

Bivariate Bernstein–gamma functions and moments of exponential functionals of subordinators

A. Barker^a, M. Savov^{b,*}

^a Department of Mathematics and Statistics, University of Reading, Reading, United Kingdom

^b Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, “Akad. Georgi Bonchev” bl. 8, Sofia 1113, Bulgaria

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Abstract

In this paper, we extend recent work on the class of Bernstein–gamma functions to the class of bivariate Bernstein–gamma functions. In the more general bivariate setting, we determine Stirling-type asymptotic bounds which generalise, improve upon, and streamline those found for univariate Bernstein–gamma functions. Then, we demonstrate the importance and power of these results through an application to exponential functionals of Lévy processes. For a subordinator (a non-decreasing Lévy process) $(X_s)_{s \geq 0}$, we study its *exponential functional*, $\int_0^t e^{-X_s} ds$, evaluated at a finite, deterministic time $t > 0$. Our main result here is an explicit infinite convolution formula for the Mellin transform (complex moments) of the exponential functional up to time t which is shown to be equivalent to an infinite series under very minor restrictions. Since exponential functionals of subordinators are part of a universal factorisation concerning all exponential functionals of Lévy processes we believe that this work may turn out to be a step towards a more in-depth study of general exponential functionals of Lévy processes on a finite time horizon.

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* Corresponding author.

E-mail addresses: Adam.Barker@pgr.reading.ac.uk (A. Barker), mladensavov@math.bas.bg (M. Savov).

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1. Introduction, background and motivation

Each Bernstein function, ϕ , has been shown to have a unique associated Bernstein–gamma function, W_ϕ , defined by the recurrent equation

$$W_\phi(z+1) = \phi(z)W_\phi(z), \quad \operatorname{Re}(z) > 0; \quad W_\phi(1) = 1,$$

see [33, Section 6] or [32]. In this work we study a suitable generalisation of the class of Bernstein–gamma functions. Each bivariate Bernstein function, κ (the Lévy–Khinchine exponent of a possibly-killed bivariate subordinator), is shown to have a unique bivariate Bernstein–gamma function, W_κ , defined by the recurrent equation

$$W_\kappa(\zeta, z+1) = \kappa(\zeta, z)W_\kappa(\zeta, z), \quad \operatorname{Re}(\zeta) \geq 0, \operatorname{Re}(z) > 0; \quad W_\kappa(\zeta, 1) = 1.$$

Our main analytical result on bivariate Bernstein–gamma functions, [Theorem 2.9](#), provides a general Stirling asymptotic representation for W_κ . It is an improvement upon [32, Theorem 3.3], which only gives a Stirling representation for the absolute value in the univariate case. Also, building upon [33, Theorem 6.1], [Theorem 2.8](#) gives a Weierstrass product representation for W_κ . The univariate Bernstein–gamma functions are intimately linked to Markovian self-similarity and other important quantities in probability and spectral theory, see [1,21,32–34] and the discussion below. In this work we will demonstrate that bivariate Bernstein–gamma functions also play a role in probability theory via the study of exponential functionals of Lévy processes up to a deterministic time, although we expect further applications to appear. In particular, we apply our results on bivariate Bernstein–gamma functions to exponential functionals of Lévy processes, as follows. For a subordinator $(X_s)_{s \geq 0}$, we study its exponential functional, defined as $I_\phi(t) := \int_0^t e^{-X_s} ds$, $t \in [0, \infty)$. Our main results concern information on the Mellin transform of $I_\phi(t)$, that is $\mathcal{M}_{I_\phi(t)}(z+1) := \mathbb{E}[I_\phi(t)^z]$, for $\operatorname{Re}(z) > 0$. Here we only highlight the following representation:

$$\mathcal{M}_{I_\phi(t)}(z+1) = \frac{\Gamma(z+1)}{W_\phi(z+1)} - \sum_{k=1}^{\infty} \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{\phi(k)} \frac{\Gamma(z+1)}{W_{\phi(k)}(z+1)}, \quad (1.1)$$

which holds under a minor regularity condition and where $\phi_{(k)}(w) := \phi(w+k) - \phi(k)$ is a Bernstein function and $W_{\phi(k)}$ is its corresponding univariate Bernstein–gamma function, see [Definitions 2.1, 2.2](#). We emphasise that $\frac{\Gamma(z+1)}{W_{\phi(k)}(z+1)}$, $k \geq 1$, are the Mellin transforms of exponential functionals of subordinators up to an infinite horizon.

The question of finding precise information on the distribution of the exponential functional of a Lévy process up to a finite, deterministic time was posed in the 1990s, see e.g. [11, Remark 3.2], yet very few works have since been able to cover this case. For the Brownian motion case an extensive study of the law of the exponential functional has been carried out in [22]. Recently, interesting results concerning moments of exponential functionals of processes with independent increments have been discussed in [35], where in particular $\mathbb{E}[I_\phi^n(t)]$, $n \geq 1$, have been computed when $\phi(0) = 0$, see [Theorem 2.17](#). In this work we provide an expression for any complex moments with positive real part. This work may turn out to be a first step for a more in-depth study of exponential functionals of more general Lévy processes up to a deterministic horizon, and the reason why the Mellin transform is a suitable starting point in such an endeavour can be explained as follows. Consider a Lévy process ξ which is killed at independent exponentially distributed random time e_q , $q > 0$. Then, in a sequence of papers [23,30–32], it has been shown gradually that in general

$$\mathbb{E} \left[\left(\int_0^{e_q} e^{-\xi_s} ds \right)^z \right] = \kappa_{q,-}(0) \frac{\Gamma(z+1)}{W_{\kappa_{q,+}}(z+1)} W_{\kappa_{q,-}}(-z), \quad \operatorname{Re}(z) \in (-1, 0],$$

where $\kappa_{q,\pm}$ are bivariate Bernstein functions corresponding to the Wiener–Hopf factors of ξ , see for example [4, Chapter VI] for an introduction to Wiener–Hopf factorisation. On the other hand

$$\frac{1}{q} \mathbb{E} \left[\left(\int_0^{e_q} e^{-\xi_s} ds \right)^z \right] = \int_0^\infty e^{-qt} \mathbb{E} \left[\left(\int_0^t e^{-\xi_s} ds \right)^z \right] dt$$

and one can try to understand $\mathbb{E} \left[\left(\int_0^t e^{-\xi_s} ds \right)^z \right]$ through standard Laplace inversion. The point where subordinators always appear is the expression $\frac{\Gamma(z+1)}{W_{\kappa_{q,+}}(z+1)}$, which corresponds to an exponential functional of a killed subordinator. Therefore, it seems likely that our results, such as (1.1), may have implications well beyond the class of subordinators.

The study of exponential functionals of Lévy processes has received much attention in recent years. Advancements in the general theory can be found in [2,3,6,8,23,26,27,30–32,35,37]. These quantities have been used in various areas of probability theory, such as branching processes and processes in random environments, see [20,25,28], spectral theory of non-self-adjoint semigroups, see [33,34], positive self-similar Markov processes, see [7,19], and financial mathematics, see [15] and [9, Section 6.3]. Exponential functionals up to a random exponential horizon have also appeared in the study of Asian options, see [14,17,29]. For Asian options, which are valued according to an integral of the form $\int_0^t e^{-X_u} du$, where X_u denotes an asset price at time u , one needs to consider exponential functionals up to a deterministic horizon, but the latter have proved to be extremely hard to deal with, and that is why researchers have focused on their Laplace transform. From this perspective, our studies, which deal with obtaining knowledge of the exponential functional up to a finite, deterministic time, are relevant to pricing of Asian options. We refer to [9,12,38] for further details on applications of exponential functionals of Lévy processes.

The remainder of the paper is structured as follows: Section 2.1 introduces notation and key quantities; Section 2.2 provides the statements of the main results on Bernstein–gamma functions and bivariate Bernstein–gamma functions; Section 2.3 contains the statements of the main results on exponential functionals, including the formula (1.1) for the Mellin transform; Sections 3 and 4 contain the proofs of the main results; Section 5 collects proofs of functional properties and results on bivariate Bernstein functions, which can be of independent interest, and are stated in Proposition 3.3; Section 6 contains proofs of the remaining lemmas. Due to the convoluted nature of some of the proofs we have adopted a non-linear style for the exposition of some of the proofs. We enlist preliminary results, prove the main ones and supply the proofs for the preliminary ones at the end. For example we prove Lemmas 4.1, 4.3, 4.4 and 4.5 in Section 6, whereas due to its higher importance Lemma 4.2 is proved in Section 4 but after the respective theorem towards which Lemma 4.2 contributes.

2. Main results

2.1. Preliminary definitions and notation

We start by defining some complex-analytical quantities. We use $\mathbb{C}$ to denote the complex plane. For any $z \in \mathbb{C}$, we write $z = \operatorname{Re}(z) + i\operatorname{Im}(z)$ and we set $z = |z|e^{i\arg z}$ with the branch of the argument function defined via the convention $\arg : \mathbb{C} \mapsto (-\pi, \pi]$. We put $\log_0 : \mathbb{C} \setminus (-\infty, 0] \mapsto \mathbb{C}$ for the main branch of the complex logarithm whereby $\log_0(z) = \ln |z| + i\arg z$. For any $-\infty \leq a < b \leq \infty$, we denote by $\mathbb{C}_{(a,b)} = \{z \in \mathbb{C} : a < \operatorname{Re}(z) < b\}$ and for any $a \in (-\infty, \infty)$ we set $\mathbb{C}_a = \{z \in \mathbb{C} : \operatorname{Re}(z) = a\}$. The notation $\mathbb{C}_{[a,b)} = \{z \in \mathbb{C} : a \leq \operatorname{Re}(z) < b\}$

and all possible variations thereof denote strips whose boundary lines are included or not in the respective subset of the complex plane. We use $A_{(a,b)}$ for the set of holomorphic functions on $\mathbb{C}_{(a,b)}$, whereas if $-\infty < a$ then $A_{[a,b)}$ stands for the holomorphic functions on $\mathbb{C}_{(a,b)}$ that can be extended continuously to $\mathbb{C}_a$. Similarly, we have the spaces $A_{[a,b]}$ and $A_{(a,b]}$. We employ $\mathbb{C}^2$ for the two dimensional complex numbers with $\mathbb{C}_{(a,b)}^2$ standing for $\mathbb{C}_{(a,b)} \times \mathbb{C}_{(a,b)}$ and $A_{(a,b)}^2$ standing for the class of bivariate holomorphic functions on $\mathbb{C}_{(a,b)}^2$.

Now, let us state key definitions for Bernstein–gamma functions, Lévy processes and exponential functionals.

Definition 2.1. A function ϕ is a Bernstein function, that is $\phi \in \mathcal{B}$, if for all $z \in \mathbb{C}_{[0,\infty)}$,

$$\phi(z) = \phi(0) + dz + \int_0^\infty (1 - e^{-zy}) \Pi(dy) \quad (2.1)$$

$$= \phi(0) + dz + z \int_0^\infty e^{-zy} \overline{\Pi}(y) dy, \quad (2.2)$$

where $\phi(0), d \in [0, \infty)$, Π denotes a non-negative measure on $[0, \infty)$ satisfying the restriction $\int_0^\infty \min\{y, 1\} \Pi(dy) < \infty$, and $\overline{\Pi}(x) := \int_x^\infty \Pi(dy)$, for $x \geq 0$. Note that in any case $\phi \in A_{[0,\infty)}$. For further background on Bernstein functions, we refer to the excellent book [36] or to the paper [32, Section 3].

With each $\phi \in \mathcal{B}$ there is an associated, possibly-killed subordinator (non-decreasing Lévy process) $X = (X_t)_{t \geq 0}$, whose Lévy –Khintchine exponent is defined by the relation

$$\mathbb{E}[e^{-zX_t}] = e^{-\phi(z)t} \text{ for } z \in \mathbb{C}_{[0,\infty)} \text{ and } t \geq 0. \quad (2.3)$$

For a subordinator with Lévy –Khintchine exponent ϕ as in (2.1), $d \geq 0$ is the linear drift, and Π is the Lévy measure, which determines the size and intensity of its jumps. If $\phi(0) > 0$, then we say that the subordinator X is killed at rate $\phi(0)$, and it follows that for an independent exponential random variable $e_{\phi(0)}$ with rate parameter $\phi(0)$,

$$X_t = \begin{cases} X_t, & t < e_{\phi(0)} \\ \infty, & t \geq e_{\phi(0)}. \end{cases}$$

If our original, unkilld subordinator X has Laplace exponent ϕ , then the process X killed at rate $q > 0$ has Laplace exponent $\phi_q(\lambda) = q + \phi(\lambda)$.

We recall that if Y is an almost surely positive random variable then $\mathcal{M}_Y(z) := \mathbb{E}[Y^{z-1}]$ is by definition its Mellin transform which is always well-defined at least for $z \in \mathbb{C}_1 = 1 + i\mathbb{R}$. Now we define Bernstein–gamma functions.

Definition 2.2. For each $\phi \in \mathcal{B}$, its associated Bernstein–gamma function W_ϕ is defined, for $z \in \mathbb{C}_{(0,\infty)}$, as the solution in the space of Mellin transforms of positive random variables of the recurrent equation

$$W_\phi(z+1) = \phi(z)W_\phi(z), \text{ for } z \in \mathbb{C}_{(0,\infty)} \text{ with } W_\phi(1) = 1. \quad (2.4)$$

The existence of W_ϕ for any $\phi \in \mathcal{B}$ is proven in [32, Section 4], where an extensive study of its complex-analytical properties has been carried out. Finally, we define the exponential functional of a subordinator:

Definition 2.3. For a subordinator with Laplace exponent $\phi \in \mathcal{B}$, its exponential functional is defined via

$$I_\phi(t) := \int_0^t e^{-X_s} ds = \int_0^{\min\{t, e_{\phi(0)}\}} e^{-X_s} ds, \quad t \in [0, \infty],$$

where the terminal value, for $t = \infty$, can also be denoted by $I_\phi := I_\phi(\infty) = \lim_{t \rightarrow \infty} I_\phi(t)$. Note that if $\phi(0) = 0$ then $e_{\phi(0)} = \infty$ almost surely.

2.2. Bivariate Bernstein–gamma functions and their stirling type approximation

To extend the theory of Bernstein–gamma functions to the bivariate setting, we first define $\mathcal{B}^2$, the class of bivariate Bernstein functions, which generalises the class $\mathcal{B}$ to the bivariate case.

Definition 2.4. We say that a function κ is a bivariate Bernstein function if for all $\zeta, z \in \mathbb{C}_{[0, \infty)}$,

$$\kappa(\zeta, z) = \kappa(0, 0) + d_1 \zeta + d_2 z + \int_0^\infty \int_0^\infty (1 - e^{-\zeta x_1 - z x_2}) \mu(dx_1, dx_2), \quad (2.5)$$

where either $\kappa(0, 0), d_1, d_2 \in [0, \infty)$ and μ is a non-negative measure on $(0, \infty) \times (0, \infty)$ such that

$$\int_0^\infty \int_0^\infty \min\{x_1, 1\} \min\{x_2, 1\} \mu(dx_1, dx_2) < \infty$$

or $\kappa(0, 0), d_1, d_2 \in [0, \infty)$ and μ is a non-negative measure on $\{0\} \times (0, \infty)$ with $\mu(dx_1, dx_2) = \delta_0(dx_1)\mu_2(dx_2)$ and $\int_0^\infty \min\{x, 1\} \mu_2(dx) < \infty$ with δ_0 being the Dirac mass at $\{0\}$.

Note that according to Lemma 3.2 we have that $\kappa \in \mathcal{A}_{[0, \infty)}^2$ and throughout this work we discard the trivial case when $\kappa(0, 0) = d_1 = d_2 = 0$ and μ is the zero measure.

Observe that $\kappa \in \mathcal{B}^2$ if and only if κ is the bivariate Laplace exponent of a possibly-killed bivariate subordinator, see [13, p. 27] for further details. For some important properties of the class $\mathcal{B}^2$, see Proposition 3.3, which collects key results on the class $\mathcal{B}^2$. These are natural extensions of known properties of the univariate class $\mathcal{B}$ but seem not to have appeared in the literature. Now, let us define the class of bivariate Bernstein–gamma functions.

Definition 2.5. We say that W_κ is a bivariate Bernstein–gamma function if

$$W_\kappa(\zeta, z+1) = \kappa(\zeta, z) W_\kappa(\zeta, z), \quad z \in \mathbb{C}_{(0, \infty)}, \quad \zeta \in \mathbb{C}_{[0, \infty)}; \quad (2.6)$$

for each $\zeta \in \mathbb{C}_{[0, \infty)}$, $W_\kappa(\zeta, 1) = 1$; $W_\kappa \in \mathcal{A}_{(0, \infty)}^2$ and for any $q \in [0, \infty)$ the function $W_\kappa(q, \cdot)$ is the Mellin transform of a positive random variable.

Remark 2.6. Taking $\zeta = q \in [0, \infty)$ in the formula in (2.5), we can write $\kappa(q, z)$ as

$$\begin{aligned} \kappa(q, z) &= \kappa(q, 0) + d_2 z + \int_0^\infty (1 - e^{-z x_2}) \left(\int_0^\infty e^{-q x_1} \mu(dx_1, dx_2) \right) \\ &=: \phi_q(0) + d_2 z + \int_0^\infty (1 - e^{-z x}) \mu_q(dx) =: \phi_q(z), \end{aligned} \quad (2.7)$$

where, crucially, $\phi_q \in \mathcal{B}$. Then, using (2.4), note that $W_\kappa(q, z) \equiv W_{\phi_q}(z)$. This gives a starting point from which we can begin to understand bivariate Bernstein–gamma functions through known univariate results, then we can extend results from $W_\kappa(q, z)$, $q \in [0, \infty)$, to $W_\kappa(\zeta, z)$, $\zeta \in \mathbb{C}_{[0, \infty)}$.

Remark 2.7. Note the reduction from the bivariate to the univariate case is simply done by taking $\kappa(q, z) = q + \phi(z)$, $\phi \in \mathcal{B}$. This directly corresponds to the killing of a one-dimensional subordinator and to the second case in [Definition 2.4](#).

It is proven in [[33](#), Section 6] that each univariate Bernstein–gamma function W_ϕ admits an absolutely convergent Weierstrass product representation. In the following [Theorem 2.8](#), we extend this infinite product representation to the class of bivariate Bernstein–gamma functions W_κ .

Theorem 2.8. *If $\kappa \in \mathcal{B}^2$, then W_κ as in [Definition 2.5](#) exists and is unique, and the following product representation, defined for $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$ by*

$$\frac{e^{-\gamma_\kappa(\zeta)z}}{\kappa(\zeta, z)} \prod_{k=1}^{\infty} \frac{\kappa(\zeta, k)}{\kappa(\zeta, k+z)} e^{\frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} z}, \quad (2.8)$$

$$\gamma_\kappa(\zeta) := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} - \log_0(\kappa(\zeta, n)) \right), \quad (2.9)$$

satisfies (2.6). In particular, it follows that for $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$

$$W_\kappa(\zeta, z) = \frac{e^{-\gamma_\kappa(\zeta)z}}{\kappa(\zeta, z)} \prod_{k=1}^{\infty} \frac{\kappa(\zeta, k)}{\kappa(\zeta, k+z)} e^{\frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} z}, \quad (2.10)$$

so that the product in (2.8) is indeed a product representation of W_κ . Moreover, $\gamma_\kappa \in \mathbb{A}_{[0, \infty)}$, and for $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$, we can express $W_\kappa(\zeta, z)$ as the limit

$$W_\kappa(\zeta, z) = \frac{1}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} e^{z \log_0 \kappa(\zeta, n)} \prod_{k=1}^n \frac{\kappa(\zeta, k)}{\kappa(\zeta, k+z)}. \quad (2.11)$$

The proof of this theorem is provided in Section 3. We proceed with the derivation of the Stirling type approximation for W_κ . For this purpose we need some more notation. Firstly, we introduce the function L_κ , which contains the main asymptotic contribution in W_κ , and is defined, for $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$, as

$$L_\kappa(\zeta, z) := \int_{1 \rightarrow 1+z} \log_0(\kappa(\zeta, \chi)) d\chi, \quad (2.12)$$

where the integral denotes the path integral along a contour starting from 1 and ending at $1+z$ which lies in the domain of analyticity of $\log_0(\kappa(\zeta, \cdot))$. If $\operatorname{Re}(z) > -1$ then there is a straight line connecting 1 to $1+z$ in the domain of analyticity of $\log_0(\kappa(\zeta, \cdot))$ and we have

$$L_\kappa(\zeta, z) = \int_{1 \rightarrow 1+z} \log_0(\kappa(\zeta, \chi)) d\chi = z \int_0^1 \log_0(\kappa(\zeta, 1+zv)) dv. \quad (2.13)$$

Now, for $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$, we have that $\operatorname{Re}(\kappa(\zeta, z)) > 0$ by (3.10), so L_κ is well-defined on $\mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$. We denote the floor function $\lfloor u \rfloor := \max \{n \in \mathbb{N} : n \leq u\}$ for $u \geq 0$, and define

$$P(u) := (u - \lfloor u \rfloor)(1 - (u - \lfloor u \rfloor)), \quad u \geq 0.$$

The function E_κ corresponds to the error term in our Stirling approximation, and is defined, for $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$, as

$$E_\kappa(\zeta, z) = \frac{1}{2} \int_1^\infty P(u) \left(\log_0 \left(\frac{\kappa(\zeta, u+z)}{\kappa(\zeta, u)} \right) \right)'' du, \quad (2.14)$$

where the two derivatives in the integral are taken with respect to u . Now we are ready to state [Theorem 2.9](#) which furnishes the Stirling asymptotic representation for W_κ . For the absolute value of the univariate case a similar, but less wieldy, asymptotic representation has been derived in [[32](#), Theorem 4.2].

Theorem 2.9. *Let $\kappa \in \mathcal{B}^2$. Then, for $(\zeta, z) \in \mathbb{C}_{[0,\infty)} \times \mathbb{C}_{(0,\infty)}$, we have that*

$$W_\kappa(\zeta, z) = \frac{\kappa^{\frac{1}{2}}(\zeta, 1)}{\kappa(\zeta, z) \kappa^{\frac{1}{2}}(\zeta, 1+z)} e^{L_\kappa(\zeta, z)} e^{-E_\kappa(\zeta, z)}, \quad (2.15)$$

$$\sup_{\kappa \in \mathcal{B}^2} \sup_{(\zeta, z) \in \mathbb{C}_{[0,\infty)} \times \mathbb{C}_{(0,\infty)}} |E_\kappa(\zeta, z)| \leq 2, \quad (2.16)$$

$$\lim_{\operatorname{Re}(z) \rightarrow \infty} E_\kappa(\zeta, z) = \frac{1}{2} \int_1^\infty P(u) \left(\left(\frac{\kappa'_z(\zeta, u)}{\kappa(\zeta, u)} \right)^2 - \frac{\kappa''_z(\zeta, u)}{\kappa(\zeta, u)} \right) du =: T_\kappa(\zeta), \quad (2.17)$$

and for each fixed $z \in \mathbb{C}_{(0,\infty)}$, we have that $\lim_{\operatorname{Re}(\zeta) \rightarrow \infty} E_\kappa(\operatorname{Re}(\zeta), z) = 0$.

The proof of this theorem is provided in Section 3. Now we state two key lemmas. [Lemma 2.10](#), is a complex generalisation of the result [[32](#), Prop. 3.1 (8)], that for all $\phi \in \mathcal{B}$, $a \in \mathbb{R}$, and $u \in (0, \infty)$, uniformly among a in compact intervals in $\mathbb{R}$,

$$\lim_{u \rightarrow \infty} \frac{\phi(u+a)}{\phi(u)} = 1. \quad (2.18)$$

Lemma 2.10. *For each $\phi \in \mathcal{B}$, $z \in \mathbb{C}$, uniformly among z in compact subsets of the complex plane $\mathbb{C}$,*

$$\lim_{u \rightarrow \infty} \frac{\phi(u+z)}{\phi(u)} = 1. \quad (2.19)$$

The next lemma is a generalisation, from the standard gamma function to the class of Bernstein–gamma functions, of the following result. For each $z \in \mathbb{C}$,

$$\lim_{u \rightarrow \infty} \frac{\Gamma(u+1)(u+1)^z}{\Gamma(u+1+z)} = 1.$$

Lemma 2.11. *For each $\phi \in \mathcal{B}$, $z \in \mathbb{C}_{(0,\infty)}$, uniformly among z in compact subsets of the complex half-plane $\mathbb{C}_{[0,\infty)}$,*

$$\lim_{u \rightarrow \infty} \frac{W_\phi(u+1)\phi^z(u+1)}{W_\phi(u+1+z)} = 1. \quad (2.20)$$

The proof of [Lemma 2.11](#) builds upon [Lemma 2.10](#). Both proofs are contained in Section 6.

2.3. Applications to exponential functionals up to a finite time

The first of our key results on exponential functionals up to a finite time is an infinite convolution formula for the Mellin transform.

Theorem 2.12. For each possibly killed subordinator with Laplace exponent $\phi \in \mathcal{B}$, for $t \in (0, \infty)$, and for $z \in \mathbb{C}_{(0, \infty)}$, the Mellin transform, $\mathcal{M}_{I_\phi(t)}(z+1) = \mathbb{E}[I_\phi^z(t)]$, satisfies

$$\begin{aligned} \mathbb{E}[I_\phi^z(t)] = 1 * \frac{zt^{z-1}}{e^{\phi(1)t}} * \bigstar_{k=1}^{\infty} \left[\delta_0(dt) + \sum_{m=1}^{\infty} \frac{(z)^{(m)}(-[\phi(k+1) - \phi(k)])^m}{(m!)^2} \frac{mt^{m-1}}{e^{\phi(k)t}} \right. \\ \left. + [\phi(z+k) - \phi(k)] \sum_{m=0}^{\infty} \frac{(z)^{(m)}(-[\phi(k+1) - \phi(k)])^m}{(m!)^2} \frac{t^m}{e^{\phi(k)t}} \right], \end{aligned} \quad (2.21)$$

where the symbol $*$ denotes convolution $(f * g)(t) := \int_0^t f(s)g(t-s)ds$, the symbol $\bigstar$ means infinite convolution, $\delta_0(dt)$ is the Dirac measure at the point 0, and $(z)^{(m)} := z(z+1) \cdots (z+m-1)$ is the rising factorial function.

While this result is interesting on its own, it makes the computations of the moments hard, and for this purpose we shall express our formula as an infinite sum rather than an infinite convolution. This requires a slight regularity condition on the underlying Bernstein function:

Definition 2.13 (Regularity Condition). We impose that the derivative of our subordinator's Laplace exponent satisfies $\beta(\phi') > -1$, where $\beta(\phi')$ denotes the lower Matuszewska index of ϕ' , defined as the infimum of $\beta \in \mathbb{R}$ for which there exists $C > 0$ such that for each $\Lambda > 1$, $\phi'(\lambda x)/\phi'(x) \geq (1 + o(1))C\lambda^\beta$, uniformly in $\lambda \in [1, \Lambda]$, as $x \rightarrow \infty$. See [10, p68] for more details but we highlight that cases like $\phi \in \mathcal{B}$ with $d > 0$ and/or $\bar{\Pi}(x) \sim x^{-\alpha}$, $x \rightarrow 0$, $\alpha \in (0, 1)$ satisfy this condition but are a very small sample of cases that fall under this definition.

Theorem 2.14. For each subordinator whose Laplace exponent ϕ satisfies the condition in Definition 2.13, for all $t \in (0, \infty]$, and for all $z \in \mathbb{C}_{(0, \infty)}$,

$$\mathcal{M}_{I_\phi(t)}(z+1) = \mathbb{E}[I_\phi^z(t)] = \frac{\Gamma(z+1)}{W_\phi(z+1)} - \sum_{k=1}^{\infty} \frac{\prod_{j=1}^k [\phi(z+j) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{\phi(k)} \frac{\Gamma(z+1)}{W_{\phi(k)}(z+1)}, \quad (2.22)$$

where $\phi_{(k)}(w) := \phi(w+k) - \phi(k)$ is a Bernstein function, and $W_{\phi(k)}$ is its corresponding Bernstein–gamma function.

Remark 2.15. Our constraint excludes cases in which ϕ is slowly varying (see [10, p6]), but it should be noted that even for e.g. $\phi(x) = \ln(1+x)$, Theorem 2.14 still holds in a region of the form $\operatorname{Re}(z) > c_{t,\phi}$, with the region depending on t and ϕ . We are unable to obtain any partial results in only the most pathological cases, e.g. $\phi(x) = \ln(\ln(e+x))$, for which the rate of growth of $\phi(x)$ to ∞ , as $x \rightarrow \infty$, is too slow. We emphasise that the Matuszewska indices of the derivatives of the Bernstein functions in this remark are of value precisely -1 .

Remark 2.16. Note that since $\int_0^{eq} e^{-X_s} ds$ has negative moments of order between $(-1, 0)$, see [Theorem 2.4][32], then clearly $I_\phi(t)$ has the same moments for any $t > 0$. Their evaluation is excluded in the statement of this theorem as there is a technical difficulty to obtain a similar expression. However, from (4.2), one can attack the problem from the fact that

$$\frac{\Gamma(z+1)}{W_{\phi_q}(z+1)} = \frac{z}{\phi_q(z)} \frac{\Gamma(z)}{W_{\phi_q}(z)},$$

where $\varphi_q(z) = \phi(z) + q$ and therefore $\mathbb{E}[I_\phi^z(t)]$ is the convolution in t of $\mathbb{E}[I_\phi^{z-1}(t)]$ with $ze^{-\phi(z)t}$ or

$$\mathbb{E}[I_\phi^z(t)] = ze^{-\phi(z)t} \int_0^t \mathbb{E}[I_\phi^{z-1}(s)] e^{\phi(z)s} ds.$$

This equation can be analysed by pre-multiplication both sides with $e^{\phi(z)t}$, subsequent differentiation in t and rearrangement.

One can quite easily obtain, using elementary methods, an explicit formula for the positive integer moments of the exponential functional of a subordinator, as it appears in the recent work [35, Corollary 1].

Theorem 2.17 (Salminen, Vostrikova 2018). *For all $\phi \in \mathcal{B}$ with $\phi(0) = 0$, and for all $n \in \mathbb{N}$,*

$$\mathbb{E}[I_\phi^n(t)] = n! \sum_{k=0}^{n-1} \frac{e^{-\phi(k)t} - e^{-\phi(n)t}}{\prod_{0 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]}. \quad (2.23)$$

Using our methodology, based on Laplace inversion, we can deduce the following formula for integer moments, which extends [35, Corollary 1] to integer moments of killed subordinators.

Corollary 2.18. *For each subordinator with Laplace exponent $\phi \in \mathcal{B}$, for $t \in (0, \infty)$ and $n \in \mathbb{N}$,*

$$\begin{aligned} \mathbb{E}[I_\phi^n(t)] &= n! \sum_{k=0}^{n-1} \frac{e^{-\phi^*(k)t} - e^{-\phi^*(n)t}}{\prod_{0 \leq j \leq n; j \neq k} [\phi^*(j) - \phi^*(k)]} \\ &= n! \sum_{k=0}^n \frac{e^{-\phi^*(k)t}}{\prod_{0 \leq j \leq n; j \neq k} [\phi^*(j) - \phi^*(k)]}, \end{aligned} \quad (2.24)$$

where $\phi^*(j) = \phi(j)$, $j \geq 1$, and $\phi^*(0) = 0$, so that $\phi^* = \phi$ when $\phi(0) = 0$.

Remark 2.19. Note that as $t \rightarrow \infty$ the second relation of (2.24) yields the well known formula

$$\mathbb{E}[I_\phi^n] = \frac{n!}{\prod_{j=1}^n \phi(j)}$$

but it also offers a full asymptotic expansion of the speed of convergence in t , the first term of which is exponential of value $e^{-\phi(1)t}$.

We proceed with the proofs of our results.

3. Proofs for bivariate Bernstein–gamma functions

Before providing the proofs for Section 2.2, we state some key results on the class of Bernstein functions $\mathcal{B}$, which can be found in [33, Section 4].

Proposition 3.1.

1. For all $\phi \in \mathcal{B}$ and $z \in \mathbb{C}_{(0,\infty)}$, we can express the derivative of ϕ as

$$\begin{aligned}\phi'(z) &= \mathbf{d} + \int_0^\infty ye^{-zy} \Pi(dy) \\ &= \mathbf{d} + \int_0^\infty e^{-zy} \bar{\Pi}(y) dy - z \int_0^\infty e^{-zy} y \bar{\Pi}(y) dy.\end{aligned}\quad (3.1)$$

2. Each $\phi \in \mathcal{B}$ is non-decreasing on $[0, \infty)$, and ϕ' is completely monotone, positive, and non-increasing on $[0, \infty)$. Hence, ϕ is strictly log-concave on $[0, \infty)$, so for all $u \in [0, \infty)$,

$$0 \leq u\phi'(u) = \phi(u) - \phi(0) - u^2 \int_0^\infty e^{-uy} y \bar{\Pi}(y) dy \leq \phi(u). \quad (3.2)$$

Following (2.7), for each $\zeta \in \mathbb{C}_{[0,\infty)}$ we can write

$$\begin{aligned}\kappa(\zeta, z) &= \kappa(\zeta, 0) + \mathbf{d}_2 z + \int_0^\infty (1 - e^{-zx_2}) \left(\int_0^\infty e^{-\zeta x_1} \mu(dx_1, dx_2) \right) \\ &= \phi_\zeta(0) + \mathbf{d}_2 z + \int_0^\infty (1 - e^{-zx}) \mu_\zeta(dx) \\ &= \phi_\zeta(0) + \mathbf{d}_2 z + z \int_0^\infty e^{-zx} \bar{\mu}_\zeta(x) dx := \phi_\zeta(z), \quad z \in \mathbb{C}_{[0,\infty)},\end{aligned}\quad (3.3)$$

where $\bar{\mu}_\zeta(y) = \int_y^\infty \mu_\zeta(dx)$, $y > 0$. We then have the following elementary claim which we provide without proof.

Lemma 3.2. Let $\kappa \in \mathcal{B}^2$. Then $\kappa \in \mathcal{A}_{[0,\infty)}^2$. Moreover, for any $\zeta \in \mathbb{C}_{[0,\infty)}$ we have that $\kappa(\zeta, z) = \phi_\zeta(z)$ and $\phi_\zeta \in \mathcal{A}_{[0,\infty)}$, see (3.3). The measure μ_ζ is a complex measure on $(0, \infty)$. If $\zeta \in \mathbb{R}$ then $\phi_\zeta \in \mathcal{B}$ if and only if $\int_0^\infty \min\{1, x\} \mu_\zeta(dx) < \infty$, which is always the case if $\zeta \geq 0$.

The next proposition collects results which may be regarded as simple extensions of results available for $\phi \in \mathcal{B}$ and which can be found in [33, Section 4] or [32, Section 3]. It is proved later in Section 5.

Proposition 3.3. Let $\kappa \in \mathcal{B}^2$. Then each of the following items holds:

1. For all $(\zeta, z) \in \mathbb{C}_{(0,\infty)}^2$,

$$\begin{aligned}\kappa'_z(\zeta, z) &= \phi'_\zeta(z) = \mathbf{d}_2 + \int_0^\infty xe^{-zx} \mu_\zeta(dx) \\ &= \mathbf{d}_2 + \int_0^\infty e^{-zx} \bar{\mu}_\zeta(x) dx - z \int_0^\infty e^{-zx} x \bar{\mu}_\zeta(x) dx \\ &= \frac{\phi_\zeta(z) - \phi_\zeta(0)}{z} - z \int_0^\infty e^{-zx} x \bar{\mu}_\zeta(x) dx.\end{aligned}\quad (3.4)$$

2. For each $\zeta \in [0, \infty)$, ϕ_ζ is non-decreasing on $[0, \infty)$, and ϕ'_ζ is completely monotone, positive, and non-increasing on $[0, \infty)$. In particular, ϕ_ζ is strictly log-concave on $[0, \infty)$.

3. For all $(\zeta, z) \in \mathbb{C}_{[0, \infty)}^2 \setminus \{(0, 0)\}$, we have that

$$\left| \frac{\kappa(\operatorname{Re}(\zeta), \operatorname{Re}(z))}{\kappa(\zeta, z)} \right| = \left| \frac{\phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))}{\phi_\zeta(z)} \right| \leq 1, \quad (3.5)$$

and moreover, the inequality (3.5) is valid for $\zeta = z = 0$ whenever $\kappa(0, 0) > 0$.

4. For all $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$, we have

$$\begin{aligned} \left| \frac{\kappa'_z(\zeta, z)}{\kappa(\zeta, z)} \right| &= \left| \frac{\phi'_\zeta(z)}{\phi_\zeta(z)} \right| \leq \frac{1}{|z|} \left(1 + \frac{|\phi_\zeta(0)|}{|\phi_\zeta(z)|} \right) + \frac{|z|}{\operatorname{Re}^2(z)} \frac{\phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))}{|\phi_\zeta(z)|} \\ &\leq \frac{1}{|z|} \left(1 + \frac{|\phi_\zeta(0)|}{\phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))} \right) + \frac{|z|}{\operatorname{Re}^2(z)} \\ &\leq \frac{1}{|z|} \left(1 + \frac{|\phi_\zeta(0)|}{\phi_{\operatorname{Re}(\zeta)}(0)} \right) + \frac{|z|}{\operatorname{Re}^2(z)} \end{aligned} \quad (3.6)$$

and

$$\frac{|\kappa''_z(\zeta, z)|}{|\kappa(\zeta, z)|} = \frac{|\phi''_\zeta(z)|}{|\phi_\zeta(z)|} \leq \left(\frac{2}{\operatorname{Re}^2(z)} + \frac{2|z|}{\operatorname{Re}^3(z)} \right) \frac{\phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))}{|\phi_\zeta(z)|} \leq \frac{2}{\operatorname{Re}^2(z)} + \frac{2|z|}{\operatorname{Re}^3(z)}. \quad (3.7)$$

In particular, taking $z = \operatorname{Re}(z) = u \in (0, \infty)$ and $\zeta \in \mathbb{C}_{[0, \infty)}$, we have the inequalities

$$\left| \frac{\kappa'_z(\zeta, u)}{\kappa(\zeta, u)} \right| = \left| \frac{\phi'_\zeta(u)}{\phi_\zeta(u)} \right| \leq \frac{1}{u} \left(\frac{|\phi_\zeta(u) - \phi_\zeta(0)|}{|\phi_\zeta(u)|} + \frac{\phi_{\operatorname{Re}(\zeta)}(u)}{|\phi_\zeta(u)|} \right) \leq \frac{2}{u}, \quad (3.8)$$

$$\frac{|\kappa''_z(\zeta, u)|}{|\kappa(\zeta, u)|} = \frac{|\phi''_\zeta(u)|}{|\phi_\zeta(u)|} \leq \frac{4}{u^2} \frac{\phi_{\operatorname{Re}(\zeta)}(u)}{|\phi_\zeta(u)|} \leq \frac{4}{u^2}. \quad (3.9)$$

5. For all $(\zeta, z) \in \mathbb{C}_{[0, \infty)}^2 \setminus \{(0, 0)\}$, we have that

$$\operatorname{Re}(\kappa(\zeta, z)) > 0, \quad (3.10)$$

and if in addition $\operatorname{Re}(\zeta) > 0$ denoting B_ζ as a ball centred at ζ with $\overline{B_\zeta} \subset \mathbb{C}_{(0, \infty)}$, we have that $\inf_{\chi \in \overline{B_\zeta}} \operatorname{Re}(\kappa(\chi, z)) > 0$.

6. For all $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$ and for all $n \geq 1$

$$|\kappa_z^{(n)}(\zeta, z)| \leq |\kappa_z^{(n)}(\operatorname{Re}(\zeta), \operatorname{Re}(z))|. \quad (3.11)$$

7. For all $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$, we have that

$$\left| \frac{\kappa'_z(\zeta, z)}{\kappa(\zeta, z)} \right| \leq \frac{2}{\operatorname{Re}(z)}, \quad (3.12)$$

$$\frac{|\kappa''_z(\zeta, z)|}{|\kappa(\zeta, z)|} = \frac{|\phi''_\zeta(u)|}{\phi_\zeta(u)} \leq \frac{4}{\operatorname{Re}^2(z)}. \quad (3.13)$$

8. For $\zeta \in \mathbb{C}_{[0, \infty)}$ and any $A > 0$ we have that

$$\lim_{x \rightarrow \infty} \sup_{0 \leq v \leq A} \left| \frac{\kappa(\zeta, x+v)}{\kappa(\zeta, x)} \right| = 1. \quad (3.14)$$

We are ready to start with the proof of the main results.

Proof of Theorem 2.8. For $\zeta = q \geq 0$ from Lemma 3.2 it holds true that $\kappa(q, \cdot) = \phi_q(\cdot) \in \mathcal{B}$ and therefore we have that $W_\kappa(q, z) = W_{\phi_q}(z)$ with $\gamma_\kappa(q)$ computed as in (2.9), see Remark 2.6. Henceforth,

$$W_\kappa(q, z) = \frac{e^{-\gamma_\kappa(q)z}}{\kappa(q, z)} \prod_{k=1}^{\infty} \frac{\kappa(q, k)}{\kappa(q, k+z)} e^{\frac{\kappa'_z(q, k)}{\kappa(q, k)} z} =: \overline{W}_\kappa(q, z), \quad z \in \mathbb{C}_{(0, \infty)}. \quad (3.15)$$

We extend analytically $\overline{W}_\kappa(\cdot, z)$ to $\mathbb{C}_{[0, \infty)}$. First, we extend $\gamma_\kappa(\cdot)$. Fix $\zeta \in \mathbb{C}_{(0, \infty)}$ and an open ball B_ζ centred at ζ with the closed ball $\overline{B}_\zeta$ satisfying $\overline{B}_\zeta \subset \mathbb{C}_{(0, \infty)}$. From item 5 of Proposition 3.3, that is $\inf_{\chi \in \overline{B}_\zeta} \operatorname{Re}(\kappa(\chi, k)) > 0$ for all $k \geq 1$, and from $\kappa \in \mathcal{A}_{(0, \infty)}^2$, see Lemma 3.2, we deduce that for any $n \in \mathbb{N}^+ := \{1, 2, \dots\}$

$$f_n(\zeta) := \sum_{k=1}^n \frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} - \log_0(\kappa(\zeta, n)) \in \mathcal{A}_{(0, \infty)}. \quad (3.16)$$

Moreover, we observe that

$$\begin{aligned} |f_n(\zeta)| &= \left| \sum_{k=1}^n \frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} - \int_1^n \frac{\kappa'_z(\zeta, x)}{\kappa(\zeta, x)} dx - \log_0(\kappa(\zeta, 1)) \right| \\ &= \left| \sum_{k=1}^n \int_0^1 \left(\frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} - \frac{\kappa'_z(\zeta, x+k)}{\kappa(\zeta, x+k)} \right) dx - \log_0(\kappa(\zeta, 1)) \right| \\ &\leq \sum_{k=1}^n \sup_{y \in [0, 1]} \left(\left| \frac{\kappa''_z(\zeta, y+k)}{\kappa(\zeta, y+k)} \right| + \left| \frac{\kappa'_z(\zeta, y+k)}{\kappa(\zeta, y+k)} \right|^2 \right) + |\log_0(\kappa(\zeta, 1))| \\ &\leq \sum_{k=1}^n \frac{8}{k^2} + |\log_0(\kappa(\zeta, 1))| \\ &< \frac{8\pi^2}{6} + |\log_0(\kappa(\zeta, 1))|, \end{aligned} \quad (3.17)$$

where in the first inequality we have used first order Taylor's expansion of $\frac{\kappa'_z(\zeta, x+k)}{\kappa(\zeta, x+k)}$ about $x = 0$ and immediate bounds, in the second we have applied (3.8) and (3.9), and the third follows from the well-known $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$. Henceforth,

$$\sup_{n \geq 1} \sup_{\chi \in \overline{B}_\zeta} |f_n(\chi)| < \frac{4\pi^2}{3} + \sup_{\chi \in \overline{B}_\zeta} |\log_0(\kappa(\chi, 1))| < \infty.$$

Therefore, f_n are uniformly bounded holomorphic functions on $\overline{B}_\zeta$ and from the dominated convergence theorem which is applicable thanks to (3.17) we deduce that

$$\lim_{n \rightarrow \infty} f_n(\chi) = \sum_{k=1}^{\infty} \int_0^1 \left(\frac{\kappa'_z(\chi, k)}{\kappa(\chi, k)} - \frac{\kappa'_z(\chi, x+k)}{\kappa(\chi, x+k)} \right) dx - \log_0(\kappa(\chi, 1)) =: f(\chi).$$

From classical result of complex analysis we conclude that $f \in \mathcal{A}_{B_\zeta}$ and since $\zeta \in \mathbb{C}_{(0, \infty)}$ we get that $f \in \mathcal{A}_{(0, \infty)}$. Since by the very definition of γ_κ , see (2.9), $\gamma_\kappa = \lim_{n \rightarrow \infty} f_n = f$ we conclude that $f = \gamma_\kappa \in \mathcal{A}_{(0, \infty)}$. The claim that $\gamma_\kappa \in \mathcal{A}_{[0, \infty)}$ is then affirmed by the fact that $\kappa(\zeta, k), \kappa'_z(\zeta, x+k), k \geq 1, x \in [0, 1]$, extend continuously to $\zeta \in i\mathbb{R}$ and that the uniform bound (3.17) is valid for $\zeta \in i\mathbb{R}$ since (3.8) and (3.9) apply respectively for each $z = k \geq 1$. In view of $\gamma_\kappa \in \mathcal{A}_{[0, \infty)}$ to analytically extend in the variable ζ the right-hand side of the infinite

product in (3.15), it suffices from Hartog's theorem, see [18, Section 2.4], to fix $z \in \mathbb{C}_{(0,\infty)}$ and show using Montel's theorem that the infinite product, which we record in (3.18), converges absolutely

$$Z_\kappa(\zeta, z) := \prod_{k=1}^{\infty} \frac{\kappa(\zeta, k)}{\kappa(\zeta, k+z)} e^{\frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} z} = \prod_{k=1}^{\infty} A_k(\zeta). \quad (3.18)$$

For any $k \geq 1$, we get using the Taylor's expansion

$$\log_0 \left(\frac{\kappa(\zeta, k+z)}{\kappa(\zeta, k)} \right) = \log_0 \left(1 + z \frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} + \frac{z^2}{2} \frac{\kappa''_z(\zeta, k+a_k z)}{\kappa(\zeta, k)} \right), \quad (3.19)$$

where $a_k \in [0, 1]$. Thus,

$$A_k(\zeta) = e^{z \frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} - \log_0 \left(1 + z \frac{\kappa'_z(\zeta, k)}{\kappa(\zeta, k)} + \frac{z^2}{2} \frac{\kappa''_z(\zeta, k+a_k z)}{\kappa(\zeta, k)} \right)}. \quad (3.20)$$

From (3.8) we have that on any ball $\overline{B_\zeta} \subset \mathbb{C}_{(0,\infty)}$ centred at ζ and any $k \geq 1$

$$\sup_{\chi \in \overline{B_\zeta}} \left| z \frac{\kappa'_z(\chi, k)}{\kappa(\chi, k)} \right| \leq 2 \frac{|z|}{k}. \quad (3.21)$$

From (3.11), (3.5), (3.9) and the fact that $|\kappa''_z(\operatorname{Re}(\zeta), \cdot)| = |\phi''_{\operatorname{Re}(\zeta)}(\cdot)| = -\phi'_{\operatorname{Re}(\zeta)}(\cdot)$ decreases on $[0, \infty)$ we also observe that for any $k \geq 1$

$$\begin{aligned} \sup_{\chi \in \overline{B_\zeta}} \sup_{v \in [0, 1]} \left| z \frac{\kappa''_z(\chi, k+vz)}{\kappa(\chi, k)} \right| &\leq \sup_{\chi \in \overline{B_\zeta}} \sup_{v \in [0, 1]} \left| z \frac{\kappa''_z(\operatorname{Re}(\chi), k+v\operatorname{Re}(z))}{\kappa(\chi, k)} \right| \\ &\leq \sup_{\chi \in \overline{B_\zeta}} \left| z \frac{\kappa''_z(\operatorname{Re}(\chi), k)}{\kappa(\chi, k)} \right| = \sup_{\chi \in \overline{B_\zeta}} \left| z \frac{\kappa''_z(\operatorname{Re}(\chi), k)}{\kappa(\chi, k)} \frac{\kappa(\operatorname{Re}(\chi), k)}{\kappa(\operatorname{Re}(\chi), k)} \right| \\ &\leq \sup_{\chi \in \overline{B_\zeta}} \left| z \frac{\kappa''_z(\operatorname{Re}(\chi), k)}{\kappa(\operatorname{Re}(\chi), k)} \right| \\ &\leq 4 \frac{|z|}{k^2}. \end{aligned} \quad (3.22)$$

Therefore, using $\log_0(1+w) = w + O(w^2)$, as $|w| \sim 0$, we get for all k large enough

$$\begin{aligned} \sup_{\chi \in \overline{B_\zeta}} |A_k(\chi)| &\leq \sup_{\chi \in \overline{B_\zeta}} \left| e^{\frac{1}{2} \left(z \frac{\kappa'_z(\chi, k)}{\kappa(\chi, k)} + \frac{z^2}{2} \frac{\kappa''_z(\zeta, k+a_k z)}{\kappa(\zeta, k)} \right)^2 - \frac{z^2}{2} \frac{\kappa''_z(\zeta, k+a_k z)}{\kappa(\zeta, k)}} \right| \leq e^{\frac{16}{k^2} (|z|^2 + |z|^4 + O(\frac{1}{k}))} \\ \inf_{\chi \in \overline{B_\zeta}} |A_k(\chi)| &\geq \inf_{\chi \in \overline{B_\zeta}} \left| e^{\frac{1}{2} \left(z \frac{\kappa'_z(\chi, k)}{\kappa(\chi, k)} + \frac{z^2}{2} \frac{\kappa''_z(\zeta, k+a_k z)}{\kappa(\zeta, k)} \right)^2 - \frac{z^2}{2} \frac{\kappa''_z(\zeta, k+a_k z)}{\kappa(\zeta, k)}} \right| \geq e^{-\frac{16}{k^2} (|z|^2 + |z|^4 + O(\frac{1}{k}))} \end{aligned}$$

and hence from (3.18)

$$\sup_{\chi \in \overline{B_\zeta}} |Z_\kappa(\chi, z)| = \sup_{\chi \in \overline{B_\zeta}} \prod_{k=1}^{\infty} |A_k(\chi)| < \infty.$$

This shows that $\overline{W_\kappa}$ is analytic in ζ for any $z \in \mathbb{C}_{(0,\infty)}$ and it equates the infinite product in (2.8) on $\mathbb{C}_{[0,\infty)} \times \mathbb{C}_{(0,\infty)}$ (or equivalently (3.15) with $q \rightarrow \zeta$). To deduce that $W_\kappa = \overline{W_\kappa}$ and therefore W_κ is represented as in (2.8) we proceed to demonstrate first that (2.11) holds. For

this it suffices to use that the infinite product in (2.8) is absolutely convergent and the form of the limit that defines γ_κ , see (2.9). Indeed, we simply write

$$\begin{aligned}\overline{W}_\kappa(\zeta, z) &= \frac{e^{-\gamma_\kappa(\zeta)z}}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{\kappa(\zeta, k)}{\kappa(\zeta, k+z)} e^{\frac{\kappa'_\zeta(\zeta, k)}{\kappa(\zeta, k)} z} \\ &= \frac{1}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} e^{z \log_0 \kappa(\zeta, n)} e^{z \sum_{k=1}^n \frac{\kappa'_\zeta(\zeta, k)}{\kappa(\zeta, k)} - z \gamma_\kappa(\zeta) - z \log_0 \kappa(\zeta, n)} \prod_{k=1}^n \frac{\kappa(\zeta, k)}{\kappa(\zeta, k+z)} \\ &= \frac{1}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} e^{z \log_0 \kappa(\zeta, n)} \prod_{k=1}^n \frac{\kappa(\zeta, k)}{\kappa(\zeta, k+z)}.\end{aligned}$$

Then

$$\begin{aligned}\overline{W}_\kappa(\zeta, z+1) &= \lim_{n \rightarrow \infty} e^{z \log_0 \kappa(\zeta, n)} \frac{\kappa(\zeta, n)}{\kappa(\zeta, n+1)} \prod_{k=1}^{n+1} \frac{\kappa(\zeta, k)}{\kappa(\zeta, k+z)} \\ &= \kappa(\zeta, z) \overline{W}_\kappa(\zeta, z)\end{aligned}$$

provided $\lim_{n \rightarrow \infty} \frac{\kappa(\zeta, n+1)}{\kappa(\zeta, n)} = 1$. This is an elementary consequence of

$$\kappa(\zeta, n+1) = \kappa(\zeta, n) + \int_n^{n+1} \kappa'_\zeta(\zeta, x) dx,$$

inequality (3.12) and item 8 of Proposition 3.3. Clearly, $\overline{W}_\kappa(\zeta, 1) = 1$. Next, for fixed $\zeta \in \mathbb{C}_{[0, \infty)}$, using (2.11) we can show that $\overline{W}_\kappa$ is analytic in z for any such ζ . This is attained by writing

$$\overline{W}_\kappa(\zeta, z) = \frac{1}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} e^{z \log_0 \kappa(\zeta, n) - \sum_{k=1}^n \log_0 \left(\frac{\kappa(\zeta, k+z)}{\kappa(\zeta, k)} \right)},$$

invoking (3.19) and (3.20) to represent the logarithm and subsequently estimate the remainder using (3.21) and (3.22) (with $\chi = \zeta$ fixed) and employ the computations leading up to (3.17) to yield the absolute convergence. Now, from Hartog's theorem, see [18, Section 2.4], $\overline{W}_\kappa \in \mathcal{A}_{(0, \infty)}^2$ and it satisfies all conditions of Definition 2.5 and therefore as it coincides with W_κ for $\zeta = q \in [0, \infty)$ we deduce that they coincide on $\mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$. The uniqueness of W_κ follows from the fact that the positive random variable associated with the Mellin transform $W_\kappa(q, z)$ is unique, for each $q \geq 0$, see e.g. [32, Section 4] and the discussion in [33, Section 6]. This concludes the proof of the theorem. $\square$

Next, we prove the Stirling asymptotic representation for a bivariate Bernstein–gamma function.

Proof of Theorem 2.9. Let us denote $f(x) := \log_0(\kappa(\zeta, x+z)/\kappa(\zeta, x))$ and $S_n := \sum_{k=1}^n f(k)$. Then applying (2.11), for each $(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}$, we can express $W_\kappa(\zeta, z)$ as

$$W_\kappa(\zeta, z) = \frac{1}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} e^{z \log_0 \kappa(\zeta, n)} \prod_{k=1}^n \frac{\kappa(\zeta, k)}{\kappa(\zeta, k+z)} \quad (3.23)$$

$$= \frac{1}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} e^{z \log_0 \kappa(\zeta, n) - \sum_{k=1}^n \log_0 \left(\frac{\kappa(\zeta, k+z)}{\kappa(\zeta, k)} \right)} \quad (3.24)$$

$$= \frac{1}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} e^{z \log_0 \kappa(\zeta, n) - S_n}. \quad (3.25)$$

Then applying [24, Section 8.2, (2.01),(2.03)] with $m = 1$ in their notation, we can write S_n as

$$S_n = \sum_{k=1}^n f(k) = \frac{1}{2} (f(1) + f(n)) + \int_1^n f(x) dx + \frac{1}{2} \int_1^n P(u) f''(u) du, \quad (3.26)$$

where recalling that $P(u) = (u - \lfloor u \rfloor) (1 - (u - \lfloor u \rfloor))$, we set

$$E_\kappa(\zeta, z, n) := \frac{1}{2} \int_1^n P(u) f''(u) du = \frac{1}{2} \int_1^n P(u) \left(\log_0 \frac{\kappa(\zeta, u+z)}{\kappa(\zeta, u)} \right)'' du. \quad (3.27)$$

Let us first show that

$$\lim_{n \rightarrow \infty} E_\kappa(\zeta, z, n) = E_\kappa(\zeta, z) \quad (3.28)$$

and estimate uniformly $|E_\kappa|$. For this purpose we estimate using (3.12) and (3.13) that

$$\begin{aligned} \left| \left(\log_0 \frac{\kappa(\zeta, u+z)}{\kappa(\zeta, u)} \right)'' \right| &\leq \left| \frac{\kappa_z''(\zeta, u+z)}{\kappa(\zeta, u+z)} \right| + \left| \frac{\kappa_z'(\zeta, u+z)}{\kappa(\zeta, u+z)} \right|^2 + \left| \frac{\kappa_z'(\zeta, u)}{\kappa(\zeta, u)} \right|^2 + \left| \frac{\kappa_z''(\zeta, u)}{\kappa(\zeta, u)} \right| \\ &\leq \frac{8}{(u + \operatorname{Re}(z))^2} + \frac{8}{u^2} \leq \frac{16}{u^2}, \end{aligned} \quad (3.29)$$

where in the last inequality we have invoked the fact that $\operatorname{Re}(z) \geq 0$. Therefore with the help of the last inequality and

$$\sup_{u \geq 1} P(u) = \sup_{u \geq 1} (u - \lfloor u \rfloor) (1 - (u - \lfloor u \rfloor)) = \frac{1}{4} \quad (3.30)$$

we obtain that

$$\sup_{\kappa \in \mathcal{B}^2} \sup_{(\zeta, z) \in \mathbb{C}_{[0, \infty)} \times \mathbb{C}_{(0, \infty)}} \sup_{n \geq 1} |E_\kappa(\zeta, z, n)| \leq \frac{1}{8} \sup_{n \geq 1} \int_1^n \frac{16}{u^2} du \leq 2$$

and hence (3.28). The bound (2.16) also follows from (3.29). The limit (2.17) is easily deduced from (3.29) wherein the first two terms in the upper bound vanish as $\operatorname{Re}(z) \rightarrow \infty$. Let us show that for any fixed z , $\lim_{\operatorname{Re}(\zeta) \rightarrow \infty} E_\kappa(\operatorname{Re}(\zeta), z) = 0$. From (2.5) we easily get that for any $z \in \mathbb{C}_{(0, \infty)}$

$$\lim_{\operatorname{Re}(\zeta) \rightarrow \infty} \kappa(\operatorname{Re}(\zeta), z) = \kappa(0, 0) + d_2 z + \infty \mathbb{I}_{\{d_1 > 0\}} + \mu((0, \infty), (0, \infty)) = \kappa(\infty, z), \quad (3.31)$$

whereas from (3.4) we get since $\lim_{\operatorname{Re}(\zeta) \rightarrow \infty} \mu_{\operatorname{Re}(\zeta)}(dy) \stackrel{w}{=} 0 dy$ and

$$\sup_{\operatorname{Re}(\zeta) \geq 1} \left| \int_0^\infty y \mu_\zeta(dy) \right| \leq \int_0^\infty y \mu_1(dy) < \infty$$

that for any $z \in \mathbb{C}_{(0, \infty)}$

$$\lim_{\operatorname{Re}(\zeta) \rightarrow \infty} \kappa_z'(\zeta, z) = d_2. \quad (3.32)$$

By the same reasoning $\lim_{\operatorname{Re}(\zeta) \rightarrow \infty} \kappa_z''(\operatorname{Re}(\zeta), z) = 0$. This together with (3.31), (3.32) and the dominated convergence theorem (applicable due to (3.29)) yield that

$$\begin{aligned} & \lim_{\operatorname{Re}(\zeta) \rightarrow \infty} E_\kappa(\zeta, z) \\ &= \frac{1}{2} \lim_{\operatorname{Re}(\zeta) \rightarrow \infty} \int_1^\infty P(u) \\ & \quad \times \left(\frac{\kappa_z''(\operatorname{Re}(\zeta), u+z)}{\kappa(\operatorname{Re}(\zeta), u+z)} - \frac{(\kappa_z'(\operatorname{Re}(\zeta), u+z))^2}{\kappa^2(\operatorname{Re}(\zeta), u+z)} - \frac{\kappa_z''(\operatorname{Re}(\zeta), u)}{\kappa(\operatorname{Re}(\zeta), u)} + \frac{(\kappa_z'(\operatorname{Re}(\zeta), u))^2}{\kappa^2(\operatorname{Re}(\zeta), u)} \right) du \\ &= \frac{1}{2} \int_1^\infty P(u) \lim_{\operatorname{Re}(\zeta) \rightarrow \infty} \left(-\frac{(\kappa_z'(\operatorname{Re}(\zeta), u+z))^2}{\kappa^2(\operatorname{Re}(\zeta), u+z)} + \frac{(\kappa_z'(\operatorname{Re}(\zeta), u))^2}{\kappa^2(\operatorname{Re}(\zeta), u)} \right) du \\ &= \frac{1}{2} \int_1^\infty P(u) \left(-\frac{d_2^2}{\kappa^2(\infty, u+z)} + \frac{d_2^2}{\kappa^2(\infty, u)} \right) du. \end{aligned}$$

Therefore, since $|\kappa(\infty, z)| = \infty$ whenever $d_2 > 0$ it follows that $\lim_{\operatorname{Re}(\zeta) \rightarrow \infty} E_\kappa(\operatorname{Re}(\zeta), z) = 0$. Next, we investigate the term $f(n)$ in (3.26). We write

$$|f(n)| = \left| \log_0 \left(1 + \frac{\kappa(\zeta, n+z) - \kappa(\zeta, n)}{\kappa(\zeta, n)} \right) \right|. \quad (3.33)$$

Clearly, from (3.11), Proposition 3.1 2 and $\operatorname{Re}(z) \geq 0$

$$\begin{aligned} |\kappa(\zeta, n+z) - \kappa(\zeta, n)| &\leq |z| \sup_{v \in [0,1]} |\kappa_z'(\zeta, n+ vz)| \leq |z| \sup_{v \in [0,1]} \kappa_z'(\operatorname{Re}(\zeta), n+ v\operatorname{Re}(z)) \\ &= |z| \sup_{v \in [0,1]} \phi_{\operatorname{Re}(\zeta)}'(n+ v\operatorname{Re}(z)) \leq |z| \phi_{\operatorname{Re}(\zeta)}'(n). \end{aligned}$$

Thus, from (3.5) and (3.8) we arrive at

$$\left| \frac{\kappa(\zeta, n+z) - \kappa(\zeta, n)}{\kappa(\zeta, n)} \right| \leq |z| \frac{\phi_{\operatorname{Re}(\zeta)}'(n)}{\phi_{\operatorname{Re}(\zeta)}(n)} \leq \frac{|z|}{n}.$$

Thus, if $|z| = o(n)$

$$|f(n)| = \left| \log_0 \left(1 + \frac{\kappa(\zeta, n+z) - \kappa(\zeta, n)}{\kappa(\zeta, n)} \right) \right| \leq \frac{2|z|}{n} = o(1) \quad (3.34)$$

and, as $n \rightarrow \infty$, $z = o(n)$, we get that

$$\begin{aligned} \lim_{n \rightarrow \infty} |f(n)| &= \lim_{n \rightarrow \infty} \left| \log_0 \left(1 + \frac{\kappa(\zeta, n+z) - \kappa(\zeta, n)}{\kappa(\zeta, n)} \right) \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{\kappa(\zeta, n+z) - \kappa(\zeta, n)}{\kappa(\zeta, n)} \right| = 0. \end{aligned} \quad (3.35)$$

Henceforth, we obtain from (3.26) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(S_n - \int_1^n f(x) dx \right) &= \frac{1}{2} f(1) + \lim_{n \rightarrow \infty} \left(E_\kappa(\zeta, z, n) + \frac{1}{2} f(n) \right) \\ &= \frac{1}{2} f(1) + E_\kappa(\zeta, z). \end{aligned} \quad (3.36)$$

Next, we estimate $I_n = \int_1^n f(x) dx$, as $n \rightarrow \infty$. From (3.10) and since $(\zeta, z, x) \in \mathbb{C}_{[0,\infty)} \times \mathbb{C}_{(0,\infty)} \times [1, \infty)$ both $\operatorname{Re}(\kappa(\zeta, x)) > 0$, $\operatorname{Re}(\kappa(\zeta, z+x)) > 0$. Therefore, $\arg\left(\frac{\kappa(\zeta, x+z)}{\kappa(\zeta, x)}\right) \in$

$(-\pi, \pi)$ and thus

$$f(x) = \log_0 \left(\frac{\kappa(\zeta, x+z)}{\kappa(\zeta, x)} \right) = \log_0(\kappa(\zeta, x+z)) - \log_0(\kappa(\zeta, x)). \quad (3.37)$$

Thus, fixing the parallelogram in $\mathbb{C}$ with vertices $1, 1+z, z+n, n$, we see from the Cauchy integral theorem applied to the function $\log_0(\kappa(\zeta, \cdot))$ which is holomorphic in an open neighbourhood of the parallelogram that

$$\begin{aligned} I_n &= \int_1^n (\log_0(\kappa(\zeta, x+z)) - \log_0(\kappa(\zeta, x))) dx \\ &= \int_{n \rightarrow n+z} \log_0(\kappa(\zeta, \chi)) d\chi - \int_{1 \rightarrow 1+z} \log_0(\kappa(\zeta, \chi)) d\chi. \end{aligned} \quad (3.38)$$

From the latter and (3.36) we get that

$$\begin{aligned} S_n &= \int_{n \rightarrow n+z} \log_0(\kappa(\zeta, \chi)) d\chi - \int_{1 \rightarrow 1+z} \log_0(\kappa(\zeta, \chi)) d\chi \\ &\quad + \frac{1}{2} \log_0 \left(\frac{\kappa(\zeta, 1+z)}{\kappa(\zeta, 1)} \right) + E_\kappa(\zeta, z) + o(1) \end{aligned} \quad (3.39)$$

and hence (3.25) can be re-expressed as follows

$$\begin{aligned} W_\kappa(\zeta, z) &= \frac{1}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} e^{z \log_0 \kappa(\zeta, n) - S_n} \\ &= \frac{1}{\kappa(\zeta, z)} \lim_{n \rightarrow \infty} \left(e^{z \log_0 \kappa(\zeta, n) - \int_{n \rightarrow n+z} \log_0(\kappa(\zeta, \chi)) d\chi + \int_{1 \rightarrow 1+z} \log_0(\kappa(\zeta, \chi)) d\chi} \right. \\ &\quad \times \left. e^{-\frac{1}{2} \log_0 \left(\frac{\kappa(\zeta, 1+z)}{\kappa(\zeta, 1)} \right) - E_\kappa(\zeta, z)} \right). \end{aligned} \quad (3.40)$$

However, as (3.37) holds true we get using (3.34) when $|z| = o(n)$ that

$$\begin{aligned} \left| z \log_0 \kappa(\zeta, n) - \int_{n \rightarrow n+z} \log_0(\kappa(\zeta, \chi)) d\chi \right| &= |z| \left| \int_0^1 \log_0 \left(\frac{\kappa(\zeta, n+ vz)}{\kappa(\zeta, n)} \right) dv \right| \\ &\leq 2 \frac{|z|^2}{n} \end{aligned}$$

and hence

$$\lim_{n \rightarrow \infty} \left| e^{z \log_0 \kappa(\zeta, n) - \int_{n \rightarrow n+z} \log_0(\kappa(\zeta, \chi)) d\chi} \right| = 1. \quad (3.41)$$

Finally, combining together the results (3.28), (3.34), (3.40) and (3.41), we deduce, as required for (2.15), that

$$W_\kappa(\zeta, z) = \frac{1}{\kappa(\zeta, z)} e^{\int_{1 \rightarrow 1+z} \log_0(\kappa(\zeta, \chi)) d\chi - \frac{1}{2} \log_0 \left(\frac{\kappa(\zeta, 1+z)}{\kappa(\zeta, 1)} \right) - E_\kappa(\zeta, z)}.$$

This completes the proof of Theorem 2.9. $\square$

4. Proofs for Exponential Functionals of Lévy Processes

Our starting point in determining the Mellin transform of the exponential functional up to a finite time is the following equation, proven in [33, Proposition 6.1.2]. For all $\phi \in \mathcal{B}$ and $z \in \mathbb{C}_{(-1, \infty)}$,

$$\mathbb{E}[I_\phi^z(\infty)] = \frac{\Gamma(z+1)}{W_\phi(z+1)}. \quad (4.1)$$

This allows us to understand $I_\phi(t)$, $t < \infty$, in terms of the exponential functionals $I_{\varphi_q}(\infty)$ of the subordinators with Lévy–Khintchine exponent $\varphi_q(w) := \phi(w) + q$, for $q \geq 0$, through the following argument:

$$\begin{aligned} \frac{\Gamma(z+1)}{W_{\varphi_q}(z+1)} &= \mathbb{E}[I_{\varphi_q}^z(\infty)] = \mathbb{E}[I_{\phi}^z(e_q)] = \int_0^\infty \mathbb{E}[I_{\phi}^z(t)] \mathbb{P}(e_q \in dt) \\ &= \int_0^\infty \mathbb{E}[I_{\phi}^z(t)] q e^{-qt} dt = q \mathcal{L}|_q \{ \mathbb{E}[I_{\phi}^z(\cdot)] \}. \end{aligned} \quad (4.2)$$

The formula in [Theorem 2.12](#) comes from inverting this Laplace transform. First, we will express the Bernstein–gamma function $W_{\varphi_q}(z+1)$ as a product of simple Laplace transforms in the variable q , which facilitates our proof of [Theorem 2.12](#).

Lemma 4.1. *For each subordinator with Laplace exponent φ_q , for all $z \in \mathbb{C}_{(-1, \infty)}$,*

$$\mathcal{L}|_q \{ \mathbb{E}[I_{\phi}^z(\cdot)] \} = \frac{\Gamma(z+1)}{q W_{\varphi_q}(z+1)} = \frac{\Gamma(z+1)}{q \varphi_q^z(1)} \prod_{k=1}^\infty \left[\frac{\varphi_q(z+k)}{\varphi_q(k)} \left(1 + \frac{\varphi_q(k+1) - \varphi_q(k)}{\varphi_q(k)} \right)^{-z} \right]. \quad (4.3)$$

We prove this lemma in [Section 6](#). Next, let us state some important facts about convolutions and Laplace transforms. We define these as $(f * g)(t) := \int_0^t f(s)g(t-s)ds$, and $\mathcal{L}|_q \{f\} := \int_0^\infty e^{-qs} f(s)ds$, respectively. Then we have

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \times \mathcal{L}\{g\}, \quad (4.4)$$

$$\mathcal{L}\{\alpha f(\cdot) + \beta g(\cdot)\} = \alpha \mathcal{L}\{f\} + \beta \mathcal{L}\{g\}, \quad \alpha, \beta \in \mathbb{C}. \quad (4.5)$$

Moreover, for $f(t) := t^{z-1}$, $g(t) := mt^{m-1}$,

$$(f * g)(t) = m \frac{\Gamma(z)\Gamma(m)}{\Gamma(z+m)} t^{z+m-1} = \frac{m!}{(z)^{(m)}} t^{z+m-1}, \quad (4.6)$$

where we recall that $(z)^{(m)} = z(z+1) \cdots (z+m-1)$ is the rising factorial function. Similarly, with $f(t) := t^{z-1}$, $g(t) := t^m$,

$$(f * g)(t) = \frac{m!}{(z)^{(m)}} \frac{t^{z+m}}{z+m} = \frac{m!}{(z)^{(m)}} \int_0^t s^{z+m-1} ds. \quad (4.7)$$

Also, we can expand brackets for convolutions of exponentials, in the sense that

$$(f \times e^c) * (g \times e^c)(t) = e^{ct} (f * g)(t). \quad (4.8)$$

Now we are ready to prove [Theorem 2.12](#).

Proof of Theorem 2.12. First, observe that since φ'_q is non-increasing, applying [\(3.2\)](#), we have that

$$\frac{\varphi_q(k+1) - \varphi_q(k)}{\varphi_q(k)} = \frac{\int_k^{k+1} \varphi'_q(x) dx}{\varphi_q(k)} \leq \frac{\varphi'_q(k)}{\varphi_q(k)} \stackrel{(3.2)}{\leq} \frac{1}{k} < 1.$$

Then using the generalised binomial series expansion for $|w| < 1$

$$(1+w)^{-z} = \sum_{m=0}^\infty \frac{(z)^{(m)}}{m!} (-w)^m,$$

we can rewrite the product in (4.3) as

$$(4.3) = \frac{\Gamma(z+1)}{q\varphi_q^z(1)} \prod_{k=1}^{\infty} \left[\frac{\varphi_q(z+k)}{\varphi_q(k)} \sum_{m=0}^{\infty} \binom{(z)^{(m)}}{m!} \left(\frac{-[\varphi_q(k+1) - \varphi_q(k)]}{\varphi_q(k)} \right)^m \right]. \quad (4.9)$$

Now, we can write each individual term in (4.9) as a well-known Laplace transform, which will then allow us to invert the Laplace transform to yield the expression in (2.21) as an infinite convolution. One easily verifies that the terms in (4.9) are the following Laplace transforms:

$$\frac{1}{q} = \mathcal{L}_q \Big|_q \{1\} \quad (4.10)$$

$$\frac{1}{\varphi_q^z(1)} = \frac{1}{(\phi(1)+q)^z} = \mathcal{L}_q \Big|_q \left\{ \frac{t^{z-1}}{\Gamma(z)e^{\phi(1)t}} \right\} \quad (4.11)$$

$$\frac{\varphi_q(z+k)}{\varphi_q(k)} = 1 + \frac{\phi(z+k) - \phi(k)}{\phi(k)+q} = \mathcal{L}_q \Big|_q \left\{ \delta_0(dt) + \frac{[\phi(z+k) - \phi(k)]}{e^{\phi(k)t}} \right\} \quad (4.12)$$

$$\left[\frac{\varphi_q(k+1) - \varphi_q(k)}{\varphi_q(k)} \right]^m = \left[\frac{\phi(k+1) - \phi(k)}{\phi(k)+q} \right]^m = \mathcal{L}_q \Big|_q \left\{ [\phi(k+1) - \phi(k)]^m \frac{t^{m-1}}{\Gamma(m)e^{\phi(k)t}} \right\}, m \geq 1, \quad (4.13)$$

and when $m = 0$, the term in (4.13) is simply the Laplace transform of the unit point mass $\delta_0(dt)$. Substituting (4.10), (4.11), (4.12), and (4.13) into Eq. (4.9), we arrive at the following formula:

$$\begin{aligned} \mathcal{L}_q \Big|_q \{ \mathbb{E}[I_{\phi}^z(t)] \} &= \Gamma(z+1) \mathcal{L}_q \Big|_q \left\{ 1 * \frac{t^{z-1}}{\Gamma(z)e^{\phi(1)t}} * \bigstar_{k=1}^{\infty} \left[\left(\delta_0(dt) + \frac{[\phi(k+z) - \phi(k)]}{e^{\phi(k)t}} \right) \right. \right. \\ &\quad \left. \left. * \left(\delta_0(dt) + \sum_{m=1}^{\infty} \frac{(z)^{(m)}(-[\phi(k+1) - \phi(k)]^m t^{m-1})}{m! \Gamma(m) e^{\phi(k)t}} \right) \right] \right\}. \end{aligned} \quad (4.14)$$

Now, by (4.5) and (4.8), noting $\Gamma(m) = (m-1)!$ and $\delta_0(\cdot) * f \equiv f$ for any function f , we have:

$$\begin{aligned} (4.14) &\stackrel{(4.5)}{=} \mathcal{L}_q \Big|_q \left\{ 1 * \frac{z t^{z-1}}{e^{\phi(1)t}} * \bigstar_{k=1}^{\infty} \left[\delta_0(dt) + \sum_{m=1}^{\infty} \frac{(z)^{(m)}(-[\phi(k+1) - \phi(k)]^m)}{(m!)^2} \frac{m t^{m-1}}{e^{\phi(k)t}} \right. \right. \\ &\quad \left. \left. + \frac{[\phi(k+z) - \phi(k)]}{e^{\phi(k)t}} + \frac{[\phi(k+z) - \phi(k)]}{e^{\phi(k)t}} * \sum_{m=1}^{\infty} \frac{(z)^{(m)}(-[\phi(k+1) - \phi(k)]^m)}{(m!)^2} \frac{m t^{m-1}}{e^{\phi(k)t}} \right] \right\} \\ &\stackrel{(4.8)}{=} \mathcal{L}_q \Big|_q \left\{ 1 * \frac{z t^{z-1}}{e^{\phi(1)t}} * \bigstar_{k=1}^{\infty} \left[\delta_0(dt) + \sum_{m=1}^{\infty} \frac{(z)^{(m)}(-[\phi(k+1) - \phi(k)]^m)}{(m!)^2} \frac{m t^{m-1}}{e^{\phi(k)t}} \right. \right. \\ &\quad \left. \left. + \frac{[\phi(k+z) - \phi(k)]}{e^{\phi(k)t}} + [\phi(k+z) - \phi(k)] \sum_{m=1}^{\infty} \frac{(z)^{(m)}(-[\phi(k+1) - \phi(k)]^m)}{(m!)^2} \frac{t^m}{e^{\phi(k)t}} \right] \right\} \\ &= \mathcal{L}_q \Big|_q \left\{ 1 * \frac{z t^{z-1}}{e^{\phi(1)t}} * \bigstar_{k=1}^{\infty} \left[\delta_0(dt) + \sum_{m=1}^{\infty} \frac{(z)^{(m)}(-[\phi(k+1) - \phi(k)]^m)}{(m!)^2} \frac{m t^{m-1}}{e^{\phi(k)t}} \right. \right. \\ &\quad \left. \left. + [\phi(k+z) - \phi(k)] \sum_{m=0}^{\infty} \frac{(z)^{(m)}(-[\phi(k+1) - \phi(k)]^m)}{(m!)^2} \frac{t^m}{e^{\phi(k)t}} \right] \right\}, \end{aligned}$$

and then the desired result (2.21) follows immediately by a simple Laplace inversion. $\square$

Lemma 4.2, which we prove further below in Section 4, builds upon [Theorem 2.12](#) by evaluating the first n terms in the infinite convolution. In the proof of [Theorem 2.14](#), we will take limits as $n \rightarrow \infty$.

Lemma 4.2. *For all $\phi \in \mathcal{B}$, $z \in \mathbb{C}_{(0,\infty)}$, and $n \in \mathbb{N}$, the following truncated convolution satisfies*

$$\begin{aligned} & \frac{zt^{z-1}}{e^{\phi(1)t}} * \bigstar_{k=1}^n \left[\delta_0(dt) + \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(k+1) - \phi(k)])^m \frac{mt^{m-1}}{e^{\phi(k)t}} \right. \\ & \quad \left. + [\phi(z+k) - \phi(k)] \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(k+1) - \phi(k)])^m \frac{t^m}{e^{\phi(k)t}} \right] \\ &= \frac{zt^{z-1}}{e^{\phi(n+1)t}} + z \sum_{k=1}^n \frac{\prod_{1 \leq j \leq n} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]} \frac{\gamma(z, [\phi(n+1) - \phi(k)]t)}{[\phi(n+1) - \phi(k)]^z e^{\phi(k)t}}, \end{aligned} \quad (4.15)$$

where $\gamma(z, u) := \int_0^u e^{-x} x^{z-1} dx$ denotes the lower incomplete gamma function.

The proof of [Theorem 2.14](#) also requires the following lemmas, which are also proven later in Section 6.

Lemma 4.3. *For each $\phi \in \mathcal{B}$ and $c > 0$, there exists a constant $C_{\phi,c} > 0$ such that for all $n \geq 1$,*

$$\frac{\phi'(n)}{\phi'(n+c)} \leq C_{\phi,c}.$$

Lemma 4.4. *For $\phi \in \mathcal{B}$ satisfying the condition in [Definition 2.13](#) and for each $z \in \mathbb{C}_{(0,\infty)}$,*

$$\sum_{k=1}^{\infty} \left| \frac{\prod_{1 \leq i \leq k} [\phi(z+i) - \phi(k)]}{\prod_{1 \leq j \leq k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{W_{\phi(k)}(z+1)} \right| < \infty, \quad (4.16)$$

and moreover there exist $c, \rho > 0$ such that for all $k \geq 1$,

$$\phi(k) \geq ck^{\rho}. \quad (4.17)$$

Lemma 4.5. *For all $\phi \in \mathcal{B}$ and $z \in \mathbb{C}_{(0,\infty)}$, there is a constant $c_{\phi,z} > 0$ such that for all $k \geq 1$,*

$$\prod_{1 \leq j \leq k-1} \frac{|\phi(z+j) - \phi(k)|}{\phi(k) - \phi(j)} \leq c_{\phi,z}.$$

The structure of the proof of [Theorem 2.14](#) is as follows: First, we will find the termwise limit, as $n \rightarrow \infty$, of each summand in (4.15). This gives a formula for the infinite convolution in (2.21), without the convolution with 1, which means this limit is an expression for $\frac{d}{dt} \mathbb{E}[I_{\phi}^z(t)]$. To show that the termwise limits correspond to the limit of the whole expression in (4.15), we employ a dominated convergence argument. We finish the proof of [Theorem 2.14](#) by integrating each term in our expression for $\frac{d}{dt} \mathbb{E}[I_{\phi}^z(t)]$, which requires a second dominated convergence argument.

Proof of Theorem 2.14. Denoting $\phi_{(k)}(w) := \phi(w+k) - \phi(k)$, we first verify that $\phi_{(k)}$ is itself a Bernstein function, with unchanged drift and rescaled Lévy measure of the form $e^{-kx} \Pi(dx)$.

Indeed, this follows by noting that (2.1) gives

$$\begin{aligned}\phi_{(k)}(w) &= \phi(0) + (k+w)d + \int_0^\infty (1 - e^{-(k+w)x})\Pi(dx) - \phi(0) - kd \\ &\quad - \int_0^\infty (1 - e^{-kx})\Pi(dx) \\ &= wd + \int_0^\infty (1 - e^{-wx})e^{-kx}\Pi(dx).\end{aligned}$$

Now, rewriting in terms of $\phi_{(k)}$, the convolution in (4.15) satisfies

$$\begin{aligned}(4.15) &= \frac{zt^{z-1}}{e^{\phi(n+1)t}} + z \sum_{k=1}^n \frac{\prod_{j=1}^{k-1} [\phi(z+j) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{\prod_{j=k}^n [\phi(z+j) - \phi(k)]}{\prod_{j=k+1}^n [\phi(j) - \phi(k)]} \\ &\quad \times \frac{\gamma(z, [\phi(n+1) - \phi(k)]t)}{[\phi(n+1) - \phi(k)]^z e^{\phi(k)t}} \\ &= \frac{zt^{z-1}}{e^{\phi(n+1)t}} + z \sum_{k=1}^n \left(\prod_{j=1}^{k-1} \frac{[\phi(z+j) - \phi(k)]}{[\phi(j) - \phi(k)]} \right) \frac{\prod_{j=0}^{n-k} \phi_{(k)}(z+j)}{\prod_{j=1}^{n-k} \phi_{(k)}(j)} \\ &\quad \times \frac{\gamma(z, [\phi(n+1) - \phi(k)]t)}{\phi_{(k)}^z(n+1-k) e^{\phi(k)t}}.\end{aligned}$$

For the termwise limit as $n \rightarrow \infty$, first observe that under the conditions of Definition 2.13, relation (4.17) implies that $\phi(\infty) = \infty$ and hence $\lim_{n \rightarrow \infty} zt^{z-1}/e^{\phi(n+1)t} = 0$, and second that $\lim_{n \rightarrow \infty} \gamma(z, [\phi(n+1) - \phi(k)]t) = \Gamma(z)$. Repeatedly using that $W_{\phi_{(k)}}(w+1) = \phi_{(k)}(w)W_{\phi_{(k)}}(w)$ from (2.4), noting $W_{\phi_{(k)}}(1) = 1$, and applying Lemma 2.11, it follows that as $n \rightarrow \infty$,

$$\begin{aligned}\frac{\prod_{j=0}^{n-k} \phi_{(k)}(z+j)}{\prod_{j=1}^{n-k} \phi_{(k)}(j)} \frac{1}{\phi_{(k)}^z(n+1-k)} &= \frac{\phi_{(k)}(z)W_{\phi_{(k)}}(z+n+1-k)}{W_{\phi_{(k)}}(z+1)W_{\phi_{(k)}}(n+1-k)} \frac{1}{\phi_{(k)}^z(n+1-k)} \\ &\stackrel{2.11}{\sim} \frac{\phi_{(k)}(z)}{W_{\phi_{(k)}}(z+1)} = \frac{[\phi(z+k) - \phi(k)]}{W_{\phi_{(k)}}(z+1)}.\end{aligned}\quad (4.18)$$

This gives us the limit, as $n \rightarrow \infty$, of each summand in (4.15). Using the dominated convergence theorem, we will now show that we can exchange the order of limits and summation, which yields the following relation

$$\frac{d}{dt} \mathbb{E}[I_\phi^z(t)] = \lim_{n \rightarrow \infty} (4.15) = \Gamma(z+1) \sum_{k=1}^\infty \frac{\prod_{j=1}^k [\phi(z+j) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{W_{\phi_{(k)}}(z+1)}. \quad (4.19)$$

In order to apply the dominated convergence theorem, we must show that the sum in (4.15) is dominated by an absolutely convergent sum. From (4.18), we can rewrite the sum in (4.15) as

$$\begin{aligned}\sum_{k=1}^\infty \mathbb{1}_{\{k \leq n\}} \frac{\prod_{j=1}^k [\phi(z+j) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{W_{\phi_{(k)}}(z+n+1-k)}{W_{\phi_{(k)}}(z+1)W_{\phi_{(k)}}(n+1-k)} \frac{\gamma(z, [\phi(n+1) - \phi(k)]t)}{\phi_{(k)}^z(n+1-k) e^{\phi(k)t}} \\ =: \sum_{k=1}^\infty F_k(n).\end{aligned}\quad (4.20)$$

Then if we can show that there exist $n_0 > 0$ and $C > 0$ such that for all $n \geq n_0$, and for all $k \geq 1$,

$$|F_k(n)| \leq C \times \left| \frac{\prod_{j=1}^k [\phi(z+j) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{W_{\phi(k)}(z+1)} \right|, \quad (4.21)$$

and moreover if

$$\sum_{k=1}^{\infty} \left| \frac{\prod_{j=1}^k [\phi(z+j) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{W_{\phi(k)}(z+1)} \right| < \infty, \quad (4.22)$$

then the dominated convergence theorem applies, so

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} F_k(n) = \sum_{k=1}^{\infty} \lim_{n \rightarrow \infty} F_k(n),$$

as required for (4.19). The absolute convergence of the sum in (4.22) follows by Lemma 4.4, so now let us prove (4.21). Firstly, one can easily verify that for all $n, k \in \mathbb{N}$, with $z = a + ib$,

$$|\gamma(z, [\phi(n+1) - \phi(k)]t)| \leq \int_0^{[\phi(n+1) - \phi(k)]t} |u^{z-1} e^{-u}| du \leq \int_0^{\infty} u^{a-1} e^{-u} du = \Gamma(a). \quad (4.23)$$

The remaining term in the expression for $F_k(n)$ from (4.20) which depends on n is

$$\frac{W_{\phi(k)}(z+n+1-k)}{W_{\phi(k)}(n+1-k)\phi_{(k)}^z(n+1-k)}. \quad (4.24)$$

Applying Theorem 2.9, we can rewrite (4.24) as

$$\frac{\phi_{(k)}(n+1-k)\phi_{(k)}^{\frac{1}{2}}(n+2-k) e^{L_{\phi(k)}(z+n+1-k) - E_{\phi(k)}(z+n+1-k) - L_{\phi(k)}(n+1-k) + E_{\phi(k)}(n+1-k)}}{\phi_{(k)}(z+n+1-k)\phi_{(k)}^{\frac{1}{2}}(z+n+2-k)\phi_{(k)}^z(n+1-k)}. \quad (4.25)$$

We will bound the absolute value of the terms in (4.25) separately. Firstly, using the result [32, Prop. 3.1.9] that $|\phi(a+ib)| \geq \phi(a)$ and the fact that ϕ is non-decreasing on $\mathbb{R}_+$, it follows that

$$\frac{\phi_{(k)}(n+1-k)\phi_{(k)}^{\frac{1}{2}}(n+2-k)}{|\phi_{(k)}(z+n+1-k)| |\phi_{(k)}^{\frac{1}{2}}(z+n+2-k)|} \leq \frac{\phi_{(k)}(n+1-k)\phi_{(k)}^{\frac{1}{2}}(n+2-k)}{\phi_{(k)}(a+n+1-k)\phi_{(k)}^{\frac{1}{2}}(a+n+2-k)} \leq 1. \quad (4.26)$$

To bound the $E_{\phi(k)}$ terms in (4.25), observe that by (2.16),

$$\sup_{n \geq 1} \sup_{1 \leq k \leq n} |E_{\phi(k)}(z+n+1-k)| \leq 2, \text{ and } \sup_{n \geq 1} \sup_{1 \leq k \leq n} |E_{\phi(k)}(n+1-k)| \leq 2,$$

so it follows immediately that for all $k \leq n$,

$$\left| e^{-E_{\phi(k)}(z+n+1-k) + E_{\phi(k)}(n+1-k)} \right| \leq e^4. \quad (4.27)$$

Next, consider the $L_{\phi(k)}$ terms in (4.25), which are defined in (2.12). Part of the integrals cancel, so

$$\begin{aligned} L_{\phi(k)}(z+n+1-k) - L_{\phi(k)}(n+1-k) &= \int_{1 \mapsto 1+(z+n+1-k)} \log_0(\phi(k)(w))dw \\ &\quad - \int_{1 \mapsto 1+(n+1-k)} \log_0(\phi(k)(w))dw \\ &= \int_{1+(n+1-k) \mapsto 1+(z+n+1-k)} \log_0(\phi(k)(w))dw = \int_{N-k \mapsto N-k+z} \log_0(\phi(k)(w))dw, \end{aligned}$$

where we substitute $N := n+2$. Using the fact that $\log_0(w) = \ln(|w|) + i \arg(w)$, it follows that

$$\begin{aligned} L_{\phi(k)}(z+n+1-k) - L_{\phi(k)}(n+1-k) &= \int_{N-k \mapsto N-k+z} \ln(|\phi(k)(w)|)dw \\ &\quad + i \int_{N-k \mapsto N-k+z} \arg(\phi(k)(w))dw. \end{aligned} \quad (4.28)$$

We consider the integrals in (4.28) along the contours γ_1, γ_2 , which are straight lines connecting $N-k$ to $N-k+a$ and $N-k+a$ to $N-k+a+ib$, respectively. Observe that along γ_1 , $\arg(\phi(k)(x)) = 0$, and note $|\gamma_2| = b$. Then for all $1 \leq k \leq n$, the second integral in (4.28) satisfies

$$\left| \int_{N-k \mapsto N-k+z} \arg(\phi(k)(w))dw \right| = \left| \int_{\gamma_2} \arg(\phi(k)(w))dw \right| \leq b \sup_{x \in \gamma_2} |\arg(\phi(k)(x))| \leq b\pi. \quad (4.29)$$

Next, we consider the first integral in (4.28) over γ_1 . Recalling $N = n+2$, this can be written for $k \leq n$ as

$$\begin{aligned} \int_{\gamma_1} \ln(|\phi(k)(w)|)dw &= \int_{N-k}^{N-k+a} \ln(\phi(k)(x))dx = \int_0^a \ln(\phi(k)(x+N-k))dx \\ &= \int_0^a \ln(\phi(x+N) - \phi(k))dx = \int_0^a \ln(\phi(x+n+2) - \phi(k))dx. \end{aligned}$$

Comparing this with the $1/\phi(k)(n+1-k) = e^{-a \ln(\phi(k)(n+1-k))}$ term from (4.25), we get that

$$\begin{aligned} &\int_0^a \ln(\phi(x+n+2) - \phi(k))dx - a \ln(\phi(k)(n+1-k)) \\ &= \int_0^a \ln \left(\frac{\phi(x+n+2) - \phi(k)}{\phi(k)(n+1-k)} \right) dx \\ &= \int_0^a \ln \left(\frac{\phi(x+n+2) - \phi(k)}{\phi(n+1) - \phi(k)} \right) dx. \end{aligned} \quad (4.30)$$

Observe that $(\phi(x+n+2) - y)/(\phi(n+1) - y)$ and $\phi(y)$ are non-decreasing in y since in the first expression $\phi(x+n+2) \geq \phi(n+1)$. So for all $k \leq n$,

$$(4.30) \leq \int_0^a \ln \left(\frac{\phi(x+n+2) - \phi(n)}{\phi(n+1) - \phi(n)} \right) dx \leq a \ln \left(\frac{\phi(a+n+2) - \phi(n)}{\phi(n+1) - \phi(n)} \right).$$

Now, since ϕ' is non-increasing on $\mathbb{R}_+$, applying Lemma 4.3, it follows that

$$(4.30) \leq a \ln \left(\frac{\int_n^{a+n+2} \phi'(y)dy}{\int_n^{n+1} \phi'(y)dy} \right) \leq a \ln \left(\frac{(a+2)\phi'(n)}{\phi'(n+1)} \right) \leq \text{constant}. \quad (4.31)$$

Finally, consider the first integral in (4.28) over the contour γ_2 . To bound (4.25), we compare it to $1/\phi_{(k)}(n+1-k)^{ib}$. Then since a real number raised to an imaginary power has absolute value 1, we arrive at

$$\left| \frac{e^{\int_{\gamma_2} \ln(|\phi_{(k)}(x)|) dx}}{\phi_{(k)}^{ib}(n+1-k)} \right| = \left| \frac{e^{\int_{N-k+a \rightarrow N-k+a+ib} \ln(|\phi_{(k)}(x)|) dx}}{\phi_{(k)}^{ib}(n+1-k)} \right| = \left| \frac{e^{i \int_0^b \ln(|\phi_{(k)}(N-k+a+iv)|) dv}}{\phi_{(k)}^{ib}(n+1-k)} \right| = 1. \quad (4.32)$$

Combining together the bounds (4.26), (4.27), (4.31), and (4.32), we conclude that there exists a constant $C(z, \phi)$ such that uniformly among $n \in \mathbb{N}$ and $1 \leq k \leq n$, $|(4.25)| \leq C(z, \phi)$, from which (4.21) follows, and therefore since Lemma 4.4 is valid we have shown that (4.19) holds. Now, applying (4.19), and recalling from (4.1) that

$$\int_0^\infty \frac{d}{ds} \mathbb{E}[I_\phi^z(s)] ds = \mathbb{E}[I_\phi^z(\infty)] = \frac{\Gamma(z+1)}{W_\phi(z+1)},$$

we can express $\mathbb{E}[I_\phi^z(t)]$ as

$$\begin{aligned} \mathbb{E}[I_\phi^z(t)] &= \int_0^t \frac{d}{ds} \mathbb{E}[I_\phi^z(s)] ds = \frac{\Gamma(z+1)}{W_\phi(z+1)} - \int_t^\infty \frac{d}{ds} \mathbb{E}[I_\phi^z(s)] ds \\ &= \frac{\Gamma(z+1)}{W_\phi(z+1)} - \Gamma(z+1) \int_t^\infty \left(\sum_{k=1}^\infty \frac{\prod_{j=1}^k [\phi(z+j) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)s}}{W_{\phi(k)}(z+1)} \right) ds. \end{aligned} \quad (4.33)$$

Now, we wish to exchange the order of integration and summation in identity (4.33). If we can show that

$$\sum_{k=1}^\infty \left(\int_t^\infty \left| \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)s}}{W_{\phi(k)}(z+1)} \right| ds \right) < \infty, \quad (4.34)$$

then by Fubini's theorem, we can exchange the order of integration and summation, yielding the result

$$\begin{aligned} \int_t^\infty \sum_{k=1}^\infty \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)s}}{W_{\phi(k)}(z+1)} ds &= \sum_{k=1}^\infty \int_t^\infty \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)s}}{W_{\phi(k)}(z+1)} ds \\ &= \sum_{k=1}^\infty \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{\phi(k) W_{\phi(k)}(z+1)}. \end{aligned} \quad (4.35)$$

Substituting this into (4.33), we conclude, as required, that

$$\mathbb{E}[I_\phi^z(t)] = \frac{\Gamma(z+1)}{W_\phi(z+1)} - \sum_{k=1}^\infty \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{\phi(k)} \frac{\Gamma(z+1)}{W_{\phi(k)}(z+1)}$$

or relation (2.22) holds true. To see that (4.34) is finite, following the same argument as in (4.35), noting that ϕ is non-decreasing,

$$\begin{aligned} (4.34) &= \sum_{k=1}^\infty \left| \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{\phi(k) W_{\phi(k)}(z+1)} \right| \\ &\leq \frac{1}{\phi(1)} \sum_{k=1}^\infty \left| \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{W_{\phi(k)}(z+1)} \right|, \end{aligned}$$

and then it follows by Lemma 4.4 that (4.34) is a finite quantity and so the proof of Theorem 2.14 is complete. $\square$

Before we prove Corollary 2.18, we state the following fact about Vandermonde matrices, see e.g. [16, p37]. For $a_1, \dots, a_n \in \mathbb{C}$, the determinant of the $n \times n$ Vandermonde matrix has the form

$$\det \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix} = \prod_{1 \leq h < l \leq n} (a_l - a_h). \quad (4.36)$$

We derive some elementary consequence of this representation, which is certainly located somewhere in the literature.

Proposition 4.6. *For any $n + 1$ different complex numbers $a_0, a_1, \dots, a_n$ it holds that*

$$\sum_{k=0}^n \frac{1}{\prod_{0 \leq j \leq n; j \neq k} [a_j - a_k]} = 0. \quad (4.37)$$

Proof of Proposition 4.6. To derive (4.37) we multiply through by $\prod_{0 \leq h < l \leq n} [a_h - a_l]$, and we see that (4.37) holds if and only if

$$\sum_{k=0}^n \left((-1)^k \prod_{\substack{0 \leq h < l \leq n; \\ h, l \neq k}} [a_h - a_l] \right) = 0. \quad (4.38)$$

Applying the formula (4.36) for Vandermonde determinants, one can verify that the sum in (4.38) equals the following determinant, evaluated using the cofactor expansion along its first column:

$$\det \begin{pmatrix} 1 & 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} \\ 1 & 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}.$$

This determinant is 0 as the first two columns are identical, so (4.38) = 0. This proves (4.37). $\square$

Proof of Corollary 2.18. We specialise (4.2) for the case $z = n$ and use the recurrent relation (2.4) to get

$$\mathcal{L}|_q \{ \mathbb{E}[I_\phi^n(\cdot)] \} = \int_0^\infty e^{-qt} \mathbb{E}[I_\phi^n(t)] dt = \frac{1}{q} \frac{n!}{\prod_{j=1}^n (q + \phi(j))}. \quad (4.39)$$

From here we immediately get that for any $t > 0$

$$\mathbb{E}[I_\phi^n(t)] = n! \left(1 * \bigstar_{k=1}^n e^{-\phi(k)t} \right).$$

Clearly, for $n = 1$,

$$\mathbb{E}[I_\phi(t)] = \int_0^t e^{-\phi(1)s} ds = \frac{1 - e^{-\phi(1)t}}{\phi(1)} = \frac{e^{-\phi^*(0)t} - e^{-\phi^*(1)t}}{\phi^*(1) - \phi^*(0)},$$

which gives the first identity of (2.24) for $n = 1$. We proceed by induction assuming that the first identity of (2.24) holds for $n = N$ and all $t > 0$. We consider $\mathbb{E}[I_\phi^{N+1}(t)]$. From above and the inductive hypothesis we have that

$$\begin{aligned} \mathbb{E}[I_\phi^{N+1}(t)] &= (N+1)! \left(1 * \bigstar_{k=1}^{N+1} e^{-\phi^*(k)t} \right) = (N+1) \int_0^t \mathbb{E}[I_\phi^N(s)] e^{-\phi(N+1)(t-s)} ds \\ &= (N+1)! \sum_{k=0}^{N-1} \frac{1}{\prod_{0 \leq j \leq N; j \neq k} [\phi^*(j) - \phi^*(k)]} \int_0^t \left(e^{-\phi^*(k)s} - e^{-\phi^*(N)s} \right) e^{-\phi^*(N+1)(t-s)} ds \\ &= (N+1)! \sum_{k=0}^{N-1} \frac{e^{-\phi^*(N+1)t}}{\prod_{0 \leq j \leq N; j \neq k} [\phi^*(j) - \phi^*(k)]} \int_0^t \left(e^{(\phi^*(N+1) - \phi^*(k))s} - e^{(\phi^*(N+1) - \phi^*(N))s} \right) \\ &\quad \times ds \\ &= (N+1)! \sum_{k=0}^{N-1} \frac{e^{-\phi^*(k)t} - e^{-\phi^*(N+1)t}}{\prod_{0 \leq j \leq N+1; j \neq k} [\phi^*(j) - \phi^*(k)]} \\ &\quad - (N+1)! \frac{e^{-\phi^*(N)t} - e^{-\phi^*(N+1)t}}{[\phi^*(N+1) - \phi^*(N)]} \sum_{k=0}^{N-1} \frac{1}{\prod_{0 \leq j \leq N; j \neq k} [\phi^*(j) - \phi^*(k)]}. \end{aligned}$$

Now, since $\phi^*(j)$, $0 \leq j \leq N$, are different numbers, from (4.37) we conclude that

$$- \sum_{k=0}^{N-1} \frac{1}{\prod_{0 \leq j \leq N; j \neq k} [\phi^*(j) - \phi^*(k)]} = \frac{1}{\prod_{0 \leq j \leq N; j \neq N} [\phi^*(j) - \phi^*(N)]}. \quad (4.40)$$

Hence, substituting above

$$\begin{aligned} \mathbb{E}[I_\phi^{N+1}(t)] &= (N+1)! \left(\sum_{k=0}^{N-1} \frac{e^{-\phi^*(k)t} - e^{-\phi^*(N+1)t}}{\prod_{0 \leq j \leq N+1; j \neq k} [\phi^*(j) - \phi^*(k)]} + \frac{e^{-\phi^*(N)t} - e^{-\phi^*(N+1)t}}{\prod_{0 \leq j \leq N+1; j \neq N} [\phi^*(j) - \phi^*(N)]} \right) \\ &= (N+1)! \sum_{k=0}^N \frac{e^{-\phi^*(k)t} - e^{-\phi^*(N+1)t}}{\prod_{0 \leq j \leq N+1; j \neq k} [\phi^*(j) - \phi^*(k)]}, \end{aligned}$$

which verifies the inductive hypothesis and thus the first identity of (2.24) is proven. For the second we use again that (4.40) holds true to deduce that for any n

$$-e^{-\phi^*(n)t} \sum_{k=0}^{n-1} \frac{1}{\prod_{0 \leq j \leq n; j \neq k} [\phi^*(j) - \phi^*(k)]} = \frac{-e^{-\phi^*(n)t}}{\prod_{0 \leq j \leq n; j \neq n} [\phi^*(j) - \phi^*(n)]}$$

and upon substitution, the second identity in (2.24) follows. $\square$

The proof of Lemma 4.2 relies upon the following lemma, which we shall prove in Section 6.

Lemma 4.7. For each $\phi \in \mathcal{B}$, and for all $n \geq 1$,

$$\sum_{k=1}^n (-1)^k \left(\prod_{\substack{1 \leq h < l \leq n; \\ h, l \neq k}} [\phi(l) - \phi(h)] \right) \left(- \prod_{1 \leq i \leq n-1} [\phi(z+i) - \phi(k)] \right) = \prod_{1 \leq h < l \leq n} [\phi(l) - \phi(h)]. \quad (4.41)$$

Proof of Lemma 4.2. We use a proof by induction. For the base case, $n = 1$, we need to verify that

$$\begin{aligned} & \frac{zt^{z-1}}{e^{\phi(1)t}} * \left[\delta_0(dt) + \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(2) - \phi(1)])^m \frac{mt^{m-1}}{e^{\phi(1)t}} \right. \\ & \quad \left. + [\phi(z+1) - \phi(1)] \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(2) - \phi(1)])^m \frac{t^m}{e^{\phi(1)t}} \right] \\ &= \frac{zt^{z-1}}{e^{\phi(2)t}} + z[\phi(z+1) - \phi(1)] \frac{\gamma(z, [\phi(2) - \phi(1)]t)}{[\phi(2) - \phi(1)]^z e^{\phi(1)t}}. \end{aligned} \quad (4.42)$$

Applying the properties of convolutions (4.6), (4.7), and (4.8), we can write

$$\begin{aligned} (4.42) &= \frac{zt^{z-1}}{e^{\phi(1)t}} + z \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(2) - \phi(1)])^m \frac{(t^{z-1} * mt^{m-1})}{e^{\phi(1)t}} \\ &\quad + z[\phi(z+1) - \phi(1)] \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(2) - \phi(1)])^m \frac{(t^{z-1} * t^m)}{e^{\phi(1)t}} \\ &= \frac{zt^{z-1}}{e^{\phi(1)t}} + \frac{z}{e^{\phi(1)t}} \sum_{m=1}^{\infty} \frac{1}{m!} (-[\phi(2) - \phi(1)])^m t^{z+m-1} \\ &\quad + \frac{z[\phi(z+1) - \phi(1)]}{e^{\phi(1)t}} \sum_{m=0}^{\infty} \frac{1}{m!} (-[\phi(2) - \phi(1)])^m \int_0^t s^{z+m-1} ds \\ &= \frac{zt^{z-1}}{e^{\phi(1)t}} \sum_{m=0}^{\infty} \frac{1}{m!} (-[\phi(2) - \phi(1)])^m t^m \\ &\quad + \frac{z[\phi(z+1) - \phi(1)]}{e^{\phi(1)t}} \int_0^t s^{z-1} \left(\sum_{m=0}^{\infty} \frac{1}{m!} (-[\phi(2) - \phi(1)])^m s^m \right) ds. \end{aligned}$$

Using the fact that $e^w = \sum_{m=0}^{\infty} \frac{w^m}{m!}$, then applying a change of variables, it follows that

$$\begin{aligned} (4.42) &= \frac{zt^{z-1}}{e^{\phi(1)t}} e^{-[\phi(2) - \phi(1)]t} + \frac{z[\phi(z+1) - \phi(1)]}{e^{\phi(1)t}} \int_0^t s^{z-1} e^{-[\phi(2) - \phi(1)]s} ds \\ &= \frac{zt^{z-1}}{e^{\phi(2)t}} + z[\phi(z+1) - \phi(1)] \frac{\gamma(z, [\phi(2) - \phi(1)]t)}{[\phi(2) - \phi(1)]^z e^{\phi(1)t}}, \end{aligned}$$

as required for the base case. For the inductive step, assume that the formula in (4.15) holds for the $(n-1)$ th convolution. Noting that

$$\frac{\gamma(z, [\phi(n) - \phi(k)]t)}{[\phi(n) - \phi(k)]^z} = \int_0^t e^{-[\phi(n) - \phi(k)]s} s^{z-1} ds,$$

this means that

$$\begin{aligned}
 & \frac{zt^{z-1}}{e^{\phi(1)t}} * \bigstar_{k=1}^{n-1} \left[\delta_0(dt) + \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} m (-[\phi(k+1) - \phi(k)])^m \frac{t^{m-1}}{e^{\phi(k)t}} \right. \\
 & \quad \left. + [\phi(z+k) - \phi(k)] \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(k+1) - \phi(k)])^m \frac{t^m}{e^{\phi(k)t}} \right] \\
 & = \frac{zt^{z-1}}{e^{\phi(n)t}} + z \sum_{k=1}^{n-1} \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n-1; j \neq k} [\phi(j) - \phi(k)]} e^{-\phi(k)t} \int_0^t e^{-[\phi(n)-\phi(k)]s} s^{z-1} ds.
 \end{aligned} \tag{4.43}$$

We are going to evaluate the convolution of (4.43) with the next term in (4.15), that is, the expression

$$\begin{aligned}
 & \left(\frac{zt^{z-1}}{e^{\phi(n)t}} + z \sum_{k=1}^{n-1} \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n-1; j \neq k} [\phi(j) - \phi(k)]} e^{-\phi(k)t} \int_0^t e^{-[\phi(n)-\phi(k)]s} s^{z-1} ds \right) \\
 & * \left[\delta_0(dt) + \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{mt^{m-1}}{e^{\phi(n)t}} \right. \\
 & \quad \left. + [\phi(z+n) - \phi(n)] \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{t^m}{e^{\phi(n)t}} \right].
 \end{aligned}$$

For brevity, we label the above convolution as

$$\left((1A) + \sum_{k=1}^{n-1} (1k) \right) * (\delta_0(dt) + (HG1) + (HG2)). \tag{4.44}$$

First, observe that

$$\left((1A) + \sum_{k=1}^{n-1} (1k) \right) * \delta_0(dt) = (1A) + \sum_{k=1}^{n-1} (1k).$$

We will see that this contribution cancels with some of the other terms arising from (4.44), which we evaluate individually.

Evaluating the $(1A) * (HG1)$ term Applying the properties (4.8) and (4.6), we can write

$$\begin{aligned}
 (1A) * (HG1) &= zt^{z-1} e^{-\phi(n)t} * \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{mt^{m-1}}{e^{\phi(n)t}} \\
 &\stackrel{(4.8)}{=} z \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{(t^{z-1} * mt^{m-1})}{e^{\phi(n)t}} \\
 &\stackrel{(4.6)}{=} z \sum_{m=1}^{\infty} \frac{1}{m!} (-[\phi(n+1) - \phi(n)])^m \frac{t^{z+m-1}}{e^{\phi(n)t}} \\
 &= \frac{zt^{z-1}}{e^{\phi(n)t}} \sum_{m=1}^{\infty} \frac{1}{m!} (-[\phi(n+1) - \phi(n)]t)^m \\
 &= \frac{zt^{z-1}}{e^{\phi(n+1)t}} - \frac{zt^{z-1}}{e^{\phi(n)t}}.
 \end{aligned} \tag{4.45}$$

Evaluating the $(1A) * (HG2)$ term Applying the properties (4.8) and (4.7), we can write

$$\begin{aligned}
 (1A) * (HG2) &= z t^{z-1} e^{-\phi(n)t} * [\phi(z+n) - \phi(n)] \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \\
 &\quad \times \frac{t^m}{e^{\phi(n)t}} \\
 &\stackrel{(4.8)}{=} z [\phi(z+n) - \phi(n)] \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{(t^{z-1} * t^m)}{e^{\phi(n)t}} \\
 &\stackrel{(4.7)}{=} z [\phi(z+n) - \phi(n)] \sum_{m=0}^{\infty} \frac{1}{m!} (-[\phi(n+1) - \phi(n)])^m \frac{\int_0^t s^{z+m-1} ds}{e^{\phi(n)t}} \\
 &= \frac{z [\phi(z+n) - \phi(n)]}{e^{\phi(n)t}} \int_0^t s^{z-1} \sum_{m=0}^{\infty} \frac{1}{m!} (-[\phi(n+1) - \phi(n)]s)^m ds \\
 &= \frac{z [\phi(z+n) - \phi(n)]}{e^{\phi(n)t}} \int_0^t s^{z-1} e^{-[\phi(n+1) - \phi(n)]s} ds. \tag{4.46}
 \end{aligned}$$

Evaluating each $(1k) * (HG1)$ term To evaluate $(1k) * (HG1)$, we first consider the quantity

$$\left(e^{-\phi(k)t} \int_0^t e^{-[\phi(n) - \phi(k)]s} s^{z-1} ds \right) * \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{m t^{m-1}}{e^{\phi(n)t}}. \tag{4.47}$$

Observe that

$$e^{-\phi(k)t} \int_0^t e^{-[\phi(n) - \phi(k)]s} s^{z-1} ds = e^{-\phi(k)t} * t^{z-1} e^{-\phi(n)t}.$$

Then by (4.8) and (4.6),

$$\begin{aligned}
 (4.47) &= (e^{-\phi(k)t} * t^{z-1} e^{-\phi(n)t}) * \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{m t^{m-1}}{e^{\phi(n)t}} \\
 &\stackrel{(4.8)}{=} e^{-\phi(k)t} * \sum_{m=1}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{(t^{z-1} * m t^{m-1})}{e^{\phi(n)t}} \\
 &\stackrel{(4.6)}{=} e^{-\phi(k)t} * \sum_{m=1}^{\infty} \frac{1}{m!} (-[\phi(n+1) - \phi(n)])^m \frac{t^{z+m-1}}{e^{\phi(n)t}}.
 \end{aligned}$$

Using the fact that $e^w - 1 = \sum_{m=1}^{\infty} \frac{w^m}{m!}$, we can evaluate this convolution as follows

$$\begin{aligned}
 (4.47) &= e^{-\phi(k)t} * \frac{t^{z-1}}{e^{\phi(n)t}} \sum_{m=1}^{\infty} \frac{1}{m!} (-[\phi(n+1) - \phi(n)]t)^m \\
 &= e^{-\phi(k)t} * \frac{t^{z-1}}{e^{\phi(n)t}} (e^{-[\phi(n+1) - \phi(n)]t} - 1) \\
 &= e^{-\phi(k)t} * \left(\frac{t^{z-1}}{e^{\phi(n+1)t}} - \frac{t^{z-1}}{e^{\phi(n)t}} \right) \\
 &= e^{-\phi(k)t} \int_0^t e^{-[\phi(n+1) - \phi(k)]s} s^{z-1} ds - e^{-\phi(k)t} \int_0^t e^{-[\phi(n) - \phi(k)]s} s^{z-1} ds. \tag{4.48}
 \end{aligned}$$

Now, multiplying (4.47) by suitable constants as in (4.44), we can express the quantity $(1k) * (HG1)$ as

$$\begin{aligned} (1k) * (HG1) &= z \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n-1; j \neq k} [\phi(j) - \phi(k)]} e^{-\phi(k)t} \int_0^t e^{-[\phi(n+1)-\phi(k)]s} s^{z-1} ds \\ &\quad - z \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n-1; j \neq k} [\phi(j) - \phi(k)]} e^{-\phi(k)t} \int_0^t e^{-[\phi(n)-\phi(k)]s} s^{z-1} ds. \end{aligned} \quad (4.49)$$

Evaluating each $(1k) * (HG2)$ term To evaluate $(1k) * (HG2)$, we first consider the quantity

$$\left(e^{-\phi(k)t} \int_0^t e^{-[\phi(n)-\phi(k)]s} s^{z-1} ds \right) * \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{t^m}{e^{\phi(n)t}}. \quad (4.50)$$

Recall that $e^{-\phi(k)t} \int_0^t e^{-[\phi(n)-\phi(k)]s} s^{z-1} ds = e^{-\phi(k)t} * t^{z-1} e^{-\phi(n)t}$. Then by (4.8) and (4.7),

$$\begin{aligned} (4.50) &= \left(e^{-\phi(k)t} * t^{z-1} e^{-\phi(n)t} \right) * \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{t^m}{e^{\phi(n)t}} \\ &\stackrel{(4.8)}{=} e^{-\phi(k)t} * \sum_{m=0}^{\infty} \frac{(z)^{(m)}}{(m!)^2} (-[\phi(n+1) - \phi(n)])^m \frac{(t^{z-1} * t^m)}{e^{\phi(n)t}} \\ &\stackrel{(4.7)}{=} e^{-\phi(k)t} * \sum_{m=0}^{\infty} \frac{1}{m!} (-[\phi(n+1) - \phi(n)])^m \frac{\int_0^t v^{z+m-1} dv}{e^{\phi(n)t}}. \end{aligned}$$

Changing the order of summation and integration, since $e^w = \sum_{m=0}^{\infty} \frac{w^m}{m!}$, it follows that

$$\begin{aligned} (4.50) &= e^{-\phi(k)t} * \left(e^{-\phi(n)t} \int_0^t v^{z-1} \sum_{m=0}^{\infty} \frac{1}{m!} (-[\phi(n+1) - \phi(n)]v)^m dv \right) \\ &= e^{-\phi(k)t} * \left(e^{-\phi(n)t} \int_0^t v^{z-1} e^{-[\phi(n+1)-\phi(n)]v} dv \right) \\ &= \int_0^t e^{-\phi(k)(t-v)} \left(e^{-\phi(n)v} \int_0^v s^{z-1} e^{-[\phi(n+1)-\phi(n)]s} ds \right) dv. \end{aligned}$$

Exchanging the order of integration, then evaluating the inner integral,

$$\begin{aligned} (4.50) &= e^{-\phi(k)t} \int_0^t s^{z-1} e^{-[\phi(n+1)-\phi(n)]s} \int_s^t e^{-[\phi(n)-\phi(k)]v} dv ds \\ &= \frac{e^{-\phi(k)t}}{[\phi(n) - \phi(k)]} \int_0^t s^{z-1} e^{-[\phi(n+1)-\phi(n)]s} \left(e^{-[\phi(n)-\phi(k)]s} - e^{-[\phi(n)-\phi(k)]t} \right) ds \\ &= \frac{e^{-\phi(k)t}}{[\phi(n) - \phi(k)]} \int_0^t s^{z-1} e^{-[\phi(n+1)-\phi(k)]s} ds \\ &\quad - \frac{e^{-\phi(n)t}}{[\phi(n) - \phi(k)]} \int_0^t s^{z-1} e^{-[\phi(n+1)-\phi(n)]s} ds. \end{aligned}$$

Now, multiplying (4.50) by suitable constants as in (4.44), we can express the quantity $(1k) * (HG2)$ as

$$(1k) * (HG2) = z \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]} \frac{[\phi(z+n) - \phi(n)]}{e^{\phi(k)t}} \int_0^t s^{z-1} e^{-[\phi(n+1) - \phi(k)]s} ds \\ - z \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]} \frac{[\phi(z+n) - \phi(n)]}{e^{\phi(n)t}} \int_0^t s^{z-1} e^{-[\phi(n+1) - \phi(n)]s} ds. \quad (4.51)$$

Combining (4.45), (4.46), (4.49), and (4.51) gives an unwieldy formula for (4.44). Observe that $[(1A) + \sum_{k=1}^{n-1} (1k)] * \delta_0(dt) = (1A) + \sum_{k=1}^{n-1} (1k)$ cancels respectively with the second terms from (4.45) and (4.49). Then

$$(4.44) = \frac{zt^{z-1}}{e^{\phi(n+1)t}} + \frac{z[\phi(z+n) - \phi(n)]}{e^{\phi(n)t}} \int_0^t s^{z-1} e^{-[\phi(n+1) - \phi(n)]s} ds \quad (4.52)$$

$$+ z \sum_{k=1}^{n-1} \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n-1; j \neq k} [\phi(j) - \phi(k)]} e^{-\phi(k)t} \int_0^t e^{-[\phi(n+1) - \phi(k)]s} s^{z-1} ds \quad (4.53)$$

$$+ z \sum_{k=1}^{n-1} \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]} \frac{[\phi(z+n) - \phi(n)]}{e^{\phi(k)t}} \int_0^t s^{z-1} e^{-[\phi(n+1) - \phi(k)]s} ds \quad (4.54)$$

$$- z \sum_{k=1}^{n-1} \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]} \frac{[\phi(z+n) - \phi(n)]}{e^{\phi(n)t}} \int_0^t s^{z-1} e^{-[\phi(n+1) - \phi(n)]s} ds. \quad (4.55)$$

Multiplying (4.53) by $1 = \frac{\phi(n) - \phi(k)}{\phi(n) - \phi(k)}$, we can combine the terms (4.53) and (4.54), then recalling the definition $\gamma(z, u) := \int_0^u x^{z-1} e^{-x} dx$, we apply a simple change of variables to get

$$(4.53) + (4.54) = z \sum_{k=1}^{n-1} \frac{\prod_{1 \leq j \leq n} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]} e^{-\phi(k)t} \int_0^t s^{z-1} e^{-[\phi(n+1) - \phi(k)]s} ds \\ = z \sum_{k=1}^{n-1} \frac{\prod_{1 \leq j \leq n} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]} \frac{\gamma(z, [\phi(n+1) - \phi(k)]t)}{[\phi(n+1) - \phi(k)]^z e^{\phi(k)t}}.$$

Therefore we see that the terms in (4.53) + (4.54) correspond to the desired sum in the range between 1 and $n-1$ as in equation (4.15), and now all that remains for the proof of Lemma 4.2 is to verify that

$$(4.52) + (4.55) = \frac{zt^{z-1}}{e^{\phi(n+1)t}} + z \frac{\prod_{1 \leq j \leq n} [\phi(z+j) - \phi(n)]}{\prod_{1 \leq j \leq n; j \neq n} [\phi(j) - \phi(n)]} \frac{\gamma(z, [\phi(n+1) - \phi(n)]t)}{[\phi(n+1) - \phi(n)]^z e^{\phi(n)t}}. \quad (4.56)$$

The first term in (4.52) cancels with the first term to the right-hand side of (4.56). Therefore, to show identity (4.56) observe that integral which appears in (4.52) and (4.55) can be written as follows

$$\int_0^t s^{z-1} e^{-[\phi(n+1) - \phi(n)]s} ds = \frac{\gamma(z, [\phi(n+1) - \phi(n)]t)}{[\phi(n+1) - \phi(n)]^z},$$

from which upon cancellation of this expression on both sides of the remaining relation, it follows that (4.56) holds if and only if

$$\begin{aligned} & [\phi(z+n) - \phi(n)] - \sum_{k=1}^{n-1} \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]} [\phi(z+n) - \phi(n)] \\ &= \frac{\prod_{1 \leq j \leq n} [\phi(z+j) - \phi(n)]}{\prod_{1 \leq j \leq n; j \neq n} [\phi(j) - \phi(n)]}. \end{aligned}$$

Then rearranging and dividing through by $[\phi(z+n) - \phi(n)]$ the last identity, we see that (4.56) holds if and only if

$$\begin{aligned} 1 &= \sum_{k=1}^{n-1} \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]} + \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(n)]}{\prod_{1 \leq j \leq n-1} [\phi(j) - \phi(n)]} \\ &= \sum_{k=1}^n \frac{\prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)]}{\prod_{1 \leq j \leq n; j \neq k} [\phi(j) - \phi(k)]}. \end{aligned}$$

Multiplying both sides by $\prod_{1 \leq h < l \leq n} [\phi(l) - \phi(h)]$, one can verify that (4.56) holds if and only if

$$\begin{aligned} & \sum_{k=1}^n (-1)^k \left(\prod_{\substack{1 \leq h < l \leq n; \\ h, l \neq k}} [\phi(l) - \phi(h)] \right) \left(- \prod_{1 \leq j \leq n-1} [\phi(z+j) - \phi(k)] \right) \\ &= \prod_{1 \leq h < l \leq n} [\phi(l) - \phi(h)]. \end{aligned}$$

But this equation holds by Lemma 4.7, and so the proof of Lemma 4.2 is complete. $\square$

5. Auxiliary results on bivariate bernstein functions: Proof of Proposition 3.3

Proof of Proposition 3.3. Item 1 follows immediately by a simple rearrangement using (3.3), and item 2 follows from the fact that $\phi_\zeta \in \mathcal{B}$ for $\zeta \in [0, \infty)$, which comes from item 2 of Proposition 3.1. Now, let us prove item 3. Let $((\xi_t, \eta_t))_{t \geq 0}$ denote the (possibly killed) bivariate subordinator associated to κ . Then for an independent exponential random variable e_q with parameter $q > 0$, we have that

$$\left| \frac{q}{q + \kappa(\zeta, z)} \right| = |\mathbb{E}[e^{-\zeta \xi_{eq} - z \eta_{eq}}]| \leq \mathbb{E}[e^{-\operatorname{Re}(\zeta) \xi_{eq} - \operatorname{Re}(z) \eta_{eq}}] = \left| \frac{q}{q + \kappa(\operatorname{Re}(\zeta), \operatorname{Re}(z))} \right|,$$

so that

$$\left| \frac{1}{q + \kappa(\zeta, z)} \right| \leq \left| \frac{1}{q + \kappa(\operatorname{Re}(\zeta), \operatorname{Re}(z))} \right|,$$

for all $q > 0$, from which it follows, taking limits as $q \rightarrow 0$, that (3.5) holds, and similarly (3.5) is verified, for $\operatorname{Re}(\zeta) = \operatorname{Re}(z) = 0$, if $\kappa(0, 0) > 0$. Let us prove item 4. We estimate the last term in (3.4) to get that

$$\begin{aligned} \left| z \int_0^\infty e^{-zy} y \bar{\mu}_\zeta(y) dy \right| &\leq |z| \int_0^\infty e^{-\operatorname{Re}(z)y} y \left| \int_y^\infty \int_0^\infty e^{-\zeta x_1} \mu(dx_1, dx_2) \right| dx_2 \\ &\leq |z| \int_0^\infty e^{-\operatorname{Re}(z)y} y \bar{\mu}_{\operatorname{Re}(\zeta)}(y) dy \leq |z| \frac{\phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))}{\operatorname{Re}^2(z)}, \end{aligned} \quad (5.1)$$

where the last inequality follows from the fact that $\phi_{\operatorname{Re}(\zeta)} \in \mathcal{B}$, $\phi'_{\operatorname{Re}(\zeta)} \geq 0$ on $[0, \infty)$ and the left-hand side of inequality (3.2). Relation (5.1) together with (3.4) show the very first inequality in (3.6) whereas the second follows from an application of (3.5) and the third from the monotonicity of $\phi_{\operatorname{Re}(\zeta)}$ on $[0, \infty)$. We proceed to establish relation (3.7). For this purpose we note from (3.4) that

$$\begin{aligned} |\phi''_{\zeta}(z)| &\leq 2 \left| \int_0^{\infty} e^{-zy} y \bar{\mu}_{\zeta}(y) dy \right| + \left| z \int_0^{\infty} e^{-zy} y^2 \bar{\mu}_{\zeta}(y) dy \right| \\ &\leq 2 \int_0^{\infty} e^{-\operatorname{Re}(z)y} y \bar{\mu}_{\operatorname{Re}(\zeta)}(y) dy + |z| \int_0^{\infty} e^{-\operatorname{Re}(z)y} y^2 \bar{\mu}_{\operatorname{Re}(\zeta)}(y) dy \\ &\leq 2 \frac{\phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))}{\operatorname{Re}^2(z)} + |z| \int_0^{\infty} e^{-\operatorname{Re}(z)y} y^2 \bar{\mu}_{\operatorname{Re}(\zeta)}(y) dy, \end{aligned}$$

where for first term in the last inequality we have invoked (5.1). It remains to bound the very last integral. However, if $\phi \in \mathcal{B}$ then ϕ' is completely monotone, see item 2 of Proposition 3.1, and hence $\phi'' \leq 0$ on $[0, \infty)$. Differentiating the last expression of (3.1) and utilising once again (5.1) we thus arrive at

$$u \int_0^{\infty} e^{-uy} y^2 \bar{\mu}(y) dy \leq -\phi''(u) + u \int_0^{\infty} e^{-uy} y^2 \bar{\mu}(y) dy = 2 \int_0^{\infty} e^{-uy} y \bar{\mu}(y) dy \leq 2 \frac{\phi(u)}{u^2}$$

and hence

$$|z| \int_0^{\infty} e^{-\operatorname{Re}(z)y} y^2 \bar{\mu}_{\operatorname{Re}(\zeta)}(y) dy \leq \frac{2|z|}{\operatorname{Re}^3(z)} \phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z)).$$

Therefore, collecting the terms we obtain that

$$|\phi''_{\zeta}(z)| \leq 2 \frac{\phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))}{\operatorname{Re}^2(z)} + \frac{2|z|}{\operatorname{Re}^3(z)} \phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))$$

and employing (3.5) we get that

$$\left| \frac{\phi''_{\zeta}(z)}{\phi_{\zeta}(z)} \right| \leq 2 \frac{\phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))}{\operatorname{Re}^2(z) |\phi_{\zeta}(z)|} + \frac{2|z|}{\operatorname{Re}^3(z)} \frac{\phi_{\operatorname{Re}(\zeta)}(\operatorname{Re}(z))}{|\phi_{\zeta}(z)|} \leq \frac{2}{\operatorname{Re}^2(z)} + \frac{2|z|}{\operatorname{Re}^3(z)}.$$

This ends the proof of (3.7). Relation (3.9) follows by a simple substitution $z = u$ in (3.7), whereas (3.8) is derived in the following manner. The first inequality is deduced by not splitting the term $\frac{\phi_{\zeta}(z) - \phi_{\zeta}(0)}{z}$ in the last identity of (3.4) and substituting in the first inequality of (3.6) with $z = \operatorname{Re}(z) = u$. To obtain the second inequality we observe from (3.5) that for any $(\zeta, u) \in \mathbb{C}_{[0, \infty)} \times (0, \infty)$

$$\left| \frac{\kappa'_{\zeta}(\zeta, u)}{\kappa_{\zeta}(\zeta, u)} \right| \leq \frac{|\phi_{\zeta}(u) - \phi_{\zeta}(0)|}{u |\phi_{\zeta}(u)|} + \frac{1}{u}. \quad (5.2)$$

Next, we note from (3.3) that

$$\begin{aligned} |\phi_{\zeta}(u) - \phi_{\zeta}(0)| &= |\kappa(\zeta, u) - \kappa(\zeta, 0)| \\ &\leq d_2 u + \int_0^{\infty} (1 - e^{-ux}) \mu_{\operatorname{Re}(\zeta)}(dx) \leq \phi_{\operatorname{Re}(\zeta)}(u). \end{aligned}$$

Therefore, (5.2) is further estimated from as (3.5)

$$\left| \frac{\kappa'_{\zeta}(\zeta, u)}{\kappa_{\zeta}(\zeta, u)} \right| \leq \frac{\phi_{\operatorname{Re}(\zeta)}(u)}{u |\phi_{\zeta}(u)|} + \frac{1}{u} \leq \frac{2}{u},$$

which proves (3.8). We proceed with item 5. Let $(\xi, \eta) := ((\xi_t, \eta_t))_{t \geq 0}$ be the possibly killed bivariate subordinator associated to κ . Then, for all $(\zeta, z) \in \mathbb{C}_{[0, \infty)}^2 \setminus \{(0, 0)\}$,

$$1 > \mathbb{E} \left[e^{-\operatorname{Re}(\zeta) \xi_1 - \operatorname{Re}(z) \eta_1} \right] \geq \left| \mathbb{E} \left[e^{-\zeta \xi_1 - z \eta_1} \right] \right| = e^{-\operatorname{Re}(\kappa(\zeta, z))}$$

and (3.10) follows. The last claim follows from the inequality above, the fact that $\inf_{\chi \in \overline{B_\zeta}} \operatorname{Re}(\zeta) > 0$ since the closed ball $\overline{B_\zeta} \subset \mathbb{C}_{(0, \infty)}$ and $\mathbb{P}(\xi_1 > 0) > 0$. Next we prove item 6. Differentiating (3.3) $n \geq 1$ times with respect to z , then taking absolute values, it follows that

$$\begin{aligned} |\kappa_z^{(n)}(\zeta, z)| &= \left| d_2 \mathbb{I}_{\{n=1\}} + (-1)^{n-1} \int_0^\infty y^n e^{-yz} \mu_\zeta(dy) \right| \\ &\leq d_2 \mathbb{I}_{\{n=1\}} + \int_0^\infty y^n e^{-\operatorname{Re}(z)y} \mu_{\operatorname{Re}(\zeta)}(dy) \\ &= \left| \phi_{\operatorname{Re}(\zeta)}^{(n)}(\operatorname{Re}(z)) \right| = \left| \kappa_z^{(n)}(\operatorname{Re}(\zeta), \operatorname{Re}(z)) \right|, \end{aligned}$$

which establishes (3.11) and item 6. For item 7, we apply first (3.11) with $n = 1, 2$, and then (3.5) to yield

$$\left| \frac{\kappa^{(n)}(\zeta, z)}{\kappa(\zeta, z)} \right| \leq \left| \frac{\kappa^{(n)}(\operatorname{Re}(\zeta), \operatorname{Re}(z))}{\kappa(\operatorname{Re}(\zeta), \operatorname{Re}(z))} \right| \left| \frac{\kappa(\operatorname{Re}(\zeta), \operatorname{Re}(z))}{\kappa(\zeta, z)} \right| \leq \left| \frac{\kappa^{(n)}(\operatorname{Re}(\zeta), \operatorname{Re}(z))}{\kappa(\operatorname{Re}(\zeta), \operatorname{Re}(z))} \right|. \quad (5.3)$$

Applying (3.8) when $n = 1$ in (5.3), we deduce that (3.12) holds. Similarly, applying (3.9) when $n = 2$ in (5.3), it follows that (3.13) holds, as required for item 7, and the proof is complete. It remains to consider item 8. We observe that

$$\begin{aligned} \lim_{x \rightarrow \infty} \sup_{0 \leq v \leq A} \left| \frac{\kappa(\zeta, x+v)}{\kappa(\zeta, x)} \right| &= \lim_{x \rightarrow \infty} \sup_{0 \leq v \leq A} \left| 1 + \frac{\kappa(\zeta, x+v) - \kappa(\zeta, x)}{\kappa(\zeta, x)} \right| \\ &= \lim_{x \rightarrow \infty} \sup_{0 \leq v \leq A} \left| 1 + \frac{\int_x^{x+v} \kappa'_z(\zeta, w) dw}{\kappa(\zeta, x)} \right|. \end{aligned}$$

However,

$$\begin{aligned} \lim_{x \rightarrow \infty} \sup_{0 \leq v \leq A} \left| \frac{\int_x^{x+v} \kappa'_z(\zeta, w) dw}{\kappa(\zeta, x)} \right| &\leq A \lim_{x \rightarrow \infty} \sup_{x \leq v \leq x+A} \frac{|\kappa'_z(\zeta, v)|}{|\kappa(\zeta, x)|} \\ &\leq A \lim_{x \rightarrow \infty} \sup_{x \leq v \leq x+A} \frac{\kappa'_z(\operatorname{Re}(\zeta), v)}{|\kappa(\zeta, x)|} \\ &= A \lim_{x \rightarrow \infty} \frac{\kappa'_z(\operatorname{Re}(\zeta), x)}{|\kappa(\zeta, x)|} \\ &\leq A \lim_{x \rightarrow \infty} \frac{\kappa'_z(\operatorname{Re}(\zeta), x)}{|\kappa(\operatorname{Re}(\zeta), x)|} = 0, \end{aligned}$$

where in the second inequality we have used (3.11) with $n = 1$, in the first identity we have used that $\kappa'_z(\operatorname{Re}(\zeta), \cdot)$ is non-increasing, see item 2 of Proposition 3.1, for the last inequality we have invoked (3.5) and the evaluation of the last limit follows from (3.8). $\square$

6. Proofs of auxiliary lemmas

Proof of Lemma 2.10. Consider the fact, see [36, Prop 3.6], that each Bernstein function $\phi \in \mathcal{B}$ preserves angular sectors, i.e. for each $w \in \mathbb{C}_{[0,\infty)}$, $|\arg(\phi(w))| \leq |\arg(w)|$, where we use the convention that $\arg(w) \in (-\pi, \pi]$. Then, write $z = a + ib$ and without loss of generality assume that $a \in [c, d]$, $b \in [c_1, d_1]$, where $-\infty < c < d < \infty$ and $-\infty < c_1 < d_1 < \infty$. It follows that for all u large enough that $u > -a$,

$$\left| \arg \left(\frac{\phi(u+z)}{\phi(u)} \right) \right| = |\arg(\phi(u+z))| \leq |\arg(u+z)| = \left| \arctan \left(\frac{b}{u+a} \right) \right|.$$

Observe that $\lim_{u \rightarrow \infty} |\arctan(b/(u+a))| = 0$, uniformly for the specified compact range of a, b . Therefore, for each $\phi \in \mathcal{B}$, as $u \rightarrow \infty$, and uniformly for $c < \operatorname{Re}(z) < d$

$$\frac{\phi(u+z)}{\phi(u)} \sim \frac{|\phi(u+z)|}{\phi(u)} \sim \frac{\operatorname{Re}(\phi(u+z))}{\phi(u)}, \quad (6.1)$$

where the very last asymptotic relation follows from the fact that

$$0 = \lim_{u \rightarrow \infty} |\arg(\phi(u+z))| = \lim_{u \rightarrow \infty} \left| \arctan \left(\frac{\operatorname{Im}(\phi(u+z))}{\operatorname{Re}(\phi(u+z))} \right) \right| = \lim_{u \rightarrow \infty} \left| \frac{\operatorname{Im}(\phi(u+z))}{\operatorname{Re}(\phi(u+z))} \right|.$$

Writing ϕ as in (2.2), since b is bounded, we have $|u+z| \sim u+a$, as $u \rightarrow \infty$, and by (2.18), as $u \rightarrow \infty$,

$$\begin{aligned} \frac{|\phi(u+z)|}{\phi(u)} &\stackrel{(2.2)}{=} \frac{|\phi(0) + (u+z)d + (u+z) \int_0^\infty e^{-(u+z)y} \overline{\Pi}(y) dy|}{\phi(0) + ud + u \int_0^\infty e^{-uy} \overline{\Pi}(y) dy} \\ &\leq \frac{\phi(0) + |u+z|d + |u+z| \int_0^\infty |e^{-(u+z)y}| \overline{\Pi}(y) dy}{\phi(0) + ud + u \int_0^\infty e^{-uy} \overline{\Pi}(y) dy} \\ &= \frac{\phi(0) + |u+z|d + |u+z| \int_0^\infty e^{-(u+a)y} \overline{\Pi}(y) dy}{\phi(0) + ud + u \int_0^\infty e^{-uy} \overline{\Pi}(y) dy} \\ &\sim \frac{\phi(0) + (u+a)d + (u+a) \int_0^\infty e^{-(u+a)y} \overline{\Pi}(y) dy}{\phi(0) + ud + u \int_0^\infty e^{-uy} \overline{\Pi}(y) dy} = \frac{\phi(u+a)}{\phi(u)} \stackrel{(2.18)}{\sim} 1. \end{aligned} \quad (6.2)$$

Now, observe that $-\cos(-by) \geq -1$ for all $b, y \in \mathbb{R}$. Then by (2.18), as $u \rightarrow \infty$,

$$\begin{aligned} \frac{\operatorname{Re}(\phi(u+z))}{\phi(u)} &= \frac{\phi(0) + \operatorname{Re}((u+z)d) + \int_0^\infty \operatorname{Re}(1 - e^{-(u+z)y}) \Pi(dy)}{\phi(0) + ud + \int_0^\infty (1 - e^{-uy}) \Pi(dy)} \\ &= \frac{\phi(0) + (u+a)d + \int_0^\infty (1 - e^{-(u+a)y} \cos(-by)) \Pi(dy)}{\phi(0) + ud + \int_0^\infty (1 - e^{-uy}) \Pi(dy)} \\ &\geq \frac{\phi(0) + (u+a)d + \int_0^\infty (1 - e^{-(u+a)y}) \Pi(dy)}{\phi(0) + ud + \int_0^\infty (1 - e^{-uy}) \Pi(dy)} = \frac{\phi(u+a)}{\phi(u)} \stackrel{(2.18)}{\sim} 1. \end{aligned} \quad (6.3)$$

Combining (6.1), (6.2), and (6.3), the proof of Lemma 2.10 is henceforth complete. $\square$

Proof of Lemma 2.11. Write $z = a + ib$ and without loss of generality assume that $a \in [c, d]$, $b \in [c_1, d_1]$, where $0 \leq c < d < \infty$ and $-\infty < c_1 < d_1 < \infty$. For $u > -a$ (so that $W_\phi(u+z+1)$ is well-defined), applying Lemma 2.10 alongside (2.15) and (2.17) from

Theorem 2.9, it follows that as $u \rightarrow \infty$,

$$\begin{aligned} \frac{W_\phi(u+1)\phi^z(u+1)}{W_\phi(u+z+1)} &\stackrel{(2.15)}{=} \phi^z(u+1) \frac{\phi(u+1+z)\phi^{\frac{1}{2}}(u+2+z)}{\phi(u+1)\phi^{\frac{1}{2}}(u+1)} \frac{e^{L_\phi(u+1)-E_\phi(u+1)}}{e^{L_\phi(u+1+z)-E_\phi(u+1+z)}} \\ &\stackrel{(2.19)}{\sim} \phi^z(u+1) e^{L_\phi(u+1)-E_\phi(u+1)-L_\phi(u+1+z)+E_\phi(u+1+z)} \\ &\stackrel{(2.17)}{\sim} \phi^z(u+1) e^{L_\phi(u+1)-L_\phi(u+1+z)}. \end{aligned} \quad (6.4)$$

Now, recalling the definition of the L_ϕ terms in (2.12), part of the integrals cancel, yielding

$$\begin{aligned} L_\phi(u+1) - L_\phi(u+1+z) &= \int_{1 \mapsto u+2} \log_0(\phi(w))dw - \int_{1 \mapsto u+2+z} \log_0(\phi(w))dw \\ &= - \int_{u+2 \mapsto u+2+z} \log_0(\phi(w))dw. \end{aligned} \quad (6.5)$$

Writing $N = u+2$ for brevity, and noting that $\log_0(\phi(w)) = \ln(|\phi(w)|) + i \arg(\phi(w))$, we have the following identity

$$(6.5) = - \int_{N \mapsto N+z} \ln(|\phi(w)|)dw - i \int_{N \mapsto N+z} \arg(\phi(w))dw. \quad (6.6)$$

We will integrate along the contours γ_1 and γ_2 , which are straight lines connecting N to $N+a$, and $N+a$ to $N+a+ib = N+z$, respectively. Note that along γ_1 , $\arg(\phi(w)) = \arg(w) = 0$, so we need only consider γ_2 for the second integral in (6.6). Now, $|\gamma_2| = b$, so

$$\left| \int_{\gamma_2} \arg(\phi(w))dw \right| \leq b \sup_{x \in \gamma_2} |\arg(\phi(w))| \leq b \sup_{w \in \gamma_2} |\arg(w)|.$$

since ϕ preserves angular sectors [36, Prop 3.6]. Now, $\sup_{w \in \gamma_2} |\arg(w)| = \arctan(b/(N+a))$, which converges to 0 as $N = u+2 \rightarrow \infty$ and b belongs to a compact interval, and we therefore conclude that

$$\lim_{n \rightarrow \infty} \int_{N \mapsto N+a+ib} \arg(\phi(w))dw = 0. \quad (6.7)$$

Now let us consider the first integral in (6.6) over γ_1 and γ_2 separately. We will compare the γ_1 contribution with $a \ln(\phi(u+1))$, and the γ_2 part with $ib \ln(\phi(u+1))$. For the γ_1 part,

$$\begin{aligned} \int_{\gamma_1} \ln(|\phi(w)|)dw - a \ln(\phi(u+1)) &= \int_0^a \ln(\phi(u+2+x))dx - a \ln(\phi(u+1)) \\ &= \int_0^a (\ln(\phi(u+2+x)) - \ln(\phi(u+1)))dx = \int_0^a \ln\left(\frac{\phi(u+2+x)}{\phi(u+1)}\right)dx. \end{aligned} \quad (6.8)$$

By Lemma 2.10, $\lim_{n \rightarrow \infty} \phi(u+2+x)/\phi(u+1) = 1$, uniformly in $x \in [0, a]$, so $\lim_{u \rightarrow \infty} (6.8) = 0$, and therefore

$$\lim_{u \rightarrow \infty} \left(\phi^a(u+1) e^{-\int_{\gamma_1} \ln(|\phi(x)|)dx} \right) = 1. \quad (6.9)$$

For the contour γ_2 , one can verify by applying the same argument as for γ_1 , that

$$\lim_{n \rightarrow \infty} \left(\phi^{ib}(u+1) e^{i \int_{\gamma_2} \ln(|\phi(w)|)dw} \right) = \lim_{u \rightarrow \infty} \left(e^{i \int_0^b \ln\left(\frac{\phi(u+2+a+iv)}{\phi(u+1)}\right)dv} \right) = 1. \quad (6.10)$$

Finally, substituting the individual limits (6.7), (6.9), and (6.10) into (6.4), we conclude that uniformly on z -compact domains of $\mathbb{C}_{[0,\infty)}$

$$\lim_{u \rightarrow \infty} \frac{W_\phi(u+1)\phi^z(u+1)}{W_\phi(u+z+1)} = 1 \quad \square$$

and this concludes the lemma.

Proof of Lemma 4.1. Firstly, by $\varphi_q(w) = q + \phi(w) \in \mathcal{B}$ and Lemma 2.11, we can rewrite

$$W_{\varphi_q}(z+1) = 1 \times W_{\varphi_q}(z+1) = \lim_{n \rightarrow \infty} \frac{W_{\varphi_q}(n+1)\varphi_q^z(n+1)W_{\varphi_q}(z+1)}{W_{\varphi_q}(n+z+1)}. \quad (6.11)$$

Recalling the relations $W_{\varphi_q}(w+1) = \varphi_q(w)W_{\varphi_q}(w)$ and $W_{\varphi_q}(1) = 1$ from (2.4), it follows that

$$\begin{aligned} (6.11) &= \lim_{n \rightarrow \infty} \frac{(\prod_{k=1}^n \varphi_q(k)) W_{\varphi_q}(1)\varphi_q^z(n+1)W_{\varphi_q}(z+1)}{(\prod_{k=1}^n \varphi_q(z+k)) W_{\varphi_q}(z+1)} \\ &= \lim_{n \rightarrow \infty} \left(\varphi_q^z(n+1) \prod_{k=1}^n \frac{\varphi_q(k)}{\varphi_q(z+k)} \right). \end{aligned}$$

Now, observing that we can rewrite $\varphi_q(n+1)$ in the form

$$\varphi_q(n+1) = \varphi_q(1) \prod_{k=1}^n \left(\frac{\varphi_q(k+1)}{\varphi_q(k)} \right) = \varphi_q(1) \prod_{k=1}^n \left(1 + \frac{\varphi_q(k+1) - \varphi_q(k)}{\varphi_q(k)} \right),$$

we conclude, as required for Lemma 4.1, that

$$W_{\varphi_q}(z+1) = \varphi_q^z(1) \prod_{k=1}^{\infty} \left[\frac{\varphi_q(k)}{\varphi_q(z+k)} \left(1 + \frac{\varphi_q(k+1) - \varphi_q(k)}{\varphi_q(k)} \right)^z \right]. \quad \square$$

Proof of Lemma 4.3. Fix $y \geq 1$ such that $\int_0^y x \Pi(dx) > 0$. Applying simple inequalities, we get

$$\begin{aligned} \frac{\phi'(n)}{\phi'(n+c)} &= \frac{d + \int_0^\infty x e^{-nx} \Pi(dx)}{d + \int_0^\infty x e^{-(n+c)x} \Pi(dx)} \leq \frac{d}{d} + \frac{\int_0^y x e^{-nx} \Pi(dx)}{\int_0^y x e^{-(n+c)x} \Pi(dx)} + \frac{\int_y^\infty x e^{-nx} \Pi(dx)}{\int_0^y x e^{-(n+c)x} \Pi(dx)} \\ &\leq 1 + e^{cy} + \frac{\int_y^\infty n x e^{-nx} \Pi(dx)}{n \int_0^y x e^{-(n+c)x} \Pi(dx)}. \end{aligned}$$

Observe that when $nx \geq y \geq 1$, the quantity $n x e^{-nx}$ is decreasing in x , and thus

$$\begin{aligned} \frac{\phi'(n)}{\phi'(n+c)} &\leq 1 + e^{cy} + \frac{n y e^{-ny} \int_y^\infty \Pi(dx)}{n \int_0^y x e^{-(n+c)x} \Pi(dx)} \\ &\leq 1 + e^{cy} + \frac{y e^{-ny} \int_y^\infty \Pi(dx)}{e^{-(n+c)y} \int_0^y x \Pi(dx)} \\ &= 1 + e^{cy} + \frac{y \int_y^\infty \Pi(dx)}{e^{-cy} \int_0^y x \Pi(dx)}, \end{aligned}$$

which is a finite constant, independent of n , as required. $\square$

Proof of Lemma 4.4. To show that the sum in (4.16) is absolutely convergent, we consider terms separately. First, consider the $1/W_{\phi(k)}(z)$ term. Applying (2.15) and (2.16) from Theorem 2.9,

$$\left| \frac{1}{W_{\phi(k)}(z+1)} \right| \leq \frac{|\phi(k)(z+1)\phi_{(k)}^{\frac{1}{2}}(z+2)|}{\phi_{(k)}(1)} \left| e^{-L_{\phi(k)}(z+1)} \right| e^2. \quad (6.12)$$

First we bound $|e^{-L_{\phi(k)}(z+1)}|$. Recalling $\log_0(w) = \ln(|w|) + i \arg(w)$, by (2.12), with $z = a + ib$,

$$\begin{aligned} L_{\phi(k)}(z+1) &= \int_{1 \rightarrow 2+z} \log_0(\phi_{(k)}(w)) dw \\ &= \int_{1 \rightarrow 2+a} \ln(|\phi_{(k)}(w)|) dw + \int_{2+a \rightarrow 2+a+ib} \ln(|\phi_{(k)}(w)|) dw \\ &\quad + i \int_{1 \rightarrow 2+a} \arg(\phi_{(k)}(w)) dw + i \int_{2+a \rightarrow 2+a+ib} \arg(\phi_{(k)}(w)) dw. \end{aligned}$$

But $\int_{2+a \rightarrow 2+a+ib} \ln(|\phi_{(k)}(w)|) dw$ is pure imaginary, and $\int_{1 \rightarrow 2+a} \arg(\phi_{(k)}(w)) dw = 0$, so

$$\left| e^{-L_{\phi(k)}(z+1)} \right| = \left| e^{-\int_1^{2+a} \ln(\phi_{(k)}(x)) dx - i \int_{2+a \rightarrow 2+a+ib} \arg(\phi_{(k)}(w)) dw} \right|.$$

As $\phi_{(k)}$ preserves angular sectors [36, Prop 3.6], it holds true that $\left| \int_{2+a \rightarrow 2+a+ib} \arg(\phi_{(k)}(w)) dw \right| \leq b\pi/2$, and so

$$\left| e^{-L_{\phi(k)}(z+1)} \right| \leq e^{\frac{b\pi}{2}} \left| e^{-\int_1^{2+a} \ln(\phi_{(k)}(x)) dx} \right| = e^{\frac{b\pi}{2}} e^{-\int_1^{2+a} \ln(\phi_{(k)}(x)) dx}.$$

Now, $\phi_{(k)}$ is non-decreasing, and $\phi'_{(k)}$ is non-increasing, so applying Lemma 4.3, we get

$$\begin{aligned} \frac{e^{-\int_1^{2+a} \ln(\phi_{(k)}(x)) dx}}{\phi_{(k)}(1)} &\leq \frac{e^{-(1+a) \ln(\phi_{(k)}(1))}}{\phi_{(k)}(1)} = \phi_{(k)}^{-(2+a)}(1) = [\phi(k+1) - \phi(k)]^{-(2+a)} \\ &= \left[\int_k^{k+1} \phi'(x) dx \right]^{-(2+a)} \leq \phi'(k+1)^{-(2+a)} \\ &\stackrel{4.3}{\leq} C_{\phi}^{2+a} \phi'(k)^{-(2+a)}, \end{aligned}$$

where C_{ϕ} depends only on ϕ . Hence there is a constant $K > 0$ so that for all $k \geq 1$,

$$\left| \frac{e^{-L_{\phi(k)}(z+1)}}{\phi_{(k)}(1)} \right| \leq K \phi'(k)^{-(2+a)},$$

from which it follows that, with a slightly different constant K_1 , for all $k \geq 1$,

$$(6.12) \leq K' \left| \phi_{(k)}(z+1) \phi_{(k)}^{\frac{1}{2}}(z+2) \right| \phi'(k)^{-(2+a)}.$$

Now, since (3.11) of Proposition 3.3.6 implies that $|\phi'(\zeta)| \leq \phi'(\operatorname{Re}(\zeta))$, $\zeta \in \mathbb{C}_{(0,\infty)}$ and ϕ' is non-increasing on $(0, \infty)$, observe that

$$|\phi_{(k)}(z+1)| = |\phi(z+1+k) - \phi(k)| = \left| \int_k^{z+1+k} \phi'(w) dw \right| \leq |z+1| \phi'(k),$$

and similarly $|\phi_{(k)}^{\frac{1}{2}}(z+2)| \leq |z+2|^{\frac{1}{2}} \phi'(k)^{\frac{1}{2}}$. Then, substituting above, we have that

$$(6.12) \leq K_1 |z+1| |z+2|^{\frac{1}{2}} \phi'(k)^{-\frac{1}{2}-a},$$

and plugging this into (4.16), it follows for another constant $K_2 > 0$, depending on z , that

$$\sum_{k=1}^{\infty} \left| \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \frac{e^{-\phi(k)t}}{W_{\phi(k)}(z+1)} \right| \leq K_2 \sum_{k=1}^{\infty} \left| \frac{\prod_{i=1}^k [\phi(z+i) - \phi(k)]}{\prod_{j=1}^{k-1} [\phi(j) - \phi(k)]} \right| \frac{e^{-\phi(k)t}}{\phi'(k)^{\frac{1}{2}+a}}. \quad (6.13)$$

Now, applying the bounds in Lemma 4.5, we see that for another constants K_3, K_4 not depending on k ,

$$(6.13) \leq K_3 \sum_{k=1}^{\infty} e^{-\phi(k)t} |\phi(z+k) - \phi(k)| \phi'(k)^{-\frac{1}{2}-a} \leq K_4 \sum_{k=1}^{\infty} e^{-\phi(k)t} \phi'(k)^{\frac{1}{2}-a}. \quad (6.14)$$

Now, recall that in Definition 2.13, we have imposed that $\beta(\phi') > -1$, where $\beta(\cdot)$ denotes the lower Matuszewska index, see [10, p68] for the formal definition. One can verify that if $\beta(\phi') > -1$, then the function $f(x) := x\phi'(x)$ has lower Matuszewska index $\beta(f) > 0$. Then applying [10, Prop 2.6.1(b)] to the function f , it follows that there exists a constant $c_1 > 0$ such that for all $k \geq 1$,

$$k\phi'(k) = f(k) \geq c_1 \int_0^k \frac{f(x)}{x} dx = c \int_0^k \phi'(x) dx = c_1(\phi(k) - \phi(0)). \quad (6.15)$$

Now, consider the lower index of the function f , defined by $\underline{\text{ind}}(f) := \liminf_{x \rightarrow \infty} \ln(f(x))/\ln(x)$, or equivalently, $\underline{\text{ind}}(f) := \sup\{\rho > 0 : \lim_{x \rightarrow \infty} x^{-\rho} f(x) = \infty\}$, see e.g. [5, p39]. From the result [10, Prop 2.2.5], $\underline{\text{ind}}(f) \geq \beta(f)$, and hence $\underline{\text{ind}}(f) \geq \beta(f) > 0$. Moreover, by (3.2), one can deduce that $\underline{\text{ind}}(\phi) \geq \underline{\text{ind}}(f) > 0$, and in particular, there exist $c, \rho > 0$ such that for all $k \geq 1$,

$$\phi(k) \geq k\phi'(k) \geq ck^{\rho}. \quad (6.16)$$

Now, if $a > 1/2$, then since ϕ is non-decreasing, by (6.15) and (6.16), for another constant K_5 ,

$$(6.14) \leq K_5 \sum_{k=1}^{\infty} e^{-ctk^{\rho}} \frac{c_1^{a-\frac{1}{2}} [\phi(k) - \phi(0)]^{\frac{1}{2}-a}}{k^{\frac{1}{2}-a}} \leq K_5 c_1^{a-\frac{1}{2}} [\phi(1) - \phi(0)]^{\frac{1}{2}-a} \sum_{k=1}^{\infty} e^{-ctk^{\rho}} k^{a-\frac{1}{2}},$$

and then one can easily verify that this sum is finite, as required. On the other hand, if $a \leq 1/2$, then since ϕ' is non-increasing,

$$(6.14) \leq K_5 \sum_{k=1}^{\infty} e^{-ctk^{\rho}} \phi'(k)^{\frac{1}{2}-a} \leq K_5 \phi'(1)^{\frac{1}{2}-a} \sum_{k=1}^{\infty} e^{-ctk^{\rho}},$$

and again one can easily verify that this sum is finite, and the proof of Lemma 4.4 is therefore complete. $\square$

Proof of Lemma 4.5. Writing $z = a + ib$, $a, b \in \mathbb{R}$, we first consider the product for $1 \leq j \leq k - C_z$, where $C_z > a$ is to be determined later. We are going to change variables in

the numerator so that the new variable ranges between 0 and $k - j$. The change of variables is motivated by the observation that

$$\phi(k) - \phi(j) = \int_j^k \phi'(x) dx = \int_0^{k-j} \phi'(j + \tau) d\tau,$$

so that after this change of variables, we can simply compare terms within each integral.

Consider first $u \in [0, 1]$ chosen so that $w = (z + j)(1 - u) + ku$. Then $(z + j)(1 - u) + ku$ ranges between $z + j$ and k , as desired. Now, τ is simply defined as $\tau := (k - j)u$, so that the ranges of integration match.

Now, changing variables to τ chosen such that $w = z + j + (1 - z/(k - j))\tau$, we can rewrite $|\phi(k) - \phi(z + j)|$ as

$$\begin{aligned} \left| \int_{z+j \mapsto k} \phi'(w) dw \right| &= \left| \int_0^{k-j} \left(1 - \frac{z}{k-j} \right) \phi' \left(z + j + \left(1 - \frac{z}{k-j} \right) \tau \right) d\tau \right| \\ &\leq \int_0^{k-j} \left| 1 - \frac{z}{k-j} \right| \left| \phi' \left(z + j + \left(1 - \frac{z}{k-j} \right) \tau \right) \right| d\tau \\ &\leq \int_0^{k-j} \left| 1 - \frac{z}{k-j} \right| \phi' \left(a + j + \left(1 - \frac{a}{k-j} \right) \tau \right) d\tau, \end{aligned}$$

where we have used for the last inequality relation (3.11) of Proposition 3.3.6. We can now compare terms. Since $\tau \leq k - j$, observe that $a - a\tau/(k - j) \geq 0$, and hence $a + j + (1 - a/(k - j))\tau \geq j + \tau$, from which it follows, since ϕ' is non-increasing on $(0, \infty)$, that

$$\phi' \left(a + j + \left(1 - \frac{a}{k-j} \right) \tau \right) \leq \phi'(j + \tau).$$

Now we want to show that $|1 - z/(k - j)| \leq 1$. We consider its square for convenience:

$$\left| 1 - \frac{z}{k-j} \right|^2 = \frac{(k-j-a)^2}{(k-j)^2} + \frac{b^2}{(k-j)^2} = 1 - \frac{2a(k-j)}{(k-j)^2} + \frac{a^2 + b^2}{(k-j)^2}.$$

This does not exceed 1 if and only if

$$\begin{aligned} -\frac{2a(k-j)}{(k-j)^2} + \frac{a^2 + b^2}{(k-j)^2} &\leq 0 \\ \iff a^2 + b^2 &\leq 2a(k-j) \iff \frac{a^2 + b^2}{2a} \leq k-j \\ \iff j &\leq k - \frac{a^2 + b^2}{2a} = k - \frac{|z|^2}{2\operatorname{Re}(z)}. \end{aligned}$$

So let us choose $C_z \geq \frac{|z|^2}{2\operatorname{Re}(z)}$. Then it follows that for all $1 \leq j \leq k - C_z$,

$$\frac{|\phi(z + j) - \phi(k)|}{\phi(k) - \phi(j)} \leq 1,$$

and hence the product simplifies substantially:

$$\begin{aligned}
 \prod_{1 \leq j \leq k-1} \frac{|\phi(z+j) - \phi(k)|}{\phi(k) - \phi(j)} &\leq \prod_{k-C_z \leq j \leq k-1} \frac{|\phi(z+j) - \phi(k)|}{\phi(k) - \phi(j)} \\
 &\leq \prod_{k-C_z \leq j \leq k-1} \frac{|\phi(z+j) - \phi(k)|}{\phi(k) - \phi(k-1)} \leq \prod_{k-C_z \leq j \leq k-1} \frac{|\int_{z+j \mapsto k} \phi'(w) dw|}{\phi'(k)} \\
 &\leq \prod_{k-C_z \leq j \leq k-1} \frac{|z+j-k| (\max_{k-C_z \leq j \leq k-1} \sup_{w \in [z+j, k]} |\phi'(w)|) \vee \phi'(k)}{\phi'(k)} \\
 &\leq \prod_{k-C_z \leq j \leq k-1} \frac{|z+j-k| (\max_{k-C_z \leq j \leq k-1} \phi'(a+j)) \vee \phi'(k)}{\phi'(k)} \\
 &\leq \prod_{k-C_z \leq j \leq k-1} \frac{|z+j-k| \phi'(k+a-C_z) \vee \phi'(k)}{\phi'(k)},
 \end{aligned}$$

where $\vee$ stands for the maximum function, we have used $|\phi'(w)| \leq \phi'(\operatorname{Re}(w))$, see (3.11), and the monotonicity of ϕ' . Now, $|z+j-k| \leq |z|+k-j \leq |z|+C_z$, because $j \geq k-C_z$, so this is

$$\prod_{1 \leq j \leq k-1} \frac{|\phi(z+j) - \phi(k)|}{\phi(k) - \phi(j)} \leq (|z|+C_z)^{C_z} \prod_{k-C_z \leq j \leq k-1} \frac{\phi'(k+a-C_z) \vee \phi'(k)}{\phi'(k)}.$$

Finally, observe that if $\phi'(k+a-C_z) \leq \phi'(k)$ then the remaining product is ≤ 1 , and on the other hand, if $\phi'(k+a-C_z) \geq \phi'(k)$, then uniformly among all k large enough, we have from Lemma 4.3

$$\frac{\phi'(k+a-C_z) \vee \phi'(k)}{\phi'(k)} = \frac{\phi'(k+a-C_z)}{\phi'(k)} \leq C,$$

where $C > 0$ is independent of k , and the proof of Lemma 4.5 is complete. $\square$

Proof of Lemma 4.7. Applying the Vandermonde identity (4.36), then using elementary row and column operations, we get

$$\begin{aligned}
 \prod_{1 \leq h < l \leq n} [\phi(l) - \phi(h)] &\stackrel{(4.36)}{=} \det \begin{pmatrix} 1 & \phi(1) & \cdots & \phi(1)^{n-2} & \phi(1)^{n-1} \\ 1 & \phi(2) & \cdots & \phi(2)^{n-2} & \phi(2)^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi(n) & \cdots & \phi(n)^{n-2} & \phi(n)^{n-1} \end{pmatrix} \\
 &= \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \phi(1) & \cdots & \phi(1)^{n-2} & \phi(1)^{n-1} \\ 0 & 1 & \phi(2) & \cdots & \phi(2)^{n-2} & \phi(2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \phi(n) & \cdots & \phi(n)^{n-2} & \phi(n)^{n-1} \end{pmatrix} \\
 &= \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \phi(1) & \cdots & \phi(1)^{n-2} & \phi(1)^{n-1} \\ 0 & 1 & \phi(2) & \cdots & \phi(2)^{n-2} & \phi(2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \phi(n) & \cdots & \phi(n)^{n-2} & \phi(n)^{n-1} \end{pmatrix}.
 \end{aligned} \tag{6.17}$$

Next, we add a scalar multiple of the first row to each of the other rows, yielding

$$(6.17) = \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ -\prod_{i=1}^{n-1} [\phi(1) - \phi(z+i)] & 1 & \phi(1) & \cdots & \phi(1)^{n-2} & \phi(1)^{n-1} - \prod_{i=1}^{n-1} [\phi(1) - \phi(z+i)] \\ -\prod_{i=1}^{n-1} [\phi(2) - \phi(z+i)] & 1 & \phi(2) & \cdots & \phi(2)^{n-2} & \phi(2)^{n-1} - \prod_{i=1}^{n-1} [\phi(2) - \phi(z+i)] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\prod_{i=1}^{n-1} [\phi(n) - \phi(z+i)] & 1 & \phi(n) & \cdots & \phi(n)^{n-2} & \phi(n)^{n-1} - \prod_{i=1}^{n-1} [\phi(n) - \phi(z+i)] \end{pmatrix}.$$

Evaluating this using the first row's cofactor expansion, we get two terms, the first of which is

$$\det \begin{pmatrix} 1 & \phi(1) & \cdots & \phi(1)^{n-2} & \phi(1)^{n-1} - \prod_{i=1}^{n-1} [\phi(1) - \phi(z+i)] \\ 1 & \phi(2) & \cdots & \phi(2)^{n-2} & \phi(2)^{n-1} - \prod_{i=1}^{n-1} [\phi(2) - \phi(z+i)] \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi(n) & \cdots & \phi(n)^{n-2} & \phi(n)^{n-1} - \prod_{i=1}^{n-1} [\phi(n) - \phi(z+i)] \end{pmatrix} = 0,$$

since the last column is a linear combination of the other columns, and the remaining term is

$$(6.17) = (-1)^n \det \begin{pmatrix} -\prod_{i=1}^{n-1} [\phi(1) - \phi(z+i)] & 1 & \phi(1) & \cdots & \phi(1)^{n-2} \\ -\prod_{i=1}^{n-1} [\phi(2) - \phi(z+i)] & 1 & \phi(2) & \cdots & \phi(2)^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\prod_{i=1}^{n-1} [\phi(n) - \phi(z+i)] & 1 & \phi(n) & \cdots & \phi(n)^{n-2} \end{pmatrix}.$$

We evaluate this using the first column's cofactor expansion. Each minor matrix is itself a Vandermonde matrix with determinant $\prod_{1 \leq h < l \leq n; h, l \neq k} [\phi(l) - \phi(h)]$, so we conclude, as required,

$$\begin{aligned} (6.17) &= (-1)^n \sum_{k=1}^n (-1)^{k+1} \left(\prod_{\substack{1 \leq h < l \leq n; \\ h, l \neq k}} [\phi(l) - \phi(h)] \right) \left(- \prod_{1 \leq i \leq n-1} [\phi(k) - \phi(z+i)] \right) \\ &= \sum_{k=1}^n (-1)^k \left(\prod_{\substack{1 \leq h < l \leq n; \\ h, l \neq k}} [\phi(l) - \phi(h)] \right) \left(- \prod_{1 \leq i \leq n-1} [\phi(z+i) - \phi(k)] \right). \quad \square \end{aligned}$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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