mathrm{e}^{\lambda t} = \frac{1}{(\mu(\lambda)-1)!} \left[D_z^{\mu(\lambda)-1} \left((z-\lambda)^{\mu(\lambda)} \mathrm{e}^{zt} b(z)/a(z) \right) \right]_{z=\lambda},$$

and D_z denotes differentiation with respect to z. (For a zero λ with $\mu(\lambda) = 1$ the last sum reduces to $b(\lambda)\mathrm{e}^{\lambda t}/a'(\lambda)$.)

Proof. Since A is a companion matrix, it follows from Theorem 2.1 in Eller [8] applied to the function $z \mapsto \mathrm{e}^{tz}$ that the (j, k)-element of the matrix e^{At} is given by

$$\frac{1}{2\pi \mathrm{i}} \int_\rho \frac{z^{j-1}\mathrm{e}^{tz} h_{k,p}(z)}{a(z)}\,\mathrm{d}z.$$

Hence the k'th element of the row vector $\mathbf{b}'\mathrm{e}^{At}$ is given by

$$\sum_{j=0}^{p-1} b_j\,(\mathrm{e}^{At})_{j+1,k} = \frac{1}{2\pi \mathrm{i}} \int_\rho \sum_{j=0}^{p-1} \frac{b_j z^j \mathrm{e}^{tz} h_{k,p}(z)}{a(z)}\,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_\rho \frac{b(z)}{a(z)} \mathrm{e}^{tz} h_{k,p}(z)\,\mathrm{d}z.$$

Eqs. (2.9) and (2.10) are immediate consequences. □

Remark. Assuming that the real parts of the eigenvalues $\lambda_1, \ldots, \lambda_p$, are strictly negative, Tsai and Chan [9] obtained an expression for $\mathbf{b}'\mathrm{e}^{At}\mathbf{e}$, $t \geq 0$, which coincides with the second expression for $\mathbf{b}'\mathrm{e}^{At}\mathbf{e}$ in the statement of Lemma 2.3, evaluated on $[0, \infty)$. □

The following lemma will be used in the proof of Proposition 2.5, when necessary conditions for the existence of stationary solutions will be established. Recall that a Lévy process L is deterministic if and only if there is a $\sigma \in \mathbb{R}$ such that $L_t = \sigma t$ for all $t \in \mathbb{R}$.

Lemma 2.4. *Suppose that $a(\cdot)$ has a zero at λ_1 of algebraic multiplicity $\mu_1 = \mu(\lambda_1)$. In the notation of Lemma 2.3 we have*

$$\mathbf{b}'\,\mathrm{e}^{At}\mathbf{e} = \sum_\lambda \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k \mathrm{e}^{\lambda t}$$

where $\mu(\lambda)$ denotes the algebraic multiplicity of λ and the coefficients $c_{\lambda k}$ were defined in the statement of the lemma. Define

$$S_0 := \sum_{r=1}^{p} \mathrm{e}^{(1-r)\lambda_1} Z_0^r,$$

where Z_n^r was defined in (2.6). The following results then hold.

(a)

$$S_0 = \int_{-1}^{0} \sum_\lambda \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} \sum_{r=1}^{p} \mathrm{e}^{(1-r)\lambda_1} \Bigg[(r-1-s)^k \mathrm{e}^{\lambda(r-1-s)} - \sum_{j=1}^{r-1} d_j (r-1-j-s)^k \mathrm{e}^{\lambda(r-1-j-s)} \Bigg]\,\mathrm{d}L_s.$$

(b) *For each $k \in \{0, \ldots, \mu_1 - 1\}$*

$$\sum_{r=1}^{p} \mathrm{e}^{(1-r)\lambda_1} \Bigg[(r-1-s)^k \mathrm{e}^{\lambda_1(r-1-s)} - \sum_{j=1}^{r-1} d_j (r-1-j-s)^k \mathrm{e}^{\lambda_1(r-1-j-s)} \Bigg] = \gamma_k \mathrm{e}^{-\lambda_1 s},$$

where γ_k is a constant such that

$$\gamma_k \begin{cases} = 0, & k < \mu_1 - 1, \\ \neq 0, & k = \mu_1 - 1. \end{cases}$$

(c) *If $b(\lambda_1) \neq 0$ and $L = (L_t)_{t\in\mathbb{R}}$ is not a deterministic process, then the support of S_0 is unbounded.*

Proof. (a) Let $h(t) = \mathbf{b}'\mathrm{e}^{At}\mathbf{e}$. Then we have by (2.6),

$$Z_0^r = \int_{-1}^{0} h(r-1-s) - \sum_{j=1}^{r-1} d_j h(r-1-j-s)\,\mathrm{d}L_s,$$

so that for $S_0 = \sum_{r=1}^{p} \mathrm{e}^{(1-r)\lambda_1} Z_0^r$ we have

$$S_0 = \int_{-1}^{0} \sum_{r=1}^{p} \mathrm{e}^{(1-r)\lambda_1} \left[h(r-1-s) - \sum_{j=1}^{r-1} d_j h(r-1-j-s) \right] \mathrm{d}L_s.$$

Inserting the specific form for $h(t)$ from Lemma 2.3 we get assertion (a).

(b) We have for $k \in \{0, \dots, \mu_1 - 1\}$

$$\begin{aligned} &\sum_{r=1}^{p} \mathrm{e}^{(1-r)\lambda_1} \left[(r-1-s)^k \mathrm{e}^{\lambda_1(r-1-s)} - \sum_{j=1}^{r-1} d_j (r-1-j-s)^k \mathrm{e}^{\lambda_1(r-1-j-s)} \right] \\ &\quad = \mathrm{e}^{-\lambda_1 s} \sum_{u=0}^{k} (-s)^{k-u} \binom{k}{u} \sum_{r=1}^{p} \left[(r-1)^u - \sum_{j=1}^{r-1} d_j (r-1-j)^u \mathrm{e}^{-\lambda_1 j} \right] \\ &\quad = \mathrm{e}^{-\lambda_1 s} \sum_{u=0}^{k} (-s)^{k-u} \binom{k}{u} \gamma_u, \end{aligned}$$

where

$$\gamma_u := \sum_{r=1}^{p} \left[(r-1)^u - \sum_{j=1}^{r-1} d_j (r-1-j)^u \mathrm{e}^{-\lambda_1 j} \right], \quad u = 0, \dots, \mu_1 - 1.$$

To establish the claim it therefore suffices to show that

$$\gamma_k \begin{cases} = 0, & k < \mu_1 - 1, \\ \neq 0, & k = \mu_1 - 1, \end{cases} \tag{2.11}$$

which will be achieved by induction. First, observe that

$$\begin{aligned} \gamma_k &= \sum_{r=0}^{p-1} r^k - \sum_{r=0}^{p-1} \sum_{j=1}^{r} d_j (r-j)^k \mathrm{e}^{-\lambda_1 j} \\ &= \sum_{r=0}^{p-1} r^k - \sum_{j=1}^{p-1} d_j \mathrm{e}^{-\lambda_1 j} \sum_{u=0}^{p-1-j} u^k. \end{aligned} \tag{2.12}$$

In particular,

$$\gamma_0 = p - (p-1) d_1 \mathrm{e}^{-\lambda_1} - \dots - d_{p-1} \mathrm{e}^{-\lambda_1 (p-1)} = \left[z^{1-p}\, D_z(z^p \Phi(z^{-1})) \right]_{z=\mathrm{e}^{\lambda_1}}. \tag{2.13}$$

If $\mu_1 = 1$ then e^{λ_1} is a zero of multiplicity one of $z \mapsto z^p \Phi(z^{-1})$ and so the derivative in (2.13) is non-zero, establishing (2.11) in this case. Now suppose that $\mu_1 > 1$. Then e^{λ_1} is a zero of multiplicity μ_1 of $z \mapsto z^p \Phi(z^{-1})$ and the derivative in (2.13) is zero, so that $\gamma_0 = 0$. Let $k \in \{1, \dots, \mu_1 - 1\}$, and make the induction hypothesis that $\gamma_j = 0$ for $j \in \{0, \dots, k-1\}$. Then according to Lemma 2.2 there exist $\delta_0, \dots, \delta_k \in \mathbb{R}$ with $\delta_k \neq 0$ such that

$$\sum_{v=0}^{k} \delta_v \sum_{u=0}^{n-1} u^v = n(n+1)\cdots(n+k), \quad n \in \mathbb{N}.$$

The induction hypothesis with (2.12) and the preceding representation give

$$\begin{aligned} \delta_k \gamma_k &= \sum_{v=0}^{k} \delta_v \gamma_v \\ &= p(p+1)\cdots(p+k) - \sum_{j=1}^{p-1} d_j e^{-\lambda_1 j}(p-j)\cdots(p-j+k). \end{aligned}$$

To complete the induction argument and establish (2.11), it now suffices to show that

$$p(p+1)\cdots(p+k) - \sum_{j=1}^{p-1} d_j e^{-\lambda_1 j}(p-j)\cdots(p-j+k) \begin{cases} = 0, & k < \mu_1 - 1, \\ \neq 0, & k = \mu_1 - 1. \end{cases} \tag{2.14}$$

Recall that $\Phi(z) = 1 - d_1 z - \cdots - d_p z^p$. Then defining

$$\Psi(z) := z^{p+k} \Phi(z^{-1}) = z^{p+k} - d_1 z^{p+k-1} - \cdots - d_p z^k,$$

we can write the derivative $\Psi^{(k+1)}(z) = D_z^{k+1} \Psi(z)$ as

$$\Psi^{(k+1)}(z) = (p+k)\cdots p\, z^{p-1} - d_1(p+k-1)\cdots(p-1)z^{p-2} - \cdots - d_{p-1}(k+1)\cdots 1 z^0.$$

Multiplying by z^{1-p} gives

$$\Psi^{(k+1)}(z) z^{1-p} = (p+k)\cdots p - d_1(p+k-1)\cdots(p-1)z^{-1} - \cdots - d_{p-1}(k+1)\cdots 1 z^{1-p},$$

from which we conclude that

$$\delta_k \gamma_k = \Psi^{(k+1)}(e^{\lambda_1})\, e^{\lambda_1(1-p)}.$$

Since λ_1 is a zero of $a(\cdot)$ with multiplicity μ_1, $e^{-\lambda_1}$ is a zero of Φ with multiplicity μ_1, and we conclude that

$$\Phi(e^{-\lambda_1}) = \Phi'(e^{-\lambda_1}) = \cdots = \Phi^{(\mu_1 - 1)}(e^{-\lambda_1}) = 0$$

and

$$\Phi^{(\mu_1)}(e^{-\lambda_1}) \neq 0.$$

Since $\Psi(z) = z^{p+k}\Phi(z^{-1})$, this shows that

$$\Psi^{(k+1)}(e^{\lambda_1}) \begin{cases} = 0, & k < \mu_1 - 1, \\ \neq 0, & k = \mu_1 - 1, \end{cases}$$

and (2.14) follows.

(c) From (a) and (b) we obtain

$$S_0 = \int_{-1}^{0} \left(\sum_{k=0}^{\mu_1 - 1} c_{\lambda_1,k} \gamma_k \mathrm{e}^{-\lambda_1 s} + f(s) \right) \mathrm{d}L_s$$
$$= \int_{-1}^{0} \left(c_{\lambda_1,\mu_1-1} \gamma_{\mu_1-1} \mathrm{e}^{-\lambda_1 s} + f(s) \right) \mathrm{d}L_s$$

for some continuous function f which is linearly independent of the function $s \mapsto \mathrm{e}^{-\lambda_1 s}$. Since $\gamma_{\mu_1-1} \neq 0$ and

$$c_{\lambda_1,\mu_1-1} = \frac{1}{(\mu_1 - 1)!} b(\lambda_1) \left[(z - \lambda_1)^{\mu_1} / a(z) \right]_{z=\lambda_1} \neq 0$$

by Lemma 2.3 and by assumption, S_0 is the integral of a non-identically zero deterministic continuous function with respect to L. Since L is not deterministic, it follows that S_0 is non-constant, and since $\Re S_0$ and $\Im S_0$ are infinitely divisible, it must have unbounded support (cf. [10], Theorem 24.3). □

The next result gives necessary conditions for a strictly stationary solution to exist.

Proposition 2.5. *Suppose that $(Y_t)_{t\in\mathbb{R}}$ is a strictly stationary CARMA process and that $(L_t)_{t\in\mathbb{R}}$ is not a deterministic process. Let λ_1 be any (possibly multiple) zero of $a(\cdot)$ which is not a zero of $b(\cdot)$. Then $\Re(\lambda_1) \neq 0$ and $E \log^+ |L_1| < \infty$.*

Proof. Since $(Y_t)_{t\in\mathbb{R}}$ is a strictly stationary CARMA process, $(Y_n)_{n\in\mathbb{Z}}$ must also be strictly stationary. Let $\widetilde{\Phi}$ be the polynomial of degree $p - 1$ defined by $\widetilde{\Phi}(z) := \Phi(z)/(1 - \mathrm{e}^{\lambda_1} z)$ and define

$$W_n := \widetilde{\Phi}(B) Y_n.$$

Then $(W_n)_{n\in\mathbb{Z}}$ is strictly stationary and

$$W_n - \mathrm{e}^{\lambda_1} W_{n-1} = Z_n, \tag{2.15}$$

where $Z_n = Z_n^1 + Z_{n-1}^2 + \cdots + Z_{n-p+1}^p$ and $Z_n^1, \ldots, Z_{n-p+1}^p$ are the independent random variables defined in Lemma 2.1. Iterating (2.15) gives

$$W_n = \mathrm{e}^{\lambda_1} W_{n-1} + Z_n = \mathrm{e}^{2\lambda_1} W_{n-2} + \mathrm{e}^{\lambda_1} Z_{n-1} + Z_n = \cdots$$
$$= \mathrm{e}^{(N+1)\lambda_1} W_{n-N-1} + \sum_{j=0}^{N} \mathrm{e}^{j\lambda_1} Z_{n-j}, \quad N \in \mathbb{N}. \tag{2.16}$$

Since

$$Z_{n-j} = \sum_{r=1}^{p} Z_{n-j-r+1}^r,$$

it follows that for $N \in \mathbb{N}$

$$\sum_{j=0}^{N} \mathrm{e}^{j\lambda_1} Z_{n-j} = \sum_{j=0}^{N} \sum_{r=1}^{p} \mathrm{e}^{j\lambda_1} Z_{n-j-r+1}^r$$

$$= \sum_{r=1}^{p} \sum_{j=-r+1}^{N-r+1} e^{j\lambda_1} Z^r_{n-j-r+1} + \sum_{r=1}^{p} \sum_{j=N-r+2}^{N} e^{j\lambda_1} Z^r_{n-j-r+1} - \sum_{r=1}^{p} \sum_{j=-r+1}^{-1} e^{j\lambda_1} Z^r_{n-j-r+1}$$

$$= \sum_{r=1}^{p} \sum_{v=0}^{N} e^{(v-r+1)\lambda_1} Z^r_{n-v} + \sum_{r=1}^{p} \sum_{j=N-r+2}^{N} e^{j\lambda_1} Z^r_{n-j-r+1} - \sum_{r=1}^{p} \sum_{j=-r+1}^{-1} e^{j\lambda_1} Z^r_{n-j-r+1}. \tag{2.17}$$

Let

$$S_n := \sum_{r=1}^{p} e^{(1-r)\lambda_1} Z^r_n, \quad n \in \mathbb{Z}. \tag{2.18}$$

Then $(S_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence and S_0 has unbounded support by Lemma 2.4 (c). We conclude from Eqs. (2.16) and (2.17) that

$$W_0 - e^{(N+1)\lambda_1} W_{-N-1} - \sum_{r=1}^{p} \sum_{j=N-r+2}^{N} e^{j\lambda_1} Z^r_{-j-r+1} + \sum_{r=1}^{p} \sum_{j=-r+1}^{-1} e^{j\lambda_1} Z^r_{-j-r+1} = \sum_{v=0}^{N} e^{v\lambda_1} S_{-v}. \tag{2.19}$$

In part (a) below we show that the assumption $\Re(\lambda_1) = 0$ leads to a contradiction. Then in parts (b) and (c) we show that $E \log^+ |L_1| < \infty$ in the cases $\Re(\lambda_1) < 0$ and $\Re(\lambda_1) > 0$ respectively.

(a) Suppose that $\Re\lambda_1 = 0$. Since $(W_n)_{n\in\mathbb{Z}}$ is strictly stationary, it is easy to see that there is some constant $K > 0$ such that

$$P\left(\left| W_0 - e^{(N+1)\lambda_1} W_{-N-1} - \sum_{r=1}^{p} \sum_{j=N-r+2}^{N} e^{j\lambda_1} Z^r_{-j-r+1} + \sum_{r=1}^{p} \sum_{j=-r+1}^{-1} e^{j\lambda_1} Z^r_{-j-r+1} \right| \le K\right) \ge \frac{1}{2}$$

for all $N \in \mathbb{N}_0$. Hence we conclude that

$$P\left(\left|\Re \sum_{v=0}^{N} e^{v\lambda_1} S_{-v}\right| \le K\right) \ge \frac{1}{2} \quad \text{and} \quad P\left(\left|\Im \sum_{v=0}^{N} e^{v\lambda_1} S_{-v}\right| \le K\right) \ge \frac{1}{2}.$$

Let $(S'_v)_{v\in\mathbb{Z}}$ be an i.i.d. sequence, independent of the sequence $(S_v)_{v\in\mathbb{Z}}$, but with the same marginal distributions. Then $\Re(e^{v\lambda_1}(S_v - S'_v))$ is the symmetrization of $\Re(e^{v\lambda_1} S_v)$ and $\Im(e^{v\lambda_1}(S_v - S'_v))$ is the symmetrization of $\Im(e^{v\lambda_1} S_v)$. It follows that for all $N \in \mathbb{N}_0$

$$P\left(\left|\Re \sum_{v=0}^{N} e^{v\lambda_1} (S_{-v} - S'_{-v})\right| \le 2K\right) \ge \frac{1}{4} \quad \text{and}$$

$$P\left(\left|\Im\sum_{v=0}^{N} \mathrm{e}^{v\lambda_1}(S_{-v}-S'_{-v})\right| \le 2K\right) \ge \frac{1}{4}.$$

In particular, neither $\left|\Re\sum_{v=0}^{N} \mathrm{e}^{v\lambda_1}(S_{-v}-S'_{-v})\right|$ nor $\left|\Im\sum_{v=0}^{N} \mathrm{e}^{v\lambda_1}(S_{-v}-S'_{-v})\right|$ converges to $+\infty$ in probability as $N \to \infty$, and since both are sums of independent symmetric terms, both terms (without the modulus) must converge almost surely (see [11], Theorem 4.17). It follows that $\sum_{v=0}^{N} \mathrm{e}^{\lambda_1 v}(S_{-v}-S'_{-v})$ converges almost surely as $N \to \infty$. The Borel–Cantelli lemma then implies that

$$\sum_{v=0}^{\infty} P(|\mathrm{e}^{v\lambda_1}(S_{-v}-S'_{-v})| > 1) = \sum_{v=0}^{\infty} P(|S_{-v}-S'_{-v}| > 1) < \infty,$$

which is impossible, since $P(|S_{-v}-S'_{-v}| > 1) = P(|S_0 - S'_0| > 1)$, which is strictly positive since S_0 has unbounded support by Lemma 2.4 (c).

(b) Now suppose that $\Re\lambda_1 < 0$. Since $(W_n)_{n\in\mathbb{Z}}$ is stationary, Slutsky's lemma and (2.19) imply that $\sum_{v=0}^{N} \mathrm{e}^{v\lambda_1} S_{-v}$ converges in probability to $W_0 + \sum_{r=1}^{p}\sum_{j=-r+1}^{-1} \mathrm{e}^{j\lambda_1} Z^r_{-j-r+1}$ as $N \to \infty$. Hence

$$W_0 + \sum_{r=1}^{p}\sum_{j=-r+1}^{-1} \mathrm{e}^{j\lambda_1} Z^r_{-j-r+1} = \sum_{v=0}^{\infty} \mathrm{e}^{v\lambda_1} S_{-v} \quad \text{a.s.,} \tag{2.20}$$

the almost sure convergence of $\sum_{v=0}^{N} \mathrm{e}^{v\lambda_1} S_{-v}$ being a consequence of the independence of the sequence (S_n). The Borel–Cantelli lemma then implies that $\sum_{v=0}^{\infty} P(|\mathrm{e}^{v\lambda_1} S_{-v}| > 1) < \infty$. From this we obtain the chain of conclusions,

$$\begin{aligned}
&\sum_{v=0}^{\infty} P(|\mathrm{e}^{v\lambda_1} S_{-v}| > 1) < \infty \\
&\Longrightarrow \sum_{v=0}^{\infty} P(|S_{-v}| > \mathrm{e}^{-v\Re\lambda_1}) < \infty \\
&\Longrightarrow \sum_{v=0}^{\infty} P(\log^+ |S_0| > -v\Re\lambda_1) < \infty \\
&\Longrightarrow \sum_{v=0}^{\infty} P(\log^+ |\Re S_0| > -v\Re\lambda_1) < \infty,
\end{aligned}$$

the last of which implies that

$$E \log^+ |\Re S_0| < \infty. \tag{2.21}$$

Similarly we find that $E\log^+ |\Im S_0| < \infty$. Recall that S_0 has unbounded support, so that at least one of $\Re S_0$ and $\Im S_0$ has unbounded support. Without loss of generality we suppose that this is the case for $\Re S_0$. (The argument which follows can easily be modified to deal with the case in which $\Im S_0$ has unbounded support.) Recall further that we can write, as in the proof of Lemma 2.4(c),

$$\Re S_0 = \int_{-1}^{0} f(s)\,\mathrm{d}L_s$$

for some continuous function f which is not identically zero. It is well known that $\Re S_0$ is infinitely divisible as an integral of a deterministic function with respect to a Lévy process, and that its Lévy measure $\nu_{\Re S_0}$ satisfies

$$\nu_{\Re S_0}(C) = \int_{-1}^{0} \int_{\mathbb{R}} \mathbf{1}_C(f(s)x)\nu(\mathrm{d}x)\,\mathrm{d}s$$

for every Borel set $C \in \mathcal{B}_1$ such that $0 \notin C$ (cf. [12], Proposition 2.6). Here ν denotes the Lévy measure of L. Now define the sets

$$C_y := (-\infty, -y] \cup [y, \infty), \qquad y > 0,$$

and choose $\varepsilon > 0$ such that

$$K := \lambda^\ell(\{s \in [-1, 0] : |f(s)| \geq \varepsilon\}) > 0,$$

where λ^ℓ denotes one dimensional Lebesgue measure. (This is possible since f is continuous and not identically zero.) It then follows that for $y > 0$

$$\begin{aligned} \nu_{\Re S_0}(C_y) &= \int_{-1}^{0} \int_{|x|\,|f(s)|\geq y} \nu(\mathrm{d}x)\,\mathrm{d}s \\ &\geq \int_{s\in[-1,0]:|f(s)|\geq\varepsilon} \int_{|x|\geq y/\varepsilon} \nu(\mathrm{d}x)\,\mathrm{d}s \\ &= K\,\nu(C_{y/\varepsilon}). \end{aligned} \tag{2.22}$$

Now since $E \log^+ |\Re S_0|$ is finite and $\Re S_0$ is infinitely divisible, it follows that

$$\int_{|x|\geq 1} \log|x|\, \nu_{\Re S_0}(\mathrm{d}x) < \infty$$

(e.g. [10], Section 25). Hence

$$\begin{aligned} \infty &> \int_{|x|\geq 1} \log|x|\, \nu_{\Re S_0}(\mathrm{d}x) \\ &= \int_{|x|\geq 1} \int_{[1,|x|]} \frac{1}{u}\,\mathrm{d}u\, \nu_{\Re S_0}(\mathrm{d}x) \\ &= \int_1^\infty \frac{1}{u}\, \nu_{\Re S_0}(C_u)\,\mathrm{d}u \\ &\overset{(2.22)}{\geq} K \int_1^\infty \frac{1}{u}\, \nu(C_{u/\varepsilon})\,\mathrm{d}u \\ &= \cdots = K \int_{|x|\geq 1/\varepsilon} \log|x|\, \nu(\mathrm{d}x). \end{aligned}$$

Again from Section 25 in Sato [10] we conclude that $E \log^+ |L_1| < \infty$.

(c) Now suppose that $\Re\lambda_1 > 0$. From Eq. (2.15) we have

$$\begin{aligned} W_n &= \mathrm{e}^{-\lambda_1} W_{n+1} - \mathrm{e}^{-\lambda_1} Z_{n+1} \\ &= \mathrm{e}^{-2\lambda_1} W_{n+2} - \mathrm{e}^{-2\lambda_1} Z_{n+2} - \mathrm{e}^{-\lambda_1} Z_{n+1} \\ &= \cdots = \mathrm{e}^{-N\lambda_1} W_{n+N} - \sum_{j=1}^{N} \mathrm{e}^{-j\lambda_1} Z_{n+j}, \end{aligned}$$

and letting $N \to \infty$ gives

$$W_n = -\mathrm{plim}_{N\to\infty} \sum_{j=1}^{N} \mathrm{e}^{-j\lambda_1} Z_{n+j},$$

where plim denotes the limit in probability. Since

$$Z_{n+j} = \sum_{r=1}^{p} Z^r_{n+j-r+1},$$

it follows that for $N \in \mathbb{N}$

$$\begin{aligned}
\sum_{j=1}^{N} \mathrm{e}^{-j\lambda_1} Z_{n+j} &= \sum_{j=1}^{N}\sum_{r=1}^{p} \mathrm{e}^{-j\lambda_1} Z^r_{n+j-r+1} \\
&= \sum_{r=1}^{p}\sum_{j=r}^{N+r-1} \mathrm{e}^{-j\lambda_1} Z^r_{n+j-r+1} - \sum_{r=1}^{p}\sum_{j=N+1}^{N+r-1} \mathrm{e}^{-j\lambda_1} Z^r_{n+j-r+1} \\
&\quad + \sum_{r=1}^{p}\sum_{j=1}^{r-1} \mathrm{e}^{-j\lambda_1} Z^r_{n+j-r+1} \\
&= \sum_{r=1}^{p}\sum_{v=1}^{N} \mathrm{e}^{-(v+r-1)\lambda_1} Z^r_{n+v} - \sum_{r=1}^{p}\sum_{j=N+1}^{N+r-1} \mathrm{e}^{-j\lambda_1} Z^r_{n+j-r+1} \\
&\quad + \sum_{r=1}^{p}\sum_{j=1}^{r-1} \mathrm{e}^{-j\lambda_1} Z^r_{n+j-r+1}.
\end{aligned}$$

Defining

$$S_n := \sum_{r=1}^{p} \mathrm{e}^{(1-r)\lambda_1} Z^r_n,$$

we find that

$$W_0 = -\sum_{v=1}^{\infty} \mathrm{e}^{-v\lambda_1} S_v - \sum_{r=1}^{p}\sum_{j=1}^{r-1} \mathrm{e}^{-j\lambda_1} Z^r_{j-r+1} \quad \text{a.s.}$$

This is the analogue of (2.20) in part (b). The remainder of the proof follows exactly the same steps as those of (b). □

If the assumption that L is not deterministic in Proposition 2.5 is dropped, then $\Re\lambda_1 \neq 0$ is no longer necessary for a strictly stationary solution to exist, see Proposition 5.1 below.

3. The stationary solution

In the previous section we established that if L is non-deterministic and the polynomials $a(\cdot)$ and $b(\cdot)$ have no common zeroes, then existence of a strictly stationary solution $(Y_t)_{t\in\mathbb{R}}$ of (1.1) and (1.2) implies that $a(\cdot)$ is non-zero on the imaginary axis and that $E\log^+|L_1| < \infty$.

In this section we show that if $a(\cdot)$ is non-zero on the imaginary axis and $E\log^+|L_1| < \infty$, then there is a unique strictly stationary solution $(Y_t)_{t\in\mathbb{R}}$ of (1.1) and (1.2) and we specify the solution explicitly as an integral with respect to L. Together with the results of Section 2, this gives necessary and sufficient conditions for the existence of a strictly stationary solution under the assumption that $a(\cdot)$ and $b(\cdot)$ have no common zeroes (Theorem 3.3). The general case in which we place no *a priori* assumptions on the zeroes of $a(\cdot)$ and $b(\cdot)$ will be dealt with in Section 4.

In order to establish uniqueness of the solution we need the following lemma. As usual, B denotes the backward shift operator.

Lemma 3.1. *Let $(V_n)_{n\in\mathbb{Z}}$ be a strictly stationary $\mathbb{C}$-valued process such that*

$$\Psi(B) = V_n - \psi_1 V_{n-1} - \cdots - \psi_p V_{n-p} = Z_n, \quad n \in \mathbb{Z},$$

where $\Psi(z) = 1 - \psi_1 z - \cdots - \psi_p z^p$ with $\psi_1, \ldots, \psi_p \in \mathbb{C}$, and $(Z_n)_{n\in\mathbb{Z}}$ is a sequence of random variables. Suppose that $\Psi(\cdot)$ has no zeroes on the unit circle. If the Laurent expansion of $\Psi^{-1}(z)$ on $\{z \in \mathbb{C} : 1 - \varepsilon \le |z| \le 1 + \varepsilon\}$ for some $\varepsilon \in (0, 1)$ is denoted by,

$$\Psi^{-1}(z) = \sum_{m\in\mathbb{Z}} c_m z^m,$$

then

$$V_n = \mathrm{plim}_{N\to\infty} \sum_{|m|\le N} (c_m B^m)\, \Psi(B) V_n = \mathrm{plim}_{N\to\infty} \sum_{|m|\le N} c_m B^m Z_n.$$

In particular, the limit in probability exists, and V_n is determined by $(Z_{n-m})_{m\in\mathbb{Z}}$ and the coefficients $\psi_1, \ldots, \psi_p$.

Proof. Define the sequence of functions,

$$f_N(z) := \sum_{|m|\le N} c_m z^m (1 - \psi_1 z - \cdots \psi_p z^p) =: \sum_{m=-N}^{N+p} b_{m,N} z^m, \quad 1 - \varepsilon \le |z| \le 1 + \varepsilon, N \in \mathbb{N}.$$

Then f_N converges uniformly to 1 on this annulus as $N \to \infty$, and it follows that the Laurent coefficients of f_N converge to those of the function 1, i.e.

$$\lim_{N\to\infty} b_{m,N} = \begin{cases} 0, & m \ne 0, \\ 1, & m = 0. \end{cases}$$

Further, observe that

$$b_{m,N} = b_{m,N'} \quad \forall\, N' \ge N > p, \quad m = -N + p, \ldots, N,$$

i.e. for fixed m, $b_{m,N}$ is constant for sufficiently large N. From the limit result, we hence see that

$$f_N(z) = 1 + \sum_{m=-N}^{-N+p-1} b_{m,N} z^m + \sum_{m=N+1}^{N+p} b_{m,N} z^m,$$

and that

$$\lim_{N\to\infty} \sup_{m\in\{-N,\ldots,-N+p-1\}\cup\{N+1,\ldots,N+p\}} |b_{m,N}| = 0$$

(due to the exponential decrease in c_m). Since $(V_n)_{n\in\mathbb{Z}}$ is stationary, it follows from Slutsky's theorem that

$$V_n = \mathrm{plim}_{N\to\infty} f_N(B) V_n = \mathrm{plim}_{N\to\infty} \sum_{|m|\le N} c_m B^m Z_n,$$

as claimed. □

The following proposition presents a sufficient condition for the existence of a strictly stationary solution.

Proposition 3.2. *Suppose that all singularities of the meromorphic function $z \mapsto b(z)/a(z)$ on the imaginary axis are removable, i.e. if $a(\cdot)$ has a zero λ_1 of multiplicity $\mu(\lambda_1)$ on the imaginary axis, then $b(\cdot)$ has also a zero at λ_1 of multiplicity greater than or equal to $\mu(\lambda_1)$. Suppose further that $E \log^+ |L_1| < \infty$. Define $\mathbf{l}(t)$, $\mathbf{r}(t)$, $\mathbf{n}(t)$ to be the sums of the residues of the column vector $\mathrm{e}^{zt} a^{-1}(z)[1\ z\ \cdots\ z^{p-1}]'$ at the zeroes of $a(\cdot)$ with strictly negative, strictly positive and zero real parts, respectively. Then*

$$\mathbf{l}(t) + \mathbf{r}(t) + \mathbf{n}(t) = \mathrm{e}^{At}\mathbf{e}, \quad t \in \mathbb{R}, \tag{3.1}$$

$$\mathbf{l}(t) = \sum_{\lambda:\Re\lambda<0} \sum_{k=0}^{\mu(\lambda)-1} \boldsymbol{\alpha}_{\lambda k} t^k \mathrm{e}^{\lambda t} = \mathrm{e}^{At}\mathbf{l}(0), \quad t \in \mathbb{R}, \tag{3.2}$$

$$\mathbf{r}(t) = \sum_{\lambda:\Re\lambda>0} \sum_{k=0}^{\mu(\lambda)-1} \boldsymbol{\beta}_{\lambda k} t^k \mathrm{e}^{\lambda t} = \mathrm{e}^{At}\mathbf{r}(0), \quad t \in \mathbb{R}, \tag{3.3}$$

for certain vectors $\boldsymbol{\alpha}_{\lambda k}, \boldsymbol{\beta}_{\lambda k} \in \mathbb{C}^p$, and

$$\mathbf{n}(t) = \mathrm{e}^{At}\mathbf{n}(0). \tag{3.4}$$

As usual, the sums are over the distinct zeroes λ of $a(\cdot)$ and $\mu(\lambda)$ denotes the multiplicity of the zero λ. Define

$$\begin{aligned}\mathbf{X}_t &:= \mathrm{e}^{At}\left(\int_{-\infty}^t \mathrm{e}^{-Au}\mathbf{l}(0)\,\mathrm{d}L_u - \int_t^\infty \mathrm{e}^{-Au}\mathbf{r}(0)\,\mathrm{d}L_u + \int_0^t \mathrm{e}^{-Au}\,\mathbf{n}(0)\,\mathrm{d}L_u\right)\\ &= \int_{-\infty}^t \mathbf{l}(t-u)\,\mathrm{d}L_u - \int_t^\infty \mathbf{r}(t-u)\,\mathrm{d}L_u + \mathrm{e}^{At}\int_0^t \mathrm{e}^{-Au}\mathbf{n}(0)\,\mathrm{d}L_u, \quad t \in \mathbb{R},\end{aligned} \tag{3.5}$$

where for $t < 0$, $\int_0^t$ is interpreted as $-\int_t^0$. Then the improper integrals over $(-\infty, t]$ and $[t, \infty)$ defining $\mathbf{X}_t$ exist as almost sure limits $\lim_{T\to\infty} \int_{-T}^t$ and $\lim_{T\to\infty} \int_t^T$, respectively, and $(\mathbf{X}_t)_{t\in\mathbb{R}}$ satisfies (1.3). Define $Y_t := \mathbf{b}'\mathbf{X}_t$, $t \in \mathbb{R}$. Then $(Y_t)_{t\in\mathbb{R}}$ is a strictly stationary solution of the CARMA equations (1.1) and (1.2), which can be written as

$$Y_t = \int_{-\infty}^\infty g(t-u)\,\mathrm{d}L_u, \quad t \in \mathbb{R}, \tag{3.6}$$

where

$$g(t) = \left(\sum_{\lambda:\Re\lambda<0} \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k \mathrm{e}^{\lambda t} \mathbf{1}_{(0,\infty)}(t) - \sum_{\lambda:\Re\lambda>0} \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k \mathrm{e}^{\lambda t} \mathbf{1}_{(-\infty,0)}(t)\right), \quad t \in \mathbb{R}, \tag{3.7}$$

with $c_{\lambda k}$ as in Lemma 2.3.

Proof. The proof of (3.1) is exactly analogous to the proof of Lemma 2.3. The first equalities in (3.2) and (3.3) are apparent from the algebraic form of the residue of the vector $\mathrm{e}^{zt} a^{-1}(z)[1\ z\ \cdots\ z^{p-1}]'$ at the zero λ of $a(\cdot)$. The right-hand sides of (3.2) and (3.3) follow from the relations,

$$\frac{\mathrm{d}\mathbf{l}(t)}{\mathrm{d}t} = A\mathbf{l}(t) \quad \text{and} \quad \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = A\mathbf{r}(t), \quad t \in \mathbb{R}, \tag{3.8}$$

which are easily verified in the case when the zeroes λ of $a(\cdot)$ are distinct, since then the residue at λ is $e^{\lambda t}[1\ \lambda\ \cdots\ \lambda^{p-1}]'/a'(\lambda)$. The general case follows from a limit argument using the differentiation lemma applied to the sum of residues. Eq. (3.4) is an immediate consequence of (3.1)–(3.3). Relations (3.2) and (3.3) imply the existence of real constants $K > 0$ and $\varepsilon > 0$ such that

$$|\mathbf{l}(-u)| \le K\,e^{-\varepsilon|u|} \quad \forall\, u \le 0 \quad \text{and}$$
$$|\mathbf{r}(-u)| \le K\,e^{-\varepsilon|u|} \quad \forall\, u \ge 0.$$

This, together with the assumption that $E\log^+|L_1| < \infty$, implies convergence in probability of the integrals defining $\mathbf{X}_t$ (see e.g. [13], Theorem 1.2 and Proposition 4.3), and the independence of the increments of L implies that there is also convergence with probability one. The following calculation shows that $\mathbf{X}_t$ satisfies (1.3). For $s \le t$ we have

$$\begin{aligned}
&e^{A(t-s)}\mathbf{X}_s + \int_s^t e^{A(t-u)}\mathbf{e}\,dL_u \\
&\overset{(3.1)\text{ and }(3.5)}{=} e^{At}\left(\int_{-\infty}^s e^{-Au}\mathbf{l}(0)\,dL_u - \int_s^\infty e^{-Au}\,\mathbf{r}(0)\,dL_u + \int_0^s e^{-Au}\mathbf{n}(0)\,dL_u\right) \\
&\quad + e^{At}\left(\int_s^t \mathbf{l}(-u)\,dL_u + \int_s^t \mathbf{r}(-u)\,dL_u + \int_s^t \mathbf{n}(-u)\,dL_u\right) \\
&\overset{(3.2)\text{–}(3.4)}{=} e^{At}\left(\int_{-\infty}^s e^{-Au}\mathbf{l}(0)\,dL_u - \int_s^\infty e^{-Au}\,\mathbf{r}(0)\,dL_u + \int_0^s e^{-Au}\mathbf{n}(0)\,dL_u\right) \\
&\quad + e^{At}\left(\int_s^t e^{-Au}\mathbf{l}(0)\,dL_u + \int_s^t e^{-Au}\mathbf{r}(0)\,dL_u + \int_s^t e^{-Au}\mathbf{n}(0)\,dL_u\right) \\
&\overset{(3.5)}{=} \mathbf{X}_t.
\end{aligned}$$

It follows that $Y_t := \mathbf{b}'\mathbf{X}_t$ is a solution of the CARMA equations. Next, observe that

$$\begin{aligned}
\mathbf{b}'e^{At}\mathbf{n}(0) &\overset{(3.4)}{=} \mathbf{b}'\mathbf{n}(t) \\
&= \sum_{\lambda:\Re\lambda=0} \mathbf{b}'\mathrm{res}_\lambda(e^{zt}a^{-1}(z)[1\ z\ \cdots\ z^{p-1}]') \\
&= \sum_{\lambda:\Re\lambda=0} \mathrm{res}_\lambda(e^{zt}a^{-1}(z)b(z)) = 0
\end{aligned}$$

by assumption, since $b(z)/a(z)$ has only removable singularities on the imaginary axis. Hence it follows from (3.5) that

$$Y_t = \mathbf{b}'\mathbf{X}_t = \mathbf{b}'\left(\int_{-\infty}^t \mathbf{l}(t-u)\,dL_u - \int_t^\infty \mathbf{r}(t-u)\,dL_u\right),$$

which is clearly strictly stationary. The representation (3.6) of Y_t is obtained by observing that $\mathbf{b}'\mathbf{l}(t)$ and $\mathbf{b}'\mathbf{r}(t)$ are precisely the sums of the residues of $z \mapsto e^{zt}b(z)/a(z)$ at the zeroes of $a(\cdot)$ with strictly negative and strictly positive parts respectively. □

We can now state the first of our main results.

Theorem 3.3. *Let L be a Lévy process which is not deterministic and suppose that $a(\cdot)$ and $b(\cdot)$ have no common zeroes. Then the CARMA equations (1.1) and (1.2) have a strictly stationary solution Y on $\mathbb{R}$ if and only if $E\log^+|L_1| < \infty$ and $a(\cdot)$ is non-zero on the imaginary axis. In* *this case the solution Y is unique and is given by (3.6) and (3.7), and the corresponding state vector $(\mathbf{X}_t)_{t\in\mathbb{R}}$ can be chosen to be strictly stationary as in (3.5).*

Proof. Suppose that a stationary solution exists. Then from Proposition 2.5 it follows that $E\log^+|L_1| < \infty$ and that $a(\cdot)$ is non-zero on the imaginary axis. Using Eq. (2.5) and applying Lemma 3.1 with $\Psi(z) = \Phi(z)$ and $Z_n = Z_n^1 + Z_{n-2}^2 + \cdots + Z_{n-p+1}^p$, where (Z_n^r) is defined by (2.6), show that $(Y_n)_{n\in\mathbb{Z}}$ is uniquely determined. The same argument shows that $(Y_{nh})_{n\in\mathbb{Z}}$ is uniquely determined for any fixed sampling interval h, and since the solution $(Y_t)_{t\in\mathbb{R}}$ is càdlàg it must be unique. Conversely, suppose that $E\log^+|L_1| < \infty$ and that all zeroes of $a(\cdot)$ have non-zero real parts. Then the existence of the strictly stationary solution Y with representation (3.6) and (3.7) and the strictly stationary state vector defined in (3.5) follows from Proposition 3.2. □

4. The general non-deterministic case

In this section we eliminate the *a priori* assumptions regarding the zeroes of $a(\cdot)$ and $b(\cdot)$ made in Theorem 3.3 and assume only that L is non-deterministic. In particular the polynomials $a(\cdot)$ and $b(\cdot)$ may have common zeroes and may have zeroes on the imaginary axis. Before we give this general necessary and sufficient condition in Theorem 4.2, we show how common zeroes in $a(\cdot)$ and $b(\cdot)$ can be factored out to give solutions of lower-order CARMA processes.

Theorem 4.1. *Let $p \geq 2$ and let $Y = (Y_t)_{t\in\mathbb{R}}$ be a $CARMA(p,q)$ process driven by L with state vector process $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$, i.e. $\mathbf{X}$ and Y satisfy (1.1) and (1.3). Suppose that $\lambda_1 \in \mathbb{C}$ is a zero of both $a(\cdot)$ and $b(\cdot)$, and define*

$$\widetilde{a}(z) := \frac{a(z)}{z-\lambda_1} = z^{p-1} + \widetilde{a}_1 z^{p-2} + \cdots + \widetilde{a}_{p-1},$$

$$\widetilde{b}(z) := \frac{b(z)}{z-\lambda_1} = \widetilde{b}_0 + \widetilde{b}_1 z + \cdots + \widetilde{b}_{p-2} z^{p-2},$$

$$\widetilde{A} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\widetilde{a}_{p-1} & -\widetilde{a}_{p-2} & -\widetilde{a}_{p-3} & \cdots & -\widetilde{a}_1 \end{bmatrix} \in \mathbb{C}^{p-1,p-1},$$

$$\widetilde{\mathbf{e}} = [0\,0\,\ldots\,0\,1]' \in \mathbb{C}^{p-1}, \quad \text{and} \quad \widetilde{\mathbf{b}} = \left[\widetilde{b}_0\,\widetilde{b}_1\,\ldots\,\widetilde{b}_{p-3}\,\widetilde{b}_{p-2}\right]' \in \mathbb{C}^{p-1}.$$

Then there exists a $\mathbb{C}^{p-1}$-valued state vector process $\widetilde{\mathbf{X}} = (\widetilde{\mathbf{X}}_t)_{t\in\mathbb{R}}$ such that

$$\widetilde{\mathbf{X}}_t = \mathrm{e}^{\widetilde{A}(t-s)}\widetilde{\mathbf{X}}_s + \int_s^t \mathrm{e}^{\widetilde{A}(t-u)}\widetilde{\mathbf{e}}\,\mathrm{d}L_u, \quad \forall\, s \leq t \in \mathbb{R}, \tag{4.1}$$

and

$$Y_t = \widetilde{\mathbf{b}}'\widetilde{\mathbf{X}}_t, \quad t \in \mathbb{R}, \tag{4.2}$$

i.e. Y is a $CARMA(p-1, q-1)$ process with the same driving Lévy process.

Proof. Observe that (1.3) and (4.1) are equivalent to

$$\mathbf{X}_t = \mathrm{e}^{At}\mathbf{X}_0 + \int_0^t \mathrm{e}^{A(t-u)}\mathbf{e}\,\mathrm{d}L_u \quad \text{and} \quad \widetilde{\mathbf{X}}_t = \mathrm{e}^{\widetilde{A}t}\widetilde{\mathbf{X}}_0 + \int_0^t \mathrm{e}^{\widetilde{A}(t-u)}\widetilde{\mathbf{e}}\,\mathrm{d}L_u \quad \forall\, t \in \mathbb{R},$$

respectively, where for $t < 0$, $\int_0^t$ is interpreted as $-\int_t^0$. Hence, using (2.10), it is enough to show that for given $\mathbf{X}_0 \in \mathbb{C}^p$ there is $\widetilde{\mathbf{X}}_0 \in \mathbb{C}^{p-1}$ such that

$$\mathbf{b}' e^{At} \mathbf{X}_0 = \widetilde{\mathbf{b}}' e^{\widetilde{A}t} \widetilde{\mathbf{X}}_0 \quad \forall\, t \in \mathbb{R}. \tag{4.3}$$

Write

$$\mathbf{X}_0 = (x_1, \ldots, x_p)' \quad \text{and} \quad \widetilde{\mathbf{X}}_0 = (\widetilde{x}_1, \ldots, \widetilde{x}_{p-1})',$$

respectively. Observe that

$$\sum_{k=1}^{p} x_k h_{k,p}(z) = \sum_{k=1}^{p} x_k \sum_{u=0}^{p-k} a_{p-k-u} z^u = \sum_{u=0}^{p-1} \left(\sum_{k=1}^{p-u} x_k a_{p-k-u} \right) z^u, \tag{4.4}$$

where $h_{k,p}(z)$ was defined in Lemma 2.3. Since $a_0 = 1$ and $z \mapsto \frac{a(z)}{z-\lambda_1}$ is a polynomial of degree $p-1$ with leading coefficient 1 we can write

$$\sum_{k=1}^{p} x_k h_{k,p}(z) = x_1 \frac{a(z)}{z-\lambda_1} + \sum_{u=0}^{p-2} \delta_u z^u, \quad z \in \mathbb{C},$$

for certain $\delta_0, \ldots, \delta_{p-2} \in \mathbb{C}$ (which, like $x_1, \ldots, x_p$, are random variables). Next, observe from $\widetilde{a}_0 = 1$ and (4.4) that

$$\sum_{k=1}^{p-1} \widetilde{x}_k h_{k,p-1}(z) = \sum_{u=0}^{p-2} \left(\widetilde{x}_{p-1-u} + \sum_{k=1}^{p-u-2} \widetilde{x}_k \widetilde{a}_{p-1-k-u} \right) z^u, \quad z \in \mathbb{C}.$$

Now define $\widetilde{x}_1, \ldots, \widetilde{x}_{p-1}$ recursively to satisfy the relations,

$$\widetilde{x}_{p-1-u} + \sum_{k=1}^{p-u-2} \widetilde{x}_k \widetilde{a}_{p-1-k-u} = \delta_u, \quad u = p-2, p-1, \ldots, 0,$$

from which we conclude that

$$\sum_{k=1}^{p} x_k h_{k,p}(z) = x_1 \frac{a(z)}{z-\lambda_1} + \sum_{k=1}^{p-1} \widetilde{x}_k h_{k,p-1}(z).$$

Since $b(z)/a(z) = \widetilde{b}(z)/\widetilde{a}(z)$, we conclude from (2.9) that

$$\mathbf{b}' e^{At} \mathbf{X}_0 = \widetilde{\mathbf{b}}' e^{\widetilde{A}t} \widetilde{\mathbf{X}}_0 + \frac{x_1}{2\pi i} \int_\rho \frac{b(z)}{a(z)} \frac{a(z)}{z-\lambda_1} e^{zt}\, dz,$$

and since $b(\lambda_1) = 0$ the integrand in the contour integral is an entire function, from which it follows that the integral term is zero, giving (4.3). $\square$

Theorem 4.2. *Suppose that $p \geq 1$, that $\mathbf{b} \neq 0$ and that the Lévy process L is not deterministic. Then the CARMA equations (1.1) and (1.2) have a strictly stationary solution Y on $\mathbb{R}$ if and only if $E \log^+ |L_1| < \infty$ and all singularities of the meromorphic function $z \mapsto b(z)/a(z)$ on the imaginary axis are removable, i.e. if $a(\cdot)$ has a zero λ_1 of multiplicity $\mu(\lambda_1)$ on the imaginary axis, then $b(\cdot)$ has also a zero at λ_1 of multiplicity greater than or equal to $\mu(\lambda_1)$. In this case, the solution is unique and is given by (3.6) and (3.7).*

Proof. If $p = 1$ then $b(z) = b_0$ is the constant polynomial, which by assumption is different from zero. The claim then follows from Theorem 3.3. So suppose that $p \geq 2$. If $a(\cdot)$ and $b(\cdot)$ have no common zeroes, the claim is true by Theorem 3.3. Now suppose that $a(\cdot)$ and $b(\cdot)$ have common zeroes. The sufficiency of the condition is then clear from Proposition 3.2. To show that it is necessary, suppose that Y is a strictly stationary solution. If λ is any zero of $a(\cdot)$ let $\mu_a(\lambda)$ denote its multiplicity and let $\mu_b(\lambda)$ be its multiplicity as a zero of $b(\cdot)$ (with $\mu_b(\lambda) := 0$ if $b(\lambda) \neq 0$). Let $\nu(\lambda) := \min(\mu_a(\lambda), \mu_b(\lambda))$ and define the polynomials,

$$\widetilde{a}(z) = \frac{a(z)}{\prod_\lambda (z-\lambda)^{\nu(\lambda)}} \quad \text{and} \quad \widetilde{b}(z) = \frac{b(z)}{\prod_\lambda (z-\lambda)^{\nu(\lambda)}},$$

where the product is over the distinct zeroes of $a(\cdot)$. From Theorem 4.1 it follows that Y is also a strictly stationary solution of a CARMA$(p-r, q-r)$ process with $r = \sum_\lambda \nu(\lambda)$ and characteristic polynomials $\widetilde{a}(\cdot)$ and $\widetilde{b}(z)$. Since $\widetilde{a}(\cdot)$ and $\widetilde{b}(\cdot)$ have no common zeroes it follows from Theorem 3.3 that $E \log^+ |L_1| < \infty$ and that the zeroes of $\widetilde{a}(\cdot)$ all have non-zero real parts. Uniqueness of the solution follows as before. □

Remark 4.3. Let L be a non-deterministic Lévy process. It is clear that a strictly stationary solution $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$ of (1.2) gives rise to a strictly stationary CARMA process Y via (1.1). Conversely, Proposition 3.2 and Theorem 4.2 imply that whenever $a(\cdot)$ and $b(\cdot)$ have no common zeroes on the imaginary axis, then to every strictly stationary solution Y there corresponds a strictly stationary state vector process $\mathbf{X}$. This is no longer true if $a(\cdot)$ and $b(\cdot)$ have common zeroes on the imaginary axis. In that case, stationary solutions Y may exist as characterised by Theorem 4.2, while a stationary state vector $\mathbf{X}$ cannot exist if $a(\cdot)$ has zeroes on the imaginary axis. The latter can be seen from Proposition 2.5, by taking another CARMA process with the same polynomial $a(\cdot)$, but a different polynomial $\widetilde{b}(\cdot)$ such that $a(\cdot)$ and $\widetilde{b}(\cdot)$ have no common zeroes.

5. The deterministic case

The characterisation of strictly stationary solutions Y of the CARMA equations (1.1) and (1.2) in the case when L is random is slightly different from the case when L is a deterministic Lévy process, in which case $a(\cdot)$ can have zeroes on the imaginary axis even if they are not factored out by the polynomial $b(\cdot)$.

Proposition 5.1. *Let L be a deterministic Lévy process, i.e. suppose there is $\sigma \in \mathbb{R}$ such that $L_t = \sigma t$ for all $t \in \mathbb{R}$. Suppose further that $\mathbf{b} \neq 0$. Denote by $\mu_a(\lambda)$ and $\mu_b(\lambda)$ the multiplicities of λ as a zero of $a(\cdot)$ and of $b(\cdot)$, respectively. Then the following results hold:*

(a) *If $a_p \neq 0$, then the CARMA equations (1.1) and (1.2) have a strictly stationary solution Y, one of which is $Y_t = \sigma b_0/a_p$ for all $t \in \mathbb{R}$. This solution is unique if and only if $\mu_b(\lambda) \geq \mu_a(\lambda)$ for every zero λ of $a(\cdot)$ such that $\Re\lambda = 0$.*

(b) *If $a_p = 0$ and $\sigma \neq 0$, then the CARMA equations (1.1) and (1.2) have a strictly stationary solution Y if and only if $\mu_b(0) \geq \mu_a(0)$. If this condition is satisfied, one solution is $Y_t = \sigma b_{\mu_a(0)}/a_{p-\mu_a(0)}$, $t \in \mathbb{R}$, and this solution is unique if and only if $\mu_b(\lambda) \geq \mu_a(\lambda)$ for all zeroes λ of $a(\cdot)$ such that $\Re\lambda = 0$.*

(c) *If $a_p = \sigma = 0$, then $Y_t = 0$, $t \in \mathbb{R}$, is a strictly stationary solution of the CARMA equations (1.1) and (1.2), and this solution is unique if and only if $\mu_b(\lambda) \geq \mu_a(\lambda)$ for all zeroes λ of $a(\cdot)$ such that $\Re\lambda = 0$.*

Proof. (a) Since $a_p \neq 0$, the matrix A is invertible. Write

$$\mathbf{X}_0 := -\sigma A^{-1}\mathbf{e} + \mathbf{V} = \left[\sigma/a_p \; 0 \; 0 \; \dots \; 0\right] + \mathbf{V}$$

for some random vector $\mathbf{V}$. Then

$$\begin{aligned}\mathbf{X}_t &= \mathrm{e}^{At}\mathbf{X}_0 + \sigma \int_0^t \mathrm{e}^{A(t-u)}\mathbf{e}\,\mathrm{d}u \\ &= \mathrm{e}^{At}\left(\mathbf{X}_0 - \sigma \mathrm{e}^{-At}A^{-1}\mathbf{e} + \sigma A^{-1}\mathbf{e}\right) \\ &= \mathrm{e}^{At}\mathbf{V} - \sigma A^{-1}\mathbf{e}.\end{aligned}$$

The choice of $\mathbf{V} = 0$ then leads to

$$Y_t = \mathbf{b}'\mathbf{X}_t = -\sigma \mathbf{b}'A^{-1}\mathbf{e} = \sigma b_0/a_p, \quad t \in \mathbb{R},$$

which is clearly stationary. Next, suppose that there is a zero λ_1 of $a(\cdot)$ with $\Re\lambda_1 = 0$ and $\mu_a(\lambda_1) > \mu_b(\lambda_1)$. Let δ be a complex-valued random variable which is uniformly distributed on the unit circle. From the form of the polynomials $h_{k,p}$ in (2.8) it is easy to see that the vector $\mathbf{V} = [V_1 \; \dots V_k]'$ can be chosen such that

$$\sum_{k=1}^{p} V_k h_{k,p}(z) = \frac{a(z)}{(z-\lambda_1)^{\mu_b(\lambda_1)+1}}\,\delta, \tag{5.1}$$

since $a(z)/(z-\lambda_1)^{\mu_b(\lambda_1)+1}$ is a polynomial of degree less than or equal to $p-1$. Let $\widetilde{b}(z) = b(z)/(z-\lambda_1)^{\mu_b(\lambda_1)}$. Then (2.9) gives

$$\mathbf{b}'\mathrm{e}^{At}\mathbf{V} = \frac{1}{2\pi \mathrm{i}}\int_\rho \frac{\widetilde{b}(z)}{z-\lambda_1}\mathrm{e}^{tz}\,\mathrm{d}z\,\delta = \widetilde{b}(\lambda_1)\mathrm{e}^{\lambda_1 t}\delta.$$

Since δ is uniformly distributed on the unit circle and $\widetilde{b}(\lambda_1) \neq 0$, $Y_t = \sigma b_0/a_p + \widetilde{b}(\lambda_1)\mathrm{e}^{\lambda_1 t}\delta$, $t \in \mathbb{R}$, gives another strictly stationary solution Y of (1.1) and (1.2), violating uniqueness. Finally, if $\mu_a(\lambda) \le \mu_b(\lambda)$ for all zeroes λ of $a(\cdot)$ such that $\Re\lambda = 0$, then these zeroes can be factored out by Theorem 4.1 and uniqueness follows as in the proof of Theorem 3.3.

(b) If $\mu_b(0) \ge \mu_a(0)$, we can factor out the common zero at 0 by Theorem 4.1, and the existence and uniqueness assertion follows from (a). So suppose that $\mu_b(0) < \mu_a(0)$. From (2.10) we conclude that

$$\sigma \int_0^t \mathbf{b}'\mathrm{e}^{A(t-u)}\mathbf{e}\,\mathrm{d}u = \frac{\sigma}{2\pi \mathrm{i}}\int_\rho \frac{b(z)}{a(z)}\int_0^t \mathrm{e}^{z(t-u)}\,\mathrm{d}u\,\mathrm{d}z = \frac{\sigma}{2\pi \mathrm{i}}\int_\rho \frac{b(z)}{a(z)}\frac{1}{z}(\mathrm{e}^{zt}-1)\,\mathrm{d}z.$$

Observe further that by (2.8) the general choice of a starting random vector $\mathbf{V} = \mathbf{X}_0$ corresponds to the general choice of a random polynomial $\sum_{k=1}^p V_k h_{k,p}(z) = \sum_{k=1}^p U_k z^{k-1}$ with random variables $U_1, \dots, U_p$. Hence we see from (2.9) that the general solution for Y_t can be written as

$$\begin{aligned}Y_t &= \mathbf{b}'\mathrm{e}^{At}\mathbf{X}_0 + \sigma \int_0^t \mathbf{b}'\mathrm{e}^{A(t-u)}\mathbf{e}\,\mathrm{d}u \\ &= \frac{1}{2\pi \mathrm{i}}\int_\rho \frac{b(z)}{a(z)z}\mathrm{e}^{zt}\left(\sigma + \sum_{k=1}^p U_k z^k\right)\mathrm{d}z - \frac{\sigma}{2\pi i}\int_\rho \frac{b(z)}{a(z)z}\,\mathrm{d}z, \quad t \in \mathbb{R}.\end{aligned}$$

By the residue theorem the latter can be written as

$$Y_t = -\frac{\sigma}{2\pi \mathrm{i}} \int_\rho \frac{b(z)}{a(z)z}\, \mathrm{d}z + \sum_{\lambda \neq 0} \sum_{k=0}^{\mu_a(\lambda)-1} \tau_{\lambda k} t^k \mathrm{e}^{\lambda t} + \sum_{k=0}^{\mu_a(0)-\mu_b(0)} \tau_{0k} t^k, \quad t \in \mathbb{R}, \tag{5.2}$$

for certain random variables $\tau_{\lambda k}$, where

$$\tau_{0,\mu_a(0)-\mu_b(0)} = \frac{\sigma}{(\mu_a(0)-\mu_b(0))!} \left[\frac{b(z) z^{\mu_a(0)}}{a(z) z^{\mu_b(0)}} \right]_{z=0} \neq 0.$$

Hence the $t^{\mu_a(0)-\mu_b(0)}$-term is multiplied by a deterministic non-zero scalar, and letting $t \to \pm\infty$ in Eq. (5.2) one can easily see that $(Y_t)_{t\in\mathbb{R}}$ cannot be stationary.

(c) That $Y_t = 0$ is a strictly stationary solution is clear, as is its uniqueness under the given condition by factoring out the common zeroes of $a(\cdot)$ and $b(\cdot)$ and applying (a). On the other hand, if there is a zero λ_1 of $a(\cdot)$ such that $\mu_a(\lambda_1) < \mu_b(\lambda_1)$, then one can choose $\mathbf{X}_0 = \mathbf{V} = [V_1 \ \ldots \ V_p]'$ such that (5.1) holds with δ being uniformly distributed on the unit circle, and as in the proof of (a) we obtain the existence of another strictly stationary solution. $\square$

6. Conclusions

We have shown that if L is any non-deterministic Lévy process then Eqs. (1.1) and (1.2) defining the corresponding Lévy-driven CARMA process have a strictly stationary solution Y if and only if $E \log^+ |L_1| < \infty$ and all the singularities of the function $z \mapsto b(z)/a(z)$ on the imaginary axis are removable. Under these conditions the strictly stationary solution is unique and is specified explicitly as an integral with respect to L by Eqs. (3.6) and (3.7). The solution is not necessarily causal (i.e. Y_t is not necessarily a measurable function of $(L_s)_{s\le t}$ for all $t \in \mathbb{R}$). From (3.7) and Theorem 4.1, it follows that the solution is causal if and only if the singularities of the function $z \mapsto b(z)/a(z)$ on or to the right of the imaginary axis are removable.

We have also given conditions for existence and uniqueness of stationary solutions in the special case in which L is deterministic.

The results represent a significant generalization of existing results which focus on causal solutions only and which, apart from more restrictive sufficient conditions for the existence of strictly stationary solutions in the general case, are restricted to solutions of the Ornstein–Uhlenbeck equation and CARMA equations driven by Lévy processes with $EL(1)^2 < \infty$.

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