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Stochastic equations of non-negative processes with jumps

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Abstract

We study stochastic equations of non-negative processes with jumps. The existence and uniqueness of strong solutions are established under Lipschitz and non-Lipschitz conditions. Under suitable conditions, the comparison properties of solutions are proved. Those results are applied to construct continuous state branching processes with immigration as strong solutions of stochastic equations.

MSC: primary 60H20; secondary 60H10

Keywords: Stochastic equation; Strong solution; Pathwise uniqueness; Comparison theorem; Non-Lipschitz condition; Continuous state branching process; Immigration

1. Introduction

Stochastic differential equations with jumps have been playing increasingly important roles in various applications. Under Lipschitz conditions, the existence and uniqueness of strong solutions of jump-type stochastic equations can be established by arguments based on Gronwall's inequality and the results on continuous-type equations; see e.g. [12]. In view of the result of [20], weaker conditions would be sufficient for the existence and uniqueness of strong solutions for one-dimensional equations. As an example of jump-type equations, let us consider the simple

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equation

$$\mathrm{d}x(t) = \phi(x(t-))\mathrm{d}z(t), \quad t \geq 0. \tag{1.1}$$

By a result of [2], if $\{z(t)\}$ is a symmetric stable process of order $1 < \alpha < 2$ and if $x \mapsto \phi(x)$ is a bounded function with modulus of continuity $x \mapsto \rho(x)$ satisfying

$$\int_{0+} \frac{1}{\rho(x)^\alpha}\mathrm{d}x = \infty, \tag{1.2}$$

then (1.1) admits a strong solution and the solution is pathwise unique. This condition is exactly the analogue of the Yamada–Watanabe criterion for the diffusion coefficient. When the integral in (1.2) is finite, Bass [2] constructed a continuous function $x \mapsto \phi(x)$ having continuity modulus $x \mapsto \rho(x)$ for which the pathwise uniqueness for (1.1) fails; see also [4]. In view of (1.2), if a power function $\rho(x) = \mathrm{const} \cdot x^\beta$ applies for all symmetric stable processes with parameters $1 < \alpha < 2$, we must have $\beta = 1$. In other words, a universal continuity modulus condition for jump-type stochastic differential equations would not be a great improvement of the Lipschitz condition.

For equations driven by non-symmetric noises, there is a new difficulty brought about by the compensators of the noises. For example, let us consider the Eq. (1.1) again with $\{z(t)\}$ being a one-sided stable process of order $1 < \alpha < 2$. For any $\varepsilon > 0$, let

$$z_\varepsilon(t) = \sum_{0<s\leq t} [z(s) - z(s-)]1_{\{z(s)-z(s-)>\varepsilon\}}$$

and let $c_\varepsilon = \mathbf{E}[z_\varepsilon(1)]$. We can define another centered Lévy process $\{w_\varepsilon(t)\}$ by

$$w_\varepsilon(t) = z(t) - z_\varepsilon(t) + c_\varepsilon t.$$

Between any two neighboring jumps of $\{z_\varepsilon(t)\}$, Eq. (1.1) reduces to

$$\mathrm{d}x(t) = \phi(x(t-))\mathrm{d}w_\varepsilon(t) - c_\varepsilon\phi(x(t-))\mathrm{d}t. \tag{1.3}$$

Then one would expect, in order that the pathwise uniqueness holds for (1.1) or (1.3), the function $x \mapsto \phi(x)$ should be as regular as the drift coefficient in the Yamada–Watanabe criterion. In other words, it should possess a continuity modulus $x \mapsto r(x)$ satisfying

$$\int_{0+} \frac{1}{r(x)}\mathrm{d}x = \infty, \tag{1.4}$$

which is much stronger than (1.2).

Continuous state branching processes with immigration (CBI-processes) constitute an important class of non-negative Markov processes with non-negative jumps. They were introduced in [13] as approximations of classical Galton–Watson branching processes with immigration. Many interesting applications of them have been found since then. In particular, CBI-processes are also known as Cox–Ingersoll–Ross models (CIR models) and have been used widely in the study of mathematical finance; see, e.g., [10,15]. Up to a minor moment assumption, a conservative CBI-process has generator A defined by

$$\begin{aligned} Af(x) = axf''(x) &+ \int_0^\infty \big[f(x+z) - f(x) - zf'(x)\big]x\nu_0(\mathrm{d}z) \\ &+ (b+\beta x)f'(x) + \int_0^\infty \big[f(x+z) - f(x)\big]\nu_1(\mathrm{d}z), \end{aligned} \tag{1.5}$$

where $a \geq 0$, $b \geq 0$ and β are constants, and $\nu_0(\mathrm{d}z)$ and $\nu_1(\mathrm{d}z)$ are σ-finite measures on $(0, \infty)$ satisfying

$$\int_0^\infty (z \wedge z^2)\nu_0(\mathrm{d}z) + \int_0^\infty (1 \wedge z)\nu_1(\mathrm{d}z) < \infty. \tag{1.6}$$

Let $\{B(t)\}$ be a standard Brownian motion and let $\{N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ and $\{N_1(\mathrm{d}s, \mathrm{d}z)\}$ be Poisson random measures with intensities $\mathrm{d}s\nu_0(\mathrm{d}z)\mathrm{d}u$ and $\mathrm{d}s\nu_1(\mathrm{d}z)$, respectively. Suppose that $\{B(t)\}$, $\{N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ and $\{N_1(\mathrm{d}s, \mathrm{d}z)\}$ are independent of each other. Let $\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ be the compensated measure of $N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$. Under a slightly stronger condition on $\nu_1(\mathrm{d}z)$ it was proved in [8] that the stochastic equation

$$\begin{aligned} x(t) = x(0) &+ \int_0^t \sqrt{2ax(s)}\mathrm{d}B(s) + \int_0^t \int_0^\infty \int_0^{x(s-)} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_0^t (b + \beta x(s))\mathrm{d}s + \int_0^t \int_0^\infty zN_1(\mathrm{d}s, \mathrm{d}z); \end{aligned} \tag{1.7}$$

has a unique non-negative strong solution. By Itô's formula one sees the solution $\{x(t)\}$ of (1.7) is a CBI-process with generator given by (1.5).

The purpose of the present paper is to study stochastic equations of non-negative processes with jumps that generalizes the CBI-processes described above. By specifying to non-negative processes we can make the best use of the first moment analysis. This kind of processes arise naturally in various applications, but they are excluded by most of the existing results in the literature because their generators possess non-Lipschitz and degenerate coefficients. We provide here some criteria for the existence and uniqueness of strong solutions of those equations. The main idea of those criteria is to assume some monotonicity condition on the kernel associated with the compensated noise so that the continuity conditions can be released. It follows from our criterion that (1.1) has a unique non-negative strong solution if $\{z(t)\}$ is a one-sided α-stable process and if $x \mapsto \phi(x)$ is a $(1/2)$-Hölder continuous non-decreasing function satisfying $\phi(0) = 0$.

To describe another consequence of our criteria, suppose that $1 < \alpha < 2$ and (a, b, β, ν_1) are given as above. Let $\{B(t)\}$ be a standard Brownian motion, $\{z_0(t)\}$ be a one-sided α-stable process with characteristic measure $z^{-1-\alpha}\mathrm{d}z$ and $\{z_1(t)\}$ be a non-decreasing pure jump Lévy process with characteristic measure $\nu_1(\mathrm{d}z)$. Suppose $\{B(t)\}$, $\{z_0(t)\}$ and $\{z_1(t)\}$ are independent of each other. We shall see for any $\sigma \geq 0$ there is a unique non-negative strong solution to

$$\mathrm{d}x(t) = \sqrt{2ax(t)}\mathrm{d}B(t) + \sqrt[\alpha]{\sigma x(t-)}\mathrm{d}z_0(t) + (\beta x(t) + b)\mathrm{d}t + \mathrm{d}z_1(t). \tag{1.8}$$

A particular case of this equation has been considered in [14], where the uniqueness of the solution was left open. The solution of (1.8) is a CBI-process with generator given by (1.5) with $\nu_0(\mathrm{d}z) = \sigma z^{-1-\alpha}\mathrm{d}z$. Of course, for any $\sigma > 0$ and $1 < \alpha < 2$ the continuity modulus of the coefficient $x \mapsto \sqrt[\alpha]{\sigma x}$ in (1.8) does not satisfy condition (1.4).

The theory of jump-type stochastic equations is not as well developed as that of continuous ones; see [3] and the references therein. We refer to [12] and [16] for the theory of stochastic analysis and to [18] for the theory of Lévy processes. Throughout this paper, we assume $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ is a filtered probability space satisfying the usual hypotheses. Moreover, we make the conventions

$$\int_a^b = \int_{(a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)}$$

for any real numbers $a \leq b$.

Some basic results for stochastic equations of non-negative processes are provided in Section 2. In particular, we give a Lipschitz condition for the existence and uniqueness of the strong solution. In Section 3 the pathwise uniqueness is studied. In Section 4 we prove a weak existence result by second moment arguments. The existence and uniqueness of strong solutions under non-Lipschitz conditions are established in Section 5, where only very weak moment conditions are required. We also prove some properties of the solutions, continuous dependence on the initial value and comparison properties. In Section 6, we illustrate some applications of the main results to stochastic differential equations driven by one-sided Lévy processes including (1.8).

2. Preliminaries

In this section, we prove some basic results on stochastic equations of non-negative processes with jumps. Suppose that U_0 and U_1 are complete separable metric spaces and that $\mu_0(\mathrm{d}u)$ and $\mu_1(\mathrm{d}u)$ are σ-finite Borel measures on U_0 and U_1, respectively. Suppose that

- $x \mapsto \sigma(x)$ and $x \mapsto b(x)$ are continuous functions on $\mathbb{R}$ satisfying $\sigma(x) = 0$ and $b(x) \geq 0$ for $x \leq 0$;
- $(x, u) \mapsto g_0(x, u)$ is a Borel function on $\mathbb{R} \times U_0$ such that $g_0(x, u) + x \geq 0$ for $x > 0$, and $g_0(x, u) = 0$ for $x \leq 0$;
- $(x, u) \mapsto g_1(x, u)$ is a Borel function on $\mathbb{R} \times U_1$ such that $g_1(x, u) + x \geq 0$ for all $x \in \mathbb{R}$ and $u \in U_1$.

Let $\{B(t)\}$ be a standard $(\mathscr{F}_t)$-Brownian motion and let $\{p_0(t)\}$ and $\{p_1(t)\}$ be $(\mathscr{F}_t)$-Poisson point processes on U_0 and U_1 with characteristic measures $\mu_0(\mathrm{d}u)$ and $\mu_1(\mathrm{d}u)$, respectively. Suppose that $\{B(t)\}$, $\{p_0(t)\}$ and $\{p_1(t)\}$ are independent of each other. Let $N_0(\mathrm{d}s, \mathrm{d}u)$ and $N_1(\mathrm{d}s, \mathrm{d}u)$ be the Poisson random measures associated with $\{p_0(t)\}$ and $\{p_1(t)\}$, respectively. Let $\tilde{N}_0(\mathrm{d}s, \mathrm{d}u)$ be the compensated measure of $N_0(\mathrm{d}s, \mathrm{d}u)$. By a *solution of* the stochastic equation

$$\begin{aligned} x(t) = x(0) &+ \int_0^t \sigma(x(s-))\mathrm{d}B(s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(\mathrm{d}s, \mathrm{d}u) \\ &+ \int_0^t b(x(s-))\mathrm{d}s + \int_0^t \int_{U_1} g_1(x(s-), u)N_1(\mathrm{d}s, \mathrm{d}u) \end{aligned} \tag{2.1}$$

we mean a càdlàg and $(\mathscr{F}_t)$-adapted process $\{x(t)\}$ that satisfies the equation almost surely for every $t \geq 0$. Since $x(s-) \neq x(s)$ for at most countably many $s \geq 0$, we can also use $x(s)$ instead of $x(s-)$ in the integrals with respect to $\mathrm{d}B(s)$ and $\mathrm{d}s$ on the right hand side of (2.1).

Proposition 2.1. *If* $\{x(t)\}$ *satisfies* (2.1) *and* $\mathbf{P}\{x(0) \geq 0\} = 1$, *then* $\mathbf{P}\{x(t) \geq 0$ *for all* $t \geq 0\} = 1$.

Proof. Suppose there exists a constant $\varepsilon > 0$ so that $\tau := \inf\{t \geq 0 : x(t) \leq -\varepsilon\} < \infty$ with strictly positive probability. The assumptions on $g_0(x, u)$ and $g_1(x, u)$ prevent $\{x(t)\}$ from jumping into the interval $[-\varepsilon, \infty)$. Then on the event $\{\tau < \infty\}$ we have $x(\tau) = x(\tau-) = -\varepsilon$ and hence $\tau > \tau_0 := \inf\{s < \tau : x(t) \leq 0 \text{ for all } s \leq t \leq \tau\}$. Here τ_0 is a random time but not a stopping time. Then we choose a deterministic time $r \geq 0$ so that $\{\tau_0 \leq r < \tau\}$ occurs with strictly positive probability. On the event $\{\tau_0 \leq r < \tau\}$ we have

$$x(t \wedge \tau) = x(r \wedge \tau) + \int_{r\wedge\tau}^{t\wedge\tau} b(x(s-))\mathrm{d}s + \int_{r\wedge\tau}^{t\wedge\tau} \int_{U_1} g_1(x(s-), u) N_1(\mathrm{d}s, \mathrm{d}u), \quad t \geq r,$$

so $t \mapsto x(t \wedge \tau)$ is non-decreasing on $[r, \infty)$. Since $x(r) > -\varepsilon$ on $\{\tau_0 \leq r < \tau\}$, we get a contradiction. $\square$

In the sequel, we shall always assume the initial variable $x(0)$ is non-negative, so Proposition 2.1 implies that any solution of (2.1) is non-negative. Then we can assume the ingredients are defined only for $x \geq 0$. For non-negative processes we can use the first moment estimates, which is essential for the CBI-process and solutions of stochastic equations driven by one-sided stable noises. Let U_2 be a Borel subset of U_1 satisfying $\mu_1(U_1 \setminus U_2) < \infty$ and consider the stochastic equation

$$\begin{aligned} x(t) = x(0) &+ \int_0^t \sigma(x(s-))\mathrm{d}B(s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(\mathrm{d}s, \mathrm{d}u) \\ &+ \int_0^t b(x(s-))\mathrm{d}s + \int_0^t \int_{U_2} g_1(x(s-), u) N_1(\mathrm{d}s, \mathrm{d}u). \end{aligned} \tag{2.2}$$

Proposition 2.2. *If there is a strong solution to (2.2), there is also a strong solution to (2.1). If the pathwise uniqueness of solutions holds for (2.2), it also holds for (2.1).*

Proof. The results hold trivially if $\mu_1(U_1 \setminus U_2) = 0$, so we assume $0 < \mu_1(U_1 \setminus U_2) < \infty$ in this proof. Suppose that (2.2) has a strong solution $\{x_0(t)\}$. Let $\{S_k : k = 1, 2, \ldots\}$ be the set of jump times of the Poisson process

$$t \mapsto \int_0^t \int_{U_1\setminus U_2} N_1(\mathrm{d}s, \mathrm{d}u).$$

We have clearly $S_k \to \infty$ as $k \to \infty$. For $0 \leq t < S_1$ set $y(t) = x_0(t)$. Suppose that $y(t)$ has been defined for $0 \leq t < S_k$ and let

$$\xi = y(S_k-) + \int_{\{S_k\}} \int_{U_1\setminus U_2} g_1(y(S_k-), u) N_1(\mathrm{d}s, \mathrm{d}u). \tag{2.3}$$

By the assumption there is also a strong solution $\{x_k(t)\}$ to

$$\begin{aligned} x(t) = \xi &+ \int_0^t \sigma(x(s-))\mathrm{d}B(S_k + s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(S_k + \mathrm{d}s, \mathrm{d}u) \\ &+ \int_0^t b(x(s-))\mathrm{d}s + \int_0^t \int_{U_2} g_1(x(s-), u) N_1(S_k + \mathrm{d}s, \mathrm{d}u). \end{aligned} \tag{2.4}$$

For $S_k \leq t < S_{k+1}$ we set $y(t) = x_k(t - S_k)$. By induction that defines a process $\{y(t)\}$, which is clearly a strong solution to (2.1). On the other hand, if $\{y(t)\}$ is a solution to (2.1), it satisfies (2.2) for $0 \leq t < S_1$ and the process $\{y(S_k + t)\}$ satisfies (2.4) for $0 \leq t < S_{k+1} - S_k$ with ξ given by (2.3). Then the pathwise uniqueness for (2.1) follows from that for (2.2) and (2.4). $\square$

The results of Proposition 2.2 can be reformulated for weak solutions of (2.1) and (2.2). In fact, some similar results were given in [19] in terms of martingale problems. Thanks to Proposition 2.2 we may focus on the existence and uniqueness of strong solutions of Eq. (2.2). For the convenience of the statements of the results, let us formulate the following conditions:

(2.a) For a constant $K \geq 0$ and a Borel subset $U_2 \subset U_1$ we have $\mu_1(U_1 \setminus U_2) < \infty$ and

$$b(x) + \int_{U_2} |g_1(x,u)|\mu_1(\mathrm{d}u) \leq K(1+x), \quad x \geq 0;$$

(2.b) There is a non-decreasing function $x \mapsto L(x)$ on $\mathbb{R}_+$ so that

$$\sigma(x)^2 + \int_{U_0} [|g_0(x,u)| \wedge g_0(x,u)^2]\mu_0(\mathrm{d}u) \leq L(x), \quad x \geq 0.$$

Proposition 2.3. *Suppose that conditions* (2.a, b) *hold. Let* $\{x(t)\}$ *be a non-negative solution of* (2.2) *and let* $\tau_m = \inf\{t \geq 0 : x(t) \geq m\}$ *for* $m \geq 1$. *Then* $\tau_m \to \infty$ *almost surely as* $m \to \infty$. *Moreover, we have*

$$\mathbf{E}[1 + x(t)] \leq \mathbf{E}[1 + x(0)]\exp\{Kt\} \tag{2.5}$$

and

$$(1+m)\mathbf{P}\{\tau_m \leq t\} \leq \mathbf{E}[1 + x(0)]\exp\{Kt\}. \tag{2.6}$$

Proof. The first assertion is immediate since $\{x(t)\}$ has càdlàg sample paths. It is easy to show that

$$\mathbf{E}\left[\left|\int_0^{t\wedge\tau_m}\int_{U_0} g_0(x(s-),u)1_{\{|g_0(x(s-),u)|>1\}}\tilde{N}_0(\mathrm{d}s,\mathrm{d}u)\right|\right]$$
$$\leq \mathbf{E}\left[\int_0^{t\wedge\tau_m}\mathrm{d}s\int_{U_0} |g_0(x(s-),u)|1_{\{|g_0(x(s-),u)|>1\}}\mu_0(\mathrm{d}u)\right].$$

Observe also that

$$\mathbf{E}\left[\left|\int_0^{t\wedge\tau_m}\sigma(x(s-))\mathrm{d}B(s)\right|^2\right] = \mathbf{E}\left[\int_0^{t\wedge\tau_m}\sigma(x(s-))^2\mathrm{d}s\right]$$

and

$$\mathbf{E}\left[\left|\int_0^{t\wedge\tau_m}\int_{U_0} g_0(x(s-),u)1_{\{|g_0(x(s-),u)|\leq 1\}}\tilde{N}_0(\mathrm{d}s,\mathrm{d}u)\right|^2\right]$$
$$= \mathbf{E}\left[\int_0^{t\wedge\tau_m}\mathrm{d}s\int_{U_0} g_0(x(s-),u)^2 1_{\{|g_0(x(s-),u)|\leq 1\}}\mu_0(\mathrm{d}u)\right].$$

Since $x(s-) \leq m$ for all $0 < s \leq \tau_m$, the above expectations are finite by (2.b), and so

$$t \mapsto \int_0^{t\wedge\tau_m}\sigma(x(s-))\mathrm{d}B(s) + \int_0^{t\wedge\tau_m}\int_{U_0} g_0(x(s-),u)\tilde{N}_0(\mathrm{d}s,\mathrm{d}u)$$

is a martingale. From (2.2) and (2.a) we get

$$\begin{aligned}\mathbf{E}[1+x(t\wedge\tau_m)] &= \mathbf{E}[1+x(0)]+\mathbf{E}\left[\int_0^{t\wedge\tau_m} b(x(s-))\mathrm{d}s\right]\\ &\quad+\mathbf{E}\left[\int_0^{t\wedge\tau_m}\mathrm{d}s\int_{U_2} g_1(x(s-),u)\mu_1(\mathrm{d}u)\right]\\ &\le \mathbf{E}[1+x(0)]+K\mathbf{E}\left[\int_0^{t\wedge\tau_m}(1+x(s-))\mathrm{d}s\right].\end{aligned}$$

Thus $t\mapsto\mathbf{E}[1+x(t\wedge\tau_m)]$ is a locally bounded function. Moreover, since $x(s-)\neq x(s)$ for at most countably many $s\ge 0$, it follows that

$$\begin{aligned}\mathbf{E}[1+x(t\wedge\tau_m)] &\le \mathbf{E}[1+x(0)]+K\mathbf{E}\left[\int_0^{t\wedge\tau_m}(1+x(s))\mathrm{d}s\right]\\ &\le \mathbf{E}[1+x(0)]+K\int_0^t\mathbf{E}[1+x(s\wedge\tau_m)]\mathrm{d}s.\end{aligned}$$

By Gronwall's lemma,

$$\mathbf{E}[1+x(t\wedge\tau_m)]\le\mathbf{E}[1+x(0)]\exp\{Kt\},\quad t\ge 0.$$

By the right continuity of $\{x(t)\}$ we have $x(\tau_m)\ge m$, so (2.6) holds, and (2.5) follows by an application of Fatou's lemma. □

Proposition 2.4. *Suppose that conditions* (2.a, b) *hold and for each* $m\ge 1$ *there is a unique strong solution to*

$$\begin{aligned}x(t) &= x(0)+\int_0^t\sigma(x(s-)\wedge m)\mathrm{d}B(s)+\int_0^t b_m(x(s-)\wedge m)\mathrm{d}s\\ &\quad+\int_0^t\int_{U_0}[g_0(x(s-)\wedge m,u)\wedge m]\tilde{N}_0(\mathrm{d}s,\mathrm{d}u)\\ &\quad+\int_0^t\int_{U_2}[g_1(x(s-)\wedge m,u)\wedge m]N_1(\mathrm{d}s,\mathrm{d}u),\end{aligned}\tag{2.7}$$

where

$$b_m(x)=b(x)-\int_{U_0}[g_0(x,u)-g_0(x,u)\wedge m]\mu_0(\mathrm{d}u).$$

Then there is a unique strong solution to (2.2).

Proof. For any $m\ge 1$ let $\{x_m(t)\}$ denote the unique strong solution to (2.7) and let $\tau_m=\inf\{t\ge 0: x_m(t)\ge m\}$. Since $0\le x_m(t)<m$ for $0\le t<\tau_m$, the trajectory $t\mapsto x_m(t)$ has no jumps larger than m on the time interval $[0,\tau_m)$. Then we have

$$\begin{aligned}x_m(t) &= x(0)+\int_0^t\sigma(x_m(s-))\mathrm{d}B(s)+\int_0^t\int_{U_0}g_0(x_m(s-),u)\tilde{N}_0(\mathrm{d}s,\mathrm{d}u)\\ &\quad+\int_0^t\mathrm{d}s\int_{U_0}[g_0(x_m(s-),u)-g_0(x_m(s-),u)\wedge m]\mu_0(\mathrm{d}u)\\ &\quad+\int_0^t b_m(x_m(s-))\mathrm{d}s+\int_0^t\int_{U_2}g_1(x_m(s-),u)N_1(\mathrm{d}s,\mathrm{d}u)\end{aligned}$$

$$= x(0) + \int_0^t \sigma(x_m(s-))\mathrm{d}B(s) + \int_0^t \int_{U_0} g_0(x_m(s-), u)\tilde{N}_0(\mathrm{d}s, \mathrm{d}u)$$
$$+ \int_0^t b(x_m(s-))\mathrm{d}s + \int_0^t \int_{U_2} g_1(x_m(s-), u)N_1(\mathrm{d}s, \mathrm{d}u)$$

for $0 \leq t < \tau_m$. In other words, $\{x_m(t)\}$ satisfies (2.2) for $0 \leq t < \tau_m$. For $n \geq m \geq 1$ let $\{y(t) : t \geq 0\}$ be the unique solution to

$$y(t) = \xi + \int_0^t \sigma(y(s-) \wedge n)\mathrm{d}B(\tau_m + s) + \int_0^t b_n(y(s-) \wedge n)\mathrm{d}s$$
$$+ \int_0^t \int_{U_0} [g_0(y(s-) \wedge n, u) \wedge n]\tilde{N}_0(\tau_m + \mathrm{d}s, \mathrm{d}u)$$
$$+ \int_0^t \int_{U_2} [g_1(y(s-) \wedge n, u) \wedge n]N_1(\tau_m + \mathrm{d}s, \mathrm{d}u),$$

where

$$\xi = x_m(\tau_m-) + \int_{\{\tau_m\}} \int_{U_0} [g_0(x_m(\tau_m-), u) \wedge n]N_0(\mathrm{d}s, \mathrm{d}u)$$
$$+ \int_{\{\tau_m\}} \int_{U_2} [g_1(x_m(\tau_m-), u) \wedge n]N_1(\mathrm{d}s, \mathrm{d}u).$$

We define $x'_n(t) = x_m(t)$ if $0 \leq t < \tau_m$ and $x'_n(t) = y(t - \tau_m)$if $t \geq \tau_m$. It is not hard to see that $\{x'_n(t)\}$ is a solution to (2.7) with the m replaced by n. By the strong uniqueness we get $x'_n(t) = x_n(t)$ for all $t \geq 0$. In particular, we infer $x_n(t) = x_m(t) < m$ for $0 \leq t < \tau_m$. Consequently, the sequence $\{\tau_m\}$ is non-decreasing. On the other hand, as in the proof of Proposition 2.3 we have

$$\mathbf{E}[1 + x_m(t \wedge \tau_m)] \leq \mathbf{E}[1 + x(0)] \exp\{Kt\}, \quad t \geq 0.$$

Then $\tau_m \to \infty$ as $m \to \infty$ first for a deterministic initial value and then for an arbitrary one. Let $\{x(t)\}$ be the process such that $x(t) = x_m(t)$ for all $0 \leq t < \tau_m$ and $m \geq 1$. It is easily seen that $\{x(t)\}$ is a strong solution of (2.2). The uniqueness of solution follows by a similar localization argument. □

Now we give a simple criterion for the existence and uniqueness of the strong solution to (2.1). We shall assume the following localized conditions:

(2.c) $x \mapsto b(x)$ is locally Lipschitz on $\mathbb{R}_+$ and for each integer $m \geq 1$ there is a constant $K_m > 0$ and a non-negative function $u \mapsto h_m(u)$ so that

$$|g_1(x, u) - g_1(y, u)| \leq K_m|x - y|h_m(u), \quad 0 \leq x, y \leq m, u \in U_2,$$

and

$$\int_{U_2} h_m(u)\mu_1(\mathrm{d}u) < \infty;$$

(2.d) $x \mapsto \sigma(x)$ is locally Lipschitz on $\mathbb{R}_+$ and for each integer $m \geq 1$ there is a constant $K_m > 0$ and a non-negative function $u \mapsto f_m(u)$ so that

$$|g_0(x, u) - g_0(y, u)| \leq K_m|x - y|f_m(u), \quad 0 \leq x, y \leq m, u \in U_0$$

and

$$\int_{U_0} [f_m(u) \wedge f_m(u)^2]\mu_0(\mathrm{d}u) < \infty.$$

Theorem 2.5. *Suppose that conditions* (2.a, c, d) *are satisfied. Then there is a unique non-negative strong solution to* (2.1).

Proof. We first note that (2.b) follows from (2.d). By Propositions 2.2 and 2.4 we only need to show there is a unique strong solution to (2.7). Let $V_0 = \{u \in U_0 : f_m(u) \leq 1/K_m\}$ and $V_2 = \{u \in U_2 : h_m(u) \leq m\}$. Then (2.c,d) imply $\mu_0(U_0 \setminus V_0) + \mu_1(U_2 \setminus V_2) < \infty$. We consider the equation

$$\begin{aligned} x(t) = x(0) &+ \int_0^t \sigma(x(s-) \wedge m)\mathrm{d}B(s) + \int_0^t b(x(s-) \wedge m)\mathrm{d}s \\ &+ \int_0^t \int_{U_0} [g_0(x(s-) \wedge m, u) \wedge m]\tilde{N}_0(\mathrm{d}s, \mathrm{d}u) \\ &- \int_0^t \mathrm{d}s \int_{U_0 \setminus V_0} [g_0(x(s-) \wedge m, u) - g_0(x(s-) \wedge m, u) \wedge m]\mu_0(\mathrm{d}u) \\ &+ \int_0^t \int_{U_2} [g_1(x(s-) \wedge m, u) \wedge m]N_1(\mathrm{d}s, \mathrm{d}u), \end{aligned} \tag{2.8}$$

which can be rewritten as a jump-type equation with compensated Poisson integrals over V_0 and V_2, non-compensated Poisson integrals over $U_0 \setminus V_0$ and $U_2 \setminus V_2$, and drift coefficient

$$\begin{aligned} x \mapsto d_m(x) := b(x \wedge m) &+ \int_{V_2} [g_1(x \wedge m, u) \wedge m]\mu_1(\mathrm{d}u) \\ &- \int_{U_0 \setminus V_0} g_0(x \wedge m, u)\mu_0(\mathrm{d}u). \end{aligned}$$

Clearly, we can choose a common sequence of constants $\{K_m\}$ for conditions (2.c) and (2.d). Then there is a new constant $C_m > 0$ so that

$$\begin{aligned} |d_m(x) - d_m(y)| &\leq |b(x \wedge m) - b(y \wedge m)| + K_m|x - y| \int_{V_2} h_m(u)\mu_1(\mathrm{d}u) \\ &\quad + K_m|x - y| \int_{U_0 \setminus V_0} f_m(u)\mu_0(\mathrm{d}u) \\ &\leq |b(x \wedge m) - b(y \wedge m)| + K_m|x - y| \int_{V_2} h_m(u)\mu_1(\mathrm{d}u) \\ &\quad + K_m|x - y| \int_{U_0 \setminus V_0} f_m(u) \wedge [K_m f_m(u)^2]\mu_0(\mathrm{d}u) \\ &\leq C_m|x - y| \end{aligned}$$

and

$$\begin{aligned} &\int_{V_0} l_0(x \wedge m, y \wedge m, u)^2\mu_0(\mathrm{d}u) + \int_{V_2} l_1(x \wedge m, y \wedge m, u)^2\mu_1(\mathrm{d}u) \\ &\quad \leq K_m^2|x - y|^2\Big(\int_{V_0} f_m(u)^2\mu_0(\mathrm{d}u) + \int_{V_2} h_m(u)^2\mu_2(\mathrm{d}u)\Big) \end{aligned}$$

$$\leq K_m^2|x-y|^2\left(\int_{V_0}[K_m^{-1}f_m(u)]\wedge f_m(u)^2\mu_0(\mathrm{d}u)+\int_{V_2}mh_m(u)\mu_2(\mathrm{d}u)\right)$$
$$\leq C_m|x-y|^2,$$

where $l_i(x,y,u)=g_i(x,u)-g_i(y,u)$. By Proposition 2.1 and the classical result it is simple to see that (2.8) has a unique strong solution; see, e.g., [12, pp. 244–245]. On the other hand, from (2.d) we have $g_0(x,u)\leq K_m x f_m(u)$ for $0\leq x\leq m$. Then $f_m(u)\leq 1/K_m$ implies $g_0(x\wedge m,u)\leq m$, and hence

$$b_m(x\wedge m)=b(x\wedge m)-\int_{U_0\setminus V_0}[g_0(x\wedge m,u)-g_0(x\wedge m,u)\wedge m]\mu_0(\mathrm{d}u).$$

Therefore (2.8) is equivalent to (2.7), so the later also has a unique non-negative strong solution. □

3. Pathwise uniqueness

In this section, we prove some results on the pathwise uniqueness of solutions to (2.1) under non-Lipschitz conditions. Suppose that $(\sigma,b,g_0,g_1,\mu_0,\mu_1)$ are given as in the last section. Given a function f defined on a subset of $\mathbb{R}$, we note

$$\Delta_z f(x)=f(x+z)-f(x)\quad\text{and}\quad D_z f(x)=\Delta_z f(x)-f'(x)z \tag{3.1}$$

if the right hand sides are meaningful. In the sequel, we assume the drift coefficient of (2.1) is written as $b(x)=b_1(x)-b_2(x)$, where $x\mapsto b_1(x)$ is continuous and $x\mapsto b_2(x)$ is continuous and non-decreasing. We consider the following condition:

(3.a) For each integer $m\geq 1$ there is a non-decreasing and concave function $z\mapsto r_m(z)$ on $\mathbb{R}_+$ such that $\int_{0+}r_m(z)^{-1}\mathrm{d}z=\infty$ and

$$|b_1(x)-b_1(y)|+\int_{U_2}|g_1(x,u)-g_1(y,u)|\mu_1(\mathrm{d}u)\leq r_m(|x-y|)$$

for all $0\leq x,y\leq m$.

Theorem 3.1. *Suppose that conditions* (2.a, b) *and* (3.a) *hold. Then the pathwise uniqueness of solutions holds for* (2.2) *if for each integer $m\geq 1$ there is a sequence of non-negative and twice continuously differentiable functions $\{\phi_k\}$ with the following properties:*

(i) $\phi_k(z)\to|z|$ *non-decreasingly as* $k\to\infty$*;*
(ii) $0\leq\phi_k'(z)\leq 1$ *for* $z\geq 0$ *and* $-1\leq\phi_k'(z)\leq 0$ *for* $z\leq 0$*;*
(iii) $\phi_k''(z)\geq 0$ *for* $z\in\mathbb{R}$ *and as* $k\to\infty$*,*

$$\phi_k''(x-y)[\sigma(x)-\sigma(y)]^2\to 0$$

uniformly in $0\leq x,y\leq m$*;*
(iv) *as* $k\to\infty$*,*

$$\int_{U_0}D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(\mathrm{d}u)\to 0$$

uniformly in $0\leq x,y\leq m$*, where* $l_0(x,y,u)=g_0(x,u)-g_0(y,u)$*.*

Proof. Suppose that $\{x_1(t)\}$ and $\{x_2(t)\}$ are two solutions of (2.2) with deterministic initial values. Proposition 2.3 implies that $t \mapsto \mathbf{E}[x_1(t)] + \mathbf{E}[x_2(t)]$ is locally bounded. Let $\zeta(t) = x_1(t) - x_2(t)$ for $t \geq 0$. From (2.2) we have

$$\begin{aligned}\zeta(t) = \zeta(0) &+ \int_0^t [\sigma(x_1(s-)) - \sigma(x_2(s-))]\mathrm{d}B(s) \\ &+ \int_0^t \int_{U_0} [g_0(x_1(s-), u) - g_0(x_2(s-), u)]\tilde{N}_0(\mathrm{d}s, \mathrm{d}u) \\ &+ \int_0^t [b(x_1(s-)) - b(x_2(s-))]\mathrm{d}s \\ &+ \int_0^t \int_{U_2} [g_1(x_1(s-), u) - g_1(x_2(s-), u)]N_1(\mathrm{d}s, \mathrm{d}u). \end{aligned} \tag{3.2}$$

Let $\tau_m = \inf\{t \geq 0 : x_1(t) \geq m \text{ or } x_2(t) \geq m\}$ for $m \geq 1$. By (3.2) and Itô's formula,

$$\begin{aligned}\phi_k(\zeta(t \wedge \tau_m)) &= \phi_k(\zeta(0)) + \int_0^{t\wedge\tau_m} \phi_k'(\zeta(s-))[b(x_1(s-)) - b(x_2(s-))]\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^{t\wedge\tau_m} \phi_k''(\zeta(s-))[\sigma(x_1(s-)) - \sigma(x_2(s-))]^2\mathrm{d}s \\ &\quad + \int_0^{t\wedge\tau_m}\int_{U_2} \Delta_{l_1(s-,u)}\phi_k(\zeta(s-))N_1(\mathrm{d}s, \mathrm{d}u) \\ &\quad + \int_0^{t\wedge\tau_m}\int_{U_0} D_{l_0(s-,u)}\phi_k(\zeta(s-))N_0(\mathrm{d}s, \mathrm{d}u) + \text{mart.} \\ &= \phi_k(\zeta(0)) + \int_0^{t\wedge\tau_m} \phi_k'(\zeta(s-))[b(x_1(s-)) - b(x_2(s-))]\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^{t\wedge\tau_m} \phi_k''(\zeta(s-))[\sigma(x_1(s-)) - \sigma(x_2(s-))]^2\mathrm{d}s \\ &\quad + \int_0^{t\wedge\tau_m}\mathrm{d}s\int_{U_2} \Delta_{l_1(s-,u)}\phi_k(\zeta(s-))\mu_1(\mathrm{d}u) \\ &\quad + \int_0^{t\wedge\tau_m}\mathrm{d}s\int_{U_0} D_{l_0(s-,u)}\phi_k(\zeta(s-))\mu_0(\mathrm{d}u) + \text{mart.}, \end{aligned} \tag{3.3}$$

where $l_i(s, u) = g_i(x_1(s), u) - g_i(x_2(s), u)$. Since $b(x) = b_1(x) - b_2(x)$ and $x \mapsto b_2(x)$ is non-decreasing, by property (ii) we have

$$\begin{aligned}\phi_k'(\zeta(s-))\big[b(x_1(s-)) - b(x_2(s-))\big] &\leq \phi_k'(\zeta(s-))\big[b_1(x_1(s-)) - b_1(x_2(s-))\big] \\ &\leq |b_1(x_1(s-)) - b_1(x_2(s-))|.\end{aligned}$$

Observe also that

$$\int_{U_2} \Delta_{l_1(s-,u)}\phi_k(\zeta(s-))\mu_1(\mathrm{d}u) \leq \int_{U_2} |g_1(x_1(s-), u) - g_1(x_2(s-), u)|\mu_1(\mathrm{d}u).$$

By condition (3.a), for any $s \leq \tau_m$ the summation of the right hand sides of the above two inequalities is no larger than $r_m(|\zeta(s-)|)$. By properties (iii) and (iv) we have

$$\phi_k''(\zeta(s-))[\sigma(x_1(s-)) - \sigma(x_2(s-))]^2 \to 0$$

and

$$\int_{U_0} D_{l_0(s-,u)}\phi_k(\zeta(s-))\mu_0(\mathrm{d}u) \to 0$$

uniformly on the event $\{s \leq \tau_m\}$. Then we can take the expectation in (3.3) and let $k \to \infty$ to get

$$\mathbf{E}[|\zeta(t \wedge \tau_m)|] \leq \mathbf{E}[|\zeta(0)|] + \mathbf{E}\left[\int_0^{t\wedge\tau_m} r_m(|\zeta(s-)|)\mathrm{d}s\right].$$

Since $\zeta(s-) < m$ for $0 < s \leq \tau_m$, we infer that $t \mapsto \mathbf{E}[|\zeta(t \wedge \tau_m)|]$ is locally bounded. Note also that $\zeta(s-) \neq \zeta(s)$ for at most countably many $s \geq 0$. Then the concaveness of $x \mapsto r_m(x)$ implies

$$\begin{aligned}\mathbf{E}[|\zeta(t \wedge \tau_m)|] &\leq \mathbf{E}[|\zeta(0)|] + \int_0^t \mathbf{E}[r_m(|\zeta(s \wedge \tau_m)|)]\mathrm{d}s \\ &\leq \mathbf{E}[|\zeta(0)|] + \int_0^t r_m(\mathbf{E}[|\zeta(s \wedge \tau_m)|])\mathrm{d}s.\end{aligned}$$

If $x_1(0) = x_2(0)$, we can use a standard argument to show $\mathbf{E}[|\zeta(t \wedge \tau_m)|] = 0$ for all $t \geq 0$; see e.g. [12, p. 184]. Since $\tau_m \to \infty$ as $m \to \infty$ by Proposition 2.3, the right continuity of $t \mapsto \zeta(t)$ implies $\mathbf{P}\{\zeta(t) = 0 \text{ for all } t \geq 0\} = 1$. □

We remark that the proof of Theorem 3.1 given above uses essentially the monotonicity of $x \mapsto b_2(x)$. A similar condition has been used in [17] for continuous-type equations. The reader may refer [6] for a thorough treatments of continuous-type stochastic differential equations with singular coefficients. For the statements of the next two theorems we introduce the following conditions:

(3.b) For every fixed $u \in U_0$ the function $x \mapsto g_0(x, u)$ is non-decreasing, and for each integer $m \geq 1$ there is a non-negative and non-decreasing function $z \mapsto \rho_m(z)$ on $\mathbb{R}_+$ so that $\int_{0+} \rho_m(z)^{-2}\mathrm{d}z = \infty$ and

$$|\sigma(x) - \sigma(y)|^2 + \int_{U_0} [|l_0(x, y, u)| \wedge l_0(x, y, u)^2]\mu_0(\mathrm{d}u) \leq \rho_m(|x - y|)^2$$

for all $0 \leq x, y \leq m$, where $l_0(x, y, u) = g_0(x, u) - g_0(y, u)$;

(3.c) For every fixed $u \in U_0$ the function $x \mapsto g_0(x, u)$ is non-decreasing, and for each integer $m \geq 1$ there is a non-negative and non-decreasing function $z \mapsto \rho_m(z)$ on $\mathbb{R}_+$ so that $\int_{0+} \rho_m(z)^{-2}\mathrm{d}z = \infty$,

$$|\sigma(x) - \sigma(y)| \leq \rho_m(|x - y|) \quad \text{and} \quad |g_0(x, u) - g_0(y, u)| \leq \rho_m(|x - y|) f_m(u)$$

for all $0 \leq x, y \leq m$ and $u \in U_0$, where $u \mapsto f_m(u)$ is a non-negative function on U_0 satisfying

$$\int_{U_0} [f_m(u) \wedge f_m(u)^2]\mu_0(\mathrm{d}u) < \infty.$$

Theorem 3.2. *Suppose that conditions* (2.a) *and* (3.a, b) *are satisfied. Then the pathwise uniqueness of solutions holds for* (2.1).

Proof. By Proposition 2.2 it suffices to prove the pathwise uniqueness for (2.2). For each integer $m \geq 1$ we shall construct a sequence of functions $\{\phi_k\}$ that satisfies the properties required in Theorem 3.1. Let $a_0 = 1$ and choose $a_k \to 0$ decreasingly so that $\int_{a_k}^{a_{k-1}} \rho_m(z)^{-2}\mathrm{d}z = k$ for $k \geq 1$. Let $x \mapsto \psi_k(x)$ be a non-negative continuous function on $\mathbb{R}$ which has support in (a_k, a_{k-1}) and satisfies $\int_{a_k}^{a_{k-1}} \psi_k(x)\mathrm{d}x = 1$ and $0 \leq \psi_k(x) \leq 2k^{-1}\rho_m(x)^{-2}$ for $a_k < x < a_{k-1}$. For each $k \geq 1$ we define the non-negative and twice continuously differentiable function

$$\phi_k(z) = \int_0^{|z|} \mathrm{d}y \int_0^y \psi_k(x)\mathrm{d}x, \quad z \in \mathbb{R}. \tag{3.4}$$

Clearly, the sequence $\{\phi_k\}$ satisfies properties (i) and (ii) in Theorem 3.1. By condition (3.b) we have

$$\phi_k''(x-y)[\sigma(x)-\sigma(y)]^2 \leq \psi_k(|x-y|)\rho_m(|x-y|)^2 \leq 2/k$$

for $0 \leq x, y \leq m$. Thus $\{\phi_k\}$ also satisfies property (iii). By Taylor's expansion,

$$D_h\phi_k(\zeta) = h^2 \int_0^1 \phi_k''(\zeta + th)(1-t)\mathrm{d}t = h^2 \int_0^1 \psi_k(|\zeta + th|)(1-t)\mathrm{d}t. \tag{3.5}$$

Consequently, the monotonicity of $z \mapsto \rho_m(z)$ implies

$$D_h\phi_k(\zeta) \leq 2k^{-1}h^2 \int_0^1 \rho_m(|\zeta + th|)^{-2}(1-t)\mathrm{d}t \leq k^{-1}h^2\rho_m(|\zeta|)^{-2} \tag{3.6}$$

if $\zeta h \geq 0$. Observe also that

$$D_h\phi_k(\zeta) = \Delta_h\phi_k(\zeta) - \phi_k'(\zeta)h \leq \Delta_h\phi_k(\zeta) \leq |h| \tag{3.7}$$

if $\zeta h \geq 0$. Since $x \mapsto g_0(x,u)$ is non-decreasing, for $0 \leq x, y \leq m$ and $n \geq 1$ we can use (3.6), (3.7) and (3.b) to get

$$\begin{aligned}
\int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(\mathrm{d}u) &\leq \frac{1}{k\rho_m(|x-y|)^2}\int_{U_0} l_0(x,y,u)^2 1_{\{|l_0(x,y,u)|\leq n\}}\mu_0(\mathrm{d}u) \\
&\quad + \int_{U_0} |l_0(x,y,u)| 1_{\{|l_0(x,y,u)|>n\}}\mu_0(\mathrm{d}u) \\
&\leq \frac{n}{k\rho_m(|x-y|)^2}\int_{U_0} |l_0(x,y,u)| \wedge l_0(x,y,u)^2\mu_0(\mathrm{d}u) \\
&\quad + \int_{U_0} g_0(m,u)1_{\{g_0(m,u)>n\}}\mu_0(\mathrm{d}u) \\
&\leq \frac{n}{k} + \int_{U_0} g_0(m,u)1_{\{g_0(m,u)>n\}}\mu_0(\mathrm{d}u).
\end{aligned}$$

Since (3.b) implies (2.b), we see that $\{\phi_k\}$ satisfies property (iv) in Theorem 3.1. That proves the pathwise uniqueness for (2.2). $\square$

Theorem 3.3. *Suppose that conditions* (2.a) *and* (3.a, c) *are satisfied. Then the pathwise uniqueness for* (2.1) *holds.*

Proof. The first part of this proof is identical with that of Theorem 3.2. Under condition (3.c) we can use (3.6) and (3.7) to see

$$D_{l_0(x,y,u)}\phi_k(x-y) \leq k^{-1}l_0(x,y,u)^2\rho_m(|x-y|)^{-2} \leq k^{-1}f_m(u)^2$$

and

$$D_{l_0(x,y,u)}\phi_k(x-y) \le |l_0(x,y,u)| \le \rho_m(|x-y|)f_m(u) \le \rho_m(m)f_m(u)$$

for all $0 \le x, y \le m$ and $u \in U_0$. By (3.c) for any $n \ge 1$ we have

$$\begin{aligned}\int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(\mathrm{d}u) &\le \frac{1}{k}\int_{U_0} f_m(u)^2 1_{\{f_m(u)\le n\}}\mu_0(\mathrm{d}u)\\ &\quad + \rho_m(m)\int_{U_0} f_m(u)1_{\{f_m(u)>n\}}\mu_0(\mathrm{d}u) \le \frac{n}{k}\int_{U_0}[f_m(u)\wedge f_m(u)^2]\mu_0(\mathrm{d}u)\\ &\quad + \rho_m(m)\int_{U_0} f_m(u)1_{\{f_m(u)>n\}}\mu_0(\mathrm{d}u).\end{aligned}$$

By letting $k \to \infty$ and $n \to \infty$ we see that $\{\phi_k\}$ satisfies property (iv) in Theorem 3.1. That proves the pathwise uniqueness first for (2.2) and then for (2.1) by an application of Proposition 2.2. □

4. Weak solutions

In this section, we prove a result on the existence of weak solutions of (2.1). The result will be used in the study of strong solutions in the next section. As for continuous-type equations, this is closely related with the corresponding martingale problems. Let us define the Lévy-type operator A from $C^2(\mathbb{R}_+)$ to $C(\mathbb{R}_+)$ by

$$\begin{aligned}Af(x) = \frac{1}{2}\sigma(x)^2 f''(x) + b(x)f'(x) + \int_{U_0} D_{g_0(x,u)}f(x)\mu_0(\mathrm{d}u)\\ + \int_{U_1} \Delta_{g_1(x,u)}f(x)\mu_1(\mathrm{d}u),\end{aligned}\tag{4.1}$$

where $(\sigma, b, g_0, g_1, \mu_0, \mu_1)$ are given as in Section 2. To simplify the statements we introduce the following conditions:

(4.a) There is a constant $K \ge 0$ such that

$$\begin{aligned}\sup_{x\ge 0}\big[|b(x)| + \sigma(x)^2\big] + \sup_{x\ge 0}\int_{U_0} g_0(x,u)^2\mu_0(\mathrm{d}u)\\ + \sup_{x\ge 0}\int_{U_1}\big[|g_1(x,u)| \vee g_1(x,u)^2\big]\mu_1(\mathrm{d}u) \le K;\end{aligned}$$

(4.b) $x \mapsto g_0(x,\cdot)$ is non-decreasing and continuous in $L^2(\mu_0)$ and there is a non-decreasing sequence $\{V_n\}$ of Borel subsets of U_0 so that $\cup_{n=1}^{\infty} V_n = U_0$ and

$$\mu_0(V_n) < \infty, \qquad \int_{V_n} g_0(x,u)\mu_0(\mathrm{d}u) < \infty$$

for every $n \ge 1$ and $x \ge 0$;

(4.c) $x \mapsto g_1(x,\cdot)$ is continuous in $L^1(\mu_1)$.

Proposition 4.1. *If condition* (4.a) *holds, for any solution* $\{x(t)\}$ *of* (2.1) *we have*

$$\mathbf{E}\Big[\sup_{0\le s\le t} x(s)^2\Big] \le 6\mathbf{E}[x(0)^2] + 24Kt + 6K^2t^2, \quad t \ge 0.\tag{4.2}$$

Proof. We first write (2.1) into

$$\begin{aligned} x(t) = x(0) &+ \int_0^t \sigma(x(s-))\mathrm{d}B(s) + \int_0^t \int_{U_0} g_0(x(s-),u)\tilde{N}_0(\mathrm{d}s,\mathrm{d}u) \\ &+ \int_0^t b(x(s-))\mathrm{d}s + \int_0^t \int_{U_1} g_1(x(s-),u)\tilde{N}_1(\mathrm{d}s,\mathrm{d}u) \\ &+ \int_0^t \mathrm{d}s \int_{U_1} g_1(x(s-),u)\mu_1(\mathrm{d}u). \end{aligned} \tag{4.3}$$

By applying Doob's inequality to the martingale terms in (4.3) we have

$$\begin{aligned} \mathbf{E}\Big[\sup_{0\le s\le t} x(s)^2\Big] &\le 6\mathbf{E}[x(0)^2] + 24\mathbf{E}\Big[\int_0^t \sigma(x(s-))^2\mathrm{d}s\Big] + 6\mathbf{E}\Big[\Big(\int_0^t |b(x(s-))|\mathrm{d}s\Big)^2\Big] \\ &\quad + 24\mathbf{E}\Big[\int_0^t \mathrm{d}s \int_{U_0} g_0(x(s-),u)^2\mu_0(\mathrm{d}u)\Big] \\ &\quad + 24\mathbf{E}\Big[\int_0^t \mathrm{d}s \int_{U_1} g_1(x(s-),u)^2\mu_1(\mathrm{d}u)\Big] \\ &\quad + 6\mathbf{E}\Big[\Big(\int_0^t \mathrm{d}s \int_{U_1} |g_1(x(s-),u)|\mu_1(\mathrm{d}u)\Big)^2\Big] \\ &\le 6\mathbf{E}[x(0)^2] + 24Kt + 6t\mathbf{E}\Big[\int_0^t b(x(s-))^2\mathrm{d}s\Big] \\ &\quad + 6t\mathbf{E}\Big[\int_0^t \Big(\int_{U_1} |g_1(x(s-),u)|\mu_1(\mathrm{d}u)\Big)^2\mathrm{d}s\Big] \\ &\le 6\mathbf{E}[x(0)^2] + 24Kt + 6K^2t^2. \end{aligned}$$

That proves (4.2). □

Proposition 4.2. *Suppose that condition* (4.a) *holds. Then a non-negative càdlàg process* $\{x(t)\}$ *is a weak solution of* (2.1) *if and only if for every* $f \in C^2(\mathbb{R}_+)$,

$$f(x(t)) - f(x(0)) - \int_0^t Af(x(s))\mathrm{d}s, \quad t \ge 0 \tag{4.4}$$

is a martingale.

Proof. Without loss of generality, we assume $x(0)$ is deterministic. If $\{x(t)\}$ is a solution of (2.1), by Itô's formula it is easy to see that (4.4) is a bounded martingale. Conversely, suppose that (4.4) is a martingale for every $f \in C^2(\mathbb{R}_+)$. By a standard stopping time argument, we have

$$x(t) = x(0) + \int_0^t b(x(s-))\mathrm{d}s + \int_0^t \mathrm{d}s \int_{U_1} g_1(x(s-),u)\mu_1(\mathrm{d}u) + M(t) \tag{4.5}$$

for a square-integrable martingale $\{M(t)\}$. Let $N(\mathrm{d}s,\mathrm{d}z)$ be the optional random measure on $[0,\infty) \times \mathbb{R}$ defined by

$$N(\mathrm{d}s,\mathrm{d}z) = \sum_{s>0} 1_{\{\Delta x(s)\ne 0\}}\delta_{(s,\Delta x(s))}(\mathrm{d}s,\mathrm{d}z),$$

where $\Delta x(s) = x(s) - x(s-)$. Let $\hat{N}(\mathrm{d}s, \mathrm{d}z)$ be the predictable compensator of $N(\mathrm{d}s, \mathrm{d}z)$ and let $\tilde{N}(\mathrm{d}s, \mathrm{d}z)$ denote the compensated random measure. By (4.5) and [9, p. 376] we have

$$x(t) = x(0) + \int_0^t b(x(s-))\mathrm{d}s + \int_0^t \mathrm{d}s \int_{U_1} g_1(x(s-), u)\mu_1(\mathrm{d}u) + M_c(t) + M_d(t), \tag{4.6}$$

where $\{M_c(t)\}$ is a continuous martingale and

$$M_d(t) = \int_0^t \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z)$$

is a purely discontinuous martingale. Let $\{C(t)\}$ denote the quadratic variation process of $\{M_c(t)\}$. By (4.6) and Itô's formula we have

$$\begin{aligned} f(x(t)) = {} & f(x(0)) + \int_0^t f'(x(s-))b(x(s-))\mathrm{d}s + \frac{1}{2}\int_0^t f''(x(s-))\mathrm{d}C(s) \\ & + \int_0^t f'(x(s-))\mathrm{d}s \int_{U_1} g_1(x(s-), u)\mu_1(\mathrm{d}u) \\ & + \int_0^t \int_{\mathbb{R}} D_z f(x(s-))\hat{N}(\mathrm{d}s, \mathrm{d}z) + \text{mart.} \end{aligned} \tag{4.7}$$

In view of (4.4) and (4.7), the uniqueness of canonical decompositions of semi-martingales implies $\mathrm{d}C(s) = \sigma(x(s))^2\mathrm{d}s$ and

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} F(s, z)\hat{N}(\mathrm{d}s, \mathrm{d}z) = {} & \int_0^t \mathrm{d}s \int_{U_0} F(s, g_0(x(s-), u))\mu_0(\mathrm{d}u) \\ & + \int_0^t \mathrm{d}s \int_{U_1} F(s, g_1(x(s-), u))\mu_1(\mathrm{d}u) \end{aligned}$$

for any non-negative Borel function F on $\mathbb{R}_+ \times \mathbb{R}$. Then we obtain the Eq. (2.1) on an extension of the probability space by applying martingale representation theorems to (4.6); see, e.g., [12, p. 84 and p. 93)]. $\square$

For simplicity we assume the initial variable $x(0)$ is deterministic in the sequel of this section. To prove the existence of a weak solution of (2.1) let $\{\varepsilon_n\}$ be a sequence of strictly positive numbers decreasing to zero. If condition (4.b) holds, for every $n \geq 0$,

$$x \mapsto \int_{V_n} g_0(x \wedge n, u)\mu_0(\mathrm{d}u)$$

is a bounded continuous non-decreasing function on $\mathbb{R}_+$. By the result on continuous-type stochastic equations, there is a weak solution to

$$\begin{aligned} x(t) = {} & x(0) + \int_0^t \sigma(x(s-))\mathrm{d}B(s) + \int_0^t b(x(s-))\mathrm{d}s \\ & - \int_0^t \mathrm{d}s \int_{V_n} g_0(x(s-) \wedge n, u)\mu_0(\mathrm{d}u); \end{aligned} \tag{4.8}$$

see, e.g., [12, p. 169]. By Theorem 3.2 the pathwise uniqueness holds for (4.8), so the equation has a unique strong solution. Let $\{W_n\}$ be a non-decreasing sequence of Borel subsets of U_1 so that $\cup_{n=1}^{\infty} W_n = U_1$ and $\mu_1(W_n) < \infty$ for every $n \geq 1$. Following the proof of Proposition 2.2 one can see for every integer $n \geq 1$ there is a weak solution $\{x_n(t) : t \geq 0\}$ to

$$\begin{aligned} x(t) = x(0) &+ \int_0^t \sigma(x(s-))\mathrm{d}B(s) + \int_0^t \int_{V_n} g_0(x(s-) \wedge n, u)\tilde{N}_0(\mathrm{d}s, \mathrm{d}u) \\ &+ \int_0^t b(x(s-))\mathrm{d}s + \int_0^t \int_{W_n} g_1(x(s-), u)N_1(\mathrm{d}s, \mathrm{d}u). \end{aligned} \tag{4.9}$$

Lemma 4.3. *Under conditions (4.a,b), the sequence $\{x_n(t) : t \geq 0\}$ is tight in the Skorokhod space $D([0, \infty), \mathbb{R}_+)$.*

Proof. Since $x(0)$ is deterministic, by Proposition 4.1 it is easy to see that

$$t \mapsto C_t := \sup_{n \geq 1} \mathbf{E}\Big[\sup_{0 \leq s \leq t} x_n(s)^2\Big]$$

is locally bounded. Then for every fixed $t \geq 0$ the sequence of random variables $x_n(t)$ is tight. Moreover, in view of (4.3), if $\{\tau_n\}$ is a sequence of stopping times bounded above by $T \geq 0$, we have

$$\begin{aligned} \mathbf{E}[|x_n(\tau_n + t) - x_n(\tau_n)|^2] &\leq 5\mathbf{E}\bigg[\bigg[\int_0^t \sigma(x_n(\tau_n + s))^2 r\bigg]\mathrm{d}s + t\int_0^t b(x_n(\tau_n + s))^2 \mathrm{d}s\bigg] \\ &\quad + 5\mathbf{E}\bigg[\int_0^t \mathrm{d}s \int_{U_0} g_0(x_n(\tau_n + s) \wedge n, u)^2 \mu_0(\mathrm{d}u)\bigg] \\ &\quad + 5\mathbf{E}\bigg[\int_0^t \mathrm{d}s \int_{U_1} g_1(x_n(\tau_n + s), u)^2 \mu_1(\mathrm{d}u)\bigg] \\ &\quad + 5t\mathbf{E}\bigg[\int_0^t \bigg(\int_{U_1} g_1(x_n(\tau_n + s), u)\mu_1(\mathrm{d}u)\bigg)^2 \mathrm{d}s\bigg] \\ &\leq 5Kt(1 + Kt), \end{aligned}$$

where the last inequality follows by (4.a). Consequently, as $t \to 0$,

$$\sup_{n \geq 1} \mathbf{E}[|x_n(\tau_n + t) - x_n(\tau_n)|^2] \to 0.$$

Then $\{x_n(t) : t \geq 0\}$ is tight in $D([0, \infty), \mathbb{R}_+)$ by the criterion of [1]; see also [11, pp. 137–138]. □

Theorem 4.4. *Under the conditions* (4.a, b, c)*, there exists a non-negative weak solution to (2.1).*

Proof. Let $\{x_n(t) : t \geq 0\}$ be the unique non-negative strong solution of (4.9). By Proposition 4.2, for every $f \in C^2(\mathbb{R}_+)$,

$$f(x_n(t)) - f(x_n(0)) - \int_0^t A_n f(x_n(s))\mathrm{d}s, \quad t \geq 0 \tag{4.10}$$

is a bounded martingale, where

$$\begin{aligned} A_n f(x) = \frac{1}{2}\sigma(x)^2 f''(x) &+ \int_{V_n} D_{g_0(x \wedge n, u)} f(x)\mu_0(\mathrm{d}u) \\ &+ b(x)f'(x) + \int_{W_n} \Delta_{g_1(x,u)} f(x)\mu_1(\mathrm{d}u). \end{aligned}$$

By Lemma 4.3 there is a subsequence $\{x_{n_k}(t) : t \geq 0\}$ of $\{x_n(t) : t \geq 0\}$ that converges to some process $\{x(t) : t \geq 0\}$ in distribution on $D([0, \infty), \mathbb{R}_+)$. By Skorokhod representation we may assume $\{x_{n_k}(t) : t \geq 0\}$ converges to $\{x(t) : t \geq 0\}$ almost surely in $D([0, \infty), \mathbb{R}_+)$. Let $D(x) := \{t > 0 : \mathbf{P}\{x(t-) = x(t)\} = 1\}$. Then the set $[0, \infty) \setminus D(x)$ is at most countable; see, e.g., [11, p. 131]. It follows that $\lim_{k\to\infty} x_{n_k}(t) = x(t)$ almost surely for every $t \in D(x)$; see, e.g., [11, p. 118]. Using conditions (4.a,b,c) it is elementary to show that (4.4) is a bounded martingale. Then the theorem follows by another application of Proposition 4.2. $\square$

As for continuous-type equations, one can derive some weak uniqueness results for (2.1) from the results on pathwise uniqueness given in the last section. The reader may refer to [19] for a systematic treatment of existence and uniqueness of solutions to martingale problems associated with Lévy-type generators. However, it seems the results of [19] do not yield immediately the results on non-negative solutions of (2.1).

5. Strong solutions

In this section, we give some criterion on the existence and uniqueness of the strong solution of Eq. (2.1). We also prove some properties of the solution, continuous dependence on the initial value and comparison properties. Recall that $b(x) = b_1(x) - b_2(x)$, where $x \mapsto b_1(x)$ is continuous and $x \mapsto b_2(x)$ is continuous and non-decreasing. Let us consider some further localizations of (3.a,b,c) as follows:

(5.a) For each integer $m \geq 1$ there is a non-decreasing and concave function $z \mapsto r_m(z)$ on $\mathbb{R}_+$ such that $\int_{0+} r_m(z)^{-1}\mathrm{d}z = \infty$ and

$$|b_1(x) - b_1(y)| + \int_{U_2} |g_1(x, u) \wedge m - g_1(y, u) \wedge m|\mu_1(\mathrm{d}u) \leq r_m(|x - y|)$$

for all $0 \leq x, y \leq m$;

(5.b) There is a non-decreasing sequence $\{V_n\}$ of Borel subsets of U_0 so that $\cup_{n=1}^{\infty} V_n = U_0$ and

$$\mu_0(V_n) < \infty, \qquad \int_{V_n} |g_0(x, u)|\mu_0(\mathrm{d}u) < \infty$$

for every $n \geq 1$ and $x \geq 0$;

(5.c) For each $u \in U_0$ the function $x \mapsto g_0(x, u)$ is non-decreasing, and for each integer $m \geq 1$ there is a non-negative and non-decreasing function $z \mapsto \rho_m(z)$ on $\mathbb{R}_+$ so that $\int_{0+} \rho_m(z)^{-2}\mathrm{d}z = \infty$ and

$$|\sigma(x) - \sigma(y)|^2 + \int_{U_0} |g_0(x, u) \wedge m - g_0(y, u) \wedge m|^2\mu_0(\mathrm{d}u) \leq \rho_m(|x - y|)^2$$

for all $0 \leq x, y \leq m$;

(5.d) For each $u \in U_0$ the function $x \mapsto g_0(x, u)$ is non-decreasing, and for each integer $m \geq 1$ there is a non-negative and non-decreasing function $z \mapsto \rho_m(z)$ on $\mathbb{R}_+$ so that $\int_{0+} \rho_m(z)^{-2}\mathrm{d}z = \infty$, $|\sigma(x) - \sigma(y)| \leq \rho_m(|x - y|)$ and

$$|g_0(x, u) \wedge m - g_0(y, u) \wedge m| \leq \rho_m(|x - y|) f_m(u)$$

for all $0 \leq x, y \leq m$ and $u \in U_0$, where $u \mapsto f_m(u)$ is a non-negative function on U_0 satisfying

$$\int_{U_0} [f_m(u) \wedge f_m(u)^2]\mu_0(\mathrm{d}u) < \infty.$$

Theorem 5.1. *Suppose that conditions* (2.a, b) *and* (5.a, b, c) *are satisfied. Then there exists a unique non-negative strong solution to* (2.1).

Proof. By Propositions 2.2 and 2.4 we only need to show that (2.7) has a unique strong solution. Since $x \mapsto g_0(x, u)$ is non-decreasing, so is

$$x \mapsto \beta_m(x) := \int_{U_0} [g_0(x, u) - g_0(x, u) \wedge m]\mu_0(\mathrm{d}u)$$

for every $m \geq 1$. One can use (2.b) and (5.c) to show that $x \mapsto \beta_m(x)$ is continuous. By conditions (5.a,c) it is easy to see that $x \mapsto g_0(x, \cdot) \wedge m$ is continuous in $L^2(\mu_0)$ and $x \mapsto g_1(x, \cdot) \wedge m$ is continuous in $L^1(\mu_1)$. Thus the ingredients of (2.7) satisfy conditions (2.a), (3.a,b) and (4.a,b,c). By Theorem 4.4 we conclude that (2.7) has a non-negative weak solution. The pathwise uniqueness of the solution follows from Theorem 3.2. Then (2.7) has a unique strong solution. □

Corollary 5.2. *There exists a unique non-negative strong solution to* (1.7).

Proof. This follows by Theorem 5.1 applied with $U_0 = (0, \infty)^2$, $U_1 = (0, \infty)$ and $U_2 = (0, 1]$. □

Theorem 5.3. *Suppose that conditions* (2.a, b) *and* (5.a, d) *are satisfied. Then there exists a unique non-negative strong solution to* (2.1).

Proof. Clearly, we may assume $\{f_m\}$ is a non-decreasing sequence in (5.d). Let $V_n := \{u \in U_0 : f_n(u) > 1/n\}$ and $V_0 = \cup_{n=1}^{\infty} V_n$. It easy to see that $g_0(x, u) = 0$ for every $x \geq 0$ and $u \in U_0 \setminus V_0$. Then we can replace U_0 by V_0 in (2.1) so that condition (5.b) is satisfied. Based on the pathwise uniqueness stated in Theorem 3.3, the remaining arguments are similar to those in the proof of Theorem 5.1. □

It is easy to see that if (2.1) does have a unique strong solution, the solution is a strong Markov process with generator A defined by (4.1). In particular, the solution of (1.7) is a CBI-process with generator given by (1.5). In the special case where $z\nu_1(\mathrm{d}z)$ is a finite measure on $(0, \infty)$, the existence and uniqueness of the non-negative strong solution to (1.7) have been established in [8]; see also [5].

We next prove some properties of the strong solutions of (2.1). In the following two theorems, we can replace conditions (5.b,c) by (5.d).

Theorem 5.4. *Suppose that* (2.a, b) *and* (5.a, b, c) *hold with* $U_2 = U_1$ *and the modulus functions* $r_m(z) \equiv r(z)$ *and* $\rho_m(z) \equiv \rho(z)$ *are independent of* $m \geq 1$. *For each integer* $n \geq 0$ *let* $\{x_n(t)\}$ *be a solution of* (2.1) *and assume* $\sup_{n\geq 0} \mathbf{E}[x_n(0)] < \infty$. *If* $\lim_{n\to\infty} \mathbf{E}[|x_n(0) - x_0(0)|] = 0$, *then*

$$\lim_{n\to\infty} \sup_{0\leq s\leq t} \mathbf{E}\big[|x_n(s) - x_0(s)|\big] = 0, \quad t \geq 0. \tag{5.1}$$

Proof. Since (5.a,c) hold with the modulus functions independent of $m \geq 1$, by Fatou's lemma it is easy to show that (3.a,b) also hold with universal modulus functions. In particular, from (5.c) we get

$$|\sigma(x) - \sigma(y)|^2 + \int_{U_0} |g_0(x, u) - g_0(y, u)|^2 \mu_0(\mathrm{d}u) \leq \rho(|x - y|)^2. \tag{5.2}$$

Let $\zeta_n(t) = x_n(t) - x_0(t)$ for $t \geq 0$. We then fix $n \geq 1$ and let $\tau_m = \inf\{t \geq 0 : x_0(t) \geq m$ or $x_n(t) \geq m\}$. By Proposition 2.3 it is easy to see that $t \mapsto \mathbf{E}[|\zeta_n(t)|]$ is uniformly bounded on each bounded interval. By the calculations in the proof of Theorem 3.1 we obtain

$$\begin{aligned}\mathbf{E}[|\zeta_n(t \wedge \tau_m)|] &\leq \mathbf{E}[|\zeta_n(0)|] + \mathbf{E}\left[\int_0^{t\wedge\tau_m} r(|\zeta_n(s-)|)\mathrm{d}s\right] \\ &\leq \mathbf{E}[|\zeta_n(0)|] + \mathbf{E}\left[\int_0^{t} r(|\zeta_n(s-)|)\mathrm{d}s\right] \\ &\leq \mathbf{E}[|\zeta_n(0)|] + \int_0^{t} r(\mathbf{E}[|\zeta_n(s)|])\mathrm{d}s.\end{aligned}$$

Since $\tau_m \to \infty$ as $m \to \infty$, an application of Fatou's lemma gives

$$\mathbf{E}[|\zeta_n(t)|] \leq \mathbf{E}[|\zeta_n(0)|] + \int_0^{t} r(\mathbf{E}[|\zeta_n(s)|])\mathrm{d}s.$$

For $n \geq 1$ and $t \geq 0$ let $R_n(t) = \sup_{m\geq n}\sup_{0\leq s\leq t}\mathbf{E}[|\zeta_m(s)|]$. By Proposition 2.3 the sequence $\{R_n(t)\}$ is uniformly bounded on each bounded interval. Then we use the monotonicity of $z \mapsto r(z)$ to get

$$R_n(t) \leq R_n(0) + \int_0^t r(R_n(s))\mathrm{d}s, \quad t \geq 0.$$

By letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} R_n(t) \leq \int_0^t r\Big(\lim_{n\to\infty} R_n(s)\Big)\mathrm{d}s, \quad t \geq 0.$$

Thus $\lim_{n\to\infty} R_n(t) = 0$ for every $t \geq 0$, as desired. □

Theorem 5.5. *Suppose that* (2.a, b) *and* (5.a, b, c) *hold with $U_2 = U_1$ and the modulus function $r_m(x) \equiv r(x)$ is independent of $m \geq 1$. In addition, we assume $x \mapsto x + g_1(x, u)$ is non-decreasing for every $u \in U_1$. If $\{x_1(t)\}$ and $\{x_2(t)\}$ are non-negative solutions of* (2.1) *satisfying* $\mathbf{P}\{x_1(0) \leq x_2(0)\} = 1$, *then we have* $\mathbf{P}\{x_1(t) \leq x_2(t)$ *for all* $t \geq 0\} = 1$.

Proof. The following arguments are similar to those in the proofs of Theorems 3.1 and 3.2. Let $\zeta(t) = x_1(t) - x_2(t)$ and $\tau_m = \inf\{t \geq 0 : x_1(t) \geq m$ or $x_2(t) \geq m\}$. Instead of (3.4), for each $k \geq 1$ we now define

$$\phi_k(z) = \int_0^z \mathrm{d}y \int_0^y \psi_k(x)\mathrm{d}x, \quad z \in \mathbb{R}. \tag{5.3}$$

Then $\phi_k(z) \to z^+ := 0 \vee z$ non-decreasingly as $k \to \infty$. Recall that $l_i(s,u) = g_i(x_1(s),u) - g_i(x_2(s),u)$ for $i = 0, 1$. Since $\phi_k(z) = 0$ for $z \leq 0$ and $x \mapsto x + g_1(x,u)$ is non-decreasing, for $\zeta(s-) \leq 0$ we have $\zeta(s-) + l_1(s-,u) \leq 0$ and hence $\Delta_{l_1(s-,u)}\phi_k(\zeta(s-)) = 0$. By (3.2) and Itô's formula,

$$\begin{aligned}\phi_k(\zeta(t\wedge\tau_m)) = &\int_0^{t\wedge\tau_m} \phi_k'(\zeta(s-))[b(x_1(s-)) - b(x_2(s-))]1_{\{\zeta(s-)>0\}}\mathrm{d}s \\ &+ \frac{1}{2}\int_0^{t\wedge\tau_m} \phi_k''(\zeta(s-))[\sigma(x_1(s-)) - \sigma(x_2(s-))]^2\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{t\wedge\tau_m} \int_{U_1} \Delta_{l_1(s-,u)} \phi_k(\zeta(s-)) 1_{\{\zeta(s-)>0\}} N_1(\mathrm{d}s, \mathrm{d}u) \\
&+ \int_0^{t\wedge\tau_m} \int_{U_0} D_{l_0(s-,u)} \phi_k(\zeta(s-)) N_0(\mathrm{d}s, \mathrm{d}u) + \text{mart.} \\
= &\int_0^{t\wedge\tau_m} \phi_k'(\zeta(s-))[b(x_1(s-)) - b(x_2(s-))] 1_{\{\zeta(s-)>0\}} \mathrm{d}s \\
&+ \frac{1}{2} \int_0^{t\wedge\tau_m} \phi_k''(\zeta(s-))[\sigma(x_1(s-)) - \sigma(x_2(s-))]^2 \mathrm{d}s \\
&+ \int_0^{t\wedge\tau_m} \mathrm{d}s \int_{U_1} \Delta_{l_1(s-,u)} \phi_k(\zeta(s-)) 1_{\{\zeta(s-)>0\}} \mu_1(\mathrm{d}u) \\
&+ \int_0^{t\wedge\tau_m} \mathrm{d}s \int_{U_0} D_{l_0(s-,u)} \phi_k(\zeta(s-)) \mu_0(\mathrm{d}u) + \text{mart.}
\end{aligned}
$$

(If $x \mapsto x + g_1(x, u)$ were allowed to decrease, we could not insert the indicator $1_{\{\zeta(s-)>0\}}$ into the integral with respect to $N_1(\mathrm{d}s, \mathrm{d}u)$.) From the above equation and the estimates in the proof of Theorem 3.2 we obtain

$$
\mathbf{E}[\zeta(t \wedge \tau_m)^+] \le \int_0^t \mathbf{E}[r(\zeta(s \wedge \tau_m)^+)] \mathrm{d}s \le \int_0^t r(\mathbf{E}[\zeta(s \wedge \tau_m)^+]) \mathrm{d}s.
$$

Then $\mathbf{E}[\zeta(t \wedge \tau_m)^+] = 0$ for all $t \ge 0$. Since $\tau_m \to \infty$ as $m \to \infty$, that proves the desired comparison result. $\square$

Example. The monotonicity of $x \mapsto g_1(x, u)$ is necessary to assure the comparison property in Theorem 5.5. To see this, let $g_1(x, u) = (1 - 2x)^+$ and let $N_1(\mathrm{d}s, \mathrm{d}u)$ be a Poisson random measure on $(0, \infty) \times (0, 1]$ with intensity $\mathrm{d}s\mathrm{d}u$. We consider the stochastic equation

$$
x(t) = x(0) + \int_0^t \int_0^1 g(x(s-), u) N_1(\mathrm{d}s, \mathrm{d}u), \quad t \ge 0. \tag{5.4}
$$

Let $\tau = \inf\{t \ge 0 : N_1((0, t] \times (0, 1]) = 1\}$. It is easy to see that

$$
x_1(t) = 1_{\{t \ge \tau\}} \quad \text{and} \quad x_2(t) = 1/2, \quad t \ge 0,
$$

defines two solutions of (5.4).

We next discuss the existence and uniqueness of the solution to a special type of stochastic equations of non-negative processes with non-negative jumps. In the sequel of this section, we suppose that

- $x \mapsto \sigma(x)$ and $x \mapsto b(x)$ are continuous functions on $\mathbb{R}_+$ satisfying $\sigma(0) = 0$ and $b(0) \ge 0$;
- $(x, u) \mapsto h_0(x, z)$ is a non-negative Borel function on $\mathbb{R}_+ \times (0, \infty)$ so that $h_0(0, z) = 0$;
- $(x, z) \mapsto h_1(x, z)$ is a non-negative Borel function on $\mathbb{R}_+ \times (0, \infty)$.

Suppose that $\mu_0(\mathrm{d}z)$ and $\mu_1(\mathrm{d}z)$ are σ-finite measures on $(0, \infty)$ satisfying

$$
\int_0^\infty h_0(x, z)(z \wedge z^2) \mu_0(\mathrm{d}z) + \int_0^\infty h_1(x, z)(1 \wedge z) \mu_1(\mathrm{d}z) < \infty, \quad x \ge 0.
$$

In view of the characterization of Feller semigroups given by Courrége [7], it is natural to consider the generator L defined by

$$Lf(x) = \frac{1}{2}\sigma(x)^2 f''(x) + \int_0^\infty [f(x+z) - f(x) - zf'(x)]h_0(x,z)\mu_0(\mathrm{d}z) + b(x)f'(x) + \int_0^\infty [f(x+z) - f(x)]h_1(x,z)\mu_1(\mathrm{d}z). \tag{5.5}$$

Let $\{B(t)\}$ be a standard $(\mathscr{F}_t)$-Brownian motion, and let $\{p_0(t)\}$ and $\{p_1(t)\}$ be $(\mathscr{F}_t)$-Poisson point processes on $(0,\infty)^2$ with characteristic measures $\mu_0(\mathrm{d}z)\mathrm{d}u$ and $\mu_1(\mathrm{d}z)\mathrm{d}u$, respectively. We assume $\{B(t)\}$, $\{p_0(t)\}$ and $\{p_1(t)\}$ are independent of each other. Let $N_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ and $N_1(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ be the Poisson random measures associated with $\{p_0(t)\}$ and $\{p_1(t)\}$, respectively. Let us consider the stochastic equation

$$x(t) = x(0) + \int_0^t \sigma(x(s-))\mathrm{d}B(s) + \int_0^t\int_0^\infty\int_0^{h_0(x(s-),z)} z\tilde{N}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t b(x(s-))\mathrm{d}s + \int_0^t\int_0^\infty\int_0^{h_1(x(s-),z)} zN_1(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \tag{5.6}$$

As before we assume $b(x) = b_1(x) - b_2(x)$, where $x \mapsto b_1(x)$ is continuous and $x \mapsto b_2(x)$ is continuous and non-decreasing. For the ingredients of the above equation we may rephrase (2.a,b) and (5.a,c) into the following conditions:

(5.e) There is a constant $K \geq 0$ so that

$$b(x) + \int_0^\infty h_1(x,z)z\mu_1(\mathrm{d}z) \leq K(1+x), \quad x \geq 0;$$

(5.f) For fixed $z > 0$ the function $x \mapsto h_0(x,z)$ is non-decreasing and there is a non-decreasing function $x \mapsto L(x)$ on $\mathbb{R}_+$ such that

$$\sigma(x)^2 + \int_0^\infty h_0(x,z)(z\wedge z^2)\mu_0(\mathrm{d}z) \leq L(x), \quad x \geq 0;$$

(5.g) For each integer $m \geq 1$ there is a non-decreasing concave function $z \mapsto r_m(z)$ on $\mathbb{R}_+$ such that $\int_{0+} r_m(z)^{-1}\mathrm{d}z = \infty$ and

$$|b_1(x) - b_1(y)| + \int_0^\infty |h_1(x,z) - h_1(y,z)|(z\wedge m)\mu_1(\mathrm{d}z) \leq r_m(|x-y|)$$

for all $0 \leq x, y \leq m$;

(5.h) For each integer $m \geq 1$ there is a non-negative and non-decreasing function $z \mapsto \rho(z)$ on $\mathbb{R}_+$ such that $\int_{0+} \rho_m(z)^{-2}\mathrm{d}z = \infty$ and

$$|\sigma(x) - \sigma(y)|^2 + \int_0^\infty |h_0(x,z) - h_0(y,z)|(z\wedge m)^2\mu_0(\mathrm{d}z) \leq \rho_m(|x-y|)^2$$

for all $0 \leq x, y \leq m$.

Theorem 5.6. *If conditions* (5.e, f, g, h) *are satisfied, there exists a unique non-negative strong solution to* (5.6) *and the solution is a strong Markov process with generator given by* (5.5).

Proof. By applying Theorem 5.1 with $U_i = (0,\infty)^2$ for $i = 0, 1, 2$ it is simple to see that (5.6) has a unique non-negative strong solution $\{x(t)\}$. In particular, to verify condition (5.b) we can simply let $V_n = [1/n, \infty) \times (0, \infty)$ for $n \geq 1$. The uniqueness implies the strong Markov property of $\{x(t)\}$. By Itô's formula it is easy to show that $\{x(t)\}$ has generator given by (5.5). □

6. Applications

In this section, we illustrate some applications of the results established in the foregoing sections to stochastic differential equations driven by one-sided Lévy processes. Let $\mu_0(\mathrm{d}z)$ and $\mu_1(\mathrm{d}z)$ be σ-finite measures on $(0, \infty)$ and let $\nu_0(\mathrm{d}z)$ and $\nu_1(\mathrm{d}z)$ be σ-finite measures on $(0, 1]$. We assume that

$$\int_0^\infty (z \wedge z^2)\mu_0(\mathrm{d}z) + \int_0^\infty (1 \wedge z)\mu_1(\mathrm{d}z) + \int_0^1 z^2\nu_0(\mathrm{d}z) + \int_0^1 z\nu_1(\mathrm{d}z) < \infty.$$

Let $\{B(t)\}$ be a standard $(\mathscr{F}_t)$-Brownian motion. Let $\{z_0(t)\}$ and $\{z_1(t)\}$ be $(\mathscr{F}_t)$-Lévy processes with exponents

$$u \mapsto \int_0^\infty \big(\mathrm{e}^{\mathrm{i}uz} - 1 - \mathrm{i}uz\big)\mu_0(\mathrm{d}z) \quad \text{and} \quad u \mapsto \int_0^\infty \big(\mathrm{e}^{\mathrm{i}uz} - 1\big)\mu_1(\mathrm{d}z),$$

respectively. Let $\{y_0(t)\}$ and $\{y_1(t)\}$ be $(\mathscr{F}_t)$-Lévy processes with exponents

$$u \mapsto \int_0^1 \big(\mathrm{e}^{\mathrm{i}uz} - 1 - \mathrm{i}uz\big)\nu_0(\mathrm{d}z) \quad \text{and} \quad u \mapsto \int_0^1 \big(\mathrm{e}^{\mathrm{i}uz} - 1\big)\nu_1(\mathrm{d}z),$$

respectively. Therefore $\{z_0(t)\}$ and $\{y_0(t)\}$ are centered and $\{z_1(t)\}$ and $\{y_1(t)\}$ are non-decreasing. Suppose that those processes are independent of each other. In addition, suppose that

- $x \mapsto \sigma(x)$ and $x \mapsto b(x)$ are continuous functions on $\mathbb{R}_+$ satisfying $\sigma(0) = 0$ and $b(0) \geq 0$;
- $x \mapsto \phi_0(x)$ and $x \mapsto \phi_1(x)$ are continuous non-negative functions on $\mathbb{R}_+$ so that $\phi_0(0) = 0$.

We consider the condition:

(6.a) There is a constant $K \geq 0$ so that $b(x) + \phi_1(x) \leq K(1 + x)$ for all $x \geq 0$.

Theorem 6.1. *Suppose that* (6.a) *holds. If, in addition, the functions σ, b, ϕ_0 and ϕ_1 are all locally Lipschitz, then there is a unique non-negative strong solution to*

$$\begin{aligned} \mathrm{d}x(t) = {} & \sigma(x(t))\mathrm{d}B(t) + \phi_0(x(t-))\mathrm{d}z_0(t) + b(x(t))\mathrm{d}t \\ & + \phi_1(x(t-))\mathrm{d}z_1(t) - x(t-)\mathrm{d}y_0(t) - x(t-)\mathrm{d}y_1(t). \end{aligned} \tag{6.1}$$

Proof. To save the notation, we extend μ_0 and μ_1 to σ-finite measures on $[-1, 0) \cup (0, \infty)$ by setting $\mu_0([-x, 0)) = \nu_0((0, x])$ and $\mu_1([-x, 0)) = \nu_1((0, x])$ for $0 < x \leq 1$. By the general result on Lévy–Itô decompositions, we have

$$z_0(t) = \int_0^t \int_0^\infty z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z), \qquad z_1(t) = \int_0^t \int_0^\infty zN_1(\mathrm{d}s, \mathrm{d}z)$$

and

$$y_0(t) = -\int_0^t \int_{[-1,0)} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z), \qquad y_1(t) = -\int_0^t \int_{[-1,0)} zN_1(\mathrm{d}s, \mathrm{d}z),$$

where $N_0(\mathrm{d}s, \mathrm{d}z)$ and $N_1(\mathrm{d}s, \mathrm{d}z)$ are independent Poisson random measures with intensities $\mathrm{d}s\mu_0(\mathrm{d}z)$ and $\mathrm{d}s\mu_1(\mathrm{d}z)$, respectively. By Theorem 2.5 applied with $U_0 = U_1 = [-1, 0)\cup(0, \infty)$ and $U_2 = [-1, 0) \cup (0, 1]$, there is a unique strong solution to

$$\begin{aligned} x(t) = x(0) &+ \int_0^t \sigma(x(s-))\mathrm{d}B(s) + \int_0^t \int_0^\infty \phi_0(x(s-))z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_0^t b(x(s-))\mathrm{d}s + \int_0^t \int_0^\infty \phi_1(x(s-))zN_1(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_0^t \int_{[-1,0)} x(s-)z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{[-1,0)} x(s-)zN_1(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

which is just another form of (6.1). □

In particular, if $\sigma \geq 0$ and b are real constants, by Theorem 6.1 there is a unique non-negative strong solution $\{S(t)\}$ to the stochastic differential equation

$$\mathrm{d}S(t) = \sigma S(t)\mathrm{d}B(t) + bS(t)\mathrm{d}t + S(t-)\mathrm{d}z_0(t) - S(t-)\mathrm{d}y_0(t).$$

The process $\{S(t)\}$ is a generalization of the geometric Brownian motion and has been used widely in mathematical finance; see, e.g., [15, p. 144].

Theorem 6.2. *Suppose that* (6.a) *holds. If, in addition, $x \mapsto \phi_0(x)$ is non-decreasing on $\mathbb{R}_+$ and for each $m \geq 1$ there is a constant $K_m \geq 0$ so that*

$$|\sigma(x) - \sigma(y)|^2 + |\phi_0(x) - \phi_0(y)|^2 + |b(x) - b(y)| + |\phi_1(x) - \phi_1(y)| \leq K_m|x - y|$$

for all $0 \leq x, y \leq m$, then there is a unique non-negative strong solution to

$$\begin{aligned} \mathrm{d}x(t) = &\ \sigma(x(t))\mathrm{d}B(t) + \phi_0(x(t-))\mathrm{d}z_0(t) + b(x(t))\mathrm{d}t \\ &+ \phi_1(x(t-))\mathrm{d}z_1(t) - x(t-)\mathrm{d}y_1(t). \end{aligned}$$

Proof. This follows from Theorem 5.3 by a simple modification of the proof of Theorem 6.1. □

Corollary 6.3. *There exists a unique non-negative strong solution to* (1.8).

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References

[1] D. Aldous, Stopping times and tightness, Ann. Probab. 6 (1978) 335–340.
[2] R.F. Bass, Stochastic differential equations driven by symmetric stable processes, in: Lecture Notes Math., vol. 1801, Springer-Verlag, Berlin, 2003, pp. 302–313.
[3] R.F. Bass, Stochastic differential equations with jumps, Probab. Surv. 1 (2004) 1–19.
[4] R.F. Bass, K. Burdzy, Z.-Q. Chen, Stochastic differential equations driven by stable processes for which pathwise uniqueness fails, Stochastic Process. Appl. 111 (2004) 1–15.

[5] J. Bertoin, J.-F. Le Gall, Stochastic flows associated to coalescent processes III: Limit theorems, Illinois J. Math. 50 (2006) 147–181.
[6] A.S. Cherny, H.-J. Engelbert, Singular Stochastic Differential Equations, in: Lecture Notes Math., vol. 1858, Springer-Verlag, Berlin, 2005.
[7] Ph. Courrége, Sur la forme intégro-différentielle des opérateurs de C_K^∞ dans C satisfaisant au principe du maximum. in: Sém. Théorie du Potentiel exposé 2, 1966/65. 38 pp.
[8] D.A. Dawson, Z.H. Li, Skew convolution semigroups and affine Markov processes, Ann. Probab. 34 (2006) 1103–1142.
[9] C. Dellacherie, P.A. Meyer, Probabilites and Potential, North-Holland, Amsterdam, 1982 (Chapters V–VIII).
[10] D. Duffie, D. Filipović, W. Schachermayer, Affine processes and applications in finance, Ann. Appl. Probab. 13 (2003) 984–1053.
[11] S.N. Ethier, T.G. Kurtz, Markov Processes: Characterization and Convergence, Wiley, New York, 1986.
[12] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, second edn., North-Holland/Kodasha, Amsterdam/Tokyo, 1989.
[13] K. Kawazu, S. Watanabe, Branching processes with immigration and related limit theorems, Theory Probab. Appl. 16 (1971) 36–54.
[14] A. Lambert, Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct, Electron. J. Probab. 12 (2007) 420–446.
[15] D. Lamberton, B. Lapeyre, Introduction to Stochastic Calculus Applied to Finance, Chapman and Hall, London, 1996.
[16] P.E. Protter, Stochastic Integration and Differential Equations, Springer-Verlag, Berlin, 2003.
[17] B.L. Rozovsky, A note on strong solutions of stochastic differential equations with random coefficients, in: Stochastic Differential Systems, Proc. IFIP-WG 7/1 Working Conf., Vilnius, 1978, in: Lecture Notes Control Inform. Sci., vol. 25, Springer-Verlag, Berlin, 1980, pp. 287–296.
[18] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge, 1999.
[19] D.W. Stroock, Diffusion processes associated with Lévy generators, Z. Wahrsch. verw. Geb. 32 (1975) 209–244.
[20] T. Yamada, S. Watanabe, On the uniqueness of solutions of stochastic differential equations, J. Math. Kyoto Univ. 11 (1971) 155–167.