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Surviving particles for subcritical branching processes in random environment

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Abstract

The asymptotic behavior of a subcritical Branching Process in Random Environment (BPRE) starting with several particles depends on whether the BPRE is strongly subcritical (SS), intermediate subcritical (IS) or weakly subcritical (WS). In the (SS + IS) case, the asymptotic probability of survival is proportional to the initial number of particles, and conditionally on the survival of the population, only one initial particle survives $a.s$. These two properties do not hold in the (WS) case and different asymptotics are established, which require new results on random walks with negative drift. We provide an interpretation of these results by characterizing the sequence of environments selected when we condition on the survival of particles. This also raises the problem of the dependence of the Yaglom quasistationary distributions on the initial number of particles and the asymptotic behavior of the Q-process associated with a subcritical BPRE.

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1. Introduction

Let f be the generating function of a random probability measure on $\mathbb{N}$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of iid copies of f which serve as a random environment. We consider a Branching Process in Random Environment (BPRE) $(Z_n)_{n\in\mathbb{N}}$ induced by $(f_n)_{n\in\mathbb{N}}$ [1–5]. This means that

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conditionally on the environment $(f_n)_{n\in\mathbb{N}}$, particles at generation n reproduce independently of each other and their offsprings have generating function f_n.

We can think of a population of plants which have a one year life-cycle. Each year the weather conditions (the environment) vary, which impacts the reproductive success of the plant. Given the climate, all the plants reproduce according to the same mechanism.

Then Z_n is the number of particles at generation n and Z_{n+1} is the sum of Z_n independent random variables with generating function f_n. That is, for every $n \in \mathbb{N}$,

$$\mathbb{E}\left(s^{Z_{n+1}}|Z_0,\dots,Z_n;\, f_0,\dots,f_n\right) = f_n(s)^{Z_n} \quad (0 \le s \le 1).$$

In the whole paper, we denote by $\mathbb{P}_k$ the probability associated with k initial particles and $F_n := f_0 \circ \cdots \circ f_{n-1}$. Then, we have for every $k \in \mathbb{N}$,

$$\mathbb{E}_k(s^{Z_{n+1}} \mid f_0,\dots,f_n) = F_{n+1}(s)^k \quad (0 \le s \le 1).$$

When the environment is deterministic (i.e. f is a deterministic generating function), this process is the Galton–Watson process (GW) and f is the generating function of the reproduction law.

In this paper, we consider the subcritical case:

$$\mathbb{E}\left(\log(f'(1))\right) < 0.$$

Then extinction occurs $a.s.$, that is

$$\mathbb{P}(\exists n \in \mathbb{N} : Z_n = 0) = 1.$$

For a subcritical GW process, if further $\mathbb{E}(Z_1 \log^+(Z_1)) < \infty$, then there exists $c > 0$ such that $\mathbb{P}(Z_n > 0) \sim cf'(1)^n$ when n tends to infinity [6]. In random environments, the asymptotic behavior depends on whether the BPRE is strongly subcritical (SS), intermediate subcritical (IS) or weakly subcritical (WS) (see [4] or the Preliminaries Section for details). A subcritical GW process is always strongly subcritical (SS).

In this paper, we study the role of the initial number of particles in such limit theorems. For a GW process, particles are independent. As a consequence, limit theorems starting with several initial particles derive from those for a single initial particle. In random environment, particles do not reproduce independently. Independence holds only conditionally on the environment and asymptotics may differ from the GW case.

First, we determine the dependence of the asymptotic survival probability in terms of the initial number of particles. In that view, we define

$$\alpha_k := \lim_{n\to\infty} \mathbb{P}_k(Z_n > 0)/\mathbb{P}_1(Z_n > 0).$$

For a GW process, $\alpha_k = k$ and the asymptotic survival probability is proportional to the initial number of particles. This equality still holds in the (SS + IS) case for BPRE, but not in the (WS) case where a different asymptotic behavior as $k \to \infty$ is established. For the proof, we need an asymptotic result on random walks with negative drift, which gives the sum of the logarithms of the mean number of offsprings for the successive environments. We refer to [7] for asymptotics of the extinction probability when the number of initial particles tends to infinity in the supercritical case.

Moreover, when the BPRE is (SS) or (IS), if we condition on the survival of the population at generation n, then only one initial particle survives at generation n when $n \to \infty$, just as for a

GW process. But this does not hold in the (WS) case, as stated in Section 3.2. Thus, (WS) BPRE conditioned to survive has a supercritical behavior, as previously observed in [2].

In Section 3.3, we give an interpretation of these results in terms of environments. Conditioning on non-extinction induces a selection of environments with high reproduction law. In the (SS + IS) case, we prove that the survival probability of the branching process in the selected environments is still zero. This is obvious if environments are $a.s.$ subcritical, i.e. $f'(1) < 1 a.s.$ But in the (WS) case, conditioning on the survival of the population selects only supercritical environments, which means that the sequence of selected environments has $a.s.$ a positive survival probability. Finally letting the initial number of particles tend to infinity, the sequence of environments selected by conditioning on the survival of the population becomes subcritical again.

Finally, in Section 3.4, we consider the size of the population conditioned to survive and we are interested in the characterization of the Yaglom quasistationary distributions starting from k particles:

$$\lim_{n\to\infty} \mathbb{P}_k(Z_n = i \mid Z_n > 0) \quad (i \geq 1).$$

In Section 3.5, we focus on the Q-process associated to the subcritical BPRE, which is defined for all $l_1, l_2, \ldots, l_n \in \mathbb{N}$, by

$$\mathbb{P}_k(Y_1 = l_1, \ldots, Y_n = l_n) = \lim_{p\to\infty} \mathbb{P}_k(Z_1 = l_1, \ldots, Z_n = l_n \mid Z_{n+p} > 0).$$

See [6] for details on the Q-process associated to GW. Again, these results depend on the subcritical regime.

2. Preliminaries

We start by recalling some known results for subcritical BPRE. Note that $s \in \mathbb{R}^+ \mapsto \mathbb{E}(f'(1)^s)$ is a convex function and define γ and α in [0, 1] such that

$$\gamma := \inf_{\theta\in[0,1]} \left\{\mathbb{E}\left(f'(1)^\theta\right)\right\} = \mathbb{E}\left(f'(1)^\alpha\right). \tag{1}$$

From now on, we assume $\mathbb{E}(f'(1)|\log(f'(1))|) < \infty$. Note that $0 < \gamma < 1$, $\gamma \leq \mathbb{E}(f'(1))$, and

$$\gamma = \mathbb{E}(f'(1)) \Leftrightarrow \mathbb{E}\left(f'(1)\log(f'(1))\right) \leq 0.$$

There are three subcritical regimes (see [4]).

- ⋆ The strongly subcritical case (SS), where $\mathbb{E}(f'(1)\log(f'(1))) < 0$. In this case, assuming further

  $$\mathbb{E}(Z_1 \log^+(Z_1)) < \infty,$$

  then there exist $c, \alpha_k > 0$ such that, as $n \to \infty$:

  $$\mathbb{P}_k(Z_n > 0) \sim c\alpha_k \mathbb{E}(f'(1))^n, \quad \alpha_1 = 1. \tag{2}$$

- ⋆ The intermediate subcritical case (IS), where $\mathbb{E}(f'(1)\log(f'(1))) = 0$. In this case, assuming further

  $$\mathbb{E}\left(f'(1)\log^2(f'(1))\right) < \infty, \qquad \mathbb{E}\left([1+\log^-(f'(1))]f''(1)\right) < \infty,$$

then there exist $c, \alpha_k > 0$ such that as $n \to \infty$:

$$\mathbb{P}_k(Z_n > 0) \sim c\alpha_k n^{-1/2}\mathbb{E}(f'(1))^n, \quad \alpha_1 = 1. \tag{3}$$

⋆ The weakly subcritical case (WS), where $0 < \mathbb{E}(f'(1)\log(f'(1))) < \infty$. In this case, assuming further

$$\mathbb{E}(f''(1)/f'(1)^{1-\alpha}) < \infty, \qquad \mathbb{E}(f''(1)/f'(1)^{2-\alpha}) < \infty,$$

then there exist $c, \alpha_k > 0$ such that as $n \to \infty$:

$$\mathbb{P}_k(Z_n > 0) \sim c\alpha_k n^{-3/2}\gamma^n, \quad \alpha_1 = 1. \tag{4}$$

In the rest of the paper, we take the integrability assumptions above for granted for each case. See [8] for asymptotics with a weaker hypothesis in the (IS) case.

It is also known that the process Z_n starting from k particles and conditioned to be non-zero converges to a finite positive random variable Υ_k, called the Yaglom quasistationary distribution (see [4]):

$$\mathbb{E}_k\left(s^{Z_n}|Z_n > 0\right) \overset{n\to\infty}{\longrightarrow} \mathbb{E}\left(s^{\Upsilon_k}\right).$$

See Section 3.3 for discussions about $(\Upsilon_k)_{k\in\mathbb{N}}$.

Actually, in [4], the result and the proof of the convergence are given for $k = 1$. It can be generalized to $k \geq 1$ with the following modifications. We borrow notations from [4]

$$f_{k,l} := \begin{cases} f_k \circ f_{k+1} \circ \cdots \circ f_{l-1}, & k < l \\ f_{k-1} \circ f_{k-2} \circ \cdots \circ f_l, & k > l \\ \text{id}, & k = l. \end{cases}$$

Then $1 - \mathbb{E}_k(s^{Z_n}|Z_n > 0) = \mathbb{E}(1 - f_{0,n}^k(s))/\mathbb{P}_k(Z_n > 0)$. Lemma 2.1 of [4] still holds replacing $f_{0,n}$ by $f_{0,n}^k$ and $\mathbb{P}(Z_n > 0)$ by $\mathbb{P}_k(Z_n > 0)$. Lemma 2.2 also still holds and results of Lemma 2.3 can now be stated as follows. By convexity of $x \in [0,1] \to x^k$ and $(f_n)_{n\in\mathbb{N}}$, for every $n \geq 0$, we have $a.s.$ $\exp(-S_i)(1 - f_{i,0}(s)^k) \leq 1$ $(0 \leq s \leq 1)$, where $S_i = \log(k) + \log(f_0'(1)) + \cdots + \log(f_{n-1}'(1))$. Moreover $\exp(-S_n)(1 - f_{n,0}(s)^k)$ converges $a.s.$ as $n \to \infty$, which is a direct consequence of the convergence for $k = 1$ given in Lemma 2.3 in [4] (noting also that this implies $f_{n,0}(s) \to 0$ $a.s.$ as $n \to \infty$).

Finally, we consider the case where the generating functions of the reproduction laws are a.s. linear fractional. Indeed in this case the survival probability in a given environment can be computed explicitly since linear fractional generating functions are stable by composition. Specifically, we suppose that

$$f(s) = 1 - \frac{A}{1-B} + \frac{As}{1-Bs} \quad \text{a.s. } (0 \leq s \leq 1), \tag{5}$$

where A, B are two r.v. such that $A \in [0,1]$, $B \in [0,1)$ and $A + B \leq 1$. In this case, setting for every $i \in \mathbb{N}$,

$$P_i := f_{n-i}'(1)\ldots f_{n-1}'(1), \qquad (P_0 = 1),$$

we have (see [9,10] or [11])

$$\mathbb{P}_1(Z_n > 0 \mid f_0, \ldots, f_{n-1}) = 1 - F_n(0) = \left(1 + \sum_{i=0}^{n-1} \frac{f_{n-i-1}''(1)}{2f_{n-i-1}'(1)} P_i\right)^{-1} P_n. \tag{6}$$

Let us label by $i \in \mathbb{N}$ the initial particles and denote by $Z_n^{(i)}$ the number of descendants of particle i at generation n. As conditionally on $(f_0, \ldots, f_{n-1})$, $(Z_n^{(i)}, i \geq 1)$ is an iid sequence, we get

$$\mathbb{P}_k(Z_n^{(1)} > 0, \ldots, Z_n^{(k)} > 0 \mid f_0, \ldots, f_{n-1}) = \left(1 + \sum_{i=0}^{n-1} \frac{f''_{n-i-1}(1)}{2 f'_{n-i-1}(1)} P_i\right)^{-k} P_n^k. \tag{7}$$

We can get now lower bounds for survival probabilities of a general BPRE by a coupling argument. We use the fact that for every probability generating function f_i, we can find a linear fractional probability generating function $\tilde{f_i}$ such that for every $s \in [0, 1]$, $\tilde{f_i}(s) \geq f_i(s)$, $\tilde{f}_i'(1) = f_i'(1)$, $\tilde{f}_i''(1) = 2 f_i''(1)$ (see [10] or [11]). Then, $\tilde{F}_n(0) \geq F_n(0)$ a.s. ensures that

$$\mathbb{P}_1(Z_n > 0 \mid f_0, \ldots, f_{n-1}) \geq \mathbb{P}_1(\tilde{Z}_n > 0 \mid \tilde{f}_0, \ldots, \tilde{f}_{n-1}) \quad \text{a.s.} \tag{8}$$

More generally, for every $k \geq 1$,

$$\begin{aligned} &\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \ldots, Z_n^{(k)} > 0 \mid f_0, \ldots, f_{n-1}) \\ &\quad = (1 - F_n(0))^k \\ &\quad \geq (1 - \tilde{F}_n(0))^k \\ &\quad = \mathbb{P}_k(\tilde{Z}_n^{(1)} > 0, \tilde{Z}_n^{(2)} > 0, \ldots, \tilde{Z}_n^{(k)} > 0 \mid \tilde{f}_0, \ldots, \tilde{f}_{n-1}) \quad \text{a.s.} \end{aligned} \tag{9}$$

3. Subcriticality starting from several particles

We specify here the asymptotics of survival probabilities starting with k particles. Then we determine how many initial particles survive conditionally on non-extinction of particles and we characterize the sequence of environments which are selected by this conditioning. Finally we consider the Yaglom quasistationary distributions of $(Z_n)_{n\in\mathbb{N}}$ and the associated Q-process. In the (SS) case, the results are those expected, *i.e.* they are analogous to those of a GW process. In the (IS) case, results are different for the Yaglom quasistationary distribution and the Q-process. In the (WS) case, all results are different.

Recall that we label by $i \in \mathbb{N}$ each particle of the initial population and denote by $Z_n^{(i)}$ the number of descendants of particle i at generation n. Thus $(Z_n^{(i)})_{n\in\mathbb{N}}$ are identically distributed BPRE ($i \in \mathbb{N}$), with common distribution $(Z_n)_{n\in\mathbb{N}}$ starting with one particle. Conditionally on the environments, these processes are independent: for all $n, k, l_i \in \mathbb{N}$,

$$\mathbb{P}_k(Z_n^{(i)} = l_i, 1 \leq i \leq k \mid f_0, \ldots, f_{n-1}) = \prod_{i=1}^{k} \mathbb{P}_1(Z_n = l_i \mid f_0, \ldots, f_{n-1}).$$

Moreover, under $\mathbb{P}_k$, $(Z_n)_{n\in\mathbb{N}}$ is a.s. equal to $\left(\sum_{i=1}^k Z_n^{(i)}\right)_{n\in\mathbb{N}}$.

3.1. Survival probabilities starting with several particles

Note that $x \mapsto \mathbb{E}\left(f'(1)^x \log(f'(1))\right)$ increases with x.

Proposition 1. *For every $k \in \mathbb{N}^*$,*

(i) *If $\mathbb{E}\left(f'(1)^k log(f'(1))\right) < 0$, then there exists $c_k > 0$ such that*

$$\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \ldots, Z_n^{(k)} > 0) \overset{n\to\infty}{\sim} c_k \mathbb{E}(f'(1)^k)^n$$

and $\mathbb{E}(f'(1)^k) < \mathbb{E}(f'(1)^{k-1}) < \cdots < \mathbb{E}(f'(1))$.

(ii) *If* $\mathbb{E}\left(f'(1)^k log(f'(1))\right) = 0$, *then there exists* $c_k > 0$ *such that*

$$\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \ldots, Z_n^{(k)} > 0) \overset{n\to\infty}{\sim} c_k n^{-1/2}\mathbb{E}(f'(1)^k)^n.$$

(iii) *If* $\mathbb{E}\left(f'(1)^k log(f'(1))\right) > 0$, *then there exists* $c_k > 0$ *such that*

$$\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \ldots, Z_n^{(k)} > 0) \overset{n\to\infty}{\sim} c_k n^{-3/2}\tilde{\gamma}^n,$$

with $\tilde{\gamma} = \inf_{u\in\mathbb{R}^+}\{\mathbb{E}(f'(1)^u)\} \in (0, 1)$ *and* $c = c_1 \geq c_2 \geq \cdots \geq c_k$.
Moreover, in the (IS + WS) *case,* $\tilde{\gamma} = \gamma$. *In the* (SS) *case,* $\tilde{\gamma} < \gamma = \mathbb{E}(f'(1))$.

The proof is given in Section 4.1 and uses the case where the probability generating function f is $a.s.$ linear fractional.

In the (SS + IS) case, the asymptotic probability of survival of particles is proportional to the number of initial particles, as stated below. This is not surprising and well known for subcritical GW process. But this does not hold in the (WS) case. Recall that α_k is defined as $\lim_{n\to\infty}\mathbb{P}_k(Z_n > 0)/\mathbb{P}_1(Z_n > 0)$.

Theorem 2. *In the* (SS + IS) *case, for every* $k \in \mathbb{N}$, $\alpha_k = k$.
In the (WS) *case,* $\alpha_k \to \infty$ *as* $k \to \infty$ *and there exists* $M_+ > 0$ *such that*

$$\alpha_k \leq M_+ k^\alpha \log(k), \quad (k \geq 2),$$

where $\alpha \in (0, 1)$ *is given by* (1).
Assuming further $\mathbb{E}(f'(1)^{1/2}\log(f'(1))) > 0$ *(i.e.* $\alpha < 1/2$*) and that* $f''(1)/f'(1)$ *is bounded by a constant, there exists* $M_- > 0$ *such that*

$$\alpha_k \geq M_- k^\alpha \log(k), \quad (k \in \mathbb{N}).$$

One can naturally conjecture that the last result still holds for $1/2 \leq \alpha < 1$. The proof also uses the linear fractional case where, conditionally on the environments, the survival probability is related to a random walk whose jumps are the log of the means of the reproduction law of the environments. This is why we need to prove a result about random walk with negative drift conditioned to be larger than $-x < 0$ (see the Appendix). One way to generalize the last result of the theorem above to the case $\mathbb{E}(f'(1)^{1/2}\log(f'(1))) > 0$ (i.e. $\alpha < 1/2$) would be to improve Lemma 11.

3.2. Survival of initial particles conditionally on non-extinction

We turn our attention to the number of particles that survive when we condition on the survival of the whole population of particles. More precisely, denote by N_n the number of particles in generation 0 whose descendant is alive at generation n. That is, starting with k particles:

$$N_n := \#\{1 \leq i \leq k : Z_n^{(i)} > 0\}.$$

We have the following elementary consequence of Proposition 1.

Proposition 3. *In the* (SS + IS) *case, for every* $k \geq 1$,

$$\lim_{n\to\infty}\mathbb{P}_k(N_n > 1 \mid Z_n > 0) = 0.$$

In the (WS) *case, for every* $k \geq 1$,

$$\lim_{n\to\infty}\mathbb{P}_k(N_n = k|Z_n > 0) > 0.$$

Thus, for (SS + IS) BPRE, conditionally on the survival of the population, only one initial particle survives, as for GW. But for (WS) BPRE, several initial particles survive with positive probability. The forthcoming Theorem 5 gives an interpretation of this property in terms of selection of favorable environments by conditioning on non-extinction. This result has an application to the branching model for cell division with parasite infection considered in [12]. In particular it ensures that the separation of descendants of parasites (see section 6.3 in [12]) holds only in the (SS + IS) case. In the same vein, we refer to [13] for results on the reduced process associated with subcritical BPRE in the linear fractional case: In the (WS) case, the number of particles of the reduced process is not $a.s.$ equal to 1 in the first generations.

We next consider the situation when the number of initial particles tends to infinity in the (WS) case. We shall see that the number of initial particles which survive conditionally on non-extinction is finite $a.s.$ but not bounded.

Theorem 4. *In the* (WS) *case, assuming* $\mathbb{E}(f'(1)^{1/2}\log(f'(1))) > 0$ *(i.e.* $\alpha < 1/2$*) and that* $f''(1)/f'(1)$ *is bounded by a constant, there exist* $A_l \downarrow_{l\to\infty} 0$ *such that for all* $k \geq l \geq 0$*,*

$$\limsup_{n\to\infty} \mathbb{P}_k(N_n \geq l \mid Z_n > 0) \leq A_l.$$

Moreover, for every $l \in \mathbb{N}^*$*,*

$$\liminf_{k\to\infty} \liminf_{n\to\infty} \mathbb{P}_k(N_n = l \mid Z_n > 0) > 0.$$

Thus, under the conditions of the theorem,

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}_k(N_n \geq l \mid Z_n > 0) \leq A_l, \quad \text{with } A_l \downarrow_{l\to\infty} 0.$$

3.3. *Selection of environments conditionally on non-extinction*

We characterize here the sequence of environments which are selected by conditioning on the survival of particles.

We denote by $\mathcal{F}$ the set of generating functions and for every $\mathbf{g}_n = (g_0, \dots, g_{n-1}) \in \mathcal{F}^n$, by $Z_{\mathbf{g}_n}$ the value at generation n of the branching process in varying environment whose reproduction law at generation $l \leq n-1$ has generating function g_l. Thus, for every $k \geq 1$,

$$\mathbb{E}_k(s^{Z_{\mathbf{g}_n}}) = [g_0 \circ g_1 \circ \cdots \circ g_{n-1}(s)]^k \quad (0 \leq s \leq 1). \tag{10}$$

Then we denote by $p(\mathbf{g}_n)$ the survival probability of a particle in environment $\mathbf{g}_n$:

$$p(\mathbf{g}_n) := \mathbb{P}_1(Z_{\mathbf{g}_n} > 0). \tag{11}$$

Denote by $\mathbf{f}_n$ the sequence of environments until time n, $i.e.$

$$\mathbf{f}_n := (f_0, f_1, \dots, f_{n-1}).$$

In the subcritical case, $p(\mathbf{f}_n) \to 0$ $a.s.$ as $n \to \infty$ since $(Z_n)_{n\in\mathbb{N}}$ becomes extinct $a.s.$ Roughly speaking, the sequences of environments have a.s. zero survival probability. In the (SS+IS) case, conditioning on the survival of particles does not change this fact, but it does in the (WS) case, as we can guess using Proposition 3. Coming back to the model of plants in random weather, the survival of flowers in the (SS + IS) case is due to the exceptional reproduction of plants (despite the weather), whereas in the (WS) case it is due to nice weather (and regular reproduction of plants).

More precisely, we prove that in the (WS) case, the sequence of environments which are selected by conditioning on $Z_n > 0$ have $a.s.$ a positive survival probability. Thus, they are 'supercritical'. In [2], the authors had already remarked this supercritical behavior of the BPRE $(Z_n)_{n\in\mathbb{N}}$ in the (WS) case by giving an analog of the Kesten–Stigum theorem, i.e. the convergence of Z_n/m^n.

Theorem 5. *In the* (SS + IS) *case, for all* $k \geq 1$, $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0) = 0.$$

In the (WS) *case, for every* $k \geq 1$,

$$\liminf_{n\to\infty} \mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0) \overset{\epsilon\to 0+}{\longrightarrow} 1.$$

This supercritical behavior in the (WS) case disappears as k tends to infinity. That is, the survival probability of selected sequences of environments tends to 0 as the number of particles grows to infinity.

Proposition 6. *In the* (WS) *case, for every* $\epsilon > 0$,

$$\limsup_{n\to\infty} \mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0) \overset{k\to\infty}{\longrightarrow} 0.$$

In other words, conditionally on the survival of Z_n, the more initial particles there are, the less environments need to be favorable to allow the survival of the population, and the less likely it is for a given particle to survive. This explains why letting the number of initial particles tend to infinity does not make the number of surviving initial particles tend to infinity, as stated in Theorem 4.

3.4. *Yaglom quasistationary distributions*

We focus now on the Yaglom quasistationary distribution of $(Z_n)_{n\in\mathbb{N}}$ (see Preliminaries for existence and references). For the GW process, this distribution does not depend on the initial number of particles and is characterized by a functional equation. This result still holds for (SS) BPRE. Indeed, starting with several particles, conditionally on the survival of one given particle, the others become extinct (see Proposition 3). Recalling that in the (SS+IS) case, $\gamma = \mathbb{E}(f'(1))$, and writing $p.g.f.$ for probability generating function, we have the following statement.

Theorem 7. *For every* $k \geq 1$, *the BPRE* Z_n *starting from* k *and conditioned to be positive converges in distribution as* $n \to \infty$ *to a r.v.* Υ_k, *whose p.g.f.* G_k *verifies*

$$\mathbb{E}(G_k(f(s))) = \gamma G_k(s) + 1 - \gamma \quad (0 \leq s \leq 1).$$

In the (SS + IS) *case, the distribution of* Υ_k *does not depend on* k.

Moreover, in the (SS) *case, the common p.g.f. of* $(\Upsilon_k : k \geq 1)$ *is the unique p.g.f.* G *which satisfies the functional equation above and* $G'(1) < \infty$.

In the (WS) case, we leave open the question of determining whether the quasistationary distribution Υ_k depends on the initial number k of particles. We know that for every $k \geq 1$, G_k

verifies the same functional equation given above but we do not know if the solution is unique. Moreover, other observations also lead us to believe that quasistationary distributions Υ_k might not depend on k. For example, we can prove that if $Z_1 \in \{0, 1, N\}$ for some $N \in \mathbb{N}^*$, then $\Upsilon_1 \stackrel{d}{=} \Upsilon_N$.

3.5. Q-process associated with a BPRE

The Q-process $(Y_n)_{n\in\mathbb{N}}$ is the BPRE $(Z_n)_{n\in\mathbb{N}}$ conditioned to survive in the distant future. See [6] for details in the case of GW processes. In the (SS) case, the Q-process converges in distribution to the size biased Yaglom distribution, as for GW process and finer results have been obtained in [14]. In the (IS + WS) case, the Q-process is transient. That is, the population needs to grow largely in the first generations so that it can survive.

Recall that for all $l_1, l_2, \ldots, l_n \in \mathbb{N}$,

$$\mathbb{P}_k(Y_1 = l_1, \ldots, Y_n = l_n) = \lim_{p\to\infty} \mathbb{P}_k(Z_1 = l_1, \ldots, Z_n = l_n | Z_{n+p} > 0).$$

Proposition 8. ⋆ *In the* (SS) *case, for every $k \in \mathbb{N}^*$, for all $l_1, l_2, \ldots, l_n \in \mathbb{N}$,*

$$\mathbb{P}_k(Y_1 = l_1, \ldots, Y_n = l_n) = [\mathbb{E}(f'(1))]^{-n} \frac{l_n}{k} \mathbb{P}_k(Z_1 = l_1, \ldots, Z_n = l_n).$$

Moreover $(Y_n)_{n\in\mathbb{N}}$ converges in distribution to the size biased Yaglom distribution.

$$\forall l \geq 0, \quad \mathbb{P}_k(Y_n = l) \stackrel{n\to\infty}{\longrightarrow} \frac{l\mathbb{P}(\Upsilon = l)}{\mathbb{E}(\Upsilon)}.$$

⋆ *In the* (IS) *case, for every $k \in \mathbb{N}^*$, for all $l_1, l_2, \ldots, l_n \in \mathbb{N}$,*

$$\mathbb{P}_k(Y_1 = l_1, \ldots, Y_n = l_n) = \mathbb{E}(f'(1))^{-n} \frac{l_n}{k} \mathbb{P}_k(Z_1 = l_1, \ldots, Z_n = l_n).$$

Moreover $Y_n \to \infty$ in probability as $n \to \infty$.

⋆ *In the* (WS) *case, for every $k \in \mathbb{N}^*$, for all $l_1, l_2, \ldots, l_n \in \mathbb{N}$,*

$$\mathbb{P}_k(Y_1 = l_1, \ldots, Y_n = l_n) = \gamma^{-n} \frac{\alpha_{l_n}}{\alpha_k} \mathbb{P}_k(Z_1 = l_1, \ldots, Z_n = l_n).$$

Moreover Y_n tends to infinity a.s.

We focus now on the environments of the Q-process. We endow $\mathcal{F}$ with distance d given by the infinity norm

$$d(f, g) = \|f - g\|_\infty$$

and we denote by $\mathcal{B}(\mathcal{F})$ the Borel σ-field.

We introduce the probability ν_k on $(\mathcal{F}^{\mathbb{N}}, \mathcal{B}(\mathcal{F})^{\otimes\mathbb{N}})$ which gives the distribution of the environments when the BPRE $(Z_n)_{n\in\mathbb{N}}$ starting from k particles is conditioned to survive. Using the Kolomogorov Theorem, it can be specified by its projection on $(\mathcal{F}^p, \mathcal{B}(\mathcal{F})^{\otimes p})$ for every $p \in \mathbb{N}$, denoted by $\nu_{k|\mathcal{F}^p}$,

$$\begin{aligned}\nu_{k|\mathcal{F}^p}(d\mathbf{g}_p) &:= \lim_{n\to\infty} \mathbb{P}_k\left(\mathbf{f}_p \in d\mathbf{g}_p | Z_{n+p} > 0\right) \\ &= \gamma^{-p}\mathbb{P}\left(\mathbf{f}_p \in d\mathbf{g}_p\right) \sum_{l=1}^{\infty} \mathbb{P}_k(Z_{\mathbf{g}_p} = l)\frac{\alpha_l}{\alpha_k},\end{aligned} \tag{12}$$

with $\mathbf{f}_p = (f_0, \ldots, f_{p-1})$ and $\gamma = \mathbb{E}(f'(1))$ in the (SS + IS) case. The limit is the weak limit of probabilities on $(\mathcal{F}^p, \mathcal{B}(\mathcal{F})^{\otimes p})$ (see [15] for the definition and Section 4.5 for the proof), which we endow with the distance d_p given by

$$d_p\left((g_0, \ldots, g_{p-1}), (h_0, \ldots, h_{p-1})\right) = \sup\{\|g_i - h_i\|_\infty : 0 \le i \le p-1\}. \tag{13}$$

For every $\mathbf{g} \in \mathcal{F}^{\mathbb{N}}$, we denote by $\mathbf{g}|n$ the first n coordinates of $\mathbf{g} \in \mathcal{F}^{\mathbb{N}}$ and we introduce the survival probability in environment $\mathbf{g} \in \mathcal{F}^{\mathbb{N}}$:

$$p(\mathbf{g}) = \lim_{n\to\infty} \mathbb{P}(Z_{\mathbf{g}|n} > 0).$$

One can naturally conjecture an analog of Theorem 5 and Proposition 6. That is, for every $k \in \mathbb{N}^*$,

In the (SS + IS) case, $\quad \nu_k(\{\mathbf{g} \in \mathcal{F}^{\mathbb{N}} : p(\mathbf{g}) = 0\}) = 1.$

In the (WS) case, $\quad \nu_k(\{\mathbf{g} \in \mathcal{F}^{\mathbb{N}} : p(\mathbf{g}) > 0\}) = 1 \quad$ and $\quad \nu_k(p(\mathbf{f}) \in \mathrm{d}x) \stackrel{k\to\infty}{\Longrightarrow} \delta_0(\mathrm{d}x).$

A perspective is to characterize the tree of particles when we condition on the survival of particles, i.e. the tree of particles of the Q-process. Informally, for a GW process, this gives a spine with finite iid subtrees (see [16,17]). This fact still holds in the (SS + IS) case but we will observe a 'multispine tree' in the (WS) case.

4. Proofs

Recall that $\mathbf{f}_n = (f_0, \ldots, f_{n-1})$ and set for every $n \in \mathbb{N}$,

$$X_n := \log(f_n'(1)), \qquad S_n := \sum_{i=0}^{n-1} X_i \quad (S_0 = 0),$$
$$L_n := \min\{S_i : 1 \le i \le n\}.$$

To get limit theorems starting from k particles, we will work conditionally on the environments so that particles reproduce independently. Thus, we need to control the asymptotic distribution of $p(\mathbf{f}_n) = \mathbb{P}_1(Z_n > 0 \mid \mathbf{f}_n)$. Roughly speaking, we prove now that $p(\mathbf{f}_n) \approx \exp(L_n)$ *a.s.* as $n \to \infty$. The proof relies on the fact that in the fractional linear case, we can compute the survival probability at time n as a function of the random walk $(S_i, 1 \le i \le n)$ (see Section 2). We use then a result on random walk with negative drift conditioned to be above $x < 0$ given in the Appendix to get the lower bound in the linear fractional case. The lower bound in the general case follows by a coupling argument, whereas the upper bound is easy.

Lemma 9. *For every $n \in \mathbb{N}$, we have*

$$p(\mathbf{f}_n) \le \exp(L_n) \quad a.s.$$

Moreover if $\mathbb{E}(f'(1)^{1/2}\log(f'(1))) > 0$ (i.e. $0 < \alpha < 1/2$) and $f''(1)/f'(1)$ is bounded, then there exists $\mu \ge 1$ such that for all $n \in \mathbb{N}$ and $x \in (0, 1]$,

$$\mathbb{P}(p(\mathbf{f}_n) \ge x) \ge \mathbb{P}(L_n \ge \log(\mu x))/4.$$

Proof. For the upper bound, note that all $n \in \mathbb{N}$ and $\mathbf{g}_n \in \mathcal{F}^n$, we have,

$$p(\mathbf{g}_n) = \mathbb{P}_1(Z_{\mathbf{g}_n} > 0) \le \mathbb{E}_1(Z_{\mathbf{g}_n}) = \prod_{i=0}^{n-1} g_i'(1).$$

Thus $p(\mathbf{f}_n) \le \mathrm{e}^{S_n} a.s.$ Adding that $p(\mathbf{f}_n)$ decreases $a.s.$ ensures that

$$p(\mathbf{f}_n) \le \mathrm{e}^{L_n} \quad \text{a.s.}$$

For the lower bound, use (8) and (6) to get

$$p(\mathbf{f}_n) \ge p(\tilde{\mathbf{f}}_n) = \frac{\tilde{P}_n}{1 + \sum_{i=0}^{n-1} \frac{\tilde{f}''_{n-i-1}(1)}{2\tilde{f}'_{n-i-1}(1)} \tilde{P}_i} = \frac{P_n}{1 + \sum_{i=0}^{n-1} \frac{f''_{n-i-1}(1)}{f'_{n-i-1}(1)} P_i} \quad \text{a.s.},$$

where $P_i := f'_{n-i}(1) \dots f'_{n-1}(1)$ $(P_0 = 1)$. Define

$$S'_i := \log(f'_{n-i}(1)) + \dots + \log(f'_{n-1}(1)) \quad (1 \le i \le n), \qquad S'_0 = 0.$$

Then $P_i = \exp(S'_i)$ and assuming that $C := \left(1 + \text{ess sup}(\frac{f''(1)}{f'(1)})\right)^{-1} > 0$, we have

$$p(\mathbf{f}_n) \ge C \frac{\mathrm{e}^{S'_n}}{2 \sum_{i=0}^{n-1} \mathrm{e}^{S'_i}} \ge \frac{C}{2} \frac{\mathrm{e}^{S'_n - \max\{S'_j : 0 \le j \le n\}}}{\sum_{i=0}^{n} \mathrm{e}^{S'_i - \max\{S'_j : 0 \le j \le n\}}} \quad \text{a.s.}$$

Thus,

$$p(\mathbf{f}_n) \ge \frac{C}{2} \frac{\mathrm{e}^{L_n}}{\sum_{i=0}^{n} \mathrm{e}^{L_n - S_i}}. \tag{14}$$

As $\alpha < 1/2$, the forthcoming Corollary 12 in the Appendix ensures that there exists $\beta > 0$ such that for all $n \in \mathbb{N}$ and $x \in (0, 1]$,

$$\begin{aligned} \mathbb{P}(p(\mathbf{f}_n) \ge x) &\ge \mathbb{P}(L_n \ge \log(2\beta x/C)) \mathbb{P}\left(\sum_{i=0}^{n} \mathrm{e}^{L_n - S_i} \le \beta \mid L_n \ge \log(2\beta x/C)\right) \\ &\ge \mathbb{P}(L_n \ge \log(\mu x))/4, \end{aligned}$$

writing $\mu = \min(1, 2\beta/C)$. $\square$

4.1. *Proofs of Section 3.1*

First we give the proof of Proposition 1, which is split into three parts. It follows the proof of Theorem 1.2 in [10]. Using also the lemma above, we are then able to prove Theorem 2.

Proof of Proposition 1(i). We follow the proof of Theorem 1.2(a) in [10] and introduce the probability $\tilde{\mathbb{P}}$ such that under $\tilde{\mathbb{P}}$, the environments still are iid and their law is given by

$$\tilde{\mathbb{P}}(f \in \mathrm{d}g) = \mathbb{E}(f'(1)^k)^{-1} g'(1)^k \mathbb{P}(f \in \mathrm{d}g).$$

Then, writing $P_n = f'_0(1) \dots f'_{n-1}(1)(P_0 = 1)$, we have

$$\mathbb{P}(Z_n^{(1)} > 0, \dots, Z_n^{(k)} > 0) = \mathbb{E}((1 - F_n(0))^k) = \mathbb{E}(f'(1)^k)^n \tilde{\mathbb{E}}(((1 - F_n(0))/P_n)^k).$$

As $\mathbb{E}(f'(1)^k \log(f'(1))) < 0$, then $\tilde{\mathbb{E}}(\log(f'(1))) < 0$ and Theorem 5 in [3] ensures that

$$C = \lim_{n\to\infty} \frac{1 - F_n(0)}{P_n}$$

exists $\tilde{\mathbb{P}}$ a.s. and belongs to $]0, 1]$. Thus, as $n \to \infty$,

$$\mathbb{P}_k(Z_n^{(1)} > 0, \ldots, Z_n^{(k)} > 0) \sim \mathbb{E}(f'(1)^k)^n \tilde{\mathbb{E}}(C^k).$$

Add that $s \mapsto \mathbb{E}(f'(1)^s)$ decreases for $s \in [0, \alpha]$ and $k < \alpha$ to complete the proof, where α is given by (1). $\square$

Proof of Proposition 1(iii). We follow the proof of Theorem 1.2 (c) in [10].

Step 1. First we consider the linear fractional case. In that case, by (7),

$$\mathbb{P}_k(Z_n^{(1)} > 0, \ldots, Z_n^{(k)} > 0 | f_0, \ldots, f_{n-1}) = \left(1 + \sum_{i=0}^{n-1} \frac{f''_{n-i-1}(1)}{2 f'_{n-i-1}(1)} P_i\right)^{-k} P_n^k.$$

Define $\tilde{\gamma}$ by

$$\tilde{\gamma} = \inf_{s\in\mathbb{R}^+} \left\{\mathbb{E}\left(f'(1)^s\right)\right\} = \mathbb{E}\left(f'(1)^{\tilde{\alpha}}\right),$$

where $0 < \tilde{\alpha} < k$ since $\mathbb{E}(f'(1)^k \log(f'(1))) > 0$. Let $\mathbb{P}_{\tilde{\alpha}}$ be the probability given by

$$\mathbb{P}_{\tilde{\alpha}}(f \in \mathrm{d}g) = \tilde{\gamma}^{-1} g'(1)^{\tilde{\alpha}} \mathbb{P}(f \in \mathrm{d}g).$$

Then

$$\mathbb{P}_k(Z_n^{(1)} > 0, \ldots, Z_n^{(k)} > 0) = \tilde{\gamma}^n \mathbb{E}_{\tilde{\alpha}} \left[\left(1 + \sum_{i=0}^{n-1} \frac{f''_i(1)}{2 f'_i(1)} P_i\right)^{-k} P_n^{k-\tilde{\alpha}}\right].$$

As $\mathbb{E}_{\tilde{\alpha}}(\log(f'(1))) = 0$, we apply Theorem 2.1 in [10] with

$$\phi(x) = x^{k-\tilde{\alpha}}, \qquad \psi(x) = (1 + x)^{-k}, \quad 0 < k - \tilde{\alpha} < k,$$

so there exists $c_k > 0$ such that, as $n \to \infty$,

$$\mathbb{P}_k(Z_n^{(1)} > 0, \ldots, Z_n^{(k)} > 0) \sim c_k \tilde{\gamma}^n n^{-3/2}.$$

Step 2. For the general case, we can use Step 1. Indeed, by (9), there exists a BPRE $(\tilde{Z}_n)_{n\in\mathbb{N}}$ such that $\tilde{f}$ is $a.s.$ linear fractional, $\tilde{f}'(1) = f'(1)$ and

$$\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \ldots, Z_n^{(k)} > 0) \geq \mathbb{P}_k(\tilde{Z}_n^{(1)} > 0, \tilde{Z}_n^{(2)} > 0, \ldots, \tilde{Z}_n^{(k)} > 0).$$

By Step 1, this yields the existence of $c_k(1) > 0$ such that

$$\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \ldots, Z_n^{(k)} > 0) \geq c_k(1)\gamma^n n^{-3/2}. \tag{15}$$

Note that by the inclusion–exclusion principle, we have

$$\mathbb{P}_k(Z_n > 0) = \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \mathbb{P}(Z_n^{(1)} > 0, \ldots, Z_n^{(i)} > 0). \tag{16}$$

Moreover, (4) ensures the convergence of $\gamma^{-n} n^{3/2} \mathbb{P}_1(Z_n > 0)$ to $c\alpha_1$. By induction, it gives the convergence of

$$\gamma^{-n} n^{3/2} \mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \ldots, Z_n^{(k)} > 0)$$

to a constant c_k, which is positive by (15).

To complete the proof note that $\gamma = \tilde{\gamma}$ iff $[\mathbb{E}(f'(1)^s)]'(1) \geq 0$, i.e. in the (IS+WS) case. □

Proof of Proposition 1(ii). The proof is close to the previous one. First, we consider the linear fractional case and the probability $\tilde{\mathbb{P}}$ defined by

$$\tilde{\mathbb{P}}(f \in \mathrm{d}g) = \mathbb{E}(f'(1)^k)^{-1} g'(1)^k \mathbb{P}(f \in \mathrm{d}g).$$

Using again (7), we get then

$$\mathbb{P}(Z_n^{(1)} > 0, \ldots, Z_n^{(k)} > 0) = \mathbb{E}(f'(1)^k)^n \tilde{\mathbb{E}}\left[\left(1 + \sum_{i=0}^{n-1} \frac{f''_{n-i-1}(1)}{2 f'_{i-i-1}(1)} P_i\right)^{-k}\right].$$

As $\tilde{\mathbb{E}}(\log(f'(1))) = 0$, we can use again Theorem 2.1 in [10] and conclude in the linear fractional case.

The general case can be proved following Step 2 in the previous proof. □

Proof of Theorem 2 (*Computation of* α_k *in the* (SS + IS) *case*). In the (SS + IS) case, Proposition 3 and (16) ensure that for every $k \in \mathbb{N}$,

$$\mathbb{P}_k(Z_n > 0) \sim k \mathbb{P}_1(Z_n > 0), \quad (n \to \infty).$$

Then, $\alpha_k = k$, which gives the first result.

Limit of α_k *in the* (WS) *case.* Note that $\mathbb{P}_1(Z_{p+n} > 0) = \sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k) \mathbb{P}_k(Z_n > 0)$. Then,

$$\frac{\mathbb{P}_1(Z_{p+n} > 0)}{\mathbb{P}_1(Z_n > 0)} = \sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k) \frac{\mathbb{P}_k(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)}. \tag{17}$$

First, $\mathbb{P}_k(\cup_{i=1}^k \{Z_n^{(i)} > 0\}) \leq \sum_{i=1}^k \mathbb{P}_k(Z_n^{(i)} > 0)$, which gives

$$\mathbb{P}_k(Z_n > 0)/\mathbb{P}_1(Z_n > 0) \leq k.$$

Moreover $\sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k) k = \mathbb{E}(Z_p) < \infty$ and $\mathbb{P}_k(Z_n > 0)/\mathbb{P}_1(Z_n > 0) \stackrel{n\to\infty}{\longrightarrow} \alpha_k$, so by bounded convergence, we get

$$\sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k) \frac{\mathbb{P}_k(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)} \stackrel{n\to\infty}{\longrightarrow} \sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k) \alpha_k.$$

Then, using again (4), letting $n \to \infty$ in (17) yields

$$\gamma^p = \sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k) \alpha_k.$$

Assuming that $(\alpha_k)_{k\in\mathbb{N}}$ is bounded by A leads to

$$\gamma^p \leq A \mathbb{P}_1(Z_p > 0).$$

Letting $p \to \infty$ leads to a contradiction with (4). Adding that α_k increases ensures that $\alpha_k \to \infty$ as $k \to \infty$.

Upper bound of α_k in the (WS) *case.* Using the independence of the particles conditionally on the environments, we have

$$\mathbb{P}_k(Z_n > 0 \mid \mathbf{f}_n) = 1 - \mathbb{P}_1(Z_n = 0 \mid \mathbf{f}_n)^k = 1 - (1 - p(\mathbf{f}_n))^k.$$

This yields the following expressions for the survival probability starting from k particles,

$$\mathbb{P}_k(Z_n > 0) = \mathbb{E}(1 - (1 - p(\mathbf{f}_n))^k) = k \int_0^1 (1-x)^{k-1} \mathbb{P}(p(\mathbf{f}_n) \geq x) \mathrm{d}x. \tag{18}$$

So we can write

$$\alpha_k = \lim_{n\to\infty} k \int_0^1 (1-x)^{k-1} \frac{\mathbb{P}(p(\mathbf{f}_n) \geq x)}{\mathbb{P}_1(Z_n > 0)} \mathrm{d}x. \tag{19}$$

Using the first inequality of Lemma 9, we have then

$$\begin{aligned}\alpha_k &\leq \limsup_{n\to\infty} k \int_0^1 (1-x)^{k-1} \frac{\mathbb{P}(\exp(L_n) \geq x)}{\mathbb{P}_1(Z_n > 0)} \mathrm{d}x \\ &\leq k . \limsup_{n\to\infty} \frac{n^{-3/2}\gamma^n}{\mathbb{P}_1(Z_n > 0)} . \limsup_{n\to\infty} \int_0^1 (1-x)^{k-1} \frac{\mathbb{P}(\exp(L_n) \geq x)}{n^{-3/2}\gamma^n}.\end{aligned}$$

By (26), we can use Fatou's Lemma and (25) ensures that there exists a linearly growing function u such that

$$\alpha_k \leq k \limsup_{n\to\infty} \frac{n^{-3/2}\gamma^n}{\mathbb{P}_1(Z_n > 0)} . \int_0^1 (1-x)^{k-1} x^{-\alpha} u(\log(1/x)) \mathrm{d}x.$$

Thus, using (4) and the fact that u is linearly growing, there exists a constant $C > 0$ such that

$$\alpha_k \leq Ck \int_0^1 (1-x)^{k-1} x^{-\alpha} [1 + \log(1/x)] \mathrm{d}x. \tag{20}$$

Finally, splitting the integral at $1/k$ and using integration by parts,

$$\begin{aligned}\int_0^1 (1-x)^{k-1} x^{-\alpha} \log(1/x) \mathrm{d}x &\leq \int_0^{1/k} x^{-\alpha} \log(1/x) \mathrm{d}x + k^{\alpha} \log(k) \int_{1/k}^1 (1-x)^{k-1} \mathrm{d}x \\ &\leq [-\alpha + 1]^{-1} \left(k^{\alpha-1} \log(k) + [-\alpha+1]^{-1} k^{\alpha-1}\right) + k^{\alpha-1} \log(k).\end{aligned}$$

Similarly $\int_0^1 (1-x)^{k-1} x^{-\alpha} \mathrm{d}x \leq [1-\alpha]^{-1} k^{\alpha-1} + k^{\alpha-1}$. Then (20) ensures that there exists $M_+ > 0$ such that for every $k > 0$, $\alpha_k \leq M_+ k^{\alpha} \log(k)$.

Lower bound of α_k in the (WS) *case* assuming further $\mathbb{E}(f'^{1/2}(1) \log(f'(1))) > 0$ (*i.e.*$\alpha < 1/2$) and $f''(1)/f'(1)$ is bounded.

By (4) and the second part of Lemma 9, there exists $\mu \geq 1$ such that for every $x \in (0, 1]$,

$$\begin{aligned}\liminf_{n\to\infty} \frac{\mathbb{P}(p(\mathbf{f}_n) \geq x)}{\mathbb{P}_1(Z_n > 0)} &= \liminf_{n\to\infty} \frac{\gamma^n n^{-3/2}}{\mathbb{P}_1(Z_n > 0)} \frac{\mathbb{P}(p(\mathbf{f}_n) \geq x)}{\gamma^n n^{-3/2}} \\ &\geq c^{-1} \liminf_{n\to\infty} \frac{\mathbb{P}(L_n \geq \log(\mu x))}{\gamma^n n^{-3/2}}.\end{aligned}$$

Using (25) and the fact that u grows linearly, there exists $D > 0$ such that

$$\liminf_{n\to\infty} \frac{\mathbb{P}(p(\mathbf{f}_n) \geq x)}{\mathbb{P}_1(Z_n > 0)} \geq D x^{-\alpha} \log(1/[x\mu]).$$

By (19) and Fatou's Lemma,

$$\alpha_k \geq D \int_0^1 (1-x)^{k-1} x^{-\alpha} [\log(1/x) + \log(1/\mu)] \mathrm{d}x.$$

For all $k \geq \mu^2$ and $x \in (0, 1/k]$, $\log(1/x) \geq 2\log(\mu)$. So for every $k \geq \mu^2$,

$$\begin{aligned}\alpha_k &\geq 2^{-1} Dk \int_0^{1/k} (1-x)^{k-1} x^{-\alpha} \log(1/x) \mathrm{d}x \\ &\geq 2^{-1} Dk \log(k) \int_0^{1/k} x^{-\alpha} \mathrm{d}x,\end{aligned}$$

which ensures that there exists M_- such that for every $k \geq 1$, $\alpha_k \geq M_- k^\alpha \log(k)$. □

4.2. *Proofs of Section 3.2*

Proof of Proposition 3. The first part (i.e. the (SS + IS) case) follows from

$$\mathbb{P}_k(\exists i \neq j, 1 \leq i, j \leq k, Z_n^{(i)} > 0, Z_n^{(j)} > 0 \mid Z_n > 0) \leq \binom{k}{2} \frac{\mathbb{P}_2(Z_n^{(1)} > 0, Z_n^{(2)} > 0)}{\mathbb{P}_k(Z_n > 0)},$$

the asymptotics given by Proposition 1(i-ii-iii) and Eqs. (2) and (3). The second part (i.e. the (WS) case) is directly derived from Proposition 1(iii) and (4). □

Proof of Theorem 4. Denote by $N(\mathbf{g}_n)$ the number of initial particles which survive until generation n where the successive reproduction laws are given by $\mathbf{g}_n$ (i.e. conditionally on $\mathbf{f}_n = \mathbf{g}_n$). Then, for all $1 \leq l \leq k$,

$$\begin{aligned}\mathbb{P}_k(N_n = l) &= \int_{\mathcal{F}^n} \mathbb{P}(\mathbf{f}_n \in \mathrm{d}\mathbf{g}_n) \mathbb{P}_k(N(\mathbf{g}_n) = l) \\ &= \int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in \mathrm{d}x) \binom{k}{l} x^l (1-x)^{k-l}.\end{aligned}$$

Note that $x \in [0,1] \mapsto x^l(1-x)^{k-l}$ is positive, increases on $[0, l/k]$ and decreases on $[l/k, 1]$.

First, we prove the upper bound. By Lemma 9, $p(\mathbf{f}_n) \leq \exp(L_n)$ a.s., so that

$$\begin{aligned}\int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in \mathrm{d}x) x^l (1-x)^{k-l} &= \int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in \mathrm{d}x, \exp(L_n) \leq l/k) x^l (1-x)^{k-l} \\ &\quad + \int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in \mathrm{d}x, \exp(L_n) > l/k) x^l (1-x)^{k-l} \\ &\leq \int_0^1 \mathbb{P}(\exp(L_n) \in \mathrm{d}x) x^l (1-x)^{k-l} \\ &\quad + \mathbb{P}(\exp(L_n) \in (l/k, 1]) (l/k)^l (1-l/k)^{k-l}.\end{aligned}$$

By (25),

$$\limsup_{n\to\infty} \frac{\mathbb{P}(\exp(L_n) \in (l/k, 1])}{\gamma^n n^{-3/2}} \leq u(\log(k/l))(k/l)^\alpha.$$

Second, using again the variations of $x \in [0,1] \mapsto x^l(1-x)^{k-l}$ and (26), we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^1 \frac{\mathbb{P}(\exp(L_n) \in \mathrm{d}x)}{n^{-3/2}\gamma^n} x^l(1-x)^{k-l} \\
&\leq \int_0^{l/k} \nu_+(\mathrm{d}x) x^l (1-x)^{k-l} + \nu_+([l/k,1])(l/k)^l(1-l/k)^{k-l} \\
&\leq c_+ \int_0^1 \log(1/x) x^{-\alpha-1} x^l (1-x)^{k-l} \mathrm{d}x \\
&\quad + c_+ \left(1 + \int_{l/k}^1 \log(1/x) x^{-\alpha-1} \mathrm{d}x\right)(l/k)^l(1-l/k)^{k-l} \\
&\leq c_+ \int_0^1 \log(1/x) x^{-\alpha-1} x^l (1-x)^{k-l} \mathrm{d}x \\
&\quad + c_+ \left(1 + \log(k/l)\frac{(k/l)^\alpha - 1}{\alpha}\right)(l/k)^l(1-l/k)^{k-l}.
\end{aligned}$$

Putting the last three inequalities together and using $u(\log(k/l)) \leq C(1+\log(k/l))$ for some $C > 0$ ensure that there exists $D > 0$ such that

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^1 \frac{\mathbb{P}(p(\mathbf{f}_n) \in \mathrm{d}x)}{n^{-3/2}\gamma^n} x^l(1-x)^{k-l} \\
&\leq c_+ \int_0^1 \log(1/x) x^{-\alpha-1} x^l (1-x)^{k-l} \mathrm{d}x + D(1 + \log(k/l)(k/l)^\alpha)(l/k)^l(1-l/k)^{k-l}.
\end{aligned}$$

Moreover, denoting by B the Beta function, we have

$$\begin{aligned}
&\int_0^1 \log(x) x^{-\alpha-1} x^l (1-x)^{k-l} \mathrm{d}x \\
&= \int_0^{1/k} \log(1/x) x^{l-\alpha-1}(1-x)^{k-l}\mathrm{d}x + \int_{1/k}^1 \log(1/x) x^{l-\alpha-1}(1-x)^{k-l}\mathrm{d}x \\
&\leq \int_0^{1/k} \log(1/x) x^{l-\alpha-1}\mathrm{d}x + \log(k) \int_{1/k}^1 x^{l-\alpha-1}(1-x)^{k-l}\mathrm{d}x \\
&\leq (l-\alpha)^{-1}\left[\log(k) k^{\alpha-l} + (l-\alpha)^{-1} k^{\alpha-l}\right] + \log(k) B(l-\alpha, k-l+1),
\end{aligned}$$

by integration by parts. By Stirling's formula, there exists $C > 0$, and then $C', C'' > 0$ such that for all $1 \leq l \leq k$,

$$\begin{aligned}
&\binom{k}{l} k^{-\alpha} B(l-\alpha, k-l+1) \\
&\leq C \frac{k^{k-\alpha+1/2}}{l^{l+1/2}(k-l)^{k-l+1/2}} \frac{(l-\alpha)^{l-\alpha-1/2}(k-l+1)^{k-l+1/2}}{(k-\alpha+1)^{k-\alpha+1/2}} \\
&\leq C' \frac{(l-\alpha)^{l-\alpha-1/2}(k-l+1)^{k-l+1/2}}{l^{l+1/2}(k-l)^{k-l+1/2}} \\
&\leq C'' \frac{1}{l^{1+\alpha}},
\end{aligned} \tag{21}$$

where the last inequality comes from the fact that $(1/x+1/2)\log(1+x)$ is bounded for $x \in [0,1]$, so that $(k-l+1/2)\log(1+1/(k-l))$ is bounded for $1 \le l < k$.

Then, combining the last three inequalities gives

$$\begin{aligned}\limsup_{n\to\infty} \frac{\mathbb{P}_k(N_n = l)}{k^\alpha \log(k) n^{-3/2}\gamma^n} &\le \limsup_{n\to\infty} \frac{\binom{k}{l}}{k^\alpha \log(k)} \int_0^1 \frac{\mathbb{P}(\exp(L_n) \in \mathrm{d}x)}{n^{-3/2}\gamma^n} x^l (1-x)^{k-l} \\ &\le (l-\alpha)^{-1}\left[\binom{k}{l} k^{-l} + (l-\alpha)^{-1} k^{-l}/\log(k) + C'' \frac{1}{l^{1+\alpha}}\right] \\ &\quad + D\binom{k}{l}(k^{-\alpha}/\log(k) + l^{-\alpha})(l/k)^l (1-l/k)^{k-l}.\end{aligned}$$

Adding that

$$\binom{k}{l} k^{-l} \le \frac{1}{l!}, \tag{22}$$

there exists $D' > 0$ such that

$$\limsup_{n\to\infty} \frac{\mathbb{P}_k(N_n = l)}{k^\alpha \log(k) n^{-3/2}\gamma^n} \le D'\left[\frac{1}{l^{1+\alpha}} + \frac{1}{l!} + \binom{k}{l} l^{-\alpha}(l/k)^l (1-l/k)^{k-l}\right].$$

Then,

$$\begin{aligned}\limsup_{n\to\infty} \frac{\mathbb{P}_k(N_n \le l)}{k^\alpha \log(k) n^{-3/2}\gamma^n} &= \limsup_{n\to\infty} \sum_{l'=l}^{k} \frac{\mathbb{P}_k(N_n = l')}{k^\alpha \log(k) n^{-3/2}\gamma^n} \\ &= \sum_{l'=l}^{k} \limsup_{n\to\infty} \frac{\mathbb{P}_k(N_n = l')}{k^\alpha \log(k) n^{-3/2}\gamma^n} \\ &\le D\sum_{l'=l}^{k}\left[\frac{1}{l'^{1+\alpha}} + \frac{1}{l'!} + \binom{k}{l'} l'^{-\alpha}(l'/k)^{l'}(1-l'/k)^{k-l'}\right] \\ &\le D\left[\sum_{l'=l}^{k}\left[\frac{1}{l'^{1+\alpha}} + \frac{1}{l'!}\right] + l'^{-\alpha}\right].\end{aligned}$$

Recalling that $\mathbb{P}_k(Z_n > 0) \sim c\alpha_k n^{-3/2}\gamma^n$, $(n \to \infty)$ and $\alpha_k \ge M_{\_}\log(k)k^\alpha$, $(k \in \mathbb{N})$ (see Theorem 2), we have

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}_k(N_n \ge l \mid Z_n > 0) &= \limsup_{n\to\infty} \frac{\mathbb{P}_k(N_n \ge l)}{c\alpha_k n^{-3/2}\gamma^n} \\ &\le (cM_{\_})^{-1} D\left[\sum_{l'=l}^{k}\left[\frac{1}{l^{1+\alpha}} + \frac{1}{l!}\right] + l^{-\alpha}\right].\end{aligned}$$

This gives the first inequality of the proposition with $A_l = (cM_{\_})^{-1} D\left[\sum_{l'=l}^{\infty}\left[\frac{1}{l^{1+\alpha}} + \frac{1}{l!}\right] + l^{-\alpha}\right]$.

We can prove similarly the lower bound. By Lemma 9, for every $x > 0$,

$$\mathbb{P}(p(\mathbf{f}_n) \ge x) \ge \mathbb{P}(L_n \ge \log(x\mu))/4.$$

Then, using also (9), for all $0 \le l < k$ and $N > 0$,

$$\begin{aligned}\mathbb{P}(p(\mathbf{f}_n) \in [l/k, Nl/k[) &= \mathbb{P}(p(\mathbf{f}_n) \geq l/k) - \mathbb{P}(p(\mathbf{f}_n) \geq Nl/k) \\ &\geq \mathbb{P}(L_n \geq \log(\mu l/k))/4 - \mathbb{P}(\exp(L_n) \geq Nl/k).\end{aligned}$$

By (25), we get

$$\begin{aligned}&\liminf_{n\to\infty} \frac{\mathbb{P}(p(\mathbf{f}_n) \in [l/k, Nl/k[)}{n^{-3/2}\gamma^n} \\ &\quad \geq (k/l)^{\alpha}[\mu^{-\alpha}u(\log(k) - \log(\mu l))/4 - N^{-\alpha}u(\log(k) - \log(Nl))].\end{aligned}$$

Then, as u is linearly growing, we can fix $N \geq 1$ so that there exists $C > 0$ such that

$$\liminf_{k\to\infty} \liminf_{n\to\infty} \frac{\mathbb{P}(p(\mathbf{f}_n) \in [l/k, Nl/k[)}{k^{\alpha} \log(k) n^{-3/2}\gamma^n} \geq l^{-\alpha} C. \tag{23}$$

Using that

$$\mathbb{P}_k(N_n = l) = \int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in \mathrm{d}x) \binom{k}{l} x^l (1-x)^{k-l},$$

and $x \to x^l(1-x)^{k-l}$ decreases on $[l/k, 1]$, we have, for every $k \geq Nl$,

$$\mathbb{P}_k(N_n = l) \geq \mathbb{P}(p(\mathbf{f}_n) \in [l/k, Nl/k[) \binom{k}{l} (Nl/k)^l (1 - Nl/k)^{k-l}.$$

Then (23) and $\lim_{k\to\infty} \binom{k}{l} (Nl/k)^l (1 - Nl/k)^{k-l} > 0$ ensure that

$$\liminf_{k\to\infty} \liminf_{n\to\infty} \frac{\mathbb{P}_k(N_n = l)}{k^{\alpha} \log(k) n^{-3/2}\gamma^n} > 0.$$

Use $\mathbb{P}_k(Z_n > 0) \sim c\alpha_k n^{-3/2}\gamma^n$ and the upper bound on α_k given in Theorem 2 to conclude. □

4.3. *Proofs of Section 3.3*

Proof of Theorem 5. Let us first consider the (WS+IS) case. Using that conditionally on $\mathbf{f}_n$, $Z_n^{(1)}$ and $Z_n^{(2)}$ are independent,

$$\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0) = \mathbb{E}(p(\mathbf{f}_n)^2).$$

Thus, for every $\epsilon > 0$, by the Markov inequality,

$$\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0 \mid Z_n > 0) \geq \epsilon^2 \mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0).$$

By Proposition 3, we get

$$\mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0) \stackrel{n\to\infty}{\longrightarrow} 0.$$

In the (WS) case, by (18), for every $\epsilon \in (0, 1]$:

$$\mathbb{P}_k(Z_n > 0) \geq \int_0^{\epsilon} \mathbb{P}(p(\mathbf{f}_n) \in \mathrm{d}x)(1 - (1-x)^k).$$

Moreover

$$\left|\int_0^{\epsilon}\mathbb{P}(p(\mathbf{f}_n)\in \mathrm{d}x)(1-(1-x)^k)-\int_0^{\epsilon}\mathbb{P}(p(\mathbf{f}_n)\in \mathrm{d}x)kx\right|$$
$$\leq k\sup_{x\in[0,\epsilon[}\left\{\left|\frac{1-(1-x)^k}{kx}-1\right|\right\}\int_0^{\epsilon}\mathbb{P}(p(\mathbf{f}_n)\in \mathrm{d}x)x$$
$$\leq k\sup_{x\in[0,\epsilon[}\left\{\left|\frac{1-(1-x)^k}{kx}-1\right|\right\}\mathbb{P}_1(Z_n>0).$$

Putting these two inequalities together yields

$$\mathbb{P}_k(Z_n>0)\geq k\int_0^{\epsilon}\mathbb{P}(p(\mathbf{f}_n)\in \mathrm{d}x)x-k\sup_{x\in[0,\epsilon[}\left\{\left|\frac{1-(1-x)^k}{kx}-1\right|\right\}\mathbb{P}_1(Z_n>0).$$

Then

$$\mathbb{P}_1(p(\mathbf{f}_n)\in[0,\epsilon),Z_n>0)=\int_0^{\epsilon}\mathbb{P}(p(\mathbf{f}_n)\in \mathrm{d}x)x$$
$$\leq \mathbb{P}_k(Z_n>0)/k+\sup_{x\in[0,\epsilon[}\left\{\left|\frac{1-(1-x)^k}{kx}-1\right|\right\}\mathbb{P}_1(Z_n>0).$$

Dividing by $\mathbb{P}_1(Z_n>0)$ and letting $n\to\infty$ ensure that

$$\limsup_{n\to\infty}\mathbb{P}_1(p(\mathbf{f}_n)\in[0,\epsilon)\mid Z_n>0)$$
$$\leq \limsup_{n\to\infty}\frac{\mathbb{P}_k(Z_n>0)}{k\mathbb{P}_1(Z_n>0)}+\sup_{x\in[0,\epsilon[}\left\{\left|\frac{1-(1-x)^k}{kx}-1\right|\right\}$$
$$\leq \frac{\alpha_k}{k}+\sup_{x\in[0,\epsilon)}\left\{\left|\frac{1-(1-x)^k}{kx}-1\right|\right\}.$$

Finally recall Theorem 2 and use

$$\alpha_k/k\overset{k\to\infty}{\longrightarrow}0,\quad \forall k\in\mathbb{N}^*,\qquad \sup_{x\in[0,\epsilon[}\left\{\left|\frac{1-(1-x)^k}{kx}-1\right|\right\}\overset{\epsilon\to 0}{\longrightarrow}0,$$

to get $\lim_{\epsilon\to 0+}\limsup_{n\to\infty}\mathbb{P}_k(p(\mathbf{f}_n)\leq\epsilon\mid Z_n>0)=0$. □

Proof of Proposition 6. Recall that for every $\mathbf{g}_n\in\mathcal{F}^n$, $\mathbb{P}_k(Z_{\mathbf{g}_n}>0)=1-(1-p(\mathbf{g}_n))^k$. Thus,

$$\mathbb{P}_k(p(\mathbf{f}_n)\in \mathrm{d}x\mid Z_n>0)=\frac{\mathbb{P}(p(\mathbf{f}_n)\in \mathrm{d}x)(1-(1-x)^k)}{\mathbb{P}_k(Z_n>0)}$$
$$=\mathbb{P}_1(p(\mathbf{f}_n)\in \mathrm{d}x\mid Z_n>0)\frac{\mathbb{P}_1(Z_n>0)}{\mathbb{P}_k(Z_n>0)}\frac{(1-(1-x)^k)}{x}.$$

Then, for every $\epsilon>0$,

$$\limsup_{n\to\infty}\mathbb{P}_k(p(\mathbf{f}_n)\geq\epsilon\mid Z_n>0)$$
$$=\frac{1}{\alpha_k}\limsup_{n\to\infty}\int_{\epsilon}^{1}\mathbb{P}_1(p(\mathbf{f}_n)\in \mathrm{d}x\mid Z_n>0)\frac{(1-(1-x)^k)}{x}$$
$$\leq\frac{1}{\epsilon\alpha_k},$$

and the left-hand part tends to zero as k tends to infinity by Theorem 2. This ends the proof. $\square$

4.4. *Proofs of Section 3.4*

We know from Section 2 that the BPRE $(Z_n)_{n\geq 0}$ starting from k particles and conditioned to be positive converges in distribution to Υ_k, and we call G_k its p.g.f:

$$G_k(s) = \lim_{n\to\infty} \mathbb{E}_k(s^{Z_n} \mid Z_n > 0) \quad (0 \leq s \leq 1).$$

Adding that by [4], $G_1'(1) < \infty$ we can split the proof of Theorem 7 into three parts.

(i) For every $k \geq 1$,

$$\mathbb{E}(G_k(f(s))) = \gamma G_k(s) + 1 - \gamma \quad (0 \leq s \leq 1).$$

(ii) In the (SS + IS) case, for every $k \geq 1$, $\Upsilon_k \stackrel{d}{=} \Upsilon_1$.

(iii) There is a unique p.g.f G which satisfies

$$\mathbb{E}(G(f(s))) = \mathbb{E}(f'(1))G(s) + 1 - \mathbb{E}(f'(1)) \quad (0 \leq s \leq 1), \quad G'(1) < \infty. \qquad (\mathcal{E})$$

One can note that (iii) proves (ii) in the (SS) case, adding that $G_k'(1) < \infty$ (whose proof for $k = 1$ in [4] can be generalized).

Proof of (i). Let f_0 be distributed as f and independent of $(Z_n)_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$,

$$\begin{aligned}
1 - \mathbb{E}_k(s^{Z_{n+1}} \mid Z_{n+1} > 0) &= \frac{\mathbb{E}_k(1 - s^{Z_{n+1}})}{\mathbb{P}_k(Z_{n+1} > 0)} \\
&= \frac{1}{\mathbb{P}_k(Z_{n+1} > 0)} \sum_{i=1}^{\infty} \mathbb{P}_k(Z_n = i)\mathbb{E}_k(1 - s^{Z_{n+1}} \mid Z_n = i) \\
&= \frac{\mathbb{P}_k(Z_n > 0)}{\mathbb{P}_k(Z_{n+1} > 0)} \sum_{i=1}^{\infty} \mathbb{P}_k(Z_n = i \mid Z_n > 0)\mathbb{E}(1 - f_0(s)^i) \\
&= \frac{\mathbb{P}_k(Z_n > 0)}{\mathbb{P}_k(Z_{n+1} > 0)} \mathbb{E}_k(1 - f_0^{Z_n}(s) \mid Z_n > 0).
\end{aligned}$$

Then letting n tend to infinity and using the asymptotics given in the Preliminaries section give

$$1 - G_k(s) = \gamma^{-1}\mathbb{E}(1 - G_k(f_0(s))),$$

where $\gamma = \mathbb{E}(f'(1))$ in the (SS + IS) case. $\square$

Proof of (ii). For every $i \geq 1$,

$$\begin{aligned}
\mathbb{P}_2(Z_n = i) = \mathbb{P}_2(Z_n^{(1)} = i, Z_n^{(2)} = 0) + \mathbb{P}_2(Z_n^{(1)} = 0, Z_n^{(2)} = i) \\
+ \mathbb{P}_2(Z_n = i, Z_n^{(1)} > 0, Z_n^{(2)} > 0).
\end{aligned}$$

Moreover $|\mathbb{P}_2(Z_n^{(1)} = i, Z_n^{(2)} = 0) - \mathbb{P}_2(Z_n^{(1)} = i)| \leq \mathbb{P}_2(Z_n^{(1)} > 0, Z_n^{(2)} > 0)$, then

$$|\mathbb{P}_2(Z_n = i) - 2\mathbb{P}_1(Z_n = i)| \leq 3\mathbb{P}_2(Z_n^{(1)} > 0, Z_n^{(2)} > 0).$$

Thus, using Proposition 3,

$$\lim_{n\to\infty} \frac{\mathbb{P}_2(Z_n = i)}{\mathbb{P}_2(Z_n > 0)} = \lim_{n\to\infty} \frac{2\mathbb{P}_1(Z_n = i)}{\mathbb{P}_2(Z_n > 0)}.$$

As $\alpha_2 = \lim_{n\to\infty} \mathbb{P}_2(Z_n > 0)/\mathbb{P}_1(Z_n > 0) = 2$, we have

$$\begin{aligned}\mathbb{P}(\Upsilon_2 = i) &= \lim_{n\to\infty} \mathbb{P}_2(Z_n = i \mid Z_n > 0)\\ &= \lim_{n\to\infty} \frac{2\mathbb{P}_1(Z_n = i \mid Z_n > 0)\mathbb{P}_1(Z_n > 0)}{\mathbb{P}_2(Z_n > 0)}\\ &= \mathbb{P}(\Upsilon_1 = i).\end{aligned}$$

Then $\Upsilon_1 \stackrel{d}{=} \Upsilon_2$ and the same argument ensure that for every $k \geq 1$, $\Upsilon_k = \Upsilon_1$. □

The proof of (iii) requires the following lemma.

Lemma 10. *If* $H : [0, 1] \to \mathbb{R}$ *is a power series continuous on* $[0, 1]$, $H(1) = 0$ *and*

$$H(s) = \frac{\mathbb{E}(H(f(s))f'(s))}{\mathbb{E}(f'(1))}, \quad (0 \leq s \leq 1), \tag{24}$$

then $H \equiv 0$.

Proof. First case: There exists $s_0 \in [0, 1)$ such that $\mathbb{E}(f'(s_0)) = \mathbb{E}(f'(1))$.

The monotonicity of f' implies

$$f'(s_0) = f'(1) \quad \text{a.s.},$$

and f' is $a.s.$ constant on $[s_0, 1]$. As it is a power series, f' is $a.s.$ constant.

Thus

$$f(s) = f'(1)s + (1 - f'(1)) \quad (0 \leq s \leq 1), \qquad f'(1) \leq 1 \quad \text{a.s.}$$

Moreover, let $|H(\alpha)| = \sup\{|H(s)|, s \in [0, 1]\}$ with $\alpha \in [0, 1)$, and note that

$$\mathbb{E}\left(f'(1)(H(\alpha) - H(f(\alpha)))\right) = 0.$$

Thus $H(f(\alpha)) = H(\alpha)$ $a.s.$ and by induction, recalling that $F_n = f_0 \circ f_1 \cdots \circ f_{n-1}$, we have

$$H(F_n(\alpha)) = H(\alpha) \quad \text{a.s.}$$

As Z_n is subcritical, then $\mathbb{E}(F_n(\alpha)) = \mathbb{E}(\alpha^{Z_n}) \to 1$ as $n \to \infty$. So $F_n(\alpha) \to 1$ in probability as $n \to \infty$. Adding that $F_n(\alpha) < 1$ $a.s.$ for every $n \in \mathbb{N}$ and that H is a power series, then H is constant and equals zero since $H(1) = 0$.

Second case: For every $s_0 \in [0, 1[$, $\mathbb{E}(f'(s_0)) < \mathbb{E}(f'(1))$.

If $H \neq 0$ then there exists $\alpha \in [0, 1[$ such that

$$\sup\{|H(s)| : s \in [0, \alpha]\} > 0$$

Let $\alpha_n \in [\alpha, 1[$ such that $\alpha_n \stackrel{n\to\infty}{\longrightarrow} 1$. Then, for every $n \in \mathbb{N}$, there exists $\beta_n \in [0, \alpha_n]$ such that:

$$\begin{aligned}\sup\{|H(s)| : s \in [0, \alpha_n]\} &= |H(\beta_n)|\\ &\leq \frac{\mathbb{E}(f'(\beta_n))}{\mathbb{E}(f'(1))} \sup\{|H(s)|, 0 \leq s \leq 1\}\\ &< \sup\{|H(s)|, 0 \leq s \leq 1\},\end{aligned}$$

since $\sup\{|H(s)|, 0 \leq s \leq 1\} > 0$ and $\mathbb{E}(f'(\beta_n)) < \mathbb{E}(f'(1))$. As $I \cap J = \varnothing$, $\sup I < \sup(I \cup J) \Rightarrow \sup I < \sup J$, we get

$$\sup\{|H(s)| : s \in [0, \alpha_n]\} < \sup\{|H(s)| : s \in]\alpha_n, 1]\}.$$

Also, $H(s) \stackrel{s\to 1}{\longrightarrow} 0$ leads to a contradiction letting $n \to \infty$. So $H = 0$. $\square$

Proof of (iii). Assume that G_1 and G_2 are two probability generating functions which verify $(\mathcal{E})$. By differentiation, G_1' and G_2' satisfy

$$\mathbb{E}(G'(f(s))f'(s)) = \mathbb{E}(f'(1))G'(s).$$

Then $H := G_2'(1)G_1' - G_1'(1)G_2'$ verifies the conditions of Lemma 10. As a consequence,

$$G_2'(1)G_1' = G_1'(1)G_2'.$$

Also, $G_1(0) = G_2(0) = 0$, $G_2(1) = G_1(1) = 1$ ensure that $G_1 = G_2$, which gives the uniqueness for $(\mathcal{E})$. $\square$

4.5. Proof of Section 3.5

Proof of Proposition 8. First, we have

$$\mathbb{P}_k(Z_1 = l_1, \ldots, Z_n = l_n | Z_{n+p} > 0) = \mathbb{P}_k(Z_1 = l_1, \ldots, Z_n = l_n)\frac{\mathbb{P}_{l_n}(Z_p > 0)}{\mathbb{P}_k(Z_{n+p} > 0)}.$$

Then, using (2)–(4), we get

$$\lim_{p\to\infty} \mathbb{P}_k(Z_1 = l_1, \ldots, Z_n = l_n | Z_{n+p} > 0) = \gamma^{-n}\frac{\alpha_{l_n}}{\alpha_k}\mathbb{P}_k(Z_1 = l_1, \ldots, Z_n = l_n)$$

and recall $\alpha_l = l$ in the (SS + IS) case to get the distribution of $(Y_n)_{n\in\mathbb{N}}$.

To get the limit distribution of $(Y_n)_{n\in\mathbb{N}}$, note that, for every $l \in \mathbb{N}^*$,

$$\mathbb{P}_k(Y_n = l) = \gamma^{-n}\frac{\alpha_l}{\alpha_k}\mathbb{P}_k(Z_n = l) = \gamma^{-n}\mathbb{P}_k(Z_n > 0)\frac{\alpha_l}{\alpha_k}\mathbb{P}_k(Z_n = l \mid Z_n > 0).$$

Use respectively (2) and (3) to get the limit in distribution in the (SS) case and the (IS).

Finally, in the (WS) case, by (4), there exists $C > 0$ such that

$$\mathbb{P}_k(Y_n \le l) \le Cn^{-3/2}\frac{\alpha_l}{\alpha_k}\mathbb{P}_k(Z_n \le l \mid Z_n > 0) \le Cn^{-3/2}\frac{\alpha_l}{\alpha_k}.$$

Then Borel–Cantelli Lemma ensures that Y_n tends $a.s.$ to infinity as $n \to \infty$. $\square$

Proof of (12). To prove the convergence and the equality, note that

$$\begin{aligned}\mathbb{P}_k\left(\mathbf{f}_p \in \mathrm{d}\mathbf{g}_p | Z_{n+p} > 0\right) &= \frac{\mathbb{P}\left(\mathbf{f}_p \in \mathrm{d}\mathbf{g}_p\right)\mathbb{E}_k(\mathbb{P}_{Z_{\mathbf{g}_p}}(Z_n > 0))}{\mathbb{P}_k(Z_{n+p} > 0)} \\ &= \frac{\mathbb{P}_1(Z_n > 0)}{\mathbb{P}_k(Z_{n+p} > 0)}\sum_{l=1}^{\infty}\mathbb{P}_k(Z_{\mathbf{g}_p} = l)\frac{\mathbb{P}_l(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)}.\end{aligned}$$

The asymptotic results given in Section 2 ensure that

$$\frac{\mathbb{P}_1(Z_n > 0)}{\mathbb{P}_k(Z_{n+p} > 0)} \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\gamma^p\alpha_k},$$

and using the bounded convergence theorem with

$$\frac{\mathbb{P}_l(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)} \stackrel{n\to\infty}{\longrightarrow} \alpha_l, \qquad \frac{\mathbb{P}_l(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)} \le l, \quad \mathbb{E}(Z_{\mathbf{g}_p}) < \infty.$$

ensures that

$$\lim_{n\to\infty} \mathbb{P}_k\left(\mathbf{f}_p \in \mathrm{d}\mathbf{g}_p | Z_{n+p} > 0\right) = \gamma^{-p}\mathbb{P}\left(\mathbf{f}_p \in \mathrm{d}\mathbf{g}_p\right)\sum_{l=1}^{\infty}\mathbb{P}_k(Z_{\mathbf{g}_p} = l)\frac{\alpha_l}{\alpha_k}.$$

This completes the proof. □

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Appendix. Random walk with negative drift

We study here the random walk $(S_n)_{n\in\mathbb{N}}$ with negative drift. Indeed, in the linear fractional case, the survival probability is a functional of the random walk obtained by summing the successive means of environments (see (6)). In the general case, the random walk appears in the lower bound of the survival probability (see (14)). More precisely, we need to control the successive values of the random walk with negative drift conditioned to stay above $-x < 0$.

More specifically, let $(X_i)_{i\in\mathbb{N}}$ be iid random variables distributed as X with

$$\mathbb{E}(X) < 0.$$

We assume that for every $z \in [0, 1]$, $\mathbb{E}(\exp(zX)) < \infty$ and $\mathbb{E}(X\exp(\alpha X)) = 0$ for some $0 < \alpha < 1$. Set $\gamma := \mathbb{E}(\exp(\alpha X))$,

$$S_n := \sum_{i=0}^{n-1} X_i, \quad (S_0 = 0),$$

and for all $n \in \mathbb{N}$, $k \in \mathbb{N}$,

$$L_n = \min\{S_i, 0 \le i \le n\}.$$

Its asymptotic behavior is given in Lemma 4.1 in [4] or Lemma 7 in [18]. There exists a linearly increasing positive function u such that, as $n \to \infty$

$$\mathbb{P}(L_n \ge -x) \sim \mathrm{e}^{\alpha x}u(x)n^{-3/2}\gamma^n, \tag{25}$$

for $x \ge 0$ if the distribution X is non-lattice, and for $x \in \lambda\mathbb{Z}$ if the distribution of X is supported by a centered lattice $\lambda\mathbb{Z}$.

Moreover for each $\theta > \alpha$, there exists $c_\theta > 0$ such that

$$\mathbb{P}(L_n \ge -x) \le c_\theta \mathrm{e}^{\theta x}n^{-3/2}\gamma^n, \quad (x \ge 0, n \in \mathbb{N}). \tag{26}$$

Finally, using (25) and the fact that u grows linearly, there exist $c_-, c_+ > 0$ such that the two following positive measures on $[0, 1]$,

$$\nu_-(\mathrm{d}x) = c_- \log(1/x)x^{-\alpha-1}\mathrm{d}x, \qquad \nu_+(\mathrm{d}x) = c_+(\delta_1(\mathrm{d}x) + \log(1/x)x^{-\alpha-1}\mathrm{d}x),$$

verify for every $x \in]0, 1]$

$$\nu_-([x, 1]) \le \lim_{n\to\infty}\frac{\mathbb{P}(\mathrm{e}^{L_n} \ge x)}{n^{-3/2}\gamma^n} \le \nu_+([x, 1]). \tag{27}$$

We need to control the successive values of the random walk conditioned to stay above $-x$ ($x \geq 0$). Under integrability conditions, it is known that the process $(S_{[nt]}/n^{1/2}|L_n \geq 0)$ converges weakly to the Brownian meander as $n \to \infty$ (see [19]). Moreover Durrett [20] has proved that if there exists $q > 2$ such that $P\{X_1 > x\} \sim x^{-q}L(x)$ as $x \to \infty$, where L is slowly varying, then $(S_{[nt]}/n|L_n \geq 0)$ converges weakly to a non-degenerate limit which has a single jump.

We prove here that the random walk conditioned to stay above $-x$ ($x \geq 0$) spends a very short time close to its minimum, by giving an upper bound of the number of visits to a level of the random walk reflected at its minimum. To be more specific, define

$$N_n(k) = \mathrm{card}\{i \in \mathbb{N}, i \leq n, k \leq S_i - L_n < k + 1\}.$$

Lemma 11. *For every $\theta > \alpha$, there exists $d > 0$ such that*

$$\limsup_{n\to\infty} \mathbb{P}(N_n(k) \geq l \mid L_n \geq -x) \leq d\mathrm{e}^{\theta k}/\sqrt{l}, \quad (k, l \in \mathbb{N}, x \geq 0).$$

Moreover for all $\theta > \alpha$ and $x \geq 0$, there exists $C > 0$ such that

$$\mathbb{P}(N_n(k) \geq l \mid L_n \geq -x) \leq C\mathrm{e}^{\theta k}/\sqrt{l}, \quad (k, n, l \in \mathbb{N}). \tag{28}$$

Moreover, we will use the following consequence of the preceding lemma.

Corollary 12. *If $\alpha < 1/2$, there exists $\beta > 0$ such that for all $x \geq 0$ and $n \in \mathbb{N}$,*

$$\mathbb{P}\left(\sum_{i=0}^{n} \exp(L_n - S_i) \leq \beta \mid L_n \geq -x\right) \geq 1/4.$$

For the sake of simplicity, we assume that $X \in \mathbb{Z}$ *a.s.* for the proof of Lemma 11. Thus

$$\forall k, n \in \mathbb{N}^2, \quad N_n(k) = \mathrm{card}\{i \in \mathbb{N}, i \leq n, S_i - L_n = k\},$$

and we denote by $(T_j : 1 \leq j \leq N_n(k))$ the successive times before n when $(S_i - L_n)_{i\in\mathbb{N}}$ visits k. That is

$$T_1 = \inf\{0 \leq i \leq n : S_i - L_n = k\}, \qquad T_{j+1} = \inf\{T_j < i \leq n : S_i - L_n = k\}.$$

First, cutting the path of the random walk between two of these passage times enables us to prove the following result.

Lemma 13. *If $X \in \mathbb{Z}$ a.s., then for all n, k, l, i and $0 \leq h \leq n$, we have*

$$\mathbb{P}(L_n \geq -i,\ N_n(k) \geq 2l,\ T_l + n - T_{N_n(k)} = h) \leq (k+1)\mathbb{P}(L_{n-h} \geq -k)\mathbb{P}(L_h \geq -i),$$

and

$$\mathbb{P}(L_n \geq -i,\ N_n(k) \geq 2l,\ T_1 + n - T_l = h) \leq (k+1)\mathbb{P}(L_{n-h} \geq -k)\mathbb{P}(L_h \geq -i).$$

Proof. We introduce the first hitting time M_n of the minimum L_n before time n and $R_n(l)$ the last passage time at l before time n

$$M_n = \inf\{j \in [1, n] : S_j = L_n\}, \qquad R_n(l) := \sup\{j \in [1, n] : S_j = l\}.$$

First, we consider the case where $M_n \in [0, T_l] \cup [T_{N_n(k)}, n]$ and split the path of the random walk between times T_l and $T_{N_n(k)}$. For all $j \leq 0$, $k \geq 0$ and $0 \leq n_1 < n_2 \leq n$, introduce then

$$\begin{aligned}
&A(j, n_1, n_2) = \{L_n = j, N_n(k) \geq 2l, T_l = n_1, T_{N_n(k)} = n_2, M_n \in [0, n_1] \cup [n_2, n]\},\\
&B(j, n_1, n_2) = \{\forall m \in [1, n_1] : S_m \geq j, S_{n_1} = S_{n_2} = j + k,\\
&\qquad \forall m \in [n_2 + 1, n] : S_m \geq j, S_m \neq j + k, \exists a \in [0, n_1] \cup [n_2, n], S_a = j\},\\
&C(j, n_1, n_2) = \{\forall m \in [n_1, n_2] : S_m \geq j, S_{n_1} = S_{n_2} = j + k\}.
\end{aligned}$$

Note that conditionally on $D(n_1, n_2) := \{S_{n_1} = S_{n_2} = j + k\}$, $B(j, n_1, n_2)$ and $C(j, n_1, n_2)$ are independent,

$$\mathbb{P}(C(j, n_1, n_2) \mid S_{n_1} = j + k) \leq \mathbb{P}(L_{n_2 - n_1} \geq -k),$$

and

$$A(j, n_1, n_2) \subset B(j, n_1, n_2) \cap C(j, n_1, n_2).$$

Then, noting also that

$$\begin{aligned}
&\mathbb{P}(C(j, n_1, n_2) \mid D(n_1, n_2))\\
&\qquad = \mathbb{P}(C(j, n_1, n_2) \mid S_{n_1} = j + k)\mathbb{P}(S_{n_1} = j + k)/\mathbb{P}(D(n_1, n_2)),
\end{aligned}$$

we have

$$\begin{aligned}
\mathbb{P}(A(j, n_1, n_2)) &\leq \mathbb{P}(D(n_1, n_2))\mathbb{P}(B(j, n_1, n_2) \mid D(n_1, n_2))\mathbb{P}(C(j, n_1, n_2) \mid D(n_1, n_2))\\
&= \mathbb{P}(S_{n_1} = j + k)\mathbb{P}(B(j, n_1, n_2) \mid D(n_1, n_2))\mathbb{P}(C(j, n_1, n_2) \mid S_{n_1} = j + k)\\
&\leq \mathbb{P}(L_{n_2 - n_1} \geq -k)\mathbb{P}(S_{n_1} = j + k)\mathbb{P}(B(j, n_1, n_2) \mid D(n_1, n_2)). \qquad (29)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\{L_n \geq -i,\ N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M_n \in [0, T_l] \cup [T_{N_k(n)}, n]\}\\
&\quad = \bigcup_{\substack{j \geq -i,\\ 1 \leq n_1 < n_2 \leq n, n_1 + n - n_2 = h}} A(j, n_1, n_2).
\end{aligned}$$

Then, using the last two relations,

$$\begin{aligned}
&\mathbb{P}(L_n \geq -i,\ N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M_n \in [0, T_l] \cup [T_{N_k(n)}, n])\\
&\quad \leq \sum_{\substack{j \geq -i,\\ 1 \leq n_1 < n_2 \leq n, n_2 - n_1 = n - h}} \mathbb{P}(A(j, n_1, n_2))\\
&\quad \leq \mathbb{P}(L_{n-h} \geq -k) \sum_{\substack{j \geq -i,\\ 1 \leq n_1 < n_2 \leq n, n_1 + n - n_2 = h}} \mathbb{P}(S_{n_1} = j + k)\mathbb{P}(B(j, n_1, n_2) \mid D(n_1, n_2)).
\end{aligned}$$

Concatenating the path of the random walk before time n_1 and after time n_2 gives

$$\begin{aligned}
&\mathbb{P}(L_n \geq -i,\ N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M_n \in [0, T_l] \cup [T_{N_k(n)}, n])\\
&\quad \leq \mathbb{P}(L_{n-h} \geq -k) \sum_{\substack{j \geq -i,\\ 1 \leq n_1 < n_2 \leq n, n_1 + n - n_2 = h}} \mathbb{P}(L_{n_1 + n - n_2} = j,\ R_{n_1 + n - n_2}(j + k) = n_1)\\
&\quad \leq \mathbb{P}(L_{n-h} \geq -k) \sum_{j \geq -i} \mathbb{P}(L_h = j)\\
&\quad = \mathbb{P}(L_{n-h} \geq -k)\mathbb{P}(L_h \geq -i). \qquad (30)
\end{aligned}$$

Second, we consider the case where $M_n \in [T_l, T_{N_n(k)}]$ and split the path of the random walk between times T_1 and T_l. For all $j, j' \leq 0$, $k \geq 0$ and $0 \leq n_1 < n_2 \leq n$, introduce then

$$\begin{aligned}
&A'(j, n_1, n_2) = \{L_n = -j, N_n(k) \geq 2l, T_l = n_1, T_{N_n(k)} = n_2, M_n \in [n_1, n_2]\},\\
&B'(j, j', n_1, n_2) = \{\forall m \in [1, n_1] : S_m \geq j', S_{n_1} = S_{n_2} = j + k,\\
&\quad \forall m \in]n_2, n] : S_m \geq j', S_m \neq j + k, \exists a \in [0, n_1] \cup [n_2, n] : S_a = j'\},\\
&C'(j, n_1, n_2) = \{\forall m \in [n_1, n_2] : S_m \geq j, S_{n_1} = S_{n_2} = k + j, \exists a \in [n_1, n_2] : S_a = j\}.
\end{aligned}$$

Note that conditionally on $D(n_1, n_2) = \{S_{n_1} = S_{n_2} = j + k\}$, $B'(j, j', n_1, n_2)$ and $C'(j, n_1, n_2)$ are independent,

$$A'(j, n_1, n_2) \subset \bigcup_{j'=j}^{j+k} B'(j, j', n_1, n_2) \cap C'(j, n_1, n_2)$$

and we get the analog of (29),

$$\mathbb{P}(A'(j, n_1, n_2)) \leq \sum_{j'=j}^{j+k} \mathbb{P}(L_{n_2-n_1} \geq -k)\mathbb{P}(S_{n_1} = j + k)\mathbb{P}(B'(j, j', n_1, n_2) \mid D(n_1, n_2)).$$

Moreover

$$\begin{aligned}
&\{L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M \in [T_l, T_{N_k(n)}]\}\\
&\quad = \bigcup_{\substack{j \geq -i,\\ 1 \leq n_1 < n_2 \leq n, n_1 + n - n_2 = h}} A'(j, n_1, n_2).
\end{aligned}$$

Then, following the proof of (30), we get

$$\begin{aligned}
&\mathbb{P}(L_n \geq -i,\ N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M_n \in [T_l, T_{N_k(n)}])\\
&\quad \leq \mathbb{P}(L_{n-h} \geq -k) \sum_{j' \geq -i, j \in [j'-k, j']} \sum_{\substack{1 \leq n_1 < n_2 \leq n,\\ n_1 + n - n_2 = h}} \mathbb{P}(S_{n_1} = j + k)\mathbb{P}(B'(j, j', n_1, n_2) \mid D(n_1, n_2))\\
&\quad \leq \mathbb{P}(L_{n-h} \geq -k) \sum_{j' \geq -i} k \max_{j \in [j'-k, j']} \sum_{\substack{1 \leq n_1 < n_2 \leq n,\\ n_1 + n - n_2 = h}} \mathbb{P}(S_{n_1} = j + k)\mathbb{P}(B'(j, j', n_1, n_2) \mid D(n_1, n_2))\\
&\quad \leq \mathbb{P}(L_{n-h} \geq -k) \sum_{j' \geq -i} k\mathbb{P}(L_h = j')\\
&\quad \leq k\mathbb{P}(L_{n-h} \geq -k)\mathbb{P}(L_h \geq -i).
\end{aligned} \tag{31}$$

Combining the inequalities (30) and (31), we get

$$\mathbb{P}(L_n \geq -i,\ N_n(k) \geq 2l,\ T_l + n - T_{N_n(k)} = h) \leq (k + 1)\mathbb{P}(L_{n-h} \geq -k)\mathbb{P}(L_h \geq -i),$$

which proves the first inequality of the lemma. The second can be proved similarly concatenating the random walk between $[0, T_1]$ and $[T_{N_n(k)}, n]$. □

Proof of Lemma 11. Let $h \in \mathbb{N}$ such that $h \geq n/2$. The first inequality of Lemma 13 ensures that

$$\mathbb{P}(L_n \geq -i,\ N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h) \leq (k + 1)\mathbb{P}(L_h \geq -i)\mathbb{P}(L_{n-h} \geq -k).$$

Using (26),

$$\begin{aligned}
&\mathbb{P}(L_n \geq -i,\ N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h)\\
&\quad \leq c_\theta (k + 1)\mathbb{P}(L_h \geq -i)\mathrm{e}^{\theta k}(n - h)^{-3/2}\gamma^{n-h}.
\end{aligned}$$

Moreover, using (25), for every $i \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for all $n_0/2 \le n/2 \le h$,

$$\mathbb{P}(L_h \ge -i) \le 2\mathrm{e}^{i\alpha} u(i) h^{-3/2} \gamma^{-h} \le 2.2^{3/2} \mathrm{e}^{i\alpha} u(i) n^{-3/2} \gamma^{h}. \tag{32}$$

Then, writing $c'_\theta = 2.2^{3/2}.c_\theta$,

$$\begin{aligned} &\mathbb{P}(L_n \ge -i,\ N_n(k) \ge 2l, T_l + n - T_{N_n(k)} = h) \\ &\quad \le c'_\theta \mathrm{e}^{\alpha i} u(i)(k+1)\mathrm{e}^{\theta k} \gamma^n n^{-3/2} (n-h)^{-3/2}. \end{aligned} \tag{33}$$

Similarly, for every h such that $n_0/2 \le n/2 \le h$, the second inequality of Lemma 13 ensures that

$$\begin{aligned} &\mathbb{P}(L_n \ge -i,\ N_n(k) \ge 2l, T_1 + n - T_l = h) \\ &\quad \le c'_\theta \mathrm{e}^{\alpha i} u(i)(k+1)\mathrm{e}^{\theta k} \gamma^n n^{-3/2} (n-h)^{-3/2}. \end{aligned} \tag{34}$$

Noting that a.s.

$$\begin{aligned} \{N_n(k) \ge 2l\} = &\bigcup_{h=n/2}^{n-l} \{N_n(k) \ge 2l, T_l + n - T_{N_n(k)} = h\} \\ &\times \bigcup_{h=n/2}^{n-l} \{N_n(k) \ge 2l, T_1 + n - T_l = h\}, \end{aligned}$$

we can combine the last two inequalities (33) and (34), which gives for every $n \ge n_0$,

$$\begin{aligned} \mathbb{P}(L_n \ge -i, N_n(k) \ge 2l) \le &\sum_{n/2 \le h \le n-l} \mathbb{P}(L_n \ge -i,\ N_n(k) \ge 2l, T_l + n - T_{N_n(k)} = h) \\ &+ \sum_{n/2 \le h \le n-l} \mathbb{P}(L_n \ge -i,\ N_n(k) \ge 2l T_1 + n - T_l = h) \\ \le\ & 2c'_\theta \mathrm{e}^{\alpha i} u(i) \gamma^n n^{-3/2} (k+1) \mathrm{e}^{\theta k} \sum_{n/2 \le h \le n-l} (n-h)^{-3/2} \\ \le\ & 2c'_\theta \mathrm{e}^{\alpha i} u(i) \gamma^n n^{-3/2} (k+1) \mathrm{e}^{\theta k} \sum_{h \ge l} h^{-3/2} \\ \le\ & 2.2c'_\theta \mathrm{e}^{\alpha i} u(i) \gamma^n n^{-3/2} (k+1) \mathrm{e}^{\theta k} / \sqrt{l}, \quad (n \ge n_0). \end{aligned}$$

Then, using again (25),

$$\limsup_{n \to \infty} \mathbb{P}(L_n \ge -i, N_n(k) \ge 2l) / \mathbb{P}(L_n \ge -i) \le 4c'_\theta c_0^{-1} (k+1) \mathrm{e}^{\theta k} / \sqrt{l}.$$

Using that $(k+1)\mathrm{e}^{\theta k} = o(\mathrm{e}^{\theta' k})$ if $\theta' > \theta$, this completes the proof of the first inequality of the lemma for $X \in \mathbb{Z}$. The general case can be proved similarly.

Note that, for every $\theta > \alpha$, when $h \ge n/2$, we can replace (32) by

$$\mathbb{P}(L_h \ge -i) \le 2^{3/2}.c_\theta \mathrm{e}^{\theta i} n^{-3/2} \gamma^h, \quad (i, h, n \in \mathbb{N}).$$

Following the proof above ensures that there exists $c''_\theta > 0$ such for all $i, n, l \in \mathbb{N}$,

$$\mathbb{P}(L_n \ge -i, N_n(k) \ge 2l) \le c''_\theta \mathrm{e}^{\theta i} \gamma^n n^{-3/2} \mathrm{e}^{\theta k} / \sqrt{l}.$$

Thus, by (25), for every $x \geq 0$, there exists $C_x > 0$ such that

$$\mathbb{P}(N_n(k) \geq l \mid L_n \geq -x) \leq 2c''_\theta C_x(k+1)\mathrm{e}^{\theta k}/\sqrt{l}, \quad (k, n, l \in \mathbb{N}),$$

which gives the second inequality of the lemma. □

Proof of Corollary 12. Let $\alpha < 1/2$ and $d > 0$ given by Theorem 2. Fix $\alpha < \theta < \mu/2 < 1/2$. Choose also $k_0 \in \mathbb{N}$ such that

$$d \sum_{k \geq k_0} \mathrm{e}^{(\theta-\mu/2)k} < 1/2.$$

By (28), for every $x \geq 0$, there exists $D > 0$ such that for every $n \in \mathbb{N}$,

$$\mathbb{P}(N_n(k) \geq \mathrm{e}^{\mu k} \mid L_n \geq -x) \leq D\mathrm{e}^{(\theta-\mu/2)k}$$

which is summable with respect to k. Thus, by Fatou's lemma,

$$\limsup_{n\to\infty} \sum_{k \geq k_0} \mathbb{P}(N_n(k) \geq \mathrm{e}^{\mu k} \mid L_n \geq -x) \leq \sum_{k \geq k_0} \limsup_{n\to\infty} \mathbb{P}(N_n(k) \geq \mathrm{e}^{\mu k} \mid L_n \geq -x).$$

By Lemma 11, this gives, for every $x > 0$,

$$\limsup_{n\to\infty} \sum_{k \geq k_0} \mathbb{P}(N_n(k) \geq \mathrm{e}^{\mu k} \mid L_n \geq -x) \leq d \sum_{k \geq k_0} \mathrm{e}^{(\theta-\mu/2)k}.$$

Then,

$$\limsup_{n\to\infty} \mathbb{P}\left(\bigcup_{k \geq k_0} \{N_n(k) \geq \mathrm{e}^{\mu k}\} \mid\ L_n \geq -x\right) < 1/2.$$

By Lemma 11 again, fix $N \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \mathbb{P}\left(\bigcup_{0 \leq k < k_0} \{N_n(k) \geq N\} \mid L_n \geq -x\right) \leq 1/4.$$

Then

$$\limsup_{n\to\infty} \mathbb{P}\left(\bigcup_{0 \leq k < k_0} \{N_n(k) \geq N\} \bigcup_{k \geq k_0} \{N_k(k) \geq \mathrm{e}^{\mu k}\} \mid L_n \geq -x\right) < 3/4.$$

Noting that

$$\sum_{i=0}^{n} \exp(L_n - S_i) \leq \sum_{k=0}^{\infty} N_n(k)\mathrm{e}^{-k},$$

this ensures that for every $x \geq 0$,

$$\liminf_{n\to\infty} \mathbb{P}\left(\sum_{i=0}^{n} \exp(L_n - S_i) \leq \beta \mid L_n \geq -x\right) > 1/4,$$

with $\beta := \sum_{0 \leq k < k_0} N\mathrm{e}^{-k+1} + \sum_{k \geq k_0} \mathrm{e}^{\mu k}\mathrm{e}^{-k+1}$. This gives the result. □

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