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Longtime behavior of a branching process controlled by branching catalysts

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Abstract

The model under consideration is a catalytic branching model constructed in Dawson and Fleischmann (1997), where the catalysts themselves undergo a spatial branching mechanism. The key result is a convergence theorem in dimension $d=3$ towards a limit with full intensity (persistence), which, in a sense, is comparable with the situation for the "classical" continuous super-Brownian motion. As by-products, strong laws of large numbers are derived for the Brownian collision local time controlling the branching of reactants, and for the catalytic occupation time process. Also, the catalytic occupation measures are shown to be absolutely continuous with respect to Lebesgue measure. © 1997 Elsevier Science B.V.

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1. Introduction and review of main results

Consider two types of "particles" situated in $\mathbb{R}^d$, one of which we call the *catalysts*, the other the *reactants*. The catalysts perform a continuous super-Brownian motion (SBM) ϱ with constant branching rate $\gamma>0$. The reactants are also super-Brownian, however given ϱ, their branching rate at time t in the volume element $\mathrm{d}b$ of $\mathbb{R}^d$ is just given by $\varrho_t(\mathrm{d}b)$. In other words, first ϱ is realized, and then a continuous SBM $X=X^\varrho=(X^\varrho,P^\varrho_{s,\mu})$ evolves with *varying* branching rates $\varrho_t(\mathrm{d}b)$ (*quenched* approach).

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More precisely, the rate of branching of a reactant with (Brownian) path W is controlled (in the sense of Dynkin's, 1991, additive functional approach to superprocesses) by the *Brownian collision local time* (BCLT) $L_{[W,\varrho]}$ of ϱ, formally described by

$$L_{[W,\varrho]}(\mathrm{d}r) := \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \delta_b(W_r), \tag{1}$$

which exists non-trivially for dimensions $d \leqslant 3$ (cf. Barlow et al., 1991). In higher dimensions, on the contrary, W and ϱ do not collide (see Barlow and Perkins, 1994, Proposition 1.3), and therefore branching should not occur, which means that X^ϱ degenerates to the heat flow. The *catalytic* SBM X^ϱ in $\mathbb{R}^d$, $d \leqslant 3$, was constructed as a continuous process in detail in Dawson and Fleischmann (1997).

It might be useful at this point to recall the longtime behavior of SBM with *constant* branching rate, starting with a (not necessarily normalized) Lebesgue measure ℓ (Dawson, 1997). In dimension one, it suffers local extinction almost surely, in dimension two in probability, whereas in $d \geqslant 3$ it converges in law to a non-trivial steady state with expectation ℓ (persistence).

The study of the longtime behavior of the catalytic SBM X^ϱ was initiated in Dawson and Fleischmann (1997), but restricted to dimension $d = 1$. In this case, X^ϱ behaves quite differently than the usual spatial branching models in low dimensions. In fact, if both the catalyst process ϱ and the catalytic SBM X^ϱ start off with the Lebesgue measure ℓ, then, for almost all catalyst process realizations, X_t^ϱ converges in probability to the starting Lebesgue measure ℓ (persistence). This is caused by the clumping features of the one-dimensional catalyst (Dawson and Fleischmann, 1988).

Here we continue the study of this model X^ϱ in the time–space catalytic medium ϱ. In dimension $d = 2$ we get only some partial results, namely, some *self-similarity* properties (Proposition 13) and a *random ergodic limit* (Theorem 15). The question whether or not persistence occurs in this "delicate" dimension is an *open* problem [1] (see also Remark 14).

But our *main result* concerns dimension $d = 3$. Here we allow ϱ to start off with the ergodic steady state (of the catalyst process) leading to a time-stationary (in law) medium. Then with respect to the *annealed* distribution (defined in Assumption 16, p. 18), X_t converges in law to some random measure of full intensity and finite variance (*convergence and persistence Theorem* 18(a)). From this point of view, the time-averaged process should obey a *strong law of large numbers* (Theorem 10). These two results can be considered as a random medium analog of properties of the classical SBM in higher dimensions. But note that it can be expected that the limit is different from the classical one.

To complete the picture, we establish *a strong law of large numbers for the* BCLT $L_{[W,\varrho]}$ (Theorem 5). We also show that the catalytic (weighted) *occupation time process* $Y_t := \int_0^t \mathrm{d}r\, X_r$ has states which are *absolutely continuous* with respect to Lebesgue measure.

[1] Persistence has meanwhile been shown in Etheridge and Fleischmann (1997).

The log-Laplace functional $v_t = v_t^\varrho \geqslant 0$ of the catalytic SBM X_t^ϱ at time t satisfies (formally) the following *reaction diffusion equation*:

$$-\frac{\partial}{\partial s} v_t(s,a) = \frac{1}{2} \Delta v_t(s,a) - \varrho_s(\mathrm{d}a)\, v_t^2(s,a), \quad s \leqslant t, \ a \in \mathbb{R}^d \tag{2}$$

with a terminal condition $v_t(s,\cdot)|_{s=t-} = f \geqslant 0$. (The backward setting reflects the fact that, for ϱ fixed, the deterministic process v^ϱ is "dual" to the stochastic process X^ϱ, where this duality is realized by the log-Laplace functional.) Via this connection, our results can also be understood as a probabilistic contribution to the study of Eq. (2) having a (random) *singular* reaction coefficient $\varrho_s(\mathrm{d}a)$, describing a spatially *heterogeneous* catalytic reaction. Actually, our results give information on the long-time behavior of the L^1-norm $\int \ell(\mathrm{d}a)\, v(s,a)$ of the solution to Eq. (2) as $s \to -\infty$ if it "starts" at time t with a finite mass $\int \ell(\mathrm{d}a)\, f(a)$. In fact, we proved in Dawson and Fleischmann (1997) that in the one-dimensional case one has convergence to the starting mass $\int \ell(\mathrm{d}a)\, f(a)$ (persistence). Dimension two is *open*.[2] But the main result of the present paper establishes in dimension three a.s. convergence to a non-zero limit (possibly depending on the medium ϱ).

Note that the one-dimensional case resembles a bit a (two-dimensional) *reaction diffusion process of electrically charged species* studied by Glitzky et al. (1996). They got convergence to an equilibrium with exponential velocity. But our three-dimensional model behaves differently in that we do not get an equilibrium at the equation level.

For background on super-Brownian motions we refer to Dawson (1993).

2. Brownian collision local time

In this section we rigorously introduce the Brownian collision local time $L = L_{[W,\varrho]}$, and state in dimension $d = 3$ a strong law of large numbers (Theorem 5).

2.1. Preliminaries

Fix a constant $p > d$ with $d \geqslant 1$ the dimension of space, and introduce the *reference function*

$$\phi_p(b) := (1 + |b|^2)^{-p/2}, \quad b \in \mathbb{R}^d. \tag{3}$$

Let $\mathscr{B}^p$ denote the space of all measurable functions f defined on $\mathbb{R}^d$ such that $|f| \leqslant c_f \phi_p$ for some constant c_f. Write $\mathscr{C}^{p,\ell}$ for the subset of all continuous functions f in $\mathscr{B}^p$ such that $f(b)/\phi_p(b)$ has a finite limit as $|b| \to \infty$. Equipped with the norm $\|f\| := \|f/\phi_p\|_\infty$, the Banach space $\mathscr{C}^{p,\ell}$ is *separable*. (Recall that the Banach space of continuous functions on a compactum with the supremum norm is separable.)

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[2] As meanwhile shown in Fleischmann and Klenke (1996), in the two-dimensional model one has convergence to a ϱ-dependent limit.

Set $I := [0, T]$, $T \geqslant 0$. Write $\mathscr{C}^{p,I}$ for the set of all continuous functions ψ defined on $I \times \mathbb{R}^d$ such that $|\psi_s| \leqslant c_\psi \phi_p$, $s \in I$, for some constant c_ψ.

Let $\mathscr{M}_p$ refer to the cone of all (non-negative) measures μ defined on $\mathbb{R}^d$ such that

$$\|\mu\|_p := \langle \mu, \phi_p \rangle := \int \mu(\mathrm{d}b)\, \phi_p(b) < +\infty. \tag{4}$$

$\mathscr{M}_p$ is endowed with the coarsest topology such that the maps $\mu \to \langle \mu, f \rangle$ are continuous where $f = \phi_p$ or $f \in \mathscr{C}_+^{\mathrm{comp}}$. Here $\mathscr{C}^{\mathrm{comp}}$ denotes the space of continuous functions on $\mathbb{R}^d$ with compact support (and the index + indicates the subset of all non-negative members). Recall that each Lebesgue measure ℓ belongs to $\mathscr{M}_p$. Write i_ℓ for the volume of the unit cube in $\mathbb{R}^d$ measured with respect to ℓ.

Let $W = (W, \Pi_{s,a})$ denote the *Brownian motion* in $\mathbb{R}^d$ on canonical path space of continuous functions, with "generator" $\frac{1}{2}\Delta$. (According to standard notation, $\Pi_{s,a}$ is the law of W if W starts at time s from a.) Furthermore, let $\mathrm{p}_t(a, b) = \mathrm{p}_t(b - a)$ refer to its continuous transition density function, and $S = \{S_t \colon t \geqslant 0\}$ to its semigroup. Set $\Pi_{s,\mu} := \int \mu(\mathrm{d}a)\, \Pi_{s,a}$. We also introduce the (time-inhomogeneous) *Brownian potential kernel*

$$\mathrm{q}_{s,t}(a, b) = \mathrm{q}_{s,t}(b - a) := \int_s^t \mathrm{d}r\ \mathrm{p}_r(a, b), \quad 0 \leqslant s \leqslant t, \ \ a, b \in \mathbb{R}^d. \tag{5}$$

2.2. Catalyst process ϱ

For convenience, we give the following definition of the *catalyst process* ϱ (see Dawson, 1993, Section 4.7).

Definition 1. (*Catalyst process* ϱ). Write $\varrho = (\varrho, \mathbb{P}_{s,\mu})$ for the continuous SBM in $\mathbb{R}^d$ with constant branching rate $\gamma > 0$. Consequently, for fixed $t \geqslant 0$, the *log-Laplace functional* of ϱ is given by

$$-\log \mathbb{P}_{s,\mu} \exp\langle \varrho_t, -f \rangle = \langle \mu, -v_t(s, \cdot) \rangle, \quad s \leqslant t, \ \ \mu \in \mathscr{M}_p, \ \ f \in \mathscr{B}^p, \tag{6}$$

where v_t is the unique non-negative solution to (2) with $\varrho_s(\mathrm{d}a)$ replaced by the constant γ, and with terminal condition $v_t(s, \cdot)|_{s=t-} = f$. Here we always work with a mild solution, that is with a solution to the equation in the integrated form, actually in Dynkin's probabilistic form

$$v_t(s, a) = \Pi_{s,a} \left[f(W_t) - \int_s^t \gamma\, \mathrm{d}r\ v_t^2(r, W_r) \right], \quad s \leqslant t, \ \ a \in \mathbb{R}^d. \tag{7}$$

ϱ is called the *catalyst process.*

Recall the *expectation* formula

$$\mathbb{P}_{s,\mu} \langle \varrho_t, f \rangle = \langle \mu, S_{t-s} f \rangle, \quad s \leqslant t, \ \ \mu \in \mathscr{M}_p, \ \ f \in \mathscr{B}^p. \tag{8}$$

Recall also, that in dimensions $d \geqslant 3$, with respect to $\mathbb{P}_{0,\ell}$, the catalyst process ϱ_t has a non-trivial limit ϱ_∞ in law as $t \uparrow \infty$ of full-intensity measure ℓ (see, e.g. Dawson and

Perkins, 1991, Proposition 6.1). Hence, here we can form the *time-stationary* continuous $\mathcal{M}_p$-valued process $\varrho = \{\varrho_t\colon t\in\mathbb{R}\}$ whose one-dimensional laws $\mathscr{L}(\varrho_t)$ coincide with $\mathscr{L}(\varrho_\infty)$. In this case we write $\mathbb{P}$ and sometimes $\mathbb{P}_{-\infty,\ell}$ for the law of ϱ.

The following mixing property is taken from a general result in Fleischmann (1982b), which was formulated in a time-discrete setting. For the present situation, a simplified proof will be given.

Lemma 2. (Time–space mixing of all orders). *In dimensions $d\geqslant 3$, the catalyst process ϱ with the time–space-shift invariant law $\mathbb{P}$ is time–space mixing of all orders. That is, for all finite sequences $B_1,\dots,B_m$ of Borel subsets of $\mathbb{R}^d$,*

$$\mathbb{P}((\varrho_{t_1}(B_1+b_1),\dots,\varrho_{t_m}(B_m+b_m))\in(\cdot))\to\prod_{i=1}^m\mathbb{P}(\varrho_0(B_i)\in(\cdot))$$

as $|(t_i,b_i)-(t_j,b_j)|\to\infty$ whenever $i\neq j$. In particular, for $f,g\in\mathscr{C}_+^{\mathrm{comp}}$, the vector $(\langle\varrho_{t_1},f\rangle,\langle\varrho_{t_2},g\rangle)$ is asymptotically independent as $|t_1-t_2|\to\infty$ (mixing in time).

Proof. First of all, recall the following *covariance formula* for ϱ:

$$\mathbb{C}\mathrm{ov}_{s,\mu}[\langle\varrho_{t_1},f\rangle,\langle\varrho_{t_2},g\rangle]=2\,\Pi_{s,\mu}\int_s^{t_1\wedge t_2}\gamma\,\mathrm{d}r\,S_{t_1-r}f(W_r)\,S_{t_2-r}g(W_r), \tag{9}$$

$s\leqslant t_1,t_2$, $\mu\in\mathcal{M}_p$, and $f,g\in\mathscr{C}_+^{\mathrm{comp}}$; see e.g. Dawson and Fleischmann (1997, Proposition 12(b), p. 230). Hence, the *covariance density* function of ϱ at $[[t_1,b_1],[t_2,b_2]]$ with respect to $\mathbb{P}_{s,\ell}$ is given by

$$2\int_s^{t_1\wedge t_2}\gamma\,\mathrm{d}r\,\mathrm{p}_{t_1+t_2-2r}(b_1,b_2).$$

Letting $s\downarrow-\infty$, we arrive at the covariance density function at $[t_1,b_1]$ and $[t_2,b_2]$ of the catalyst process ϱ with respect to $\mathbb{P}$. Since $\mathbb{P}$ is invariant with respect to the time–space shift and *infinitely divisible*, it *suffices* to show that this covariance density function converges to 0 as $|[t_1,b_1]-[t_2,b_2]|\to\infty$ on the sets $\mathbb{R}_+\times\{|b_1-b_2|\geqslant\varepsilon\}$, $\varepsilon>0$; see the remark after Theorem 2.0.2 in Fleischmann (1982a). Here we may set $[t_2,b_2]=0$ without loss of generality. Thus, it is sufficient to demonstrate that

$$\int_{-\infty}^0\mathrm{d}r\,\mathrm{p}_{t-r}(b)\to 0\quad\text{as } |(t,b)|\to+\infty\quad\text{on } \mathbb{R}_+\times\{|b|\geqslant\varepsilon\},\ \varepsilon>0.$$

But the latter integral equals $\int_t^\infty\mathrm{d}r\,\mathrm{p}_r(b)$ and can be estimated from above by $\leqslant\mathrm{const}[|b|^{2-d}\wedge t^{-1/2}]$ with a constant *const* depending on ε. (We also later use the symbol *const* to denote a constant which might be different at different places.) This finishes the proof. □

2.3. Brownian collision local time $L_{[W,\varrho]}$

Assumption 3. (Catalyst process). *From now on we restrict our attention to dimensions $d\leqslant 3$, and assume, if not otherwise indicated, that the catalyst process ϱ is distributed according to $\mathbb{P}_{0,\ell}$ or to the stationary $\mathbb{P}$, the latter of course only if $d=3$.*

For $\varepsilon > 0$ and given ϱ, consider the following continuous additive functional $L^{\varepsilon}_{[W,\varrho]}$ of Brownian motion W:

$$L^{\varepsilon}(\mathrm{d}r) := L^{\varepsilon}_{[W,\varrho]}(\mathrm{d}r) := \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \mathrm{p}_{\varepsilon}(W_r, b), \tag{10}$$

describing the collision local time of the measure-valued path ϱ with the "ε-vicinity" of the Brownian path W.

Lemma 4. (Brownian collision local time $L_{[W,\varrho]}$). *Suppose Assumption* 3 *is satisfied, and fix a constant* $\xi \in (0, \frac{1}{4})$. *Then for almost all paths* ϱ *of the catalyst process, there exists a continuous additive functional* $L = L_{[W,\varrho]}$ *of the Brownian motion* W, *called the* Brownian collision local time (BCLT) *of* ϱ, *with the following properties.*
(a) (Convergence) *If* ψ *is a (strictly) positive function in* $\mathscr{C}^{p,I}$, $I = [0, T]$, $T > 0$, *then*

$$\sup_{s \in I, a \in \mathbb{R}^d} \Pi_{s,a} \sup_{s \leqslant t \leqslant T} \left| \int_s^t L^{\varepsilon}(\mathrm{d}r)\, \psi_r(W_r) - \int_s^t L(\mathrm{d}r)\, \psi_r(W_r) \right|^2 \underset{\varepsilon \downarrow 0}{\longrightarrow} 0.$$

(b) (First two moments) *For measurable* $\psi : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}_+$, *and* $s \leqslant t$, $a \in \mathbb{R}^d$,

$$\Pi_{s,a} \int_s^t L_{[W,\varrho]}(\mathrm{d}r)\, \psi_r(W_r) = \int_s^t \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \mathrm{p}_{r-s}(a,b)\, \psi_r(b),$$

$$\Pi_{s,a} \left[\int_s^t L_{[W,\varrho]}(\mathrm{d}r)\, \psi_r(W_r) \right]^2$$

$$= 2 \int_s^t \mathrm{d}r \int_r^t \mathrm{d}r' \int \varrho_r(\mathrm{d}b) \int \varrho_{r'}(\mathrm{d}b')\, \mathrm{p}_{r-s}(a,b)\, \mathrm{p}_{r'-r}(b,b')\, \psi_r(b)\, \psi_{r'}(b').$$

Proof. This follows from Proposition 6, p. 256 and Theorem 4, p. 259 in Dawson and Fleischmann (1997). □

2.4. A strong law of large numbers for $L_{[W,\varrho]}$ *in* $d = 3$

In this subsection we assume that Brownian motion W is distributed according to $\Pi_{0,0}$. First we recall that in dimension $d = 1$ the *total* BCLT $L_{[W,\varrho]}(\mathbb{R}_+)$ of ϱ is *finite*, for almost all $[W, \varrho]$ (Dawson and Fleischmann, 1997, Proposition 7, p. 264). Next we mention that in $d = 2$ we have a self-similarity property for $L = L_{[W,\varrho]}$, see Corollary 12 below. But in dimension $d = 3$, a strong law of large numbers holds (recall i_{ℓ} denotes the volume of the unit cube):

Theorem 5. (Strong LLN for the BCLT). *If* $d = 3$, *then*

$$T^{-1} L_{[W,\varrho]}[0, T] \underset{T \uparrow \infty}{\longrightarrow} \mathrm{i}_{\ell}, \qquad \Pi_{0,0} \times \mathbb{P}_{0,\ell}\text{-}a.s. \quad and \quad \Pi_{0,0} \times \mathbb{P}\text{-}a.s.$$

Proof (1) (*Expectation*) First of all, for $s \leqslant 0$, by the expectation formula in Lemma 4(b),

$$\Pi_{0,0} \times \mathbb{P}_{s,\ell}\, T^{-1} L[0, T] = \mathbb{P}_{s,\ell}\, T^{-1} \int_0^T \mathrm{d}r \langle \varrho_r, \mathrm{p}_r \rangle \equiv \mathrm{i}_{\ell} \tag{11}$$

since by the expectation formula (8)

$$\mathbb{P}_{s,\ell}\, \varrho_r \equiv \ell \tag{12}$$

(independent of the dimension d).

(2) (*Variance estimate*) Next we show that $T^{-1}L[0,T]$ has a $\Pi_{0,0} \times \mathbb{P}_{s,\ell}$-variance of order $\mathrm{O}(T^{-1/2})$ as $T \uparrow \infty$, uniformly in $s \leqslant 0$ (covering the cases $s=0$ and $s=-\infty$ corresponding to $\mathbb{P}_{0,\ell}$ and $\mathbb{P}$, respectively, we are interested in).

In view of the second moment formula in Lemma 4(b),

$$\Pi_{0,0}(L[0,T])^2 = 2\int_0^T \mathrm{d}r \int_r^T \mathrm{d}r' \int \varrho_r(\mathrm{d}b) \int \varrho_{r'}(\mathrm{d}b')\, \mathrm{p}_r(b)\, \mathrm{p}_{r'-r}(b,b').$$

Therefore, by (12), by

$$\int \ell(\mathrm{d}b) \int \ell(\mathrm{d}b')\, \mathrm{p}_r(b)\, \mathrm{p}_{r'-r}(b,b') = \mathrm{i}_\ell^2$$

and by step (1),

$$\begin{aligned} &\Pi_{0,0} \times \mathbb{P}_{s,\ell}\, |T^{-1}L[0,T] - \mathrm{i}_\ell|^2 \\ &\quad = 2\,T^{-2} \int_0^T \mathrm{d}r \int_r^T \mathrm{d}r'\, \mathrm{Cov}_{s,\ell}[\varrho_r(\mathrm{d}b), \varrho_{r'}(\mathrm{d}b')]\, \mathrm{p}_r(b)\, \mathrm{p}_{r'-r}(b,b'). \end{aligned} \tag{13}$$

But by (9), the latter covariance expression equals

$$2\,\Pi_{s,\ell} \int_s^r \gamma\, \mathrm{d}t \int \mathrm{d}b\, \mathrm{p}_{r-t}(W_t, b)\, \mathrm{p}_r(b)\, \mathrm{p}_{2r'-r-t}(W_t, b).$$

Therefore, we may continue the r.h.s. of (13) with

$$= 4\,\mathrm{i}_\ell\, T^{-2} \int_0^T \mathrm{d}r \int_r^T \mathrm{d}r' \int_s^r \gamma\, \mathrm{d}t\, \mathrm{p}_{2r'-2t}(0).$$

However, the internal integral is of order $\mathrm{O}((r'-r)^{-1/2})$, uniformly in $s \leqslant 0$. Hence, altogether we get

$$\Pi_{0,0} \times \mathbb{P}_{s,\ell}|T^{-1}L[0,T] - \mathrm{i}_\ell|^2 \leqslant \text{const}\, T^{-1/2} \tag{14}$$

uniformly in $s \leqslant 0$.

(3) (*Conclusion*) Now the proof can easily be completed. From (14),

$$\Pi_{0,0} \times \mathbb{P}_{s,\ell}\{|T^{-1}L[0,T] - \mathrm{i}_\ell| \geqslant \varepsilon\} \leqslant \text{const}\, \varepsilon^{-2}\, T^{-1/2}.$$

By Borel–Cantelli,

$$S^{-4}L[0,S^4] \to \mathrm{i}_\ell \text{ along the integers } S \to \infty, \text{ a.s.}$$

But for $S^4 \leqslant T < (S+1)^4$,

$$(S+1)^{-4}L[0,S^4] \leqslant T^{-1}L[0,T] \leqslant S^{-4}L[0,(S+1)^4],$$

and the claimed a.s. convergence follows, finishing the proof. □

3. Occupation times

Here we rigorously introduce the catalytic SBM X^{ϱ}, verify that its catalytic occupation time Y^{ϱ} has absolutely continuous states, and satisfies a strong law of large numbers, the latter in the case $d=3$.

3.1. Catalytic SBM

Since the BCLT $L=L_{[W,\varrho]}$ of Lemma 4 is a locally admissible additive functional of Brownian motion W with "small" increments, one can conclude for the existence of the *catalytic SBM* X^{ϱ} *in the catalytic medium* ϱ:

Lemma 6. (Catalytic SBM X^{ϱ}). *Under Assumption* 3, *for almost all realizations* ϱ *of the catalyst process, the following statements hold*:

(a) (Existence). *There exists the continuous SBM* $X=X^{\varrho}=(X^{\varrho},P^{\varrho}_{s,\mu})$ *with branching functional given by the BCLT* $L=L_{[W,\varrho]}$.

(b) (Log-Laplace functional). *The log-Laplace functional of* X^{ϱ} *is given by*

$$-\log P^{\varrho}_{s,\mu}\exp\langle X_t,-f\rangle=\langle\mu,-v_t(s,\cdot\,)\rangle, \tag{15}$$

$s\leqslant t$, $\mu\in\mathcal{M}_p$, $f\in\mathcal{B}^p$, *where* v_t *is the unique non-negative solution to Dynkin's log-Laplace equation*

$$v_t(s,a)=\Pi_{s,a}\left[f(W_t)-\int_s^t L(\mathrm{d}r)v_t^2(r,W_r)\right], \tag{16}$$

$s\leqslant t$, $a\in\mathbb{R}^d$.

(c) (Moments). *Expectation and covariance of* X^{ϱ} *are given by*

$$P^{\varrho}_{s,\mu}\langle X_t,f\rangle=\langle\mu,S_{t-s}f\rangle,\quad s\leqslant t,\ \mu\in\mathcal{M}_p,\ f\in\mathcal{B}^p,$$

and

$$\begin{aligned}&\mathrm{Cov}^{\varrho}_{s,\mu}[\langle X_{t_1},f\rangle,\langle X_{t_2},g\rangle]\\&\quad=2\int\mu(\mathrm{d}a)\int_s^{t_1\wedge t_2}\mathrm{d}r\int\varrho_r(\mathrm{d}b)\mathrm{p}(r-s,a,b)S_{t_1-r}f(b)S_{t_2-r}g(b),\end{aligned}$$

$s\leqslant t_1,t_2$, $\mu\in\mathcal{M}_p$, *and* $f,g\in\mathcal{C}_+^{\mathrm{comp}}$.

Proof. See (Dawson and Fleischmann, 1997, Definition 5, p. 261) which is based on Theorem 1(b) (p. 235), and formula (5.10). □

Note that (16) is the precise meaning of the catalytic reaction diffusion Eq. (2) with reaction rate $\varrho_s(\mathrm{d}a)$.

3.2. Absolutely continuous occupation time states

Since X^{ϱ} is pathwise continuous, we may introduce the *catalytic* (*weighted*) *occupation time process* $Y=Y^{\varrho}=\{Y_t^{\varrho}\colon t\geqslant 0\}$ related to $X=X^{\varrho}$, defined by $Y_t:=\int_0^t \mathrm{d}r\,X_r$.

Recall that for the "classical" continuous SBM, say X^{ℓ}, in dimensions $d \leqslant 3$, the related occupation measures Y_t^{ℓ} are absolutely continuous, i.e. density functions y_t^{ℓ} exist (see e.g., Fleischmann, 1988). This property is shared also by the SBM X^{ϱ} in the catalytic medium ϱ. For a convenient formulation, we introduce the *annealed* law $\mathscr{P}_{\ell} := \mathbb{P}_{0,\ell} P_{0,\ell}^{\varrho}$. That is, the laws $P_{0,\ell}^{\varrho}$ of X^{ϱ} given ϱ are averaged by means of the distribution $\mathbb{P}_{0,\ell}$ of ϱ.

Theorem 7. (Occupation densities). *Under Assumption* 3, *for* $T>0$ *and* $z \in \mathbb{R}^d$ *fixed, the following statements hold.*

(a) (Densities of Y_t) *With respect to the annealed law* $\mathscr{P}_{\ell}$ *we obtain*:

(a1) (L^2-densities) *The* $L^2(\mathscr{P}_{\ell})$*-limit of* $\langle Y_T, \mathrm{p}_{\varepsilon}(z,\cdot)\rangle$ *as* $\varepsilon \downarrow 0$ *exists and is denoted by* $y_T(z)$.

(a2) (Absolutely continuous states) *With respect to* $\mathscr{P}_{\ell}$, *the random measure* Y_T *is absolutely continuous with density function* y_T:

$$\mathscr{P}_{\ell}(Y_T(\mathrm{d}b) = y_T(b)\,\mathrm{d}b) = 1.$$

(a3) (First two moments) *The following formulas hold*:

$$\mathscr{P}_{\ell}\, y_T(z) \equiv \mathrm{i}_{\ell},$$

$$\mathscr{V}\mathrm{ar}_{\ell}\, y_T(z) = 2\,\mathrm{i}_{\ell}^2 \int_0^T \mathrm{d}r \int_0^{T-r} \mathrm{d}t \int_0^{T-r} \mathrm{d}t'\, \mathrm{p}_{t+t'}(0) > 0.$$

(b) (Densities of Y_t^{ϱ}) *For* $\mathbb{P}_{0,\ell}$*-a.a. catalyst process realizations* ϱ *we obtain*:

(b1) (L^2-densities) *The* $L^2(P_{0,\ell}^{\varrho})$*-limit of* $\langle Y_T^{\varrho}, \mathrm{p}_{\varepsilon}(z,\cdot)\rangle$ *as* $\varepsilon \downarrow 0$ *exists and is denoted by* $y_T^{\varrho}(z)$.

(b2) (Absolutely continuous states) *With respect to* $P_{0,\ell}^{\varrho}$, *the random measure* Y_T^{ϱ} *is absolutely continuous with density function* y_T^{ϱ}:

$$P_{0,\ell}^{\varrho}(Y_T^{\varrho}(\mathrm{d}b) = y_T^{\varrho}(b)\,\mathrm{d}b) = 1.$$

(b3) (First two moments) *The following formulas hold*:

$$P_{0,\ell}^{\varrho}\; y_T^{\varrho}(z) \equiv \mathrm{i}_{\ell},$$

$$V\mathrm{ar}_{0,\ell}^{\varrho}\; y_T^{\varrho}(z) = 2\,\mathrm{i}_{\ell} \int_0^T \mathrm{d}r \int \varrho_r(\mathrm{d}b)\mathrm{q}_{0,T-r}^2(b,z)$$

(*recall definition* (5) *of the Brownian potential kernel* q).

Proof. We start with the proof of (b). According to (Dawson and Fleischmann, 1997, Proposition 5, p. 240) it suffices to show that for almost all ϱ,

$$\Pi_{0,\ell} \int_0^T L_{[W,\varrho]}(\mathrm{d}r)\mathrm{q}^2_{r',\varepsilon+r'}(W_r,z) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0 \quad \text{for } r'=0 \text{ and } r'=T-r. \tag{17}$$

By the expectation formula in Lemma 4(b)

$$\Pi_{s,\ell} \int_s^T L_{[W,\varrho]}(\mathrm{d}r)\psi_r(W_r) = \mathrm{i}_\ell \int_s^T \mathrm{d}r \int \varrho_r(\mathrm{d}b)\psi_r(b). \tag{18}$$

Hence, the l.h.s. in (17) equals

$$\mathrm{i}_\ell \int_0^T \mathrm{d}r \int \varrho_r(\mathrm{d}b)\mathrm{q}^2_{r',\varepsilon+r'}(b,z).$$

Since this is monotone in ε, the limit as $\varepsilon \downarrow 0$ exists (for each ϱ). Thus, it is sufficient to show that the expectation over ϱ converges to 0 as $\varepsilon \downarrow 0$. But by (12), the latter $\mathbb{P}_{s,\ell}$-expectation, $s \leqslant 0$, is independent of s and equals

$$\mathrm{i}_\ell \int_0^T \mathrm{d}r \int \ell(\mathrm{d}b)\mathrm{q}^2_{r',\varepsilon+r'}(b,z) = \mathrm{i}_\ell^2 T \int_{r'}^{\varepsilon+r'} \mathrm{d}t \int_{r'}^{\varepsilon+r'} \mathrm{d}t'\, \mathrm{p}_{t+t'}(0).$$

Because the integrand is monotone decreasing in t and t', we may replace r' by 0, and since T is fixed we continue the latter formula line with

$$\leqslant \mathrm{const} \int_0^\varepsilon \mathrm{d}t \int_0^\varepsilon \mathrm{d}t'(t+t')^{-d/2} \leqslant \mathrm{const}\ \varepsilon^{1/2} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0 \ \text{since}\ d \leqslant 3.$$

This finishes the proof of (b). But then (a) immediately follows by averaging over the medium. □

In dimension $d=3$, let $\mathscr{P}$ denote the *annealed* law $\mathbb{P}P^\varrho_{0,\ell}$ of the catalytic SBM concerning the time-space stationary medium ϱ with law $\mathbb{P}$.

Remark 8. (*Occupation densities in the stationary case*). Theorem 7 remains valid, if in dimension $d=3$ we replace $\mathbb{P}_{0,\ell}$ by $\mathbb{P}$, thus $\mathscr{P}_\ell$ by $\mathscr{P}$.

Remark 9. (*Occupation density field*). It can be expected that for the catalytic occupation time process Y^ϱ in all dimensions ($d \leqslant 3$) a jointly continuous catalytic occupation density field y^ϱ exists,[3] as it does for the "classical" continuous SBM, established by Sugitani (1989) and reproved in Dawson and Fleischmann (1997, Lemma 7, p. 243).

3.3. *A strong law of large numbers for Y^ϱ in $d=3$*

First we recall that in dimension $d=1$, the catalytic SBM X^ϱ_t converges in $P^\varrho_{0,\ell}$-probability to ℓ as $t \uparrow \infty$, for $\mathbb{P}_{0,\ell}$-almost all ϱ, see Dawson and Fleischmann (1997, Theorem 6, p. 273). This of course implies a law of large numbers for the related catalytic occupation time process Y^ϱ. In $d=2$, self-similarity properties hold instead, see Proposition 13 below. Here now we restrict our attention to dimension $d=3$. Recall that $\mathscr{P}_\ell$ and $\mathscr{P}$ denote the annealed laws of the catalytic SBM arising by averaging $P^\varrho_{0,\ell}$ over the medium ϱ by means of $\mathbb{P}_{0,\ell}$ and the stationary $\mathbb{P}$, respectively.

[3] As recently shown in Fleischmann and Klenke (1997), in dimension $d=2,3$ the catalytic SBM X^ϱ itself has actually a smooth density field defined on a time–space set of full Lebesgue measure.

Theorem 10. (LLN for Y). *If $d=3$, then*

$$T^{-1}Y_T^{\varrho} \underset{T\uparrow\infty}{\longrightarrow} \ell, \quad \mathscr{P}_\ell\text{-a.s.} \quad \text{and} \quad \mathscr{P}\text{-a.s.}$$

Consequently, in dimension $d=3$, the time-averaged X^{ϱ}-process behaves for almost all ϱ just as for the "classical" SBM. That is, here the *averaging principle* holds: Finally, only the expectation ℓ of the medium ϱ_t is "effective", leading to the expectation of X^{ϱ}.

Proof. Since $P_{0,\ell}^{\varrho}T^{-1}Y_T^{\varrho}\equiv \ell$ by the expectation formula in Lemma 6(c), as in step (3) of the proof of Theorem 5, it suffices to show that for fixed $f\geqslant 0$ in the separable Banach space $\mathscr{C}^{p;\ell}$,

$$\mathscr{P}_{s,\ell}\,|T^{-1}\langle Y_T, f\rangle - \langle \ell, f\rangle|^2 \leqslant \text{const}\, T^{-1/2}, \tag{19}$$

uniformly, in $s\leqslant 0$. Here we wrote $\mathscr{P}_{s,\ell}$ for $\mathbb{P}_{s,\ell}P_{0,\ell}^{\varrho}$, $s\in[-\infty,0]$. But

$$Var_{0,\ell}^{\varrho}\langle Y_T, f\rangle = 2\,\Pi_{0,\ell}\int_0^T L(\mathrm{d}r)\left[\int_r^T \mathrm{d}t\, \Pi_{r,W_r} f(W_t)\right]^2, \tag{20}$$

see Dawson and Fleischmann (1997, formula (3.22)). Hence, the l.h.s. in (19) equals

$$2\,T^{-2}\mathbb{P}_{s,\ell}\Pi_{0,\ell}\int_0^T L(\mathrm{d}r)\left[\int_r^T \mathrm{d}t\, \Pi_{r,W_r} f(W_t)\right]^2.$$

Using the expectation formulas (18) and (12), we may continue with

$$= 2\,\ell\, T^{-2}\int_0^T \mathrm{d}r \int \ell(\mathrm{d}b)\left[\int_r^T \mathrm{d}t \int \mathrm{d}z\, \mathrm{p}_{t-r}(b,z)\, f(z)\right]^2.$$

Interchanging the order of integrations, and calculating the $\ell(\mathrm{d}b)$-integral, this can be estimated from above by

$$\leqslant 2\langle \ell, f\rangle^2 T^{-2}\int_0^T \mathrm{d}r \int_r^T \mathrm{d}t \int_r^T \mathrm{d}t'\, \mathrm{p}_{t+t'-2r}(0).$$

As in step (2) of the proof of Theorem 5, the internal integral can be estimated by $\leqslant \text{const}(t-r)^{-1/2}$, and the claim follows. $\square$

4. Self-similarities in dimension $d=2$

Recall that in dimension $d=2$ the "classical" SBM ϱ with law $\mathbb{P}_{0,\ell}$ is *self-similar*: For $K>0$,

$$\{K^{-1}\varrho_{Kt}(K^{1/2}\cdot)\colon\, t\geqslant 0\} \overset{\mathscr{L}}{=} \{\varrho_t\colon\, t\geqslant 0\}. \tag{21}$$

This has some consequences for the BCLT $L_{[W,\varrho]}$ and the catalytic SBM X^{ϱ}.

4.1. A scaling property of $L_{[W,\varrho]}$

We start with a scaling property of the Brownian collision local time:

Lemma 11. (Scaling of the BCLT). *Fix* $d=2$, $s\geqslant 0$, $a\in\mathbb{R}^2$, *and* $K>0$. *Then for* $\Pi_{s,a}\times\mathbb{P}_{0,\ell}$*-almost all* $[W,\varrho]$, *and measurable* $g:\mathbb{R}_+\mapsto\mathbb{R}_+$,

$$\int L_{[W,\varrho]}(\mathrm{d}r)K^{-1}g(K^{-1}r)=\int L_{[K^{-1/2}W_{K\cdot},K^{-1}\varrho_{K\cdot}(K^{1/2}\cdot)]}(\mathrm{d}r)g(r). \tag{22}$$

Proof. Recalling the definition (10) of L^{ε}, by definition of the BCLT $L=L_{[W,\varrho]}$ it suffices to verify the claim with L replaced by $L^{K\varepsilon}$ and L^{ε}, respectively. Then by (10),

$$\int L^{K\varepsilon}_{[W,\varrho]}(\mathrm{d}r)K^{-1}g(K^{-1}r)=\int \mathrm{d}r\int \varrho_r(\mathrm{d}b)\mathrm{p}_{K\varepsilon}(W_r,b)K^{-1}g(K^{-1}r).$$

By a change of variables, and using the self-similarity of the Brownian transition density p, the r.h.s. can be written as

$$\int \mathrm{d}r\int K^{-1}\varrho_{Kr}(K^{1/2}\mathrm{d}b)\,\mathrm{p}_{\varepsilon}(K^{-1/2}W_{Kr},b)\,g(r).$$

Again by (10), we arrive at the r.h.s. of (22) with L replaced by L^{ε}, finishing the proof. □

Combining Lemma 11 with the self-similarities of Brownian motion W and ϱ (recall (21)) we get the following result.

Corollary 12. (Self-similarity of the BCLT). *For* $d=2$ *and* $K>0$, *with respect to* $\Pi_{0,0}\times\mathbb{P}_{0,\ell}$,

$$K^{-1}L_{[W,\varrho]}(K\cdot)\overset{\mathscr{L}}{=}L_{[W,\varrho]}.$$

4.2. Self-similarities of $[X^{\varrho},Y^{\varrho}]$

Instead of the well-known self-similarity of the "classical" SBM in $d=2$ (as in (21)), for the catalytic SBM we have the following versions (recall the catalytic occupation time process Y introduced in the beginning of section 3.2):

Proposition 13. (Scaling and self-similarity of $[X,Y]$). *Suppose* $d=2$, $K>0$, *and* $T\geqslant 0$.

(a) (Scaling) *For* $\mathbb{P}_{0,\ell}$*-almost all* ϱ *the following holds. If* $[X,Y]$ *is formed with respect to* $P^{\varrho}_{0,\ell}$, *then the pair*

$$[K^{-1}X_{KT}(K^{1/2}\cdot),K^{-2}Y_{KT}(K^{1/2}\cdot)]$$

has the same law as $[X_T,Y_T]$ *formed with respect to* $P^{K^{-1}\varrho_{K\cdot}(K^{1/2}\cdot)}_{0,\ell}$.

(b) (Self-similarities) *Formed either with respect to the* random *law* $P^{\varrho}_{0,\ell}$, *or the* annealed *law* $\mathscr{P}_{\ell}=\mathbb{P}_{0,\ell}P^{\varrho}_{0,\ell}$,

$$[K^{-1}X_{KT}(K^{1/2}\cdot), K^{-2}Y_{KT}(K^{1/2}\cdot)] \stackrel{\mathscr{L}}{=} [X_T, Y_T].$$

Proof. By Dawson and Fleischmann (1997, Hypothesis 2, p. 231, and notation 3.22), for f, g in $\mathscr{B}_+^p$, and $\mathbb{P}_{0,\ell}$-a.a. ϱ,

$$-\log P_{0,\ell}^{\varrho} \exp[\langle X_T, -f\rangle + \langle Y_T, -g\rangle] = \langle \ell, v_T(0,\cdot)\rangle$$

with

$$v_T(s,a) = \Pi_{s,a}\left[f(W_T) + \int_s^T \mathrm{d}r\, g(W_r) - \int_s^T L_{[W,\varrho]}(\mathrm{d}r)\, v_T^2(r, W_r)\right],$$

$0 \leqslant s \leqslant T$, $a \in \mathbb{R}^2$. Hence

$$\begin{aligned} &-\log P_{0,\ell}^{\varrho} \exp[\langle X_{KT}, -K^{-1}f(K^{-1/2}\cdot)\rangle + \langle Y_{KT}, -K^{-2}g(K^{-1/2}\cdot)\rangle] \\ &= \langle \ell, v_{KT}(0,\cdot)\rangle = \langle \ell, Kv_{KT}(0, K^{1/2}\cdot)\rangle. \end{aligned} \tag{23}$$

with

$$\begin{aligned} Kv_{KT}(Ks, K^{1/2}a) = \Pi_{Ks,K^{1/2}a}&\left[f(K^{-1/2}W_{KT}) + \int_{Ks}^{KT} \mathrm{d}r\, K^{-1}g(K^{-1/2}W_r)\right. \\ &\left. - K^{-1}\int_{Ks}^{KT} L_{[W,\varrho]}(\mathrm{d}r) K^2 v_{KT}^2(r, W_r)\right]. \end{aligned}$$

Setting $u_T(s,a) := Kv_{KT}(Ks, K^{1/2}a)$ (for the fixed K), by a change of variables and using the scaling of BCLT Lemma 11 (with $[s,a]$ replaced by $(Ks, K^{1/2}a)$), the latter equation can be written as

$$\begin{aligned} u_T(s,a) = \Pi_{Ks,K^{1/2}a}&\left[f(K^{-1/2}W_{KT}) + \int_s^T \mathrm{d}r\, g(K^{-1/2}W_{Kr})\right. \\ &\left. - \int_s^T L_{[K^{-1/2}W_{K\cdot}, K^{-1}\varrho_{K\cdot}(K^{1/2}\cdot)]}(\mathrm{d}r) u_T^2(r, K^{-1/2}W_{Kr})\right]. \end{aligned}$$

But W distributed according to $\Pi_{Ks,K^{1/2}a}$ implies by scaling that the process $t \to K^{-1/2}W_{Kt}$ has the law $\Pi_{s,a}$. Therefore, the latter formula lines can be continued with

$$= \Pi_{s,a}\left[f(W_T) + \int_s^T \mathrm{d}r\, g(W_r) - \int_s^T L_{[W, K^{-1}\varrho_{K\cdot}(K^{1/2}\cdot)]}(\mathrm{d}r)\, u_T^2(r, W_r)\right].$$

Hence, by uniqueness of the solution to the log-Laplace equation (see Dawson and Fleischmann, 1997, Proposition 1(a), p. 255), we conclude that the r.h.s. of (23) equals

$$-\log P_{0,\ell}^{K^{-1}\varrho_{K\cdot}(K^{1/2}\cdot)} \exp[\langle X_T, -f\rangle + \langle Y_T, -g\rangle].$$

This proves the claim (a). The statement (b) concerning the random law $P_{0,\ell}^{\varrho}$ then immediately follows from the self-similarity (21) of ϱ, and the claim concerning the annealed law $\mathscr{P}_\ell$ results by integration with $\mathbb{P}_{0,\ell}$. This finishes the proof. □

Remark 14. (*Open problems*). By the self-similarity of X^ϱ with respect to the annealed law $\mathscr{P}_\ell$, the distribution of X_T^ϱ coincides with the law of $TX_1^\varrho(T^{-1/2}\cdot)$. Passing formally to $T\uparrow\infty$, we arrive at X^{ϱ}_{∞} and $x^{\varrho}_{1}(0)\ell$. This relates the questions of existence of a limit X^{ϱ}_{∞} with full expectation and of the existence of a density $x^{\varrho}_{1}(0)$ of X^{ϱ}_{1} at the origin with full expectation, in the critical dimension $d=2$. But whether or not a non-trivial limit X^{ϱ}_{∞} exists remains *open.*[4]

4.3. A random ergodic limit

Recall that for the continuous SBM X^{ℓ} in $\mathbb{R}^2$ with constant branching rate and with law $P^{\ell}_{0,\ell}$ we have the following "random" ergodic limit:

$$T^{-1}Y^{\ell}_{T} \underset{T\uparrow\infty}{\longrightarrow} y^{\ell}_{1}(0)\,\ell \quad \text{in law,}$$

where $y^{\ell}_{1}(0)$ is the random density of the occupation measure Y^{ℓ}_{1} at time 1 at the origin 0; see e.g. Fleischmann (1988). The two-dimensional catalytic SBM X^{ϱ} has a similar property:

Theorem 15. (Random ergodic limit). *Let $d=2$ and consider the catalytic SBM X^{ϱ} with annealed law $\mathscr{P}_{\ell}=\mathbb{P}_{0,\ell}P^{\varrho}_{0,\ell}$, or with $\mathbb{P}_{0,\ell}$-random law $P^{\varrho}_{0,\ell}$.*

(a) (Annealed approach) *$T^{-1}Y^{\varrho}_{T}$ converges in $\mathscr{P}_{\ell}$-law as $T\uparrow\infty$ towards the multiple $y_1(0)\ell$ of Lebesgue measure ℓ, where $y_1(0)$, is the $L^2(\mathscr{P}_{\ell})$-density at 0 of the catalytic occupation measure Y_1 at time* 1, *according to Theorem* 7(a1).

(b) (Quenched approach) *The $\mathbb{P}_{0,\ell}$-random law of $T^{-1}Y^{\varrho}_{T}$ converges in $\mathbb{P}_{0,\ell}$-law as $T\uparrow\infty$ towards the $\mathbb{P}_{0,\ell}$-random law of the multiple $y^{\varrho}_{1}(0)\ell$ of Lebesgue measure ℓ, where $y^{\varrho}_{1}(0)$, given ϱ, is the $L^2(P^{\varrho}_{0,\ell})$-density at 0 of the catalytic occupation measure Y^{ϱ}_{1} at time* 1, *according to Theorem* 7(b1).

Proof. We start by proving part (b). Using the random self-similarity in Proposition 13(b), the $\mathbb{P}_{0,\ell}$-random law of $T^{-1}Y^{\varrho}_{T}$ coincides with that of $TY_1(T^{-1/2}\cdot)$. But by Theorem 7(b1), for $f\in\mathscr{C}^{p;\ell}_{+}$ and $\mathbb{P}_{0,\ell}$-almost all ϱ

$$\langle TY^{\varrho}_{1}(T^{-1/2}\cdot),f\rangle \underset{T\uparrow\infty}{\longrightarrow} y^{\varrho}_{1}(0)\langle\ell,f\rangle \quad \text{in } L^2(P^{\varrho}_{0,\ell}),$$

implying the claim.

Part (a) follows analogously from the annealed self-similarity in Proposition 13(b), and Theorem 7(a1). $\square$

Note that as opposed to dimensions 1 and 3, here the limit remains random after the averaging procedure, since the $\mathscr{P}_{\ell}$-variance of $y_1(0)$ is (strictly) positive, and the $P^{\varrho}_{0,\ell}$-variance of $y^{\varrho}_{1}(0)$ is positive with $\mathbb{P}_{0,\ell}$-probability one.

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[4] These questions are meanwhile positively answered in Etheridge and Fleischmann (1997) and Fleischmann and Klenke (1997).

5. Persistence in dimension $d=3$

In this final section we deal with the following situation (which occurred already in Remark 8).

Assumption 16. (Time–space-shift invariance). *Let $d=3$ and assume that the catalyst process ϱ is distributed according to the time–space-shift invariant distribution $\mathbb{P}$ (introduced in section 2.2), and consider the annealed law $\mathscr{P}=\mathbb{P}P^{\varrho}_{0,\ell}$.*

Remark 17. (*Approximation*). Working with the non-stationary catalyst process ϱ distributed according to $\mathbb{P}_{0,\ell}$ would require some additional approximations.

5.1. Main result

Now we are in a position to formulate our *main result*:

Theorem 18. (Convergence and persistence of second order). *Impose Assumption* 16.

(a) (Annealed convergence) *With respect to the annealed distribution $\mathscr{P}$, the catalytic SBM X_T converges in law as $T\uparrow\infty$ to some limit X_∞ with full intensity ℓ and finite variance (persistence of second-order).*

(b) (Random convergence) *The $\mathbb{P}$-random distribution of*

$$P^{\varrho}_{0,\ell}(X_T\in\cdot\,)=:Q^{\varrho}_T$$

converges in $\mathbb{P}$-law as $T\uparrow\infty$ to some $\mathbb{P}$-random distribution Q^{ϱ}_∞ with full intensity and finite variance (persistence of second order): With $\mathbb{P}$-probability one,

$$\int Q^{\varrho}_\infty(\mathrm{d}v)v=\ell,\qquad \int Q^{\varrho}_\infty(\mathrm{d}v)|\langle v,f\rangle-\langle \ell,f\rangle|^2<\infty,\quad f\in\mathscr{B}^p_+.$$

Consequently, at the first sight, our catalytic SBM X^{ϱ} behaves similarly to the classical continuous SBM X^{ℓ}. However, the main difference should be that a new limit occurs. For instance, the limiting random measure X_∞ of (a) should be *different* from the classical steady-state X^{ℓ}_∞.

5.2. Proof of the main theorem

The key of proof will be a *backward technique*: By the time-stationarity of the random medium ϱ we may start X^{ϱ} at time $-T$ with ℓ, and observe the state at time 0. Then we may continue for *fixed* realization ϱ, sending $-T$ to $-\infty$, by exploiting some backward monotonicities.

(1) (*Convergence*) First of all, for $\mathbb{P}$-almost all ϱ, the law Q^{ϱ}_T coincides with the law $P^{\varrho_{T+(\cdot)}}_{-T,\ell}(X_0\in\cdot\,)$. Here $\varrho_{T+(\cdot)}$ is the catalyst process shifted by time T. Hence, by the time-shift invariance of the catalyst process ϱ, the distribution of the random law Q^{ϱ}_T coincides with that of the random law $P^{\varrho}_{-T,\ell}(X_0\in\cdot\,)$. (Note that here we made a transition in $\mathbb{P}$-law, but after this we will continue for a fixed ϱ.)

Given ϱ, we turn to the log-Laplace functional according to Lemma 6(b): For $f \in \mathscr{B}_+^p$, writing $\langle \ell, f\rangle =: \|f\|_1$,

$$-\log P^{\varrho}_{-T,\ell} \exp \langle X_0, -f\rangle = \|v_0(-T,\cdot)\|_1,$$

where by the log-Laplace equation (16),

$$\|v_0(-T,\cdot)\|_1 = \Pi_{-T,\ell}\left[f(W_0) - \int_{-T}^{0} L_{[W,\varrho]}(\mathrm{d}r) v_0^2(r, W_r)\right].$$

Using the expectation formula in Lemma 4(b), this formula line can be continued with

$$= \|f\|_1 - \mathrm{i}_\ell \int_{-T}^{0} \mathrm{d}r \int \varrho_r(\mathrm{d}b) v_0^2(r,b).$$

But this non-negative expression is non-increasing[5] in the variable T. Hence, the limit of $\|v_0(-T,\cdot)\|_1$ exists (for the fixed ϱ) and determines a log-Laplace functional of a random measure, its law denoted by Q^{ϱ}_{∞}. (In fact, note that the family $\{Q^{\varrho}_T\colon T \geqslant 0\}$ is relatively compact since all laws have expectation measure ℓ, see (24) below.) This gives the convergence claim in (b). By averaging over ϱ, the convergence statement of (a) also follows.

(2) (*Expectation bounds*) For almost all ϱ, from the expectation formula in Lemma 6(c),

$$\int Q^{\varrho}_T(\mathrm{d}\mu)\mu = P^{\varrho}_{0,\ell} X_T \equiv \ell, \tag{24}$$

which implies for the limits that

$$\int Q^{\varrho}_{\infty}(\mathrm{d}\mu)\mu \leqslant \ell, \quad \text{hence} \quad \int \mathbb{P} Q^{\varrho}_{\infty}(\mathrm{d}\mu)\mu \leqslant \ell. \tag{25}$$

Consequently, the limiting intensity measures in (b) and (a) are bounded by ℓ.

(3) (*Variance bounds*) Let again $f \in \mathscr{B}_+^p$. Given ϱ, by the variance formula in Lemma 6(c),

$$\mathit{Var}^{\varrho}_{-T,\ell}\langle X_0, f\rangle = 2\mathrm{i}_\ell \int_{-T}^{0} \mathrm{d}r \int \varrho_r(\mathrm{d}b)[S_{-r} f(b)]^2,$$

which monotonically converges to

$$2\mathrm{i}_\ell \int_{-\infty}^{0} \mathrm{d}r \int \varrho_r(\mathrm{d}b)[S_{-r} f(b)]^2 \quad \text{as } T \uparrow \infty. \tag{26}$$

Integrating ϱ with $\mathbb{P}$, by the expectation formula (12) we get the monotone convergence

$$\mathbb{P}\, \mathit{Var}^{\varrho}_{0,\ell}\langle X_T, f\rangle \underset{T\uparrow\infty}{\nearrow} 2 \int \ell(\mathrm{d}x) f(x) \int \ell(\mathrm{d}y) f(y) \int_0^{\infty} \mathrm{d}r\, \mathrm{p}_{2r}(x,y). \tag{27}$$

Note that by (12), the l.h.s. in (27) is the variance of $\langle X_T, f\rangle$ with respect to the annealed law $\mathscr{P}$. On the other hand, the r.h.s. is the variance expression related to the

[5] Note that this monotonicity would be violated if we started ϱ at time $-T$ with $\varrho_{-T} = \ell$. That is, the present method only works for the time-stationary process ϱ on the whole time axis $\mathbb{R}$.

classical steady-state X^{ℓ}_{∞} (see, for instance, Dawson, 1977), hence is finite. Therefore, also the limit (26) is finite $\mathbb{P}$-a.s. But this implies that in (25) equalities must hold (persistence). Moreover, the variance of $\langle X_{\infty}, f\rangle$ is finite with respect to $\mathbb{P}Q^{\varrho}_{\infty}$, hence, also with respect to the laws Q^{ϱ}_{∞}, given ϱ. In other words, in both cases (a) and (b), we get persistence of second order. This finishes the proof. □

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