On the eigenvalue process of a matrix fractional Brownian motion

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Abstract

We investigate the process of eigenvalues of a symmetric matrix-valued process which upper diagonal entries are independent one-dimensional Hölder continuous Gaussian processes of order $\gamma \in (1/2, 1)$. Using the stochastic calculus with respect to the Young integral we show that these eigenvalues do not collide at any time with probability one. When the matrix process has entries that are fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$, we find a stochastic differential equation in a Malliavin calculus sense for the eigenvalues of the corresponding matrix fractional Brownian motion. A new generalized version of the Itô formula for the multidimensional fractional Brownian motion is first established.
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1. Introduction

In a pioneering work in 1962, the nuclear physicist Freeman Dyson [11] studied the stochastic process of eigenvalues of a Hermitian matrix Brownian motion. The case of the (real) symmetric

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matrix Brownian motion was first considered by McKean [22] in 1969. In both cases the corresponding processes of eigenvalues are called Dyson Brownian motion and are governed by a noncolliding system of Itô Stochastic Differential Equations (SDEs) with non-smooth diffusion coefficients; see the modern treatments, for example, in the books by Anderson, Guionnet and Zeitouni [1] and Tao [30].

More specifically, for the symmetric case, let $\{B(t)\}_{t\geq 0} = \{(b_{jk}(t))\}_{t\geq 0}$ be a $d \times d$ symmetric matrix Brownian motion. That is, $(b_{jj}(t))_{j=1}^{d}$, $(b_{jk}(t))_{j<k}$, is a set of $d(d+1)/2$ independent one-dimensional Brownian motions with parameter $(1 + \delta_{jk})t$. For each $t > 0$, $B(t)$ is a Gaussian Orthogonal (GO) random matrix with parameter t [1,23]. Let $\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_d(t)$, $t \geq 0$, be the d-dimensional stochastic process of eigenvalues of B. If the eigenvalues start at different positions $\lambda_1(0) > \lambda_2(0) > \cdots > \lambda_d(0)$, then they never meet at any time ($\lambda_1(t) > \lambda_2(t) > \cdots > \lambda_d(t)$ almost surely $\forall t > 0$) and furthermore they form a diffusion process satisfying the Itô SDE

$$\lambda_i(t) = \sqrt{2}W_t^i + \sum_{j\neq i} \int_0^t \frac{\mathrm{d}s}{\lambda_i(s) - \lambda_j(s)}, \quad 1 \leq i \leq d, \ t \geq 0, \tag{1.1}$$

where $W_t^1, \ldots, W_t^d$ are independent one-dimensional standard Brownian motions.

Thus, Dyson Brownian motion can be thought as a model for the evolution of d Brownian motions $\lambda_1(t), \ldots, \lambda_d(t)$ that are restricted to never intersect and having a repulsion force which is inversely proportional to the distance between any two eigenvalues. For that reason $\lambda(t) = (\lambda_1(t), \ldots, \lambda_d(t))$ is also called a Dyson non-colliding Brownian motion.

Different aspects of the Dyson Brownian motion have been considered by several authors [2,6,7,12,14,20,18,27,29]. The corresponding eigenvalues processes of other matrix stochastic processes and their associated Itô SDEs have been studied in [4,5,10,9,19,21,28], among others.

In the literature, sometimes a Dyson process means a $d \times d$ matrix process in which the entries undergo diffusion, see [31]. The evolution of the singular values of a matrix process is often called Laguerre process, being an important example the Wishart case [4,5,10,21,28].

The purpose of this paper is to study the eigenvalue process of a symmetric matrix Gaussian process $\{G(t)\}_{t\geq 0}$ which entries in the upper diagonal part are independent one-dimensional zero mean Gaussian processes with Hölder continuous paths of order $\gamma \in (1/2, 1)$. In this case $G(t)$ is still a GO random matrix of parameter t, for each $t > 0$, and therefore the eigenvalues of $G(t)$ are distinct for each $t > 0$ a.s. However $\{G(t)\}_{t\geq 0}$ is not a matrix process with independent increments nor a matrix diffusion. We prove that the corresponding eigenvalues do not collide at any time with probability one. The proof of this fact is based on the stochastic calculus with respect to the Young integral [32]. Our noncolliding method of proof is different from the case of the Brownian matrix and other matrix diffusions, for which one first has to find a diffusion process governing the eigenvalue process as done, for example, in [1] for the Hermitian Brownian motion. Our result does not include the latter as a special case.

We study in detail the case when the matrix process has entries that are fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$. For this matrix process we find a stochastic differential equation for the eigenvalue process, similar to Eq. (1.1), where instead of $\sqrt{2}W_t^i$ we obtain processes Y_t^i which are expressed as Skorohod indefinite integrals. This equation is derived applying a generalized version of the Itô formula in the Skorohod sense for the multidimensional fractional Brownian motion, which has its own interest. Each process Y^i has the same H-self-similarity and $1/H$-variation properties as a one-dimensional fractional Brownian motion, although there is no reason for them to be fractional Brownian motions. This phenomenon

is related to the representation of fractional Bessel processes, established by Guerra and Nualart in [13]. In the case of the fractional Bessel process, the fact that the indefinite Skorohod integral appearing in the representation is not a one-dimensional standard Brownian motion was proved by Hu and Nualart in [16]. In our case we conjecture that each Y^i is not a one-dimensional fractional Brownian motion, but at this moment we are not able to give a proof of this fact.

The paper is organized as follows. Section 2 contains preliminaries on Malliavin calculus for the fractional Brownian motion. Section 3 establishes a generalized version of the Itô formula for the multidimensional fractional Brownian motion, in the case of functions that are smooth only on a dense subset of the Euclidean space, as needed to describe the evolution of the eigenvalues, a result does not covered in the literature. It also recalls a property on the $1/H$-variation of a divergence integral and establishes a result that allows to compute the $1/H$-variation of a Skorohod integral wrt a multidimensional Brownian motion. In Section 4 we prove that if the matrix process $\{G(t)\}_{t \geq 0}$ has entries that are Hölder continuous Gaussian processes of order $\gamma \in (1/2, 1)$, then the corresponding eigenvalues processes do not collide at any time with probability one. Finally, Section 5 considers the SDE of the eigenvalues of a matrix fractional Brownian notion similar to (1.1) but in the Malliavin calculus sense.

2. Malliavin calculus for the fBm

In this section we present some basic facts on the Malliavin calculus, or stochastic calculus of variations, with respect to the fractional Brownian motion. We refer the reader to [26] for a detailed account of this topic.

Suppose that $B = \{B_t, t \geq 0\}$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. That is, B is a zero mean Gaussian process with covariance

$$R(t, s) = \mathbb{E}(B_t B_s) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right).$$

The process B is H-self-similar, that is, for any $a > 0$, $\{B_{at}, t \geq 0\}$ and $\{a^H B_t, t \geq 0\}$ have the same law. On the other hand, it possesses a finite $1/H$-variation on any time interval of length t (see Definition 3.2) equal to $t\mathbb{E}(|Z|^{1/H})$, where Z is a $N(0, 1)$ random variable.

Fix a time interval $[0, T]$ and let $\mathcal{H}$ be the Hilbert space defined as the closure of the set of step functions with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R(s, t).$$

Given two step functions φ, ψ on $[0, T]$, its inner product can be expressed as

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T \varphi_t \psi_s |t - s|^{2H-2} dt ds,$$

where $\alpha_H = H(2H - 1)$. We denote by $|\mathcal{H}|$ the space of measurable functions φ on $[0, T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 := \alpha_H \int_0^T \int_0^T |\varphi_t||\varphi_s||t - s|^{2H-2} dt ds < \infty.$$

This space is a Banach space, which is isometric to a subspace of $\mathcal{H}$ and it will be identified with this subspace. Moreover, we have the following continuous embeddings (see [24]).

$$L^{1/H}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{2.1}$$

The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space generated by B. We denote this isometry by $h \mapsto B(h)$.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f\left(B(h_1), \ldots, B(h_n)\right),$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded) and $h_i \in \mathcal{H}$. The derivative operator is defined in $\mathcal{S}$ as the $\mathcal{H}$-valued random variable

$$DF := \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(B(h_1), \ldots, B(h_n)\right) h_i.$$

The derivative operator is a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$. We denote by D^k the iteration of D. For any $p \geq 1$, we define the Sobolev space $\mathbb{D}^{k,p}$ as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p := \mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\sum_{j=1}^k \left\|D^j F\right\|_{\mathcal{H}^{\otimes j}}^p\right].$$

In a similar way, given a Hilbert space V, we can define the Sobolev space of V-valued random variables, denoted by $\mathbb{D}^{k,p}(V)$.

The divergence operator δ is the adjoint of the derivative operator, defined by means of the duality relationship

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathcal{H}}\right], \tag{2.2}$$

where u is a random variable in $L^2(\Omega; \mathcal{H})$. The domain $\mathrm{Dom}(\delta)$ is the set of random variables $u \in L^2(\Omega; \mathcal{H})$ such that

$$|\mathbb{E}\left(\langle DF, u\rangle_{\mathcal{H}}\right)| \leq c\|F\|_2$$

for all $F \in \mathbb{D}^{1,2}$. We have the inclusion $\mathbb{D}^{1,2}(\mathcal{H}) \subset \mathrm{Dom}(\delta)$, and for any $u \in \mathbb{D}^{1,2}(\mathcal{H})$, the variance of the divergence of u can be computed as follows

$$\mathbb{E}\left[\delta(u)^2\right] = \mathbb{E}\left[\|u\|_{\mathcal{H}}^2\right] + \mathbb{E}\left[\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}}\right] \leq \|u\|_{\mathbb{D}^{1,2}(\mathcal{H})}^2,$$

where $(Du)^*$ is the adjoint of Du in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

By Meyer's inequalities, for all $p > 1$, the divergence operator can be extended to the space $\mathbb{D}^{1,p}(\mathcal{H})$ and the mapping

$$\delta : \mathbb{D}^{1,p}(\mathcal{H}) \to L^p(\Omega)$$

is a continuous operator, that is,

$$\|\delta(u)\|_p \leq C_p \|u\|_{\mathbb{D}^{1,p}(\mathcal{H})}. \tag{2.3}$$

Denote by $|\mathcal{H}| \otimes |\mathcal{H}|$ the space of measurable functions φ defined on $[0,T]^2$ such that

$$\|\varphi\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 := \alpha_H^2 \int_{[0,T]^4} \left|\varphi_{r,\theta}\right| \left|\varphi_{u,\eta}\right| |r-u|^{2H-2} |\theta-\eta|^{2H-2} \, dr du d\theta d\eta < \infty.$$

As in the case of functions of one variable, this space is a Banach space, which is isometric to a subspace of $\mathcal{H} \otimes \mathcal{H}$ and it will be identified with this subspace. For any $p \geq 1$, we denote by

$\mathbb{D}^{1,p}(|\mathcal{H}|)$ the subspace of the Sobolev space $\mathbb{D}^{1,p}(\mathcal{H})$ whose elements u are such that $u \in |\mathcal{H}|$ a.s., $Du \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s. and

$$\mathbb{E}\left[\|u\|_{|\mathcal{H}|}^p\right] + \mathbb{E}\left[\|Du\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^p\right] < \infty.$$

Then, using the embedding (2.1) one can show that for any $p > 1$, we have the continuous embedding $\mathbb{L}_H^{1,p} \subset \mathbb{D}^{1,p}(|\mathcal{H}|)$, where $\mathbb{L}_H^{1,p}$ is the set of processes $u \in \mathbb{D}^{1,p}(|\mathcal{H}|)$ such that

$$\|u\|_{\mathbb{L}_H^{1,p}}^p := \mathbb{E}\left[\|u\|_{L^{1/H}([0,T])}^p\right] + \mathbb{E}\left[\|Du\|_{L^{1/H}([0,T]^2)}^p\right] < \infty. \tag{2.4}$$

As a consequence, Meyer's inequalities (2.3) imply

$$\|\delta(u)\|_p \le C_p' \|u\|_{\mathbb{L}_H^{1,p}} \tag{2.5}$$

if $u \in \mathbb{L}_H^{1,p}$ and $p > 1$.

Let $N \in \mathbb{N}$, $d \ge 2$, $H \in (1/2, 1)$ and consider the N-dimensional fractional Brownian motion

$$B = \{B_t, t \ge 0\} = \left\{B_t^{(1)}, \ldots, B_t^{(N)}, \ t \in [0,T]\right\},$$

with Hurst parameter H, defined on the probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is generated by B. That is, the components $B^{(i)}$, $i = 1, \ldots, d$, are independent fractional Brownian motions with Hurst parameter H. We can define the derivative and divergence operators, $D^{(i)}$ and $\delta^{(i)}$, with respect to each component $B^{(i)}$, as before. Denote by $\mathbb{D}_i^{1,p}(\mathcal{H})$ the associated Sobolev spaces. We assume that these spaces include functionals of all the components of B and not only of component i. Similarly, we introduce the spaces $\mathbb{L}_{H,i}^{1,p}$. That is, $\mathbb{L}_{H,i}^{1,p}$ is the set of processes $u \in \mathbb{D}_i^{1,p}(\mathcal{H})$ such that

$$\|u\|_{\mathbb{L}_{H,i}^{1,p}}^p := \mathbb{E}\left[\|u\|_{L^{1/H}([0,T])}^p\right] + \mathbb{E}\left[\left\|D^{(i)}u\right\|_{L^{1/H}([0,T]^2)}^p\right] < \infty.$$

The processes $u \in \mathbb{L}_{H,i}^{1,p}$ are, in general, functionals of all the components of the process B.

If a process $u = \{u_t, t \in [0,T]\}$ belongs to the domain of $\delta^{(i)}$, we call $\delta^{(i)}(u)$ the Skorohod integral of u with respect to the fractional Brownian motion $B^{(i)}$ and we will make use of the notation

$$\delta^{(i)}(u) = \int_0^T u_s \delta B_s^{(i)}.$$

3. Stochastic calculus for the fBm

There exist a huge literature on the stochastic calculus for the fBm. There are essentially two types of stochastic integrals: path-wise integrals defined using the Young integral in the case $H > 1/2$, and the Skorohod integral which is the adjoint of the divergence operator introduced in the previous section. We refer the reader to the monographs by Biagini, Hu, Øksendal and Zhang [3] and Mishura [25] and the references therein.

In order to describe the evolution of the eigenvalues of a matrix fractional Brownian motion we need a multidimensional version of the Itô formula for the Skorohod integral, in the case of

functions that are smooth only on a dense open subset of the Euclidean space and satisfy some growth requirements. This type of formula is not covered by the existing literature on the subject and we provide below a proof based on a duality argument, following the approach developed in the paper [8].

Theorem 3.1. *Suppose that B is an N-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$. Consider a function $F : \mathbb{R}^N \to \mathbb{R}$ such that:*

(1) *There exists an open set $G \subset \mathbb{R}^N$ such that G^c has zero Lebesgue measure and F is twice continuously differentiable in G.*

(2) *$|F(x)| + \left|\frac{\partial F}{\partial x_i}(x)\right| \leq C(1 + |x|^M)$, for some constants $C > 0$ and $M > 0$ and for all $x \in G$ and $i = 1, \ldots, N$.*

(3) *For each $i = 1, \ldots, N$ and for each $s > 0$ and $p \geq 1$,*

$$\mathbb{E}\left[\left|\frac{\partial^2 F}{\partial x_i^2}(B_s)\right|^p\right] \leq C s^{-pH},$$

for some constant $C > 0$.

Then, for each $i = 1, \ldots, N$ and $t \in [0, T]$, the process $\{\frac{\partial F}{\partial x_i}(B_s)\mathbf{1}_{[0,t]}(s), \ s \in [0, T]\}$ belongs to the space $\mathbb{L}_{H,i}^{1,1/H}$ and

$$F(B_t) = F(0) + \sum_{i=1}^{N} \int_0^t \frac{\partial F}{\partial x_i}(B_s)\delta B_s^i + H \sum_{i=1}^{N} \int_0^t \frac{\partial^2 F}{\partial x_i^2}(B_s)s^{2H-1}ds. \tag{3.1}$$

Proof. Notice first that the processes $F(B_t)$, $\frac{\partial F}{\partial x_i}(B_t)$ and $\frac{\partial^2 F}{\partial x_i^2}(B_t)$ are well defined because the probability that B_t belongs to G^c is zero. On the other hand, $F(0)$ is also well defined as the limit in $L^1(\Omega)$ of $F(B_t)$ as t tends to zero. This limit exists because for any $s < t$ we can write

$$|F(B_t) - F(B_s)| = \left|\int_0^1 \sum_{i=1}^{N} \frac{\partial F}{\partial x_i}(B_s + \sigma(B_t - B_s))(B_t^i - B_s^i)d\sigma\right|$$

$$\leq C\left(1 + \frac{1}{M+1}|B_t - B_s|^M\right).$$

Conditions (2) and (3) imply that for each $i = 1, \ldots, N$, the process $u_i(s) = \frac{\partial F}{\partial x_i}(B_s)\mathbf{1}_{[0,t]}(s)$ belongs to the space $\mathbb{L}_{H,i}^{1,1/H}$. In fact,

$$\mathbb{E}\left[\int_0^T |u_i(s)|^{1/H}\,ds\right] \leq C^{1/H}\mathbb{E}\left[\int_0^T (1 + |B_s|^M)^{1/H}ds\right] < \infty,$$

and

$$\mathbb{E}\left[\int_0^T \int_0^T |D_r^{(i)}u_i(s)|^{\frac{1}{H}}drds\right] = \mathbb{E}\left[\int_0^T s\left|\frac{\partial^2 F}{\partial x_i^2}(B_s)\right|^{\frac{1}{H}}ds\right] \leq CT.$$

On the other hand, taking $p = 1$ in condition (3), we also have for each $i = 1, \ldots, N$,

$$\mathbb{E}\left[\int_0^t \left|\frac{\partial^2 F}{\partial x_i^2}(B_s)\right|s^{2H-1}ds\right] < \infty.$$

As a consequence, all terms in Eq. (3.1) are well defined. Then, to prove the equality it suffices to show that for any random variable of the form

$$G = g(B_{t_1}, \ldots, B_{t_n}),$$

where $0 < t_1 < \cdots < t_i < t < t_{i+1} < \cdots t_n$, for some $i = 0, 1, \ldots, n$, where g is C^∞ with compact support, we have

$$
\mathbb{E}\left[GF(B_t) - GF(0)\right] = \sum_{i=1}^N \mathbb{E}\left[G\int_0^t \frac{\partial F}{\partial x_i}(B_s)\delta B_s^i\right]
$$

$$
+ H\sum_{i=1}^N \mathbb{E}\left[G\int_0^t \frac{\partial^2 F}{\partial x_i^2}(B_s)s^{2H-1}ds\right]. \tag{3.2}
$$

Denote by $p_{t,\tau}(y, x)$ the joint density of the vector $(B_{t_1}, \ldots, B_{t_n}, B_t)$, where $\tau = (t_1, \ldots, t_n)$. By the duality between the Skorohod integral and the derivative operator (see (2.2)), we can write

$$
\mathbb{E}\left[G\int_0^t \frac{\partial F}{\partial x_i}(B_s)\delta B_s^i\right] = \mathbb{E}\left[\left\langle D^i G, \frac{\partial F}{\partial x_i}(B.)\mathbf{1}_{[0,t]}\right\rangle_{\mathcal{H}}\right]
$$

$$
= \sum_{j=1}^n \mathbb{E}\left[\frac{\partial g}{\partial y_{ij}}(B_{t_1}, \ldots, B_{t_n})\left\langle \mathbf{1}_{[0,t_j]}, \frac{\partial F}{\partial x_i}(B.)\mathbf{1}_{[0,t]}\right\rangle_{\mathcal{H}}\right]
$$

$$
= \int_0^t \sum_{j=1}^n \mathbb{E}\left[\frac{\partial g}{\partial y_{ij}}(B_{t_1}, \ldots, B_{t_n})\frac{\partial F}{\partial x_i}(B_s)\right]\frac{\partial R}{\partial s}(s, t_j)ds
$$

$$
= \int_0^t \sum_{j=1}^n \int_{\mathbb{R}^{N(n+1)}} \frac{\partial g}{\partial y_{ij}}(y)\frac{\partial F}{\partial x_i}(x)\frac{\partial R}{\partial s}(s, t_j)p_{s,\tau}(x, y)dxdyds.
$$

We can integrate by parts in the above expression, for each fixed $s \notin \{0, t_1, \ldots, t_n\}$. We know that $\frac{\partial F}{\partial x_i}$ is only differentiable in G, but using condition (2), and a regularization procedure, we can proof rigorously this integration by parts argument. In that way we obtain

$$
\mathbb{E}\left[G\int_0^t \frac{\partial F}{\partial x_i}(B_s)\delta B_s^i\right] = -\int_0^t \sum_{j=1}^n \int_{\mathbb{R}^{N(n+1)}} \frac{\partial g}{\partial y_{ij}}(y)F(x)\frac{\partial R}{\partial s}(s, t_j)\frac{\partial p_{s,\tau}}{\partial x_i}(x, y)dxdyds.
$$

On the other hand,

$$
\mathbb{E}\left[GF(B_t) - GF(0)\right] = \int_{\mathbb{R}^{N(n+1)}} F(x)g(y)p_{t,\tau}(x, y)dxdy - \mathbb{E}(G)F(0)
$$

$$
= \int_0^t \int_{\mathbb{R}^{N(n+1)}} F(x)g(y)\frac{\partial p_{s,\tau}}{\partial s}(x, y)dxdyds.
$$

The second term in (3.2) follows by regularizing F with an approximation of the identity. Also, integrating by parts,

$$
\mathbb{E}\left[G\int_0^t \frac{\partial^2 F}{\partial x_i^2}(B_s)s^{2H-1}ds\right] = \int_0^t \int g(y)\frac{\partial^2 F}{\partial x_i^2}(x)p_{s,\tau}(x, y)s^{2H-1}dxdyds
$$

$$
= \int_0^t \int g(y)F(x)\frac{\partial^2 p_{s,\tau}}{\partial x_i^2}(x, y)s^{2H-1}dxdyds.
$$

Finally, the result follows from

$$\frac{\partial p_{s,\tau}}{\partial s} = \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\partial R}{\partial s}(s, t_j) \frac{\partial^2 p_{s,\tau}}{\partial y_{ij} \partial x_i} + \sum_{i=1}^{N} H s^{2H-1} \frac{\partial^2 p_{s,\tau}}{\partial x_i^2}. \quad \square$$

In the rest of this section we will recall a property on the $1/H$-variation of the divergence integral established in [13]. Fix $T > 0$ and set $t_i^n := \frac{iT}{n}$, where n is a positive integer and $i = 0, 1, \ldots, n$. Given a stochastic process $X = \{X_t, t \in [0, T]\}$ we define for each $p \geq 1$,

$$V_n^p(X) := \sum_{i=0}^{n-1} \left| X_{t_{i+1}^n} - X_{t_i^n} \right|^p.$$

Definition 3.2. The p-variation, $p \geq 1$, of a stochastic process X is defined as the limit, in $L^1(\Omega)$, if it exists, of $V_n^p(X)$ as $n \to \infty$.

Then, following result (see Theorem 4.8 in [13]), allows one to compute the $1/H$-variation of a Skorohod integral with respect to a multidimensional fractional Brownian motion.

Theorem 3.3. *Let $\frac{1}{2} < H < 1$ and $u^i \in \mathbb{L}_{H,i}^{1,1/H}$ for each $i = 1, \ldots, N$. Set $X_t := \sum_{i=1}^{N} \int_0^t u_s^{(i)} \delta B_s^{(i)}$, for each $t \in [0, T]$. Then*

$$V_n^{1/H}(X) \xrightarrow[n \to \infty]{L^1(\Omega)} \int_{\mathbb{R}^N} \left[\int_0^T |\langle u_s, \xi \rangle|^{1/H} \, ds \right] \nu(d\xi),$$

where ν is the standard normal distribution on $\mathbb{R}^N$.

4. No colliding of eigenvalues of a matrix Gaussian Hölder continuous

In this section we will show that for the random symmetric matrix corresponding to a general Gaussian process with Hölder continuous trajectories of order larger than $1/2$, the eigenvalues do not collide almost surely.

Suppose that $x = \{x(t), \ t \geq 0\}$ is a zero mean Gaussian process satisfying

$$\mathbb{E}(|x(t) - x(s)|^2) \leq C_T |t - s|^{2\gamma}, \tag{4.1}$$

for any $s, t \in [0, T]$, where $\gamma \in (1/2, 1)$. Suppose also that $x(0) = 0$ and the variance of $x(t)$ is positive for any $t > 0$. We know that, by Kolmogorov continuity theorem, the trajectories of x are Hölder continuous of order β for any $\beta < \gamma$.

Consider a symmetric random matrix of the form

$$X(t) = X(0) + \widehat{X}(t),$$

where $X(0)$ is a fixed deterministic symmetric matrix and $\widehat{X}_{ij}(t) = x_{ij}(t)$ if $i < j$ and $\widehat{X}_{ii}(t) = \sqrt{2} x_{i,i}(t)$, where $\{x_{ij}, \ i \leq j\}$ are independent copies of the process x. For any $t > 0$ the matrix $X(t)$ has full rank a.s. The following is the main result of this section.

Theorem 4.1. *Denote by $\lambda_i(t)$ the eigenvalues of the random matrix $X(t)$, $i = 1, \ldots, n$. We can assume that $\lambda_1(t) \geq \cdots \geq \lambda_d(t)$. Then,*

$$P(\lambda_1(t) > \cdots > \lambda_d(t), \ \forall t > 0) = 1. \tag{4.2}$$

Proof. The proof will be done in several steps. Fix $t_0 > 0$. From the well-known results for the Gaussian Orthogonal random matrix we know that $\lambda_1(t_0) > \cdots > \lambda_d(t_0)$ almost surely.

Applying the Hoffman–Wielandt inequality (see [15]), we deduce

$$\sum_{i=1}^{d}(\lambda_i(t) - \lambda_i(s))^2 \leq \frac{1}{d} \sum_{i,j=1}^{d} (X_{ij}(t) - X_{ij}(s))^2 = \frac{1}{d} \sum_{i,j=1}^{d} (\widehat{X}_{ij}(t) - \widehat{X}_{ij}(s))^2$$

for any $s, t \geq 0$. This implies that for each i and each real $p \geq 1$,

$$\mathbb{E}(|\lambda_i(t) - \lambda_i(s)|^p) \leq C|t - s|^{p\gamma}, \tag{4.3}$$

and choosing p such that $p\gamma > 1$ we deduce that the trajectories of $\lambda_i(t)$ are Hölder continuous of order β for any $\beta < \gamma$.

Consider the stopping time

$$\tau = \inf\{t \geq t_0 : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j\}.$$

Notice that $\tau > t_0$ almost surely. On the random interval $[t_0, \tau)$ the function $\log(\lambda_i(t) - \lambda_j(t))$, where $i \neq j$, is well defined and we can use the stochastic calculus with respect to the Young integral to write for any $t_0 \leq t < \tau$,

$$\log(\lambda_i(t) - \lambda_j(t)) = \log(\lambda_i(t_0) - \lambda_j(t_0)) + \int_{t_0}^{t} \frac{1}{\lambda_i(s) - \lambda_j(s)} \left(d\lambda_i(s) - d\lambda_j(s) \right). \tag{4.4}$$

The Riemann–Stieltjes integral

$$I_{i,j}^i(t) := \int_{t_0}^{t} \frac{1}{\lambda_i(s) - \lambda_j(s)} d\lambda_i(s),$$

can be expressed in terms of fractional derivative operators, following the approach by Zäble [33]. Choosing α such that $1 - \gamma < \alpha < \frac{1}{2}$, we obtain

$$I_{i,j}^i(t) = \int_{t_0}^{t} D_{t_0+}^{\alpha}(\lambda_i - \lambda_j)_{t_0}^{-1}(s) D_{t-}^{1-\alpha}\lambda_{i,t-}(s)ds + (\lambda_i(t_0) - \lambda_j(t_0))^{-1}(\lambda_i(t) - \lambda_i(t_0)),$$

where

$$I_{i,j}(s) := D_{t_0+}^{\alpha}(\lambda_i - \lambda_j)_0^{-1}(s) = \frac{1}{\Gamma(1-\alpha)} \left(s^{-\alpha} \left(\frac{1}{\lambda_i(s) - \lambda_j(s)} - \frac{1}{\lambda_i(t_0) - \lambda_j(t_0)} \right) \right.$$
$$\left. + \alpha \int_{t_0}^{s} \frac{(\lambda_i(s) - \lambda_j(s))^{-1} - (\lambda_i(y) - \lambda_j(y))^{-1}}{(s-y)^{\alpha+1}} dy \right),$$

and

$$J_j(s) := D_{t-}^{1-\alpha}\lambda_{i,t-}(s) = \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda_i(s) - \lambda_i(t)}{(t-s)^{1-\alpha}} + (1-\alpha)\int_s^t \frac{\lambda_i(s) - \lambda_i(y)}{(y-s)^{2-\alpha}} dy \right).$$

We claim that

$$P\left(\int_{t_0}^{t} |I_{i,j}(s)||J_j(s)|ds < \infty, \text{ for all } t \geq t_0, \ i \neq j \right) = 1. \tag{4.5}$$

This claim implies that for all $i \neq j$, $\int_{t_0}^{\tau} |I_{i,j}(s)||J_j(s)|ds < \infty$ almost surely on the set $\{\tau < \infty\}$. Therefore $P(\tau = \infty) = 1$, otherwise we would get a contradiction with $\log(\lambda_i(\tau) - \lambda_j(\tau)) = -\infty$.

In order to prove the claim (4.5) we are going to show that for all $t \geq t_0$, and for all $i \neq j$,

$$\mathbb{E}\left(\int_{t_0}^t |I_{i,j}(s)||J_j(s)|\right) ds < \infty. \tag{4.6}$$

In order to show (4.6), we first apply Hölder's inequality with exponents $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and we get

$$\mathbb{E}(|I_{i,j}(s)||J_j(s)|) \leq [E(|I_{i,j}(s)|^p)]^{\frac{1}{p}} [E(|J_j(s)|^q)]^{\frac{1}{q}}.$$

By the estimate (4.3) for any fixed β such that $1 - \alpha < \beta < \gamma$ there exists a random variable G with moments of all orders such that for all i

$$|\lambda_i(s) - \lambda_i(y)| \leq CG|s - y|^{\beta}, \tag{4.7}$$

for all $s, y \in [t_0, t]$, which leads to the estimate

$$E(|J_j(s)|^q) \leq C_q E(G^q), \tag{4.8}$$

for all $q > 1$ and for some constant $C > 0$. In order to estimate $E(|I_{i,j}(s)|^p)$ we consider first the second summand in the definition of $I_{i,j}(s)$ denoted by

$$K_{i,j}(s) := \int_{t_0}^s \frac{(\lambda_i(s) - \lambda_j(s))^{-1} - (\lambda_i(y) - \lambda_j(y))^{-1}}{(s - y)^{\alpha+1}} dy.$$

This term can be expressed as

$$K_{i,j}(s) = \int_{t_0}^s \frac{(\lambda_i(y) - \lambda_j(y) - \lambda_i(s) + \lambda_j(s))}{(s - y)^{\alpha+1} [(\lambda_i(s) - \lambda_j(s))][(\lambda_i(y) - \lambda_j(y))]} dy.$$

Then, if $a + b = 1$, using the estimate (4.7), we obtain

$$|K_{i,j}(s)| \leq \int_{t_0}^s \frac{|\lambda_i(y) - \lambda_j(y) - \lambda_i(s) + \lambda_j(s)|^a |\lambda_i(y) - \lambda_j(y) - \lambda_i(s) + \lambda_j(s)|^b}{(s - y)^{\alpha+1} |(\lambda_i(s) - \lambda_j(s))||(\lambda_i(y) - \lambda_j(y))|} dy$$

$$\leq (2G)^a \int_{t_0}^s \frac{|\lambda_i(y) - \lambda_j(y) - \lambda_i(s) + \lambda_j(s)|^b}{|(\lambda_i(s) - \lambda_j(s))||(\lambda_i(y) - \lambda_j(y))|} (s - y)^{a\beta - \alpha - 1} dy$$

$$\leq (2G)^a \int_{t_0}^s \left(|\lambda_i(y) - \lambda_j(y)|^{b-1} |(\lambda_i(s) - \lambda_j(s))|^{-1} \right.$$

$$\left. + |\lambda_i(y) - \lambda_j(y)|^{-1} |(\lambda_i(s) - \lambda_j)(s)|^{b-1} \right) (s - y)^{a\beta - \alpha - 1} dy.$$

Therefore

$$\|K_{i,j}(s)\|_p \leq 2^a \|G^a\|_{p_1} \int_{t_0}^s \left(\||\lambda_i(y) - \lambda_j(y)|^{b-1}\|_{p_2} \||\lambda_i(s) - \lambda_j(s)|^{-1}\|_{p_3} \right.$$

$$\left. + \||\lambda_i(y) - \lambda_j(y)|^{-1}\|_{p_3} \||\lambda_i(s) - \lambda_j(s)|^{b-1}\|_{p_2} \right) (s - y)^{a\beta - \alpha - 1} dy,$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$, with $p_i > 1$ for $i = 1, 2, 3$. We choose a, p_1, p_2 and p_3 such that

$$a > \frac{\alpha}{\beta}, \qquad p_3 < 2, \qquad p_2 < \frac{2\beta}{\alpha},$$

which is possible by taking p and p_1 close to 1 and using the inequality $\alpha < \frac{1}{2} < \beta$. Finally, to complete the proof we need to estimate the expectation

$$\mathbb{E}(|(\lambda_i(s) - \lambda_j(s))|^{-q}) \tag{4.9}$$

when $q < 2$. The joint density of the eigenvalues $\lambda_1(s) > \cdots > \lambda_d(s)$ is given by

$$c_d \prod_{k<h} |(\lambda_k - \lambda_h)| \, \sigma(s)^{-d(d+1)/2} \exp\left(-\sum_{i=1}^d \frac{\lambda_i^2}{4\sigma^2(s)}\right), \tag{4.10}$$

where c_d is a constant depending only on d and $\sigma^2(s)$ is the variance of $x(s)$; see [1, Thm. 2.5.2]. Then, the expectation $\mathbb{E}(|(\lambda_i(s) - \lambda_j(s))|^{-q})$ can be estimated up to a constant by

$$\int_{\mathbb{R}^d} \prod_{k<h} (\lambda_k - \lambda_h) |\lambda_i - \lambda_j|^{-q} \sigma(s)^{-d(d+1)/2} \exp\left(-\sum_{i=1}^d \frac{\lambda_i^2}{4\sigma^2(s)}\right) d\lambda.$$

Making the change of variable $\lambda_i = \sigma(s)\mu_i$ we get

$$\mathbb{E}(|(\lambda_i(s) - \lambda_j(s))|^{-q}) \leq C\sigma(s)^{-q}.$$

Our hypothesis of positivity of the variance, together with its continuity, imply that $\sigma(s)$ is bounded away from zero in the interval $[t_0, t]$. Therefore, $\mathbb{E}(|(\lambda_i(s) - \lambda_j(s))|^{-q})$ is uniformly bounded on $[t_0, t]$. This allows us to complete the proof of (4.6). So, the claim (4.5) holds, and this implies that $P(\tau = \infty) = 1$. Finally, letting t_0 tend to zero we obtain the desired result. $\qquad\square$

Remark 4.2. Theorem 4.1 also holds in the case of a random Hermitian matrix corresponding to a general Gaussian process with Hölder continuous trajectories of order γ. Namely, consider a Hermitian random matrix

$$X(t) = X(0) + \widehat{X}(t),$$

where $X(0)$ is a fixed deterministic Hermitian matrix and

$$\widehat{X}_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}} \left\{ \mathrm{Re}(x_{ij}(t)) + \sqrt{-1}\,\mathrm{Im}(x_{ij}(t)) \right\} & \text{if } i \neq j, \\ x_{ii}(t) & \text{if } i = j, \end{cases}$$

where $\{\mathrm{Re}(x_{ij}(t)), \mathrm{Im}(x_{ij}(t)), \ i < j, x_{ii}(t)\}$ are independent copies of a zero mean Gaussian process $x = \{x(t), \ t \geq 0\}$ satisfying (4.1) for any $s, t \in [0, T]$, where $\gamma \in (1/2, 1)$. Since for each $t > 0$ $X(t)$ is a Gaussian unitary ensemble, then the matrix $X(t)$ has full rank and distinct eigenvalues a.s. In this case the joint density of the eigenvalues $\lambda_1(s) > \cdots > \lambda_d(s)$ is given by (see [1, Thm. 2.5.2])

$$\widetilde{c}_d \prod_{k<h} |\lambda_k - \lambda_h|^2 \, \sigma(s)^{-d^2} \exp\left(-\sum_{i=1}^d \frac{\lambda_i^2}{2\sigma^2(s)}\right), \tag{4.11}$$

where $\widetilde{c}_d$ is a constant depending only on d and $\sigma^2(s)$ is the variance of $x(s)$, which is assumed to be positive. Then, proceeding as in the proof of Theorem 4.1 one can prove that if $\lambda_i(t)$ denote the eigenvalues of the random matrix $X(t)$, $i = 1, \ldots, n$ and $\lambda_1(t) \geq \cdots \geq \lambda_d(t)$, then (4.2) also holds.

5. Stochastic differential equation for the eigenvalues of a matrix fBm

We first consider several needed facts about eigenvalues as functions of entries of a symmetric matrix. We denote by $\mathcal{H}_d$ the collection of symmetric d-dimensional matrices. For a matrix $X = (x_{ij}) \in \mathcal{H}_d$ we use the coordinates $x_{ij}, i \leq j$, and in this way we identify $\mathcal{H}_d$ with $\mathbb{R}^{d(d+1)/2}$. We denote by $\mathcal{H}_d^{vg}$ the set of matrices $X \in \mathcal{H}_d$ such that there is a factorization

$$X = UDU^*,$$

where D is a diagonal matrix with entries $\lambda_i = D_{ii}$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_d$, U is an orthogonal matrix, with $U_{ii} > 0$ for all i, $U_{ij} \neq 0$ for all i, j and all minors of U have non zero determinants. The matrices in the set $\mathcal{H}_d^{vg}$ are called *very good* matrices, and we can identify $\mathcal{H}_d^{vg}$ as an open subset of $\mathbb{R}^{d(d+1)/2}$. It is known that the complement of $\mathcal{H}_d^{vg}$ has zero Lebesgue measure.

Denote by $\mathcal{U}_d^{vg}$ the set of all orthogonal matrices U, with $U_{ii} > 0$ for all i, $U_{ij} \neq 0$ for all i, j and all minors of U have non zero determinants. Let $\mathcal{S}_d$ be the open simplex

$$\mathcal{S}_d = \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d : \lambda_1 > \lambda_2 > \cdots > \lambda_d\}. \tag{5.1}$$

For any $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathcal{S}_d$, let D_λ be the diagonal matrix such that $D_{ii} = \lambda_i$. On the other hand, consider the mapping $T : \mathcal{U}_d^{vg} \to \mathbb{R}^{d(d-1)/2}$ defined by

$$T(U) = \left(\frac{U_{12}}{U_{11}}, \ldots, \frac{U_{1d}}{U_{11}}, \ldots, \frac{U_{d-1,d}}{U_{d-1,d-1}} \right).$$

It is known that T is bijective and smooth. Then, the mapping $\hat{T} : \mathcal{S}_d \times T(\mathcal{U}_d^{vg}) \to \mathcal{H}_d^{vg}$ given by $\hat{T}(\lambda, z) = T^{-1}(z) D_\lambda T^{-1}(z)^*$ is a smooth bijection. Denote by Ψ the inverse of $\hat{T}$. Then,

$$\Psi(X) = (\Phi(X), T(U)).$$

As a consequence of these results, $\lambda(X) = \Phi(X)$ is a smooth function of $X \in \mathcal{H}_d^{vg}$.

Suppose that X is a smooth function of a parameter $\theta \in \mathbb{R}$. Then, we know that

$$\partial_\theta \lambda_i = (U^* \partial_\theta X U)_{ii}$$

and

$$\partial_\theta^2 \lambda_i = (U^* \partial_\theta^2 X U)_{ii} + 2 \sum_{j \neq i} \frac{|(U^* \partial_\theta X U)_{ij}|^2}{\lambda_i - \lambda_j}.$$

In particular if $\theta = x_{kh}$ with $k \leq h$, then

$$\frac{\partial \lambda_i}{\partial x_{kh}} = 2 U_{ik} U_{ih} \mathbf{1}_{\{k \neq h\}} + U_{ik}^2 \mathbf{1}_{\{k=h\}},$$

and

$$\frac{\partial^2 \lambda_i}{\partial x_{kh}^2} = 2 \sum_{j \neq i} \frac{|U_{ik} U_{jh} + U_{ih} U_{jk}|^2}{\lambda_i - \lambda_j} \mathbf{1}_{\{k \neq h\}} + 2 \sum_{j \neq i} \frac{|U_{ik} U_{jk}|^2}{\lambda_i - \lambda_j} \mathbf{1}_{\{k=h\}}.$$

Consider now a family of independent fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$, $b = \{\{b_{ij}(t), t \geq 0\}, 1 \leq i \leq j \leq d\}$. We define the *symmetric matrix fractional*

Brownian motion (with parameter H) $B(t)$ by $B_{i,j}(t) = b_{ij}(t)$ if $i < j$ and $B_{ii}(t) = \sqrt{2}b_{i,i}(t)$. We identify $B(t)$ as an element in $\mathbb{R}^{d(d+1)/2}$.

As a consequence of the previous discussion, we have the following result.

Lemma 5.1. *For any $i = 1, \ldots, d$, there exists a function $\Phi_i : \mathbb{R}^{d(d+1)/2} \to \mathbb{R}$, which is C^∞ in an open subset $G \subset \mathbb{R}^{d(d+1)/2}$, with $|G^c| = 0$, such that $\lambda_i(t) = \Phi_i(b(t))$. Moreover, for any $k \leq h$*

$$\frac{\partial \Phi_i}{\partial b_{kh}} = 2U_{ik}U_{ih}\mathbf{1}_{\{k \neq h\}} + \sqrt{2}U_{ik}^2\mathbf{1}_{\{k=h\}}, \tag{5.2}$$

and

$$\frac{\partial^2 \Phi_i}{\partial b_{kh}^2} = 2\sum_{j \neq i} \frac{|U_{ik}U_{jh} + U_{ih}U_{jk}|^2}{\lambda_i - \lambda_j}\mathbf{1}_{\{k \neq h\}} + 4\sum_{j \neq i} \frac{|U_{ik}U_{jk}|^2}{\lambda_i - \lambda_j}\mathbf{1}_{\{k=h\}}.$$

On the other hand, the joint density of the eigenvalues $\lambda_1(t) > \cdots > \lambda_d(t)$ of $B(t)$ can be obtained from [1, Thm. 2.5.2] as follows

$$c_d \prod_{k<h} |\lambda_k - \lambda_h| \, t^{-Hd(d+1)/2} \exp\left(-\sum_{i=1}^d \frac{\lambda_i^2}{4t^{2H}}\right), \tag{5.3}$$

where c_d is a constant depending only on d.

Then we can prove the following analogous of (1.1) for the evolving of the eigenvalues processes of a matrix fractional Brownian motion. It is given in terms of the Skorohod integral of the functions $\frac{\partial \Phi_i}{\partial b_{kh}}$, $i = 1, \ldots, d$.

Theorem 5.2. *Let $H \in (1/2, 1)$ and $\{B(t), t \geq 0\}$ be a matrix fractional Brownian motion of parameter H as above. Let $X(0)$ be an arbitrary deterministic symmetric matrix and $B(0) = X(0)$. For each $t \geq 0$, let $\lambda_1(t), \ldots, \lambda_d(t)$ be the eigenvalues of $B(t)$. Then, for any $t > 0$ and $i = 1, \ldots, d$,*

$$\lambda_i(t) = \lambda_i(0) + Y_t^i + 2H \sum_{j \neq i} \int_0^t \frac{s^{2H-1}}{\lambda_i(s) - \lambda_j(s)} ds, \tag{5.4}$$

where

$$Y_t^i = \sum_{k \leq h} \int_0^t \frac{\partial \Phi_i}{\partial b_{kh}}(b(s))\delta b_{kh}(s). \tag{5.5}$$

Proof. Without loss of generality we can consider $X(0) = 0$. Consider the function $\Phi : \mathbb{R}^{d(d+1)/2} \to \mathbb{R}^d$ introduced in Lemma 5.1, which is C^∞ in an open subset $G \subset \mathbb{R}^{d(d+1)/2}$ whose complement has zero Lebesgue measure. We claim that this function satisfies the assumptions of Theorem 3.1. First notice that

$$\sum_{i=1}^d \Phi_i^2 \leq \frac{1}{d} \sum_{i=1}^d b_{ij}^2 + \frac{1}{2}d \sum_{i<j}^d b_{ij}^2,$$

and

$$\left|\frac{\partial \Phi_i}{\partial b_{kh}}\right| \leq 2 + \sqrt{2}.$$

On the other hand, taking into account that $\lambda_i(t) = \Phi_i(b(t))$ and using the density of the eigenvalues (5.3), we can write

$$
\begin{aligned}
\mathbb{E}\left[\left|\frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(b_s)\right|^p\right] &\leq C_p \sum_{j \neq i} \mathbb{E}[|\lambda_i(s) - \lambda_j(s)|^{-p}] \\
&= C_p \sum_{j \neq i} \int_{\mathcal{S}_d} \prod_{k<h} (\lambda_k - \lambda_h)|\lambda_i - \lambda_j|^{-p} s^{-\frac{Hd(d+1)}{2}} \exp\left(-\sum_{i=1}^d \frac{\lambda_i^2}{4s^{2H}}\right) d\lambda \\
&\leq C_p s^{-pH},
\end{aligned} \tag{5.6}
$$

where we have made the change of variable $\lambda_i = s^H \mu_i$.

Therefore, conditions (1), (2) and (3) if Theorem 3.1 hold for Φ. Notice also that

$$
\sum_{k \leq h} \frac{\partial^2 \Phi_i}{\partial b_{kh}^2} = 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}.
$$

Then, Theorem 3.1 yields

$$
\lambda_i(t) = Y_t^i + 2H \sum_{j \neq i} \int_0^t \frac{s^{2H-1}}{\lambda_i(s) - \lambda_j(s)} ds,
$$

where Y_t^i is given by (5.5). We remark that by Theorem 4.1 $(\lambda_1(t), \ldots, \lambda_d(t))$ belongs to the open simplex $\mathcal{S}_d$ for each $t > 0$ with probability one. □

Remark 5.3. (a) Notice that Eq. (5.4) is similar to the Itô SDE (1.1) satisfied for the eigenvalues of the matrix Brownian motion, in the case $H = 1/2$. A natural question is to ask whether the processes Y^i are one-dimensional independent fractional Brownian motions in the general case $H > 1/2$. In the case $H = 1/2$ this is obtained applying Lévy's characterization theorem. Unfortunately, although there is a version of this theorem for the fractional Brownian motion (see [17]), it cannot be used here due to the lack of martingale properties.

(b) We will show below that for each $i = 1, \ldots, d$, the process Y^i has the same self-similar and variation properties as the fractional Brownian motion. However, from these properties we cannot conclude that Y^i is a fractional Brownian motion because we do not know if these processes are Gaussian. We conjecture that these processes are not Gaussian (see [16] for a related problem concerning the fractional Bessel process, where again this type of non Gaussian process appears).

Proposition 5.4. *Assuming $B(0) = 0$, the process $Y = (Y^1, \ldots, Y^d)$ is H-self-similar.*

Proof. Let $a > 0$. By the self-similarity of the fractional Brownian motion it follows that $\{\lambda(at), \ t \geq 0\}$ has the same law as $\{a^H \lambda(t), \ t \geq 0\}$. Then, the result follows from the equation

$$
Y_t^i = \lambda_i(t) - 2H \sum_{j \neq i} \int_0^t \frac{s^{2H-1}}{\lambda_i(s) - \lambda_j(s)} ds,
$$

for $i = 1, \ldots, d$. □

Finally, as an application of Theorem 3.3, we show that for each $i = 1, \ldots, d$, the process Y^i has the same $1/H$ variation as a fBm with variance $2t^{2H}$.

Proposition 5.5. *For each $i = 1, \ldots, d$, the process Y^i has a $1/H$ variation equal to $\sqrt{2t}\mathbb{E}\left[|Z|^{1/H}\right]$, where Z is $N(0,1)$ random variable.*

Proof. By Theorem 5.2, we have that for each k, h the process $\left\{ \frac{\partial \Phi_i}{\partial b_{kh}}(b(s)), \ s \in [0, T] \right\}$ belongs to the space $\mathbb{L}_{H,i}^{1,1/H}$ for each $i = 1, \ldots, N = \frac{d(d+1)}{2}$. Therefore, by Theorem 3.3 the $1/H$-variation of Y^i in the time interval $[0, t]$ is given by

$$\mathbb{E}^{\Theta} \left[\int_0^t \left| \sum_{k \leq h} \frac{\partial \Phi_i}{\partial b_{kh}}(b(s)) \Theta_{kh} \right|^{1/H} ds \right],$$

where Θ is an N-dimensional standard normal random variable. Let us denote by S^N the unitary N-dimensional sphere and let σ_N be the uniform probability measure defined on S^{N-1}. Notice that the vector $\frac{1}{\sqrt{2}} \left(\frac{\partial \Phi_i}{\partial b_{kh}}(b(s)) \right)_{k \leq h}$, denoted by R_s^i, takes values in S^{N-1} because from (5.2) we obtain

$$\sum_{k \leq h} \left| \frac{\partial \Phi_i}{\partial b_{kh}}(b(s)) \right|^2 = 2.$$

Therefore,

$$2\mathbb{E}^{\Theta} \left[\int_0^t |\langle R_s^i, \Theta \rangle|^{1/H} ds \right] = 2\mathbb{E}[|\Theta|] \int_{S^{d-1}} \left[\int_0^T |\langle R_s^i, \eta \rangle|^{1/H} ds \right] \sigma_N(d\eta).$$

Moreover, if $e \in S^{d-1}$, the integral $\int_{S^{N-1}} |\langle e, \eta \rangle|^{1/H} \sigma_N(d\eta)$ does not depend on the vector e. Therefore, choosing $e = (1, 0, \ldots, 0)$, yields

$$2\mathbb{E}^{\Theta} \left[\int_0^t |\langle R_s^i, \Theta \rangle|^{1/H} ds \right] = 2t\mathbb{E}[|\Theta|] \int_{S^{N-1}} |\eta_1|^{1/H} \sigma_N(d\eta) = 2t\mathbb{E} \left[|Z|^{1/H} \right],$$

where Z is a one-dimensional $N(0, 1)$ random variable. This completes the proof of the proposition. $\quad\square$

Remark 5.6. Consider the Hermitian matrix fractional Brownian motion $B(t) = (B_{ij}(t))$ where

$$B_{ij} = \begin{cases} \frac{1}{\sqrt{2}} \mathrm{Re}(b_{ij}(t)) + \sqrt{-1} \mathrm{Im}(b_{ij}(t)) & \text{if } i \neq j, \\ b_{i,i}(t), & \text{if } i = j, \end{cases}$$

where $\left\{ \mathrm{Re}(b_{ij}(t)), \mathrm{Im}(b_{ij}(t)), \ i < j, b_{i,i}(t) \right\}$ is a family of independent fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$. We identify $B(t)$ as an element in $\mathbb{R}^{d^2}$.

(a) In this case one can prove that if $X(0)$ is an arbitrary deterministic Hermitian matrix and $B(0) = X(0)$ and for each $t \geq 0$, $\lambda_1(t), \ldots, \lambda_d(t)$ denote the eigenvalues of $B(t)$, then, for any $t > 0$ and $i = 1, \ldots, d$,

$$\lambda_i(t) = \lambda_i(0) + Y_t^i + 2H \sum_{j \neq i} \int_0^t \frac{s^{2H-1}}{\lambda_i(s) - \lambda_j(s)} ds, \tag{5.7}$$

where

$$Y_t^i = \sum_{k \leq h} \int_0^t \frac{\partial \Upsilon_i}{\partial b_{kh}}(b(s)) \delta b_{kh}(s). \tag{5.8}$$

The process Y_t^i given by (5.8) is such that for each $i = 1, \ldots, d$, Υ_i is a smooth function of $X \in \mathcal{H}_d^{(2)vg}$, the set of very good Hermitian matrices consisting of matrices with a decomposition

$X = UDU^*$, where D is a diagonal matrix with entries $\lambda_i = D_{ii}$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_d$, U is an unitary matrix, with $U_{ii} > 0$ for all i, $U_{ij} \neq 0$ for all i, j and all minors of U have non zero determinants ($U \in \mathcal{U}_d^{(2)vg}$). This follows from Lemma 2.5.6 in [1] which gives that the mapping $T^{(2)} : \mathcal{U}_d^{(2)vg} \to \mathbb{R}^{d(d-1)}$ defined by

$$T^{(2)}(U) = \left(\frac{U_{12}}{U_{11}}, \ldots, \frac{U_{1d}}{U_{11}}, \frac{U_{2,3}}{U_{22}}, \ldots, \frac{U_{2,d}}{U_{2,2}}, \cdots \frac{U_{d-1,d}}{U_{d-1,d-1}} \right),$$

is bijective and smooth and the complement of $\mathcal{H}_d^{(2)vg}$ has Lebesgue measure zero. Therefore the mapping $\hat{T}^{(2)} : \mathcal{S}_d \times T^{(2)}(\mathcal{U}_d^{(2)vg}) \to \mathcal{H}_d^{(2)vg}$ given by $\hat{T}^{(2)}(\lambda, z) = \left(T^{(2)} \right)^{-1}(z) D_\lambda \left(T^{(2)} \right)^{-1}(z)^*$ is a smooth bijection. As a consequence, $\lambda(X) = \Upsilon(X)$ is a smooth function of $X \in \mathcal{H}_d^{(2)vg}$.

(b) The proof of (5.7), analogous to that of Theorem 5.2, requires to consider estimates like (5.6) but now using the joint density of the eigenvalues $(\lambda_1(t), \ldots, \lambda_d(t))$ in the Hermitian case (from (4.11)) given by

$$\widetilde{c}_d \prod_{k<h} |\lambda_k - \lambda_h|^2 \, t^{-Hd^2} \exp\left(-\sum_{i=1}^d \frac{\lambda_i^2}{2t^{2H}} \right),$$

where $\widetilde{c}_d$ is a constant that depends only on d.

Acknowledgments

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